Some Results on Generalized Fuzzy $\mathcal{F}$-Metric Spaces

Dipti Barman and Tarapada Bag

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 3
Pages: 117-144

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2025.2069608.2299



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 3 (2026), 117-144
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2025.2069608.2299

Some Results on Generalized Fuzzy $\mathscr{F}$-Metric Spaces

Dipti Barman[1] and Tarapada Bag[2*]

Abstract. In this paper, the notion of generalized fuzzy $\mathscr{F}$-metric space is introduced and some examples are given. Some topological properties as well as some basic properties are studied in such space. A comparative study among generalized fuzzy $\mathscr{F}$-metric space, fuzzy $\mathscr{F}$-metric space, fuzzy metric space and b-fuzzy metric space have been done.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [10] in 1965 to address the unclear or imprecise situations in everyday life. A fuzzy set A in a nonempty set X is defined as a function with domain X and values in the interval $[0, 1]$.

Fuzzification of Mathematics is going on in different directions such as fuzzy set theory, fuzzy number theory, fuzzy algebra, fuzzy topology, fuzzy functional analysis, etc. having their applications in real life problems in the field of Artificial intelligence, Robotics, Pattern Recognition, Decision Making etc.

It is well known that metric structure plays pivotal role in functional analysis. So in order to take care about the suitable fuzzifications of this structure. In 1975, Kramosil & Michalek [15] first introduced the concept of fuzzy metric space. After that George & Veeramani [2] modified the definition of Kramosil & Michalek in order to develop topological results as well as fixed point results in such space. Grabiec [11] was a praioneer author to develop fixed point results in fuzzy metric spaces.

2020 *Mathematics Subject Classification.* 46S40, 54E35.
Key words and phrases. t-norm, Fuzzy metric, b-fuzzy metric, Fuzzy $\mathscr{F}$-metric, Generalized fuzzy $\mathscr{F}$-metric.
Received: 2025-08-22, Accepted: 2025-11-05.
* Corresponding author.

Following the definition of fuzzy metric space introduced by George & Veeramani, several authors introduced the concept of generalized fuzzy metric spaces viz. b-fuzzy metric space [20], fuzzy b-metric space [19], extended fuzzy b-metric space [14], fuzzy 2-metric space [21], fuzzy cone metric space [22], G-fuzzy metric space [8], etc. in different approaches in order to develop more results of functional analysis in fuzzy setting. Various fixed-point theorems for different types of contraction mappings in generalized fuzzy metric spaces have been established (for references please see [6, 9, 13, 17]).

In 2018, Jleli et al. [12] gave an idea of a new generalization of metric spaces. It is more general than other generalized metric spaces. Following this definition [1] in 2024, Das et al. introduced the idea of fuzzy $\mathscr{F}$-metric space which is a completely different type of generalization of fuzzy metric space. It is proved that George & Veeramani-type fuzzy metric space is a fuzzy $\mathscr{F}$-metric space but fuzzy b-metric space is not a fuzzy $\mathscr{F}$-metric space, that's why we think that it is not a proper generalization of fuzzy metric spaces.

Our main motivation of this paper is to introduce a new concept of generalized fuzzy $\mathscr{F}$-metric space which is a generalization of various fuzzy metric spaces. We show that every fuzzy $\mathscr{F}$-metric space, b-fuzzy metric space and George & Veeramani type fuzzy metric space are generalized fuzzy $\mathscr{F}$-metric space. In this context it is worth mentaining the work of Abdelhamid Moussaoui et al. [3–5] for fixed point results in fuzzy metric spaces.

Some topological properties of generalized fuzzy $\mathscr{F}$-metric spaces have been studied. We give an example to show that an open ball is not necessarily an open set and a closed ball is not necessarily a closed set. We also define convergent sequences and Cauchy sequences, along with some related results.

This manuscript is divided into the following sections. In Section 2, we provide some basic notations and results from existing literature that help develop the main results. In Section 3, definition, examples of generalized fuzzy $\mathscr{F}$-metric spaces are given and their relations with other generalized fuzzy metric spaces are studied. This section includes the open ball, Hausdorff topology, some topological results, convergent sequence and Cauchy sequence, along with related results.

2. Preliminaries

In this section, we introduce some basic concepts that will help in the main results of this paper.

Definition 2.1 ([7])**.** A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $\star$ is associative and commutative;
(ii) $\alpha \star 1 = \alpha, \forall\ \alpha \in [0,1]$;
(iii) $\alpha \star \gamma \leq \beta \star \delta$ whenever $\alpha \leq \beta$ and $\gamma \leq \delta, \forall\ \alpha, \beta, \gamma, \delta \in [0,1]$.

If $\star$ is continuous, then it is called continuous t-norm.
The following are examples of some t norms.

(i) Standard intersection : $\alpha \star \beta = \min\{\alpha, \beta\}$;
(ii) Algebraic product : $\alpha \star \beta = \alpha\beta$;
(iii) Bounded difference : $\alpha \star \beta = \max\{0, \alpha + \beta - 1\}$.

We recall the definition of b-metric space as follows.

Definition 2.2 ([18])**.** Let X be a nonempty set and $k \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if for all $x, y, z \in X$, the following conditions hold:

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \leq k[d(x,y) + d(y,z)]$.

Then the triple (X, d, k) is called a b-metric space with constant coefficient k.

Some examples of b-metric space which is important for Section 3 as follows.

Example 2.3 ([16])**.** Let $X = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ and D be a function defined by

$$D(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \left\{\frac{1}{2n} : n = 1,2,3,\ldots\right\} \\ 4, & \text{otherwise.} \end{cases}$$

Then D is a b-metric space on X with constant coefficient $k = \frac{8}{3}$.

We recall the definition of fuzzy metric space introduced by Kramosil & Michalek [15].

Definition 2.4 ([15])**.** The 3-tuple $(X, M, \star)$ is said to be a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x,y,0) = 0$;
(ii) $(M(x,y,t) = 1 \ \text{ for all } \ t > 0)$ if and only if $x = y$;
(iii) $M(x,y,t) = M(y,x,t)$;
(iv) $M(x,y,t) \star M(y,z,s) \leq M(x,z,t+s)$;

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

for all $x, y, z \in X$ and for all $t, s > 0$.

Next we recall the definition fuzzy metric space introduced by George & Veeraani [2].

Definition 2.5 ([2])**.** The 3-tuple $(X, M, \star)$ is said to be a fuzzy metric space if the fuzzy set M on $X^2 \times (0, \infty)$ satisfies the following conditions:

(i) $M(x, y, t) > 0$;
(ii) $(M(x, y, t) = 1, \forall\ t > 0)$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, s + t) \geq M(x, y, t) \star M(y, z, s)$;
(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous;

for all $x, y, z \in X$ and $t, s > 0$.

Lemma 2.6 ([11])**.** *George & Veeramani type fuzzy metric $M(x, y, .)$ over a nonempty set X is non-decreasing with respect to $t > 0$ for all $x, y \in X$.*

Remark 2.7 ([2])**.** In George & Veramani's type fuzzy metric $(X, M, \star)$, the following results hold:

(i) If $M(x, y, t) > 1 - r$ for all $x, y \in X, t > 0, 0 < r < 1$, we can find $t_0, 0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$.
(ii) For any $r_1 > r_2$ in $(0, 1)$, we can find $r_3 \in (0, 1)$ such that $r_1 \star r_3 \geq r_2$ and for any $r_4 \in (0, 1)$ we can find $r_5 \in (0, 1)$ such that $r_5 \star r_5 \geq r_4$.

Now we recall the definition of b-fuzzy metric space introduced by Sedghi and Shobe [20].

Definition 2.8 ([20])**.** A 3-tuple $(X, M, \star)$ is called a b-fuzzy metric space if X is an arbitrary (non-empty) set, $\star$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and for each $s, t > 0$ and $b \geq 1$ be a given real number,

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M\left(x, y, \frac{s}{b}\right) \star M\left(y, z, \frac{t}{b}\right) \leq M(x, z, s + t)$;
(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.9 ([20])**.** Let $M(x, y, t) = \exp\left(\frac{|x-y|^p}{t}\right)$, where $p > 1$ is a real number, $x, y \in X$ and $t > 0$. Then M is a b-fuzzy metric with coefficient $p = 2^{p-1}$ and t-norm = 'product'.

Recently, Das et al. proposed [1] a new generalization of the George and Veeramani type fuzzy metric space concept. Using a certain class

of functions, the authors defined the notion of a fuzzy $\mathscr{F}$-metric space [1]. First of all, we recall the set of functions which use in the definition of fuzzy $\mathscr{F}$-metric.

Let $\mathscr{F}$ denote the class of functions $f : [0,1] \to [0,1]$ that satisfy the following conditions:

($\mathscr{F}1$) f is strictly increasing in $[0,1)$;
($\mathscr{F}2$) For every sequence $\{t_n\}$ in $[0,1]$, we have

$$\lim_{n\to\infty} t_n = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} f(t_n) = 1.$$

Utilizing these functions, the authors proposed an extension of the George and Veeramani type fuzzy metric space and introduced the idea of fuzzy $\mathscr{F}$-metric spaces as follows:

Definition 2.10 ([1])**.** Let X be a nonempty set and $M : X \times X \times (0,\infty) \to [0,1]$ be a mapping and $\star$ be a continuous t-norm. If there exists $(f,\alpha) \in \mathscr{F} \times (0,1]$ that M satisfies the following conditions:

(i) $M(x,y,t) > 0$;
(ii) $(M(x,y,t) = 1,\ \forall\ t > 0)$ if and only if $x = y$;
(iii) $M(x,y,t) = M(y,x,t)$;
(iv) For every $(x,y) \in X \times X$, for every $N \in \mathbb{N}, N \geq 2$ and every $\{u_i\}_i^N \subseteq X$ with $u_1 = x$ and $u_N = y$, we have $M(x,y,t) < 1$ implies

$$(f(M(x,y,t)))^{\alpha} \geq f(M(u_1,u_2,t_1) \star M(u_2,u_3,t_2) \star \cdots \star M(u_{N-1},u_N,t_{N-1}))$$

where $t = t_1 + t_2 + \cdots + t_{N-1}$. Then M is said to be a fuzzy $\mathscr{F}$- metric on X and the 5-tuple $(X,M,f,\alpha,\star)$ is said to be a fuzzy $\mathscr{F}$-metric space.

Example 2.11 ([1])**.** Let $X = \mathbb{R}$ and define a function $M : X \times X \times (0,\infty) \to [0,1]$ by $M(x,y,t) = \left(\frac{t}{t+1}\right)^{|x-y|^2}$ for all $x,y \in X$ and for all $t \in (0,\infty)$. Then M is fuzzy $\mathscr{F}$-metric on X with respect to $f(x) = x^2, x \in [0,1], \alpha = \frac{1}{2}$ and $\star =$ 'product'.

Example 2.12 ([1])**.** Consider two metrics d_1 and d_2 on $\mathbb{R}$, defined by

$$d_1(x,y) = \max\{|x_i - y_i| : i = 1,2\}$$
$$d_2(x,y) = \left\{\sum_{i=1}^{2} |x_i - y_i|, i = 1,2\right\}.$$

for all $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. Then, clearly $d_1(x,y) \leq d_2(x,y)$ for all $x,y \in \mathbb{R}^2$. Now, define a function $M : \mathbb{R}^2 \times \mathbb{R}^2 \times (0,\infty) \to [0,1]$

by

$$M(x,y,t) = \begin{cases} \frac{1}{2}, & 0 < t \leq d_1(x,y) \\ \frac{1}{4}, & d_1(x,y) < t \leq d_2(x,y) \\ \frac{t}{t+d_2(x,y)}, & d_2(x,y) < t < \infty \end{cases}$$

for all $x, y \in \mathbb{R}^2$ and for all $t > 0$. Then M is fuzzy $\mathscr{F}$-metric on $\mathbb{R}^2$ with respect to $f(x) = x, x \in [0,1], \alpha = \frac{1}{2}$ and $\star = \min$.

Proposition 2.13 ([1]). *Every fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is a Hausdorff space.*

Proposition 2.14 ([1]). *The limit of $\mathscr{F}$-convergent sequence in fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is unique.*

3. A Generalized Fuzzy $\mathscr{F}$-Metric Space

Let $\mathcal{F}$ be the set of all functions $f : [0,1] \to [0,1]$ satisfying the following conditions:

($\mathcal{F}1$) f is strictly increasing in $[0,1)$;
($\mathcal{F}2$) For every sequence $\{t_n\}$ in $[0,1]$, we have

$$\lim_{n\to\infty} t_n = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} f(t_n) = 1.$$

The following examples are the members of $\mathcal{F}$.

Example 3.1. (i) $f(x) = x^n$, $n \in \mathbb{N}, \forall\ x \in [0,1]$.
(ii) $f(x) = x^{\frac{1}{n}}$, $n \in \mathbb{N}, \forall\ x \in [0,1]$.
(iii) $f(x) = \frac{x+1}{2}$, $x \in [0,1]$.

Following the notation of fuzzy $\mathscr{F}$-metric space, we now generalize Definition 2.10 by introducing the idea of generalized fuzzy $\mathscr{F}$-metric space as follows.

Definition 3.2. Let X be a nonempty set and $M : X \times X \times (0,\infty) \to [0,1]$ be a mapping and $\star$ be a continuous t-norm. If there exists $(f, \alpha) \in \mathcal{F} \times (0,1]$ and a constant $k \geq 1$ such that M satisfies the following conditions:

(i) $M(x,y,t) > 0$;
(ii) $(M(x,y,t) = 1, \forall\ t > 0)$ if and only if $x = y$;
(iii) $M(x,y,t) = M(y,x,t)$;
(iv) $(f(M(x,z,k(s+t))))^{\alpha} \geq f(M(x,y,s) \star M(z,y,t)), \forall\ x, y \in X, \forall\ s, t > 0$.

Then M is said to be a generalized fuzzy $\mathscr{F}$-metric on X and the 6-tuple $(X, M, f, \alpha, \star, k)$ is said to be a generalized fuzzy $\mathscr{F}$-metric space.

The following Propositions show that fuzzy metric space, b-fuzzy metric space and fuzzy $\mathscr{F}$-metric space are generalized fuzzy $\mathscr{F}$- fuzzy metric spaces.

Proposition 3.3. *Every George & Veeramani type fuzzy metric space $(X, M, \star)$ is a generalized fuzzy $\mathscr{F}$-metric space.*

Proof. Since $(X, M, \star)$ is a fuzzy metric space, so clearly M satisfies Definition 3.2 ((i)-(iii)). We only verify the condition Definition 3.2 (iv). Since M is a fuzzy metric, so $M(x, y, t)$ is non-decreasing with respect to t. For $x, y, z \in X$ and $s, t > 0$, we have

$$\begin{aligned} & M(x, y, k(s+t)) \geq M(x, y, s+t) \geq M(x, z, s) \star M(z, y, t) \\ \text{or} \quad & f(M(x, y, k(s+t))) \geq f(M(x, z, s) \star M(z, y, t)) \\ \text{or} \quad & (f(M(x, y, k(s+t))))^{\alpha} \geq f(M(x, z, s) \star M(z, y, t)). \end{aligned}$$

Therefore, M is a generalized fuzzy $\mathscr{F}$-metric on X for any $k \geq 1$, $f \in \mathcal{F}$, and $\alpha \in (0, 1]$. □

Proposition 3.4. *Every b-fuzzy metric space $(X, M, \star)$ is a generalized fuzzy $\mathscr{F}$-metric space.*

Proof. We only prove that M satisfies the condition Definition 3.2 (iv). For, $x, y, z \in X, s, t > 0$ and there exists $k \geq 1$ we have from Definition 2.8 (iv)

$$\begin{aligned} & M(x, y, k(s+t)) \geq M(x, z, s) \star M(z, y, t) \\ \text{or} \quad & f(M(x, y, k(s+t))) \geq f(M(x, z, s) \star M(z, y, t)) \\ \text{or} \quad & (f(M(x, y, k(s+t))))^{\alpha} \geq f(M(x, z, s) \star M(z, y, t)). \end{aligned}$$

Hence, M is a generalized fuzzy $\mathscr{F}$-metric on X. □

Next, we introduce a condition that aids in the proof of Proposition 3.7.

($\mathscr{F}M5$)**:** Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. We assume the following condition:

$$\lim_{t \to \infty} M(x, y, t) = 1, \quad \forall\ x, y \in X.$$

Some Observations on the Condition ($\mathscr{F}M5$):

In the following example, we show that the condition ($\mathscr{F}M5$) is not necessary in a fuzzy $\mathscr{F}$-metric space.

Example 3.5. We consider two metrics D_1 and D_2 on $\mathbb{R}^2$ defined by $D_1(x, y) = \max\{|x_i - y_i| : i = 1, 2\}$ and $D_2(x, y) = \sum_{i=1}^{2} |x_i - y_i|$ for all $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Then clearly $D_1(x, y) \leq D_2(x, y)$ for all $x, y \in \mathbb{R}^2$.

Now we have defined a function $M : \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} \frac{1}{2}, & 0 < t \leq D_1(x, y) \\ \frac{1}{4}, & D_1(x, y) < t \leq D_2(x, y) \\ \frac{1}{8}, & D_2(x, y) < t < \infty \\ 1, & x = y, \forall\ t > 0 \end{cases}$$

for all $x, y \in \mathbb{R}^2$ and for all $t > 0$. First of all, we show that M is a fuzzy $\mathscr{F}$-metric on $\mathbb{R}^2$ with respect to the t-norm $\star =$ 'product', $f(x) = x, x \in [0, 1]$ and $\alpha = \frac{1}{4}$. we can easily see that Definition 2.10 ((i)-(iii)) are satisfied. We only verify the condition Definition 2.10 (iv).

Let $x, y, z \in \mathbb{R}^2$ and $s, t > 0$ us have the following cases:

Case-I: $s > D_2(x, z), t > D_2(z, y)$

Therefore, $s + t > D_2(x, z) + D_2(z, y) > D_2(x, y)$ which implies

$$M(x, y, s + t) = \frac{1}{8}.$$

Again, we have $M(x, z, s) = \frac{1}{8}$ and $M(z, y, t) = \frac{1}{8}$.

Now, $f(M(x, y, s + t))^\alpha = \frac{1}{8}^{\frac{1}{4}}$ and $f(M(x, z, s) \cdot M(z, y, t)) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$. Therefore,

$$f(M(x, y, s + t))^\alpha > f(M(x, z, s) \cdot M(z, y, t)).$$

Case-II: If $s > D_2(x, z)$ and $D_1(z, y) < t \leq D_2(z, y)$ then

$$M(x, z, s) = \frac{1}{8}, \qquad M(z, y, t) = \frac{1}{4}.$$

Therefore, $s + t > D_2(x, z) + D_1(z, y) \geq D_1(x, z) + D_1(z, y) \geq D_1(x, y)$ and hence

$$M(x, y, s + t) = \begin{cases} \frac{1}{4}, & D_1(x, y) < s + t < D_2(x, y) \\ \frac{1}{8}, & D_2(x, y) < s + t < \infty \end{cases}$$

Here, $f(M(x, z, s) \cdot M(z, y, t)) = \frac{1}{32}$.

If $M(x, y, s + t) = \frac{1}{4}$ then

$$f(M(x, y, s + t))^\alpha = \frac{1}{4}^{\frac{1}{4}}.$$

If $M(x, y, s + t) = \frac{1}{8}$ then $f(M(x, y, s + t))^\alpha = \frac{1}{4}^{\frac{1}{4}}$. Therefore,

$$f(M(x, y, s + t))^\alpha > f(M(x, z, s) \cdot M(z, y, t)).$$

Case-III: If $s > D_2(x,z)$ and $0 < t \leq D_1(z,y)$ then $M(x,z,s) = \frac{1}{8}$ and $M(z,y,t) = \frac{1}{2}$. Therefore, $s+t > D_2(x,z) + 0 > 0$ which implies

$$M(x,y,s+t) = \begin{cases} \frac{1}{2}, & 0 < s+t \leq D_1(x,y) \\ \frac{1}{4}, & D_1(x,y) < s+t \leq D_2(x,y) \\ \frac{1}{8}, & D_2(x,y) < s+t < \infty. \end{cases}$$

Here $f(M(x,z,s) \star M(z,y,t)) = \frac{1}{16}$.

If $M(x,y,s+t)$ then

$$f(M(x,y,s+t))^{\alpha} = \frac{1}{2}^{\frac{1}{4}} > \frac{1}{16}.$$

If $M(x,y,s+t) = \frac{1}{4}$ then $f(M(x,y,s+t))^{\alpha} = \frac{1}{4}^{\frac{1}{4}} > \frac{1}{16}$. If

$$M(x,y,s+t) = \frac{1}{8}$$

then $f(M(x,y,s+t))^{\alpha} = \frac{1}{8}^{\frac{1}{4}}$. Therefore,

$$f(M(x,y,s+t))^{\alpha} > f(M(x,z,s) \cdot M(z,y,t)).$$

If $x \neq y$ then $\lim_{t\to\infty} M(x,y,t) = \frac{1}{8}$. If $x = y$ then $\lim_{t\to\infty} M(x,y,t) = 1$. Thus $\lim_{t\to\infty} M(x,y,t) \neq 1, \forall\ x, y \in \mathbb{R}^2$.

We observe in Examples (2.11) and (2.12) that

$$\lim_{t\to\infty} M(x,y,t) = 1, \quad \forall\ x,\ y \in X.$$

Before proving that every fuzzy $\mathscr{F}$-metric space is a generalized fuzzy $\mathscr{F}$-metric space, we prove a Lemma.

Lemma 3.6. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space satisfying $(\mathscr{F}M5)$. Then there exists $k \geq 1$ such that*

$$M(x,y,kt) \geq M(x,y,t), \quad \forall x, y \in X \text{ and } \forall t > 0.$$

Proof. Let us assume that for all $k \geq 1$ there exists $x_0, y_0 \in X$, $t_0 > 0$ such that $M(x_0,y_0,kt_0) < M(x_0,y_0,t_0)$.

Let $k = n, n \in \mathbb{N}$ then

$$M(x_0,y_0,nt_0) < M(x_0,y_0,t_0) \tag{3.1}$$

Case I: If $x_0 = y_0$, then from (3.1) we get

$$1 < 1, \quad \text{a contradiction.}$$

Case-II: If $x_0 \neq y_0$, then

$$M(x_0,y_0,nt_0) < M(x_0,y_0,t_0)$$

$$\begin{aligned} &\text{or } \lim_{n\to\infty} M(x_0, y_0, nt_0) \leq M(x_0, y_0, t_0) \\ &\text{or } 1 \leq M(x_0, y_0, t_0), \text{ (using } (\mathscr{F}M5)) \\ &\text{or } M(x_0, y_0, t_0) = 1 \\ &\text{or } x_0 = y_0 \quad \text{(a contracdiction)}. \end{aligned}$$

So, our assumption is wrong. Hence, there exists $k \geq 1$ such that $M(x, y, kt) \geq M(x, y, t), \forall x, y \in X$ and $\forall t > 0$. □

Proposition 3.7. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space satisfying $(\mathscr{F}M5)$ then every fuzzy $\mathscr{F}$-metric space is a generalized fuzzy $\mathscr{F}$-metric space.*

Proof. We only prove that M satisfies the condition Definition 3.2 (iv). Since M is a fuzzy $\mathscr{F}$-metric, then there exists $(f, \alpha) \in \mathcal{F} \times (0, 1]$ for all $x, y, z \in X$ and for all $s, t > 0$ such that

$$(f(M(x, y, s + t)))^{\alpha} \geq f(M(x, z, s) \star M(z, y, t))$$

From Lemma 3.6, for all $x, y \in X$ and for all $s, t > 0$ there exists $k \geq 1$ such that

$$\begin{aligned} &M(x, y, k(s + t)) \geq M(x, y, s + t) \\ \text{or } &f(M(x, y, k(s + t)))^{\alpha} \geq f(M(x, y, s + t))^{\alpha} \\ \text{or } &f(M(x, y, k(s + t)))^{\alpha} \geq f(M(x, z, s) \star M(z, y, t)). \end{aligned}$$

This shows that M is a generalized fuzzy $\mathscr{F}$-metric on X. □

In this context, we present the following example of a generalized fuzzy $\mathscr{F}$-metric space, which is not a fuzzy $\mathscr{F}$-metric space.

Example 3.8. We consider a b-metric on $\mathbb{R}$ define by $d(x, y) = |x - y|^2$, $\forall\, x, y \in X$. Now we have defined a function $M : X \times X \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} \frac{1}{2}, & 0 < t \leq 2d(x, y) \\ \frac{1}{4}, & 2d(x, y) < t \leq 4d(x, y) \\ \frac{t}{t+|x-y|^2}, & 4d(x, y) < t < \infty. \end{cases}$$

We claim that M is a generalized fuzzy $\mathscr{F}$-metric on $\mathbb{R}$ with respect to t-norm $\star = \min$. Now we observe that M easily satisfies Definition 3.2 ((i)-(iii)). We only verify the condition Definition 3.2 (iv) for $k = 2, f(x) = x, x \in [0, 1], \alpha = \frac{1}{2}$.

Case-I: $s > 4d(x, z), t > 4d(z, y)$

Therefore,

$$2(s + t) > 2(4d(x, z) + 4d(z, y)) = 4(2d(x, z) + 2d(z, y)) > 4d(x, y)$$

which implies $M(x,y,2(s+t)) = \frac{2(s+t)}{2(s+t)+|x-y|^2}$. Again, we have

$$M(x,z,s) = \frac{s}{s+|x-z|^2}, \qquad M(z,y,t) = \frac{t}{t+|z-y|^2}.$$

Let $x,y,z \in \mathbb{R}$ and $s,t>0$. Without restraining the generality, we assume that $M(z,y,t) \leq M(x,z,s)$, i.e. $t|x-z|^2 \leq s|y-z|^2$. Now, we have

$$\begin{aligned}
&M(x,y,2(s+t)) - \min\{M(z,y,t), M(x,z,s)\} \\
&= M(x,y,2(s+t)) - M(z,y,t) \\
&= \frac{2(s+t)}{2(s+t)+|x-y|^2} - \frac{t}{t+|z-y|^2} \\
&= \frac{2s|z-y|^2 + 2t|z-y|^2 - t|x-y|^2}{\left(2(s+t)+|x-y|^2\right)\left(t+|z-y|^2\right)} \\
&\geq \frac{2t|x-z|^2 + 2t|z-y|^2 - t|x-y|^2}{\left(2(s+t)+|x-y|^2\right)\left(t+|z-y|^2\right)} \\
&\geq 0.
\end{aligned}$$

So, $M(x,y,2(s+t)) \geq M(x,z,s) \star M(z,y,t)$. Taking $f(x) = x, x \in [0,1], \alpha = \frac{1}{2}$, we have

$$f(M(x,y,2(s+t)))^{\alpha} \geq f(M(x,z,s) \star M(z,y,t)).$$

Case-II: If $s > 4d(x,z)$ and $2d(z,y) < t \leq 4d(z,y)$ then $M(x,z,s) = \frac{s}{s+|x-z|^2}$ and $M(z,y,t) = \frac{1}{4}$. Therefore,

$$2(s+t) > 2(4d(x,z) + 2d(z,y)) > 2d(x,y)$$

and hence

$$M(x,y,2(s+t)) = \begin{cases} \frac{1}{4}, & 2d(x,y) < 2(s+t) \\ \frac{2(s+t)}{2(s+t)+|x-y|^2}, & 4d(x,y) < 2(s+t) < \infty. \end{cases}$$

So,

$$\begin{aligned}
\min\{M(x,z,s), M(z,y,t)\} &= \min\left\{\frac{1}{4}, \frac{s}{s+|x-z|^2}\right\} \\
&= \frac{1}{4} \leq M(x,y,2(s+t)).
\end{aligned}$$

Therefore

$$(f(M(x,y,2(s+t))))^{\alpha} \geq f(M(x,z,s) \star M(z,y,t)),$$

where $f(x) = x,\ x \in [0,1],\ \alpha = \frac{1}{2}$.

Case-III: If $s > 4d(x,z)$ and $0 < t \leq 2d(z,y)$ then

$$M(x,z,s) = \frac{s}{s+|x-z|^2}, \qquad M(z,y,t) = \frac{1}{2}.$$

Now, $s+t > 4d(x,z) > 0$. Therefore,

$$M(x,y,2(s+t)) = \begin{cases} \frac{1}{2}, & 0 < 2(s+t) \leq 2d(x,y) \\ \frac{1}{4}, & 2d(x,y) < 2(s+t) \leq 4d(x,y) \\ \frac{2(s+t)}{2(s+t)+|x-y|^2}, & 4d(x,y) < 2(s+t) < \infty. \end{cases}$$

So, $\min\{M(x,z,s), M(z,y,t)\} = \frac{1}{2}$. Thus, if $0 < 2(s+t) \leq 2d(x,y)$ or $4d(x,y) < 2(s+t) < \infty$ then $M(x,y,2(s+t)) \geq M(x,z,s) \star M(z,y,t)$. But if $2d(x,y) < 2(s+t) \leq 4d(x,y)$ then

$$(M(x,y,2(s+t)))^{\frac{1}{2}} = M(x,z,s) \star M(z,y,t).$$

Therefore, for all $x, y \in X$ and $s, t > 0$, we have

$$(f(M(x,y,2(s+t))))^{\alpha} \geq f(M(x,z,s) \star M(z,y,t))$$

where $f(x) = x$, $x \in [0,1]$, $\alpha = \frac{1}{2}$. Hence M is a generalized fuzzy $\mathscr{F}$-metric on $\mathbb{R}$ with respect to $f(x) = x, x \in [0,1], \alpha = \frac{1}{2}, k = 2$ and $\star = \min$.

Now, we have to show that M is not a fuzzy $\mathscr{F}$-metric on $\mathbb{R}$.
If possible, suppose that M is fuzzy $\mathscr{F}$-metric space. Then there exists $(f,\alpha) \in \mathcal{F} \times (0,1]$ such that M satisfies Definition 2.10 (iv). Let $n \in \mathbb{N}$ and $u_i = \frac{1}{n}$, for $i = 1, 2, \ldots, n$.

We take $u_0 = 0, u_n = 1$. Let

$$t_0 > 4d(u_0,u_1), t_1 > 4d(u_1,u_2), \ldots, t_{n-1} > 4d(u_{n-1},u_n).$$

Therefore,
$t_0 + t_1 + \cdots + t_{n-1} > 4d(u_0,u_1) + 4d(u_1,u_2) + \cdots + 4d(u_{n-1},u_n) > 0$
Then

$$M(u_0,u_n,t_0+\cdots+t_{n-1}) = \begin{cases} \frac{1}{2}, & 0 < t_0+\cdots+t_{n-1} \leq 2d(u_0,u_n) \\ \frac{1}{4}, & 2(u_0,u_n) < t_0+\cdots+t_{n-1} \leq 4d(u_0,u_n) \\ \frac{(t_0+\cdots+t_{n-1})}{(t_0+\cdots+t_{n-1})+|u_0-u_n|^2}, & 4d(u_0,u_n) < t_0+\cdots+t_{n-1} < \infty. \end{cases}$$

Again,

$$M(0,1,t_0+t_1+\cdots+t_{n-1})$$

$$= \begin{cases} \frac{1}{2}, & 0 < t_0 + t_1 + \cdots + t_{n-1} \leq 2 \\ \frac{1}{4}, & 2 < t_0 + t_1 + \cdots + t_{n-1} \leq 4 \\ \frac{(t_0+t_1+\cdots+t_{n-1})}{(t_0+t_1+\cdots+t_{n-1})+1}, & 4 < t_0 + t_1 + \cdots + t_{n-1} < \infty. \end{cases}$$

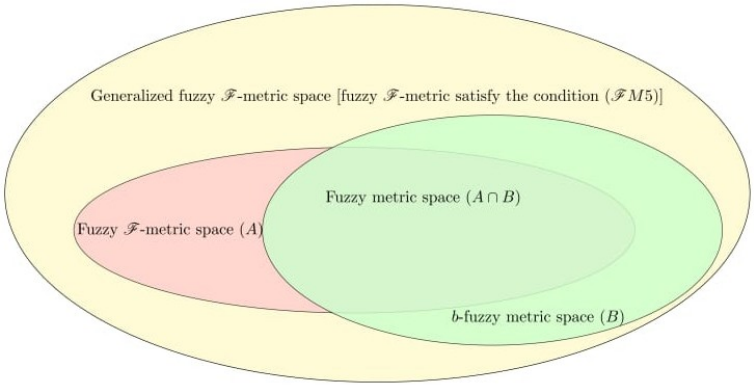


FIGURE 1. A simple schematic figure showing the relationships among fuzzy metric space, b-fuzzy metric space, fuzzy $\mathscr{F}$-metric space and generalized fuzzy $\mathscr{F}$-metric space

Again, now

$$\begin{aligned} M(u_i, u_{i+1}, t_i) &= \frac{t_i}{t_i + \left|\frac{i}{n} - \frac{i+1}{n}\right|^2} \\ &= \frac{t_i}{t_i + \frac{1}{n^2}}, \quad i = 0, 1, 2, \ldots, n \end{aligned}$$

implies

$$\min\{M(u_0, u_1, t_0), M(u_1, u_2, t_1), \ldots, M(u_{n-1}, u_n, t_{n-1})\} = \frac{t_p}{t_p + \frac{1}{n^2}},$$

for some $i = p$.

From Definition 2.10 (iv), we have

$$\begin{aligned} &(f(0, 1, t_0 + t_1 + \cdots t_{n-1}))^\alpha \\ &\quad \geq f(\min\{M(u_0, u_1, t_0), \ldots, M(u_{n-1}, u_n, t_{n-1})\}) \\ &\quad = f\left(\frac{t_p}{t_p + \frac{1}{n^2}}\right) \end{aligned}$$

for some $i = p$.

Since $\lim_{n\to\infty} \frac{t_p}{t_p+\frac{1}{n^2}} = 1$, so By $(\mathcal{F}2)$, we get $\lim_{n\to\infty} f\left(\frac{t_p}{t_p+\frac{1}{n^2}}\right) = 1$.
Therefore, $(f(0, 1, t_0 + t_1 + \cdots t_{n-1}))^\alpha \geq 1$.
This implies that $M(0, 1, t_0 + t_1 + \cdots t_{n-1}) = 1$.

By Definition 2.10 (ii), we get $0 = 1$. This is a contradiction. Hence, M is not a fuzzy $\mathscr{F}$-metric on $\mathbb{R}$.

Next, we present a generalized fuzzy $\mathscr{F}$-metric space which is not a b-fuzzy metric space.

Example 3.9. We consider two metrics D_1 and D_2 on $\mathbb{R}^2$ define by $D_1(x, y) = \max\{|x_i - y_i| : i = 1, 2\}$ and $D_2(x, y) = \sum_{i=1}^{2} |x_i - y_i|$, $\forall\ x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then clearly $D_1(x, y) \leq D_2(x, y)$ for all $x, y \in \mathbb{R}^2$. Now, we have defined a function $M : \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} \frac{1}{2}, & 0 < t \leq D_1(x, y) \\ \frac{1}{4}, & D_1(x, y) < t \leq D_2(x, y) \\ \frac{1}{8}, & D_2(x, y) < t < \infty \\ 1, & x = y, \forall\ t > 0 \end{cases}$$

$\forall\ x, y \in X$ and $\forall\ t > 0$.

First of all, we show that M is a generalized fuzzy $\mathscr{F}$-metric on $\mathbb{R}^2$ with respect to t-norm = 'product', $f(x) = x, x \in [0, 1], \alpha = \frac{1}{4}, k = 2$. We can easily see that Definition 3.2 ((i)-(ii)) are satisfied. We only verify the condition Definition 3.2 (iv).
Let $x, y, z \in \mathbb{R}^2$ and $s, t > 0$ us have the following cases:
Case-I: If $s > D_2(x, z)$, $t > D_2(z, y)$, then $M(x, z, s) = \frac{1}{8}$, $M(z, y, t) = \frac{1}{8}$. Now,

$$2(s + t) > 2(D_2(x, z) + D_2(z, y)) \geq D_2(x, z) + D_2(z, y) \geq D_2(x, y)$$

$\therefore\ 2(s + t) > D_2(x, y)$.
Then $M(x, y, 2(s + t)) = \frac{1}{8}$. Therefore,

$$f(M(x, z, s) \cdot M(z, y, t)) = f\left(\frac{1}{8} \cdot \frac{1}{8}\right) = \frac{1}{64}.$$

Hence,

$$(f(M(x, y, 2(s + t))))^\alpha > f(M(x, z, s) \cdot M(z, y, t)).$$

Case-II: If $s > D_2(x,z)$ and $D_1(z,y) < t \leq D_2(z,y)$ then $M(x,z,s) = \frac{1}{8}$ and $M(z,y,t) = \frac{1}{4}$. Therefore,

$$2(s+t) > 2[D_2(x,z) + D_1(z,y)] \geq 2[D_1(x,z) + D_1(z,y)] \geq D_1(x,y).$$

$\therefore\ 2(s+t) > D_1(x,y)$. Then

$$M(x,y,2(s+t)) = \begin{cases} \frac{1}{4}, & D_1(x,y) < 2(s+t) \leq D_2(x,y) \\ \\ \frac{1}{8}, & D_2(x,y) < 2(s+t) < \infty. \end{cases}$$

$\therefore\ f(M(x,z,s) \cdot M(z,y,t)) = f\left(\frac{1}{8} \cdot \frac{1}{4}\right) = \frac{1}{32}$.

If $M(x,y,2(s+t)) = \frac{1}{4}$, then $(f(M(x,y,2(s+t))))^{\alpha} = \frac{1}{4}^{\frac{1}{4}} > \frac{1}{32}$. If $M(x,y,2(s+t)) = \frac{1}{8}$, then $(f(M(x,y,2(s+t))))^{\alpha} > \frac{1}{32}$. Hence,

$$(f(M(x,y,2(s+t))))^{\alpha} > f(M(x,z,s) \cdot M(z,y,t)).$$

Case-III: If $s > D_2(x,z)$ and $0 < t \leq D_1(z,y)$ then $M(x,z,s) = \frac{1}{8}$ and $M(z,y,t) = \frac{1}{2}$. Therefore, $2(s+t) > 2(D_2(x,z) + 0) > 0$. Then

$$M(x,y,2(s+t)) = \begin{cases} \frac{1}{2}, & 0 < 2(s+t) \leq D_1(x,y) \\ \\ \frac{1}{4}, & D_1(x,y) < 2(s+t) \leq D_2(x,y) \\ \\ \frac{1}{8}, & D_2(x,y) < 2(s+t) < \infty. \end{cases}$$

Here, $f(M(x,z,s) \cdot M(z,y,t)) = f\left(\frac{1}{8} \cdot \frac{1}{2}\right) = \frac{1}{16}$.

If $M(x,y,2(s+t)) = \frac{1}{2}$ then $(f(M(x,y,2(s+t))))^{\alpha} = \frac{1}{2}^{\frac{1}{4}} > \frac{1}{16}$. If $M(x,y,2(s+t)) = \frac{1}{4}$ then $(f(M(x,y,2(s+t))))^{\alpha} = \frac{1}{4}^{\frac{1}{4}} > \frac{1}{16}$. If $M(x,y,2(s+t)) = \frac{1}{8}$, then $(f(M(x,y,2(s+t))))^{\alpha} = \frac{1}{8}^{\frac{1}{4}} > \frac{1}{16}$.

Hence M is a generalized fuzzy $\mathscr{F}$-metric on X with respect to $f(x) = x, x \in [0,1], \alpha = \frac{1}{4}, k = 2$.

Now we want to show that M is not b-fuzzy metric space.

We choose $x_0 = (0,0), y_0 = z_0 = (2,2), s_0 = 1.5, t_0 = 1.5$. Then $M(x_0,z_0,1.5) = \frac{1}{2}$ and $M(y_0,z_0,t_0) = 1$.

$\therefore\ M(x_0,z_0,s_0) \cdot M(z_0,y_0,t_0) = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

For all $k \geq 1$, we have

$$M(x_0,y_0,k(s_0+t_0)) = M((0,0),(2,2),k(1.5+1.5))$$

$$= \begin{cases} \frac{1}{2}, & 0 < 3k \leq 2 \\ \\ \frac{1}{4}, & 2 < 3k \leq 4 \\ \\ \frac{1}{8}, & 4 < 3k < \infty. \end{cases}$$

For all $k \geq 1$, only two cases arise, either $M((0,0),(2,2),3k) = \frac{1}{4} <$ or $M((0,0),(2,2),3k) = \frac{1}{8}$. In both cases, the following relation holds

$$M((0,0),(2,2),3k) \ngeq M((0,0),(2,2),1.5) \cdot M((2,2),(2,2),1.5).$$

Hence, M is not a b-fuzzy metric on $\mathbb{R}^2$.

Remark 3.10. The family of generalized fuzzy $\mathscr{F}$-metric spaces is larger than the family of fuzzy $\mathscr{F}$-metric spaces over a non-empty set.

We present an example that plays a vital role in explaining that every open ball does not form an open set in a generalized fuzzy $\mathscr{F}$-metric space as follows.

Example 3.11. We consider the b-metric space in the Example 2.3. Then by Example 2.3, we define a mapping $M : X \times X \times (0,\infty) \to [0,1], X = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ by

$$M(x,y,t) = \frac{t}{t + D(x,y)}.$$

Where D is defined in the Example 2.3.
Now we will prove that M is a generalized fuzzy $\mathscr{F}$-metric with $\star = \min, k = \frac{8}{3}, f(x) = x^{\frac{1}{2}}, x \in [0,1], \alpha = \frac{1}{2}$.

(i) $M(x,y,t) > 0, \forall\ t > 0, \forall\ x, y \in X$.

(ii) $M(x,y,t) = 1, \forall\ t > 0$

$$\begin{aligned} &\iff \frac{t}{t + D(x,y)\,1} = 1 \\ &\iff t = t + D(x,y) \\ &\iff x = y. \end{aligned}$$

(iii) $M(x,y,t) = M(y,x,t)$

(iv) Let $x, y, z \in X$ and $s, t > 0$. Without restraining the generality, we assume that $M(x,z,s) \leq M(z,y,t)$, i.e. $sD(z,y) \leq tD(x,z)$. Now, we have

$$\begin{aligned} &M(x,y,k(s+t)) - \min\{M(x,z,s), M(z,y,t)\} \\ &\quad = M(x,y,k(s+t)) - M(x,z,s) \\ &\quad = \frac{k(s+t)}{(s+t) + D(x,y)} - \frac{s}{s + D(x,z)} \\ &\quad = \frac{k(s+t)(s + D(x,z)) - s(k(s+t) + D(x,y))}{(s + D(x,z))(k(s+t) + D(x,y))} \\ &\quad \geq \frac{ksD(x,z) + ksD(z,y) - sD(x,y)}{(s + D(x,z))(k(s+t) + D(x,y))} \geq 0. \end{aligned}$$

Thus, $M(x,y,k(s+t)) \geq \min\{M(x,z,s), M(z,y,t)\}$, for $k = \frac{8}{3}$.

Therefore, for all $x,y,z \in X$ and for all $s,t>0$, we have

$$\left(f\left(M\left(x,y,\frac{8}{3}(s+t)\right)\right)\right)^{\frac{1}{2}} \geq f\left(\min\{M(x,z,s), M(z,y,t)\}\right)$$

where $f(x) = x^{\frac{1}{2}}, x \in [0,1]$.

Hence, M is a generalized fuzzy $\mathscr{F}$-metric on X with respect $f(x) = x^{\frac{1}{2}}$, $x \in [0,1]$, $\alpha = \frac{1}{2}$, $k = \frac{8}{3}$ and $\star = \min$.

Now we present an example that plays a vital role in explaining that every closed ball does not form a closed set in a generalized fuzzy $\mathscr{F}$-metric space.

Example 3.12. Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\right\}$ and

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \in \{0,1\} \\ |x-y|^2 & x \neq y \{0\} \cup \left\{\frac{1}{2n} : n = 1,2,3,\ldots\right\} \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Let $x,y,z \in X$ us have the following cases:
Case I: If $d(x,y) = d(0,1) = 1$ then

$$d(x,z) + d(z,y) = \begin{cases} d\left(0,\frac{1}{2n}\right) + d\left(\frac{1}{2n}\right) = \frac{1}{4n^2} + \frac{1}{4}, & z = \frac{1}{2n} \\ d\left(0,\frac{1}{2n+1}\right) + d\left(\frac{1}{2n+1}\right) = \frac{1}{2n}, & z = \frac{1}{2n+1} \end{cases}$$

Case-II: If $d(x,y) = d\left(0,\frac{1}{2n}\right) = \frac{1}{4n^2}$ then

$$d(x,z) + d(z,y) = \begin{cases} d\left(0,\frac{1}{2m}\right) + d\left(\frac{1}{2m},\frac{1}{2n}\right) = \left|\frac{1}{2m}\right|^2 + \left|\frac{1}{2m} - \frac{1}{2n}\right|^2, & z = \frac{1}{2m} \\ d\left(0,\frac{1}{2m+1}\right) + d\left(\frac{1}{2m+1},\frac{1}{2n}\right), & z = \frac{1}{2m+1} \\ d(0,1) + d\left(1,\frac{1}{2n}\right) = 1 + \frac{1}{4}, & z = 1 \end{cases}$$

By the above calculations, we get

$$d(x,y) \leq 4\left[d(x,z) + d(z,y)\right]. \tag{3.2}$$

The other cases can be verified similarly. This proves that d is a b-metric on X with $k = 4$.

We consider the b-metric space discussed above. We define a mapping $M : X \times X \times (0, \infty) \to [0, 1]$, $X = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$ by

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and for all $t > 0$.

We claim that M is a generalized fuzzy $\mathscr{F}$-metric on X concerning the t-norm $=$ min and $k = 4$. We observe that M satisfies Definition 3.2 ((i)-(iii)) easily. We only verify the condition Definition 3.2 (iv). Without loss of generality, we assume that

$$M(x, z, s) \leq M(z, y, t) \quad \Longrightarrow \quad sd(z, y) \leq td(x, z).$$

Now, we have

$$\begin{aligned}
&M(x, y, k(s + t)) - \min\{M(x, z, s), M(z, y, t)\} \\
&= M(x, y, k(s + t)) - M(x, z, s) \\
&= \frac{k(s + t)}{k(s + t) + d(x, y)} - \frac{s}{s + d(x, z)} \\
&= \frac{k(s + t)(s + d(x, z)) - s(k(s + t) + d(x, y))}{(k(s + t) + d(x, y))(s + d(x, z))} \\
&= \frac{ks(s + t) + k(s + t)d(x, z) - sk(s + t) - sd(x, y)}{(k(s + t) + d(x, y))(s + d(x, z))} \\
&= \frac{ksd(x, z) + ktd(x, z) - sd(x, y)}{(k(s + t) + d(x, y))(s + d(x, z))} \\
&= \frac{4sd(x, z) + 4sd(z, y) - sd(x, y)}{(k(s + t) + d(x, y))(s + d(x, z))} \geq 0. \quad \text{(using 3.2)}
\end{aligned}$$

Therefore, $M(x, y, k(s + t)) \geq \min\{M(x, z, s) \star M(z, y, t)\}$, $k = 4$. Therefore, for all $x, y, z \in X$ and for all $s, t > 0$ as well as for all $f \in \mathcal{F}$, $\alpha \in (0, 1)$, M forms a generalized fuzzy $\mathscr{F}$- metric on X.

Now we study the topology on a generalized fuzzy $\mathscr{F}$-metric space that is induced by the open ball.

Definition 3.13. Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy space. For some $x \in X$, $0 < r < 1, t > 0$, we define an open ball as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Proposition 3.14. *Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space. Then*

$$\tau = \{A \subseteq X : \textit{for each } x \in A, \textit{ there exist } 0 < r < 1, t > 0$$

such that $B(x,r,t) \subseteq A\}$

Proof. The proof is similar to the topology in [1]. □

Next, we define the definition of open ball and closed ball in a generalized fuzzy $\mathscr{F}$-metric space.

Definition 3.15. A subset G in a generalized fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star, k)$ is said to be open if $G \in \tau$ and $F \subseteq X$ is said to be closed if $X \backslash F \in \tau$.

The following example shows that the open ball $B(x,r,t)$ does not form an open set.

Example 3.16. From Example 3.11, we have

$$M(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ \frac{t}{t+1}, & \text{if } x \neq y \in \{0,1\} \\ \frac{t}{t+|x-y|}, & \text{if } x \neq y \in \{0,1\} \cup \left\{\frac{1}{2n} : n = 1,2,3,\ldots\right\} \\ \frac{t}{t+4}, & \text{otherwise.} \end{cases}$$

We consider an open ball $B\left(1, \frac{1}{2}, 2\right) = \left\{y \in X : M(1,y,2) > \frac{1}{2}\right\}$. Now, for $y \in \{0,1\}$, $M(1,y,2) = \frac{2}{3} > \frac{1}{2}$ and $y \in \left\{\frac{1}{2n} : n = 1,2,3,\ldots\right\}$, $M\left(1, \frac{1}{2n}, 2\right) = \frac{2}{2+4} < \frac{1}{2} \implies \frac{1}{2n} \notin B\left(1, \frac{1}{2}, 2\right)$, for all $n \in \mathbb{N}$. Therefore, $B\left(1, \frac{1}{2}, 2\right) = \{0,1\}$.

We consider another open ball

$$B(0,r,t) = \{y \in X : M(0,y,t) > 1 - r\}, \quad \forall r \in (0,1), \forall t > 0.$$

For $y \in \left\{\frac{1}{2n} : n = 1,2,3,\ldots\right\}$, $M\left(0, \frac{1}{2n}, t\right) > 1 - r$ or $\frac{t}{t+\left|0-\frac{1}{2n}\right|} > 1 - r$

$$\therefore \frac{t}{t + \frac{1}{2n}} > 1 - r \tag{3.3}$$

Now the above relation (3.3) satisfies for all $t > 0$ if $\frac{t}{t+\frac{1}{2n}} = 1$. This implies that $\frac{1}{2n} = 0$, it is not possible. This proves that $B(0,r,t) \nsubseteq B\left(1, \frac{1}{2}, 2\right)$. Hence, $B\left(1, \frac{1}{2}, 2\right)$ is not an open set. So $B\left(1, \frac{1}{2}, 2\right) \notin \tau$.

Lemma 3.17. *Let* $\{x_n\}$ *be a sequence in* $(X, M, f, \alpha, \star, k)$ *such that for any* $0 < r < 1$ *and* $t > 0$ *there exists a natural number* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x, t) > 1 - r, \forall\ n \geq n_0$ *if and only if* $x_n \in B(x,r,t), \forall\ n \geq n_0$, *for any* $r \in (0,1)$ *and* $t > 0$, *for some* $x \in X$.

Proof. For any $r \in (0,1)$ and any $t > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & M(x_n, x, t) > 1 - r, \quad \forall\ n \geq n_0 \\ \text{or} \quad & M(x, x_n, t) > 1 - r, \quad \forall\ n \geq n_0 \\ \text{or} \quad & x_n \in B(x, r, t), \quad \forall\ n \geq n_0. \end{aligned}$$

Conversely, let $x_n \in B(x, r, t)$, $\forall\ n \geq n_0$ for any $r \in (0,1)$ and $t > 0$.

$$\implies \quad M(x, x_n, t) > 1 - r, \quad \forall\ n \geq n_0 \text{ for any } r \in (0,1), \forall\ t > 0.$$

□

Proposition 3.18. *Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space. Then for any non-empty subset A of X, the following statements are equivalent*

(i) *A is closed.*
(ii) *For any $\{x_n\} \subset A$,*

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \implies \quad x \in A.$$

Proof. Assume that A is closed and $\{x_n\}$ is a sequence in A such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad x \in X. \tag{3.4}$$

If possible, suppose that $x \in X \backslash A$. Since A is closed, then $X \backslash A$ is open. Therefore, there exists some $r \in (0,1), t > 0$ such that $B(x, r, t) \subset X \backslash A$. i.e. $B(x, r, t) \cap A = \emptyset$.

On the other hand, by (3.4), there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} & M(x_n, x, t) > 1 - r, \quad \forall\ n \geq N \\ \text{or } & x_n \in B(x, r, t), \quad \forall\ n \geq \mathbb{N}. \end{aligned}$$

Hence $x_N \in B(x, r, t) \cap A$, which is a contradiction. Therefore, we deduce that $x \in A$.

Conversely, (ii) holds. Next suppose $x \in X \backslash A$.

We have to prove that there is some $r \in (0,1)$, $t > 0$ such that $B(x, r, t) \subset X \backslash A$.

If possible, suppose that for every $r \in (0,1)$ and $t > 0$ there exists $x_r \in B(x, r, t) \cap A$. For any $\{a_n\}$ $0 < a_n < 1$ such that $a_n \to 0$, $n \to \infty$ there exists $x_n \in B(x, r, t) \cap A$.

This implies that $x_n \in A$ and $x_n \in B(x, a_n, t)$

$$\begin{aligned} & \text{or} \quad M(x_n, x, t) > 1 - a_n \\ & \text{or} \quad \lim_{n\to\infty} M(x_n, x, t) \geq \lim_{n\to\infty} (1 - a_n) \\ & \text{or} \quad \lim_{n\to\infty} M(x_n, x, t) \geq 1 \\ & \text{or} \quad \lim_{n\to\infty} M(x_n, x, t) = 1. \end{aligned}$$

By (ii) implication $x \in A$, which contradicts the fact that $x \in X \backslash A$. Hence A is closed. □

Now we define the definition of closed ball in a generalized fuzzy $\mathscr{F}$-metric space.

Definition 3.19. Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space. For some $x \in X$, $0 < r < 1, t > 0$, we define a closed ball as

$$\overline{B(x, r, t)} = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

The following example shows that the closed ball $\overline{B(x, r, t)}$ does not form a closed set.

Example 3.20. From Example 3.12, we have

$$M(x, y, t) = \begin{cases} 1, & x = y \\ \frac{t}{t+1}, & x \neq y \in \{0, 1\} \\ \frac{t}{t+|x-y|^2}, & x \neq y \in \{0, 1\} \cup \left\{\frac{1}{2n} : n = 1, 2, 3, \ldots\right\} \\ \frac{t}{t+\frac{1}{4}}, & \text{otherwise.} \end{cases}$$

We consider a closed ball

$$\overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)} = \left\{y \in X : M(1, y, 2) \geq \frac{1}{2}\right\}.$$

For $y = \frac{1}{2n}, n = 1, 2, 3, \ldots$, $M\left(1, \frac{1}{2n}, \frac{1}{2}\right) = \frac{0.5}{0.5+0.25} > \frac{1}{2}$. So, $\frac{1}{2n} \in \overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)}$. For $y = 0, M\left(0, 0, \frac{1}{2}\right) = \frac{0.5}{0.5+1} \not\geq \frac{1}{2}$. So $0 \notin \overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)}$.

We take a sequence $\{x_n\} = \left\{\frac{1}{2n}\right\} \in \overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)}, \forall\ n \in \mathbb{N}$. Now, we have

$$\begin{aligned} \lim_{n\to\infty} M(x_n, 0, t) &= \lim_{n\to\infty} M\left(\frac{1}{2n}, 0, t\right) \\ &= \lim_{n\to\infty} \frac{t}{t + \left|0 - \frac{1}{2n}\right|} \\ &= 1, \quad \forall\ t > 0. \end{aligned}$$

But $0 \notin \overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)}$. Hence, $\overline{B\left(1, \frac{1}{2}, \frac{1}{2}\right)}$ does not form a closed set.

Next, we introduce a sufficient condition that ensures that $\overline{B(x, r, t)}$ is a closed set.

Proposition 3.21. *Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space, $a \in X$ and $0 < r < 1, t > 0$. We denote by $\overline{B(a,r,t)} = \{x \in X : M(a,x,t) \geq 1-r\}$. Suppose for every sequence $\{x_n\} \subset X$,*

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall\ t > 0$$
$$\text{or } M(x,y,t) \geq \lim_{n\to\infty} \sup M(x_n, y, t), \quad \forall\ y \in X.$$

Then $\overline{B(a,r,t)}$ is closed.

Proof. Let $\{x_n\} \in \overline{B(a,r,t)}$ be a sequence such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \text{for some } x \in X.$$

We have to prove that $x \in \overline{B(a,r,t)}$.

By the definition $\overline{B(a,r,t)}$, we have

$$M(x_n, a, t) \geq 1-r, \quad n \in \mathbb{N} \text{ and } t > 0.$$

By the condition $M(x,y,t) \geq \lim_{n\to\infty} \sup M(x_n, y, t), \forall\ y \in X$, we obtain

$$M(x,a,t) \geq \lim_{n\to\infty} \sup M(x_n, a, t) \geq 1-r$$
$$\text{or } \ M(x,a,t) \geq 1-r$$
$$\text{or } \ x \in \overline{B(a,r,t)}.$$

□

Proposition 3.22. *Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space. Then for a non-empty subset A of X, for each $x \in \overline{A}$, there exists $0 < r < 1$ and there exists $t > 0$ such that $B(x,r,t) \cap A \neq \emptyset$.*

Proof. Let $(f, \alpha) \in (\mathcal{F} \times (0,1])$, $k \geq 1$ be such that Definition 3.2 (iv) satisfies. Let us defined

$$A' = \{x \in X : \text{ for any } 0 < r < 1, t > 0, \ \exists\ a \in A$$
$$\text{such that } M(x,a,t) > 1-r\}.$$

From Definition 3.2 (ii), it can easily be seen that $A \subset A'$.

Next, we shall show that A' is closed. Let $\{x_n\}$ be a sequence in A' such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad x \in X, \forall\ t > 0.$$
$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad x \in X, \forall\ t > 0 \tag{3.5}$$

By $(\mathcal{F}2)$, let $0 < \epsilon < 1$ there exist $0 < \delta < 1$ such that

$$1-\delta < t \leq 1 \quad \Longrightarrow \quad 1-\epsilon < f(t) \leq 1.$$

For any $\delta \in (0,1)$, we can find $\gamma \in (0,1)$ such that

$$(1-\gamma) \star (1-\gamma) \geq 1-\delta.$$

On the other hand, by (3.5) there exists $N \in \mathbb{N}$ such that $M(x_n, x, t) > 1-\gamma, \forall\ n \geq N,\ \forall\ t > 0$.

Since $x_N \in A'$ there exists $a \in A$ such that $M(a, x_N, t) > 1-\gamma$. Let us suppose that $x \notin A'$. Then for all $a \in A$, there exists $0 < r_0 < 1, t_0 > 0$ such that $M(x, a, t_0) \leq 1 - r_0$. Now, we have

$$M\left(x, x_N, \frac{t_0}{2k}\right) \star M\left(x_N, a, \frac{t_0}{2k}\right) > (1-\gamma) \star (1-\gamma) \geq (1-\delta)$$

$$\text{or}\quad M\left(x, x_N, \frac{t_0}{2k}\right) \star M\left(x_N, a, \frac{t_0}{2k}\right) > 1-\delta$$

$$\text{or}\quad f\left(M\left(x, x_N, \frac{t_0}{2k}\right) \star M\left(x_N, a, \frac{t_0}{2k}\right)\right) > 1-\epsilon.$$

Since $\epsilon > 0$ is arbitrary, so we have

$$\begin{aligned}
&(f(M(x, a, t_0)))^\alpha \geq 1\\
\text{or}\quad &(f(1-r_0))^\alpha \geq (f(M(x, a, t_0)))^\alpha \geq 1\\
\text{or}\quad &(f(1-r_0))^\alpha = 1\\
\text{or}\quad &f(1-r_0) = 1\\
\text{or}\quad &(1-r_0) = 1, \text{ using } (\mathcal{F}2)\\
\text{or}\quad &r_0 = 0, \text{ a contradiction.}
\end{aligned}$$

So our assumption is wrong. Therefore, $x \in A'$. □

Next, we prove a lemma which is important for proving the Theorem 3.24.

Lemma 3.23. *Suppose $f : [0,1] \to [0,1]$ be a function satisfying the following conditions $(\mathcal{F}1)$ and $(\mathcal{F}2)$. Then $f(1) = 1$.*

Proof. Suppose that $f(1) \neq 1$. So, $f(1) < 1$ or $f(1) > 1$.

Case I : If $f(1) > 1$, then $f(1) \notin [0,1]$. So, this case is not possible.

Case II : If $f(1) < 1$. Let us consider a sequence $\{t_n\}$ such that $t_n = 1$ for all $n \in \mathbb{N}$. So, by condition $(\mathcal{F}2)$ $\lim\limits_{n\to\infty} f(t_n) = 1$. Again, we have

$$\begin{aligned}
&t_n = 1, \quad \forall\ n \in \mathbb{N}\\
\text{or}\quad &f(t_n) = f(1), \quad \forall\ n \in \mathbb{N}\\
\text{or}\quad &\lim_{n\to\infty} f(t_n) = f(1) < 1, \quad \forall\ n \in \mathbb{N}\\
\text{or}\quad &\lim_{n\to\infty} f(t_n) < 1 \quad \text{(contradiction to the condition } (\mathcal{F}2)).
\end{aligned}$$

So, our assumption is wrong. Hence, $f(1) = 1$. □

Next, we prove that τ is a Hausdorff topology.

Theorem 3.24. *Every generalized fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star, k)$ is a Hausdorff space.*

Proof. Let $x, y \in X$ with $x \neq y$. Then there exists $t_0 > 0$ such that $M(x, y, t_0) < 1$. Let $a_n = 1 - \frac{M(x,y,t_0)}{n}, \forall\ n \in \mathbb{N}$ then $a_n \to 1$ as $n \to \infty$. Since for each n, $0 < a_n < 1$, so by Remark 2.7 (ii) there exists $0 < b_n < 1$ such that

$$b_n \star b_n \geq a_n.$$

Now, for some $p \in \mathbb{N}$, we consider two open sets $A = \text{int}\left(B\left(x, 1 - b_p, \frac{t_0}{2}\right)\right)$, $B = int\left(B\left(y, 1 - b_p, \frac{t_0}{2}\right)\right)$ containing x and y respectively. If $A \cap B = \emptyset$, then the proof is done.
If possible, suppose that there exists $z \in A \cap B$. Then $z \in A$ and $z \in B$. This implies that $z \in B\left(x, 1 - b_p, \frac{t_0}{2}\right)$ and $z \in B\left(y, 1 - b_p, \frac{t_0}{2}\right)$.

$$\implies \quad M\left(x, z, \frac{t_0}{2}\right) > b_p \quad \text{and} \quad M\left(z, y, \frac{t_0}{2}\right) > b_p.$$

Now, we have

$$\begin{aligned}(f(M(x, y, kt_0)))^\alpha &\geq f\left(M\left(x, z, \frac{t_0}{2}\right) \star M\left(z, y, \frac{t_0}{2}\right)\right) \\ &> f(b_p \star b_p) \geq f(a_p).\end{aligned}$$

Letting $p \to \infty$, we obtain

$$\begin{aligned}&(f(M(x, y, kt_0)))^\alpha \geq f(1) = 1 \quad \text{(using Lemma (3.23))} \\ \text{or}\quad &(f(M(x, y, kt_0)))^\alpha = 1 \\ \text{or}\quad &f(M(x, y, kt_0)) = 1 \\ \text{or}\quad &M(x, y, kt_0) = 1 \quad \text{(using Lemma (3.23))} \\ \text{or}\quad &x = y \quad \text{(using Lemma (3.23))}.\end{aligned}$$

This contradicts our assumption. Hence, the proof is done. □

Next, we define convergent sequences and Cauchy sequences in a generalized fuzzy $\mathscr{F}$-metric space.

Definition 3.25. Let $\{x_n\}$ be a sequence in a generalized fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star, k)$. Then $\{x_n\}$ is said to be

(i) convergent if there exists $x \in X$ such that for any $0 < r < 1$, there exists a natural number $N \in \mathbb{N}$ such that for all $t > 0$, $M(x_n, x, t) > 1 - r, \forall\ n \geq N$.
(ii) Cauchy sequence if for each $t > 0$, $0 < r < 1$, there exists a natural number $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r, \forall\ m, n \geq N$.
(iii) X is said to be complete if every Cauchy sequence in X converges to some point in X.

The following proposition can be easily proved as follows.

Proposition 3.26. *Let $(X, M, f, \alpha, \star, k)$ be a generalized fuzzy $\mathscr{F}$-metric space and $\{x_n\}$ be a sequence in X and $x \in X$*

(i) *$\{x_n\}$ is convergent to x if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1, \forall\, t > 0$.*

(ii) *$\{x_n\}$ is Cauchy if and only if $\lim_{m,n\to\infty} M(x_n, x_m, t) = 1, \forall\, t > 0$.*

Proof. (i) Let $\{x_n\}$ be a sequence in X and converges to $x \in X$. Then for any $0 < r < 1$ there exists a natural number $N \in \mathbb{N}$ such that for all $t > 0$ $M(x_n, x, t) > 1 - r, \forall\, n \geq N$. This implies that $\lim_{n\to\infty} M(x_n, x, t) = 1, t > 0$.

Conversely, if for each $t > 0$, $M(x_n, x, t) \to 1$ as $n \to \infty$ then $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that $1 - M(x_n, x, t) < r$, $\forall\, n \geq n_0$. It follows that $M(x_n, x, t) > 1 - r, \forall\, n \geq n_0$.

(ii) Let $\{x_n\}$ be a Cauchy sequence in X. Then for each $t > 0$, $r \in (0, 1)$, there exists a natural number $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r, \forall\, m, n \geq N$. This implies that $\lim_{n\to\infty} M(x_n, x_m, t) = 1, \forall\, t > 0$.

Conversely, if for each $t > 0$, $M(x_n, x_m, t) \to 1$ as $m, n \to \infty$ then $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that $1 - M(x_n, x_m, t) < r, \forall\, n \geq n_0$. It follows that $M(x_n, x, t) > 1 - r, \forall n \geq n_0$. □

Proposition 3.27. *The limit of a sequence in a generalized fuzzy $\mathscr{F}$-metric space, if it exists, is unique.*

Proof. Let $\{x_n\}$ be a convergent sequence in a generalized fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star, k)$. Let $x, y \in X$ be such that $\{x_n\}$ converges to x and y. Then

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall\, t > 0, \qquad \lim_{n\to\infty} M(x_n, y, t) = 1, \quad \forall\, t > 0.$$

Since M is a generalized fuzzy $\mathscr{F}$-metric, there exists $(f, \alpha) \in \mathcal{F} \times (0, 1], k \geq 1$ satisfying Definition 3.2 (iv).

Now suppose $x \neq y$. Then there exists $t_0 > 0$ such that $M(x, y, t_0) < 1$ and hence by Definition 3.2 (iv)

$$(f(M(x, y, kt_0)))^\alpha \geq f\left(M\left(x, x_n, \frac{t_0}{2}\right) \star M\left(x_n, y, \frac{t_0}{2}\right)\right)$$

$$\text{or} \quad (f(M(x, y, kt_0)))^\alpha \geq \lim_{n\to\infty} f\left(M\left(x, x_n, \frac{t_0}{2}\right) \star M\left(x_n, y, \frac{t_0}{2}\right)\right).$$

By $(\mathcal{F}2)$, we have $\lim_{n\to\infty} M\left(x, x_n, \frac{t_0}{2}\right) \star M\left(x_n, y, \frac{t_0}{2}\right) = 1 \star 1 = 1$ which implies

$$(f(M(x, y, kt_0)))^\alpha \geq 1$$

$$\begin{aligned}
&\text{or} \quad (f(M(x,y,kt_0)))^{\alpha} = 1\\
&\text{or} \quad f(M(x,y,kt_0)) = 1\\
&\text{or} \quad M(x,y,kt_0) = 1 \quad (\text{by}\,(\mathcal{F}2))\\
&\text{or} \quad x = y, \quad \text{a contradiction.}
\end{aligned}$$

So, our assumption is wrong. Hence, limit of a convergent sequence in a generalized fuzzy $\mathscr{F}$-metric space is unique. □

Proposition 3.28. *Every convergent sequence in a generalized fuzzy $\mathscr{F}$-metric space is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence in a generalized fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star, k)$ converging to $x \in X$. Then

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall\ t > 0. \tag{3.6}$$

Let $0 < \epsilon < 1$. By $(\mathcal{F}2)$ there exist $0 < \delta < 1$ such that

$$1 - \delta < t \leq 1 \quad \Longrightarrow \quad 1 - \epsilon < f(t) \leq 1. \tag{3.7}$$

By Remark 2.7 (ii), for any $\delta \in (0,1)$, we can find $\gamma \in (0,1)$ such that $(1-\gamma) \star (1-\gamma) \geq 1 - \delta$.

Again, from (3.6) implies that for $t_1 \geq 0$ and $t_2 > 0$ there exists $N_1(t_1, \gamma)$ and $N_2(t_2, \gamma)$ such that

$$M(x_n, x, t_1) > 1 - \gamma, \quad \forall\ n \geq N_1(t_1, \gamma)$$

and

$$M(x_m, x, t_2) > 1 - \gamma, \quad \forall\ m \geq N_2(t_2, \gamma).$$

Let $t = t_1 + t_2$ and $N(t, \gamma) = \max\{N_1(t_1, \gamma), N_2(t_2, \gamma)\}$. Then for all $m, n \geq N(t, \gamma)$, for all $t > 0$, we have

$$\begin{aligned}
&M(x_n, x, t_1) \star M(x_m, x, t_2) \geq (1-\gamma) \star (1-\gamma) \geq (1-\delta),\\
&\text{or} \quad f(M(x_n, x, t_1) \star M(x_m, x, t_2)) > 1 - \epsilon, \ (\text{using } (3.7))\\
&\text{or} \quad (f(x_n, x_m, k(t_1 + t_2)))^{\alpha} > 1 - \epsilon,\\
&\text{or} \quad \lim_{n,m\to\infty} f(M(x_n, x_m, k(t_1 + t_2))) = 1,\\
&\text{or} \quad \lim_{m,n\to\infty} M(x_n, x_m, kt) = 1, \ (\text{using}\,(\mathcal{F}2)).
\end{aligned}$$

Therefore, $\lim_{n,m\to\infty} M(x_n, x_m, T), \forall\ T > 0$, say $kt = T$. This proves that $\{x_n\}$ is a Cauchy sequence in $(X, M, f, \alpha, \star, k)$. □

4. Conclusions

In this paper, a new concept of generalized $\mathscr{F}$- fuzzy metric space is introduced and some topological properties of such spaces are discussed. Since the concept of generalized fuzzy $\mathscr{F}$-metric space is a new approach, there is a wide scope of research to develop the results of fuzzy functional analysis by using such generalized fuzzy $\mathscr{F}$-metric space.

Acknowledgment. The author D.B. acknowledges financial support awarded by CSIR-UGCNET (DECEMBER 2022/JUNE 2023), University Grants Commission (UGC), New Delhi, India, through research fellowship [Ref. No231610065558] for carrying out research work leading to the preparation of this manuscript. The authors are also thankful to the Department of Mathematics, Siksha-Bhavana, Visva-Bharati, India.

References

1. A. Das, D. Barman and T. Bag, *A new generalization of George & Veeramani type fuzzy metric space*, Probl. Anal. Issues Anal., 13 (31) 3 (2024), pp. 23-42.
2. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (3) (1994), pp. 395-399.
3. A. Moussaoui, *Exploring fixed points via admissibility criteria for fuzzy θ_f-weak contractions*, Nonlinear Analysis: Modelling and Control, 30 (2025), pp. 1-14.
4. A. Moussaoui, N. Hussain, S. Melliani, H. Nasr and M. Imdad, *Fixed point results via extended $\mathcal{FZ}$-simulation functions in fuzzy metric spaces*, J. Inequal. Appl., (2022), 2022:69.
5. A. Moussaoui and S. Radenovic, *Fuzzy metric spaces: a survey on fixed point results, contraction principles and simulation functions*, Fixed Point Theory Algorithms Sci. Eng., (2025), 25:24.
6. D. Poovaragavan, M. Jeyaraman, V. Gupta and N. Muthulakshmi, *A novel approach on extended b-fuzzy metric spaces and fixed point results with application*, Pesqui. Oper., 45: e290557 (2025), pp. 1-15.
7. G.J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Printice-Hall of India Private Limited, New Delhi-110001, 1997.
8. G. Sun and K. Yang, *Generalized fuzzy metric spaces with properties*, Research Journal of Applied Sciences, Engineering and Technology, 7 (2) (2010), pp. 673-678.
9. I. Khan, M. Shaheryar, F.U. Din, U. Ishtiaq and I. Popa, *Fixed-point results in fuzzy S-metric space with applications to fractals and satellite web coupling problem*, Fractal and Fractional, 9 (2025), 164.

10. L. A. Zadeh, *Fuzzy sets*, Inf. Control, 8 (1965), pp. 338-353.
11. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets Syst., 27 (1988), pp. 385-389.
12. M. Jleli and B. Samet, *On a new generalization of metric spaces*, J. Fixed Point Theory Appl., 20 (2018), Article ID: 128.
13. M. Raji, L. Rathour, L.N. Mishra and V.N. Mishra, *Fuzzy cone metric spaces and fixed point theorems for fuzzy type contraction*, Transactions on Fuzzy Sets and Systems, 3 (2) (2024), pp. 82-99.
14. M.S. Sezen, *Controlled fuzzy metric spaces and some related fixed point results* , Numer. Methods Partial Differ. Equations, 37 (2021), pp. 583-593.
15. O. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (5) (1975), pp. 326-334.
16. P. Kumam, N.V. Dung and V.T.L. Hang, *Some equivalences between cone b-metric spaces and b-metric spaces*, Abstr. Appl. Anal., (2013).
17. R. Tiwari, N. Sharma, A. Fulga and R. Patel, *Fixed point results in controlled fuzzy metric spaces with an application to the conversion of solar energy into electric power*, Advances in Fixed Point Theory, (2025), 15:10.
18. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta. Math. Inform. Univ. Os traviensis, 1 (1993), pp. 5-11.
19. S. Nadaban, *Fuzzy b-Metric space*, International Journal of Computers Communications & Control, 11 (2) (2016), pp. 273-281.
20. S. Sedghi and N. Shobe, *Common fixed point theorems in b-fuzzy metric spaces*, Nonlinear Funct. Anal. Appl., 17 (3) (2012), pp. 349-359.
21. S. Sharma, *On fuzzy metric space*, Southeast Asian Bull. Math., 26 (2002), pp. 133-145.
22. T. Oner, M.B. Kandemir and B. Tanay, *Fuzzy cone metric spaces*, J. Nonlinear Sci. Appl., 8 (2015), pp. 610-616.

[1] Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, Birbhum, West Bengal, India.
Email address: diptibarmanhmt@gmail.com

[2] Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, Birbhum, West Bengal, India.
Email address: tarapadavb@gmail.com