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From Support Metrics to Closure Operators: A Topological Approach to Nominal Sets

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Abstract. We introduce and study a topology on nominal sets induced by a support closure operator, which arises naturally from the action of finitary permutations on a countable set of atoms. This topology is generated by support classes and is closely related to a metric defined via symmetric differences of supports. We characterize continuous maps between nominal sets in this topological setting, investigate compactness and Hausdorffness properties and analyze the interplay between equivariance, support closure and topological structure. Our results provide new insights into the algebraic and topological aspects of nominal sets, enriching their theory and potential applications in areas such as logic, computer science and algebraic structures with name-binding.

1. Introduction and Preliminaries

Nominal sets, introduced to computer science by Gabbay and Pitts in the early 2000s [9], give us a mathematical framework for reasoning about names and binding in formal languages and syntax. They model sets equipped with an action of the group of finitary permutations on a countably infinite set of atomic names, together with a notion of finite support that captures the dependency of elements on finitely many atoms. This approach presents a sound alternative to traditional treatments of variable binding and α-equivalence, which often rely on representative choices and can be cumbersome in formal reasoning.

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Andrew M. Pitts's seminal work systematically develops the theory of nominal sets, exploring their algebraic and categorical properties and demonstrating their applications in programming language semantics, logic and functional programming. His monograph [15] is a comprehensive reference that has shaped much of the contemporary research in this area. Gabbay, Pitts and their collaborators have contributed foundational papers that collectively lay the groundwork for understanding supports, freshness and name abstraction—key concepts central to the theory (see, e.g., [7–9]).

Subsequent research has extended these ideas to various domains, including nominal automata, nominal rewriting systems and nominal logic, highlighting the versatility of nominal techniques in handling syntax with binding (see, e.g., [2, 4, 10, 12, 16]). The interplay between algebraic structures and topological or metric properties on nominal sets has also attracted attention, helping us better understand continuity, convergence and compactness phenomena in semantics and computation [14].

In this paper, the proposed approach is inspired by our earlier studies (specifically, the idea originates from our previous work [6]). We contribute to this line of research by introducing and studying a topology on nominal sets induced by a support closure operator. This closure operator, defined via support classes, generates a natural topology that reflects the underlying permutation symmetry and finite support conditions. We analyze the properties of this topology, including its relation to a metric defined by symmetric differences of supports and characterize continuous maps, compactness and separation axioms within this framework.

Our results provide new observations about the mathematical structure of nominal sets. By bridging algebraic, metric and topological perspectives, this work suggests a few possible directions for applications in logic, semantics and algebraic models of name-binding. The main contribution of this paper is the introduction of a topology on nominal sets induced by a support-based closure operator (Theorem 3.12). Unlike the considered topology for data languages in [3], which uses permutation orbits for language recognition, our support closure topology $\overline{\tau}$ (Theorem 3.12) uses support-equivalence classes X_t, yielding compactness (Theorem 3.21) that is not equivalent to the orbit-finite compactness in [3].

In contrast, [14] studies categorical limits over nominal sets, whereas we develop an *intrinsic point-set topology* from support closures, introducing the support metric $d(x, x') = |\operatorname{supp} x \,\Delta\, \operatorname{supp} x'|$ absent in both [3, 14].

A key point we show is that topological continuity and metric continuity are not the same (Example 4.23). This turns out to be useful for resolving quotient behavior in nominal categories beyond equivariant functors [14].

Table 1 summarizes our key contributions relative to prior nominal topology approaches [3, 14].

TABLE 1. Comparison with prior nominal topologies

Aspect	**Our Work**	**[3], [14]**
Topology	Topology with basis $\{X_t\}_{t\in X}$	Nominal topology (given in Def. 1.11)
Compactness	Uniform finitely supported, Thm. 3.21	Orbit-finite, [3, Def. 3.10, Lem. 3.11]
Continuity	Metric & topo., Ex. 4.23	Continuous w.r.t. Nominal topology
Metric	$d(x,x') = \lvert\operatorname{supp} x \,\Delta\, \operatorname{supp} x'\rvert$	Pseudo-metric, [3, Example 3.2(3)]

The paper is organized as follows. After a preliminary review of nominal sets in Section 1, Section 2 introduces the support metric on nominal sets and investigates its fundamental properties, including its relationship with permutation symmetries and continuous metric equivariant maps. In Section 3, we define the support closure operator and develop the induced topology on nominal sets, examining key topological properties such as closure, compactness and the structure of support classes. Section 4 focuses on the study of continuous equivariant maps between nominal spaces, exploring notions of topological continuity, support-covered maps and their connections to compactness and separation axioms.

1.1. **Nominal sets.** Suppose $\mathbb{D}$ is a set, then a permutation π of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. The permutations of $\mathbb{D}$ with composition and identity form a group, called the *symmetric group* on the set $\mathbb{D}$ and denoted by $\operatorname{Sym}(\mathbb{D})$. A permutation $\pi \in \operatorname{Sym}(\mathbb{D})$ is finitary if the set $\{a \in \mathbb{D} \mid \pi a \neq a\}$ is a finite subset of $\mathbb{D}$. It is clear that the permutations $\mathrm{id}_{\mathbb{D}}$ is finitary and that the composition and the inverse of finitary permutations are finitary. Therefore, we get a subgroup of $\operatorname{Sym}(\mathbb{D})$ of finitary permutations, denoted by $\operatorname{Perm}(\mathbb{D})$. We fix a countable infinite set $\mathbb{D}$ whose elements are denoted by $a, b, c, \ldots$ and called *atomic names*. Let X be a set equipped with an action of the group $\operatorname{Perm}(\mathbb{D})$, $\operatorname{Perm}(\mathbb{D}) \times X \to X$ mapping (π, x) to πx. We call X a $\operatorname{Perm}(\mathbb{D})$-*set*,

whenever for every $\pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$ and every $x \in X$ the following hold: (1) $\pi_1(\pi_2 x) = (\pi_1\pi_2)\, x$ and (2) $\mathrm{id}_{\mathbb{D}}\, x = x$.

A subset Y of a $\mathrm{Perm}(\mathbb{D})$-set X is called *equivariant* if $\pi y \in Y$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and $y \in Y$. $\mathrm{Perm}(\mathbb{D})$-sets are the objects of a category, denoted by $\mathrm{Perm}(\mathbb{D})$-**Set** whose morphisms are equivariant maps, i.e. maps subject to the rule $f(\pi x) = \pi f(x)$, for all $x \in X, \pi \in \mathrm{Perm}(\mathbb{D})$, whose compositions and identities are as in the category **Set** of sets and maps.

An element x of a $\mathrm{Perm}(\mathbb{D})$-set X is called a *fixed* (or equivariant) element if $\pi x = x$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$. The set of all fixed elements of the $\mathrm{Perm}(\mathbb{D})$-set X is denoted by $\mathrm{Fix}(X)$. A $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are fixed is called *discrete*.

Given a $\mathrm{Perm}(\mathbb{D})$-set X, a set of atomic names $S \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and for every $a \in S$,

$$\pi(a) = a \quad \Rightarrow \quad \pi x = x.$$

Given a $\mathrm{Perm}(\mathbb{D})$-set X, we say an element $x \in X$ is finitely supported, if there is some finite set of atomic names that is, a support for the element x.

Definition 1.1 ([15])**.** A *nominal set* is a $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are finitely supported. Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose compositions and identities are as in the category of $\mathrm{Perm}(\mathbb{D})$-**Set**.

Example 1.2.
(1) The set $\mathbb{D}$ is a nominal set with the natural action $\pi a = \pi(a)$.
(2) The action of $\mathrm{Perm}(\mathbb{D})$ on $\mathbb{D}$ extends pointwise to action of $\mathrm{Perm}(\mathbb{D})$ on tuples $\mathbb{D}^n$ and $\mathbb{D}^{(n)}$. So, the sets

$$\mathbb{D}^n = \{(a_1, a_2, \ldots, a_n) \in \mathbb{D}^n \mid a_i \in \mathbb{D}\} \text{ and}$$
$$\mathbb{D}^{(n)} = \{(a_1, a_2, \ldots, a_n) \in \mathbb{D}^n \mid a_i \neq a_j \text{ for } i \neq j\}$$

are nominal sets.

Example 1.3. Given a $\mathrm{Perm}(\mathbb{D})$-set X, the power set of X, $\mathcal{P}(X)$, with the action

$$\begin{aligned} \mathrm{Perm}(\mathbb{D}) \times \mathcal{P}(X) &\to \mathcal{P}(X) \\ (\pi, Y) &\rightsquigarrow \{\pi(x) : x \in Y\} \end{aligned}$$

is a $\mathrm{Perm}(\mathbb{D})$-set.

A set of atomic names D supports $Y \in \mathcal{P}(X)$ if and only if

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))\,((\forall a \in D)\ \pi(a) = a) \quad \Rightarrow \quad (\forall\, x \in Y)\ \pi x \in Y.$$

Notation 1.4. For a given nominal set X, we will frequently write $\mathcal{P}_{\mathrm{fs}}(X)$ for the set consisting of all finitely supported subsets of X and $\mathcal{P}_{\mathrm{f}}(X)$ for the set consisting of all finite subsets of X.

Definition 1.5 ([15])**.** A nominal set X is called *uniformly finitely supported* if there exists a finite subset $S \subseteq \mathbb{D}$ such that S is a finite support of all elements of X.

Remark 1.6. (i) Suppose X is a nominal set and $x \in X$. The intersection of two (finite) supports of x is also a (finite) support of x [15, Propositions 2.1 and 2.3]. Therefore, each element $x \in X$ has a least (finite) support, denoted by $\operatorname{supp}_X x$, or simply $\operatorname{supp} x$ when there is no ambiguity. In fact, $\operatorname{supp} x = \bigcap \{C : C \text{ is a support of } x\}$.

(ii) An element x of a nominal set X is a fixed element if and only if $\operatorname{supp} x = \emptyset$.

(iii) A subset A of a nominal set X is equivariant if and only if $\operatorname{supp} A = \emptyset$.

Proposition 1.7 ([15])**.** *Suppose X is a $\operatorname{Perm}(\mathbb{D})$-set and $x \in X$. A subset $A \subseteq \mathbb{D}$ supports x if and only if, for all $a_1, a_2 \in \mathbb{D} \setminus A$, we have $(a_1\ a_2)\, x = x$.*

Remark 1.8. Every finite nominal set is discrete.

Lemma 1.9 ([6, Lemma 3.4])**.** *Let X be a nominal set and $x, x' \in X$. Then, there exists π with $\operatorname{supp} \pi x \subseteq \operatorname{supp} x'$ or $\operatorname{supp} \pi x' \subseteq \operatorname{supp} x$.*

Lemma 1.10 ([15, Lemma 2.12])**.** *Let $f : X \to Y$ be an equivariant map between nominal sets and $x \in X$. Then, $\operatorname{supp} f(x) \subseteq \operatorname{supp} x$ and if f is also injective, then $\operatorname{supp} f(x) = \operatorname{supp} x$.*

Definition 1.11. Let X be a nominal set and $\tau \subseteq \mathcal{P}_{\text{fs}}(X)$ an equivariant subset. Then τ is a *nominal topology* on X if τ satisfies the following properties;

(i) $\emptyset, X \in \tau$;
(ii) if $A, B \in \tau$, then $A \cap B \in \tau$;
(iii) if $\mathcal{U}$ is a finitely supported subset of τ, then $\bigcup \mathcal{U} \in \tau$.

2. The Support Metric and Equivariant Metric Continuous Maps on Nominal Sets

This section introduces a metric on nominal sets that quantifies the distance between elements by measuring the symmetric difference of their finite supports. It shows how this metric naturally captures the inherent permutation symmetries of nominal sets and provides a framework for analyzing continuous equivariant maps.

Definition 2.1. Let X be a nominal set. The map $d\colon X \times X \to \mathbb{R}^+$, defined by

$$d\left(x, x'\right) = |\operatorname{supp} x \,\Delta\, \operatorname{supp} x'|, \quad \text{for all } x, x' \in X,$$

where Δ denotes the symmetric difference between sets, is a metric on X. The topology induced by d is denoted by τ_d and is called *the support metric topology* and the topological space X is called *a nominal metric space*.

If more than one nominal set is under consideration, we use the notation $d_X(x, x')$, instead of $d(x, x')$ to explicitly indicate the metric associated with the nominal set X. This avoids ambiguity when working with multiple spaces.

Note 2.2. The support metric quantifies how much two elements $x, x' \in X$ differ in their atomic name dependencies ($d(x, x') = |\operatorname{supp} x \,\Delta\, \operatorname{supp} x'|$). The symmetric difference $|\operatorname{supp} x \,\Delta\, \operatorname{supp} x'|$ counts atoms that support exactly one of x or x'.

This provides a natural measure of "name-sensitivity" similarity, with potential applications in nominal rewriting, logical semantics and permutation-invariant algorithms.

Intuitive Example: Imagine friends with contact lists of phone numbers (atoms). Alice's list: $\{a, b\}$ (depends on Alice/Bob). Bob's list: $\{b, c\}$ (depends on Bob/Charlie). Their "support distance" is $|\{a, b\} \,\Delta\, \{b, c\}| = |\{a, c\}| = 2$: they differ by exactly two unique contacts. Friends with identical lists have distance 0, like elements in the same support-class X_t.

This metric groups "essentially equivalent" elements (distance zero) that behave identically under permutations fixing their shared support.

Example 2.3. Let X be a nominal set. Then,

(1) if X is discrete, then $d(x, x') = 0$, for all $x, x' \in X$,
(2) if $X = \mathbb{D}^{(n)}$, then $d(x, x') = 2m$, where $0 \leq m \leq n$.

Note 2.4. Let X be a nominal metric space and $x, x' \in X$. Then, $d(x, x') = 0$ if and only if $\operatorname{supp} x = \operatorname{supp} x'$.

Definition 2.5. An equivariant map $f : X \to Y$ between two nominal metric spaces (X, d_X) and (Y, d_Y) is called *metric continuous* if for all $x, x' \in X$,

$$d_Y\left(f(x), f\left(x'\right)\right) \leq d_X\left(x, x'\right).$$

Example 2.6.

(1) By Lemma 1.10, every injective equivariant map is metric continuous.
(2) Constant maps are metric continuous.
(3) Let $X = \mathbb{D}^{(2)} \cup \mathbb{D}^{(3)}$ and $Y = \mathbb{D}^{(2)} \cup \{\theta\}$. Define the map $f\colon X \to Y$ by

$$f(d_1, d_2) = (d_2, d_1), \quad \text{for all } (d_1, d_2) \in \mathbb{D}^{(2)} \text{ and}$$

$$f(d_1, d_2, d_3) = \theta, \quad \text{for all } (d_1, d_2, d_3) \in \mathbb{D}^{(3)}.$$

Then f is equivariant but not metric continuous, as it fails to satisfy the metric continuity condition:

$$d_Y(f(d_1, d_2), f(d_1, d_2, d_3)) > d_X((d_1, d_2), (d_1, d_2, d_3)).$$

Remark 2.7. Let $f : X \to Y$ be an equivariant map. Then X admits a disjoint decomposition $X = X_+ \dot{\cup} X_0$, where:

$$X_+ = \{x \in X \mid \operatorname{supp} f(x) \neq \emptyset\}, \qquad X_0 = \{x \in X \mid \operatorname{supp} f(x) = \emptyset\}$$

are equivariant subsets. Furthermore, $X_+ \cap \operatorname{Fix}(X) = \emptyset$ and $\operatorname{Fix}(X) \subseteq X_0$.

Proposition 2.8. *Let $f : X \to Y$ be an equivariant map. Then, $f|_{X_+}$ is metric continuous if and only if $\operatorname{supp} f(x) = \operatorname{supp} x$, for all $x \in X_+$.*

Proof. ($\Rightarrow$) Suppose f is an equivariant metric continuous map between nominal metric spaces. First, observe that since f is equivariant, Lemma 1.10 implies $\operatorname{supp} f(x) \subseteq \operatorname{supp} x$ for all $x \in X$.

To prove the reverse inclusion $\operatorname{supp} x \subseteq \operatorname{supp} f(x)$, assume for contradiction that there exists $a \in \operatorname{supp} x \setminus \operatorname{supp} f(x)$. Considering an arbitrary $a' \in \operatorname{supp} f(x)$ we have

$$a = (a\ a')(a') \in \operatorname{supp}(a\ a') f(x) \setminus \operatorname{supp} f(x),$$

which implies $(a\ a') f(x) \neq f(x)$. Since $a, a' \in \operatorname{supp} x$, we have

$$\operatorname{supp} x = \operatorname{supp}(a\ a') x$$

and thus by Note 2.4 we have $d(x, (a\ a') x) = 0$. By the continuity of f, $d(f(x), f((a\ a') x)) = 0$. Lemma 2.4 then yields $\operatorname{supp} f(x) = \operatorname{supp} f((a\ a') x)$. However, this contradicts the fact that $a \in \operatorname{supp} f((a\ a') x) \setminus \operatorname{supp} f(x)$.

($\Leftarrow$) The other direction follows directly. □

Corollary 2.9. *Let $f : X \to Y$ be an equivariant and $X_0 = \operatorname{Fix}(X)$. Then,*

(i) *f is metric continuous if and only if $f|_{X_+}$ is metric continuous.*
(ii) *If $f : X \to Y$ is a metric continuous map then for every $x \in X_+$, $\operatorname{supp} f(x) = \operatorname{supp} x$.*
(iii) *An equivariant map $f : \mathbb{D}^{(n)} \to \mathbb{D}^{(m)}$ is metric continuous if and only if $n = m$.*

Proof. (i) To prove the non-trivial part, let $x \in X_+$ and $x' \in X_0$. Then, $d_Y(f(x), f(x')) = |\operatorname{supp} f(x)| \leq |\operatorname{supp} x| = d_X(x, x')$.

Parts (ii) and (iii) follow from Proposition 2.8. □

Remark 2.10. Let X be a nominal set. Then,

(i) $\mathrm{S}_X = \{|\mathrm{supp}\, x| \mid x \in X\}$ is a non-empty subset of $\mathbb{N}^0$ and has the least element.
(ii) the set $X_{_k} = \{x \in X : |\mathrm{supp}\, x| < k\}$ is an equivariant subset of X.
(iii) the set $\{|\mathrm{supp}\, x| \mid x \in X_k\}$ is a bounded subset of $\mathbb{N}$ and thus has a maximum element.

Proposition 2.11. *Suppose $f : X \to Y$ is an equivariant metric continuous map and $\emptyset \neq X_0 \neq \mathrm{Fix}\,(X)$. Let k be the least element of*

$$\{|\mathrm{supp}\, x| \mid x \in X_0 \setminus \mathrm{Fix}\,(X)\}.$$

Then:

(i) *$|\mathrm{supp}\, x| < k$, for all $x \in X_+$.*
(ii) *$X_+ = \{x \in X \setminus \mathrm{Fix}\,(X) \mid |\mathrm{supp}\, x| < k\}$.*

Proof. (i) Assume there exists $x \in X_+$ such that $|\mathrm{supp}\, x| \geq k$. By definition of k, there exists $x' \in X_0 \setminus \mathrm{Fix}\,(X)$ with $|\mathrm{supp}\, x'| = k$ and we have $|\mathrm{supp}\, x| \geq |\mathrm{supp}\, x'| = k$. By Lemma 1.9, there exists a permutation π such that $\mathrm{supp}\,(\pi x') \subseteq \mathrm{supp}\, x$. Consequently, $|\mathrm{supp}\,(\pi x') \setminus \mathrm{supp}\, x| = 0$ and the distance between x and $\pi x'$ in X is

$$d_X\left(x, \pi x'\right) = \left|\mathrm{supp}\, x \setminus \mathrm{supp}\left(\pi x'\right)\right| = |\mathrm{supp}\, x| - \left|\mathrm{supp}\left(\pi x'\right)\right|.$$

Since f is metric continuous, Proposition 2.8 implies $\mathrm{supp}\, f\,(x) = \mathrm{supp}\, x$. From the inequality

$$d_Y\left(f\,(x), f\left(\pi x'\right)\right) \leq d_X\left(x, \pi x'\right),$$

we deduce

$$|\mathrm{supp}\, f\,(x)| = |\mathrm{supp}\, x| \leq \left|\mathrm{supp}\, x \setminus \mathrm{supp}\left(\pi x'\right)\right|.$$

This forces

$$|\mathrm{supp}\, x| = \left|\mathrm{supp}\, x \setminus \mathrm{supp}\left(\pi x'\right)\right|,$$

which is a contradiction because $\mathrm{supp}\,(\pi x') \neq \emptyset$.

(ii) This follows immediately from part (i). □

Theorem 2.12. *Let $f : X \to Y$ be an equivariant metric continuous map. Then $k \geq 2n$, where*

$$k = \mathrm{Min}\,\{|\mathrm{supp}\, x| \mid x \in X_0 \setminus \mathrm{Fix}\,(X)\} \text{ and } n = \mathrm{Max}\,\{|\mathrm{supp}\, x| \mid x \in X_+\}.$$

Proof. Let $x \in X_+$ with $|\mathrm{supp}\, x| = n$ and $x' \in X_0 \setminus \mathrm{Fix}\,(X)$ with $|\mathrm{supp}\, x'| = k$. By Proposition 2.11 and Lemma 1.9, there exists a permutation π such that $\mathrm{supp}\,(\pi x) \subseteq \mathrm{supp}\, x'$. Thus,

$$d_Y\left(f\,(\pi x), f\left(x'\right)\right) = |\mathrm{supp}\, f\,(\pi x)| = |\mathrm{supp}\, f\,(x)|$$

and

$$d_X\left(\pi x, x'\right) = \left|\operatorname{supp} x' \setminus \operatorname{supp}\left(\pi x\right)\right|.$$

Since f is metric continuous, we have $\operatorname{supp} f\left(x\right) = \operatorname{supp} x$ and consequently

$$\left|\operatorname{supp} x\right| \leq \left|\operatorname{supp} x' \setminus \operatorname{supp}\left(\pi x\right)\right|.$$

Therefore, $n \leq k - n$. □

Following the definitions of k and n in Theorem 2.12, we establish the following theorem:

Theorem 2.13. *Let $f : X \to Y$ be an equivariant map. Then, f is metric continuous if and only if $f|_{X_+}$ is metric continuous and $k \geq 2n$.*

Proof. Since f is metric continuous, its restriction $f|_{X_+}$ is automatically metric continuous. Moreover, by Theorem 2.12, we have $k \geq 2n$.

Now, suppose the restriction $f|_{X_+}$ is metric continuous and $k \geq 2n$. To prove that f is metric continuous, it suffices to show that for any $x \in X_+$ and $x' \in X_0$, we have $d_Y\left(f\left(x\right), f\left(x'\right)\right) \leq d_X\left(x, x'\right)$. Let $t \in X_0 \setminus \operatorname{Fix}\left(X\right)$ with $\left|\operatorname{supp} t\right| = k$. Then, for our given x and x', we have $\left|\operatorname{supp} x\right| \leq n$ and $\left|\operatorname{supp} x'\right| \geq k$.

The following inequalities hold:

$$\begin{aligned} d_Y\left(f\left(x\right), f\left(x'\right)\right) &= \left|\operatorname{supp} f\left(x\right)\right| \\ &= \left|\operatorname{supp} x\right| \\ &\leq n \\ &\leq k - n \\ &\leq \left|\operatorname{supp} t \setminus \operatorname{supp} x\right| \\ &\leq \left|\operatorname{supp} x' \setminus \operatorname{supp} x\right| \\ &\leq \left|\operatorname{supp} x \setminus \operatorname{supp} x'\right| + \left|\operatorname{supp} x' \setminus \operatorname{supp} x\right| \\ &= d_X\left(x, x'\right). \end{aligned}$$

□

3. The support closure Operator and Its Induced Topology on Nominal Sets

We define the support closure operator on nominal sets and examine the induced topology. This operator captures closure behavior linked to support equivalence classes, reflecting the nominal structure's symmetries. The section systematically develops foundational results about this topology, including compactness, separation axioms and its relation to the support metric topology.

Definition 3.1. Let X be a nominal set. For any $x \in X$ and $A, B \subseteq X$, we define:

(i) The *distance from x to A*:

$$d(x,A) := \min\{d(x,t) \mid t \in A\}.$$

(ii) The *distance from A to B*:

$$d(A,B) := \min\left\{d\left(x,x'\right) \mid x \in A,\, x' \in B\right\}.$$

(iii) The *support-class* of $t \in X$:

$$X_t := \{x \in X \mid \operatorname{supp} x = \operatorname{supp} t\}.$$

(iv) The *closure* of A:

$$\begin{aligned}\overline{A} &:= \{x \in X \mid d(x,A) = 0\} \\ &= \{x \in X \mid \exists t \in A,\, \operatorname{supp} x = \operatorname{supp} t\}.\end{aligned}$$

Standard nominal topology relies on orbit-closures, which mix elements with different supports and ignore atomic dependencies. Our support closure $cl(A) = \bigcup_{t\in\operatorname{supp} A} X_t$ addresses this by preserving support structure. Example 3.3 demonstrates its nontrivial behavior, while Example 3.2 shows that for any nominal set X and $t \in X$, we have $t \in \operatorname{Orb}(t) \cap X_t \neq \emptyset$, but neither inclusion holds in general. This closure applies in name-binding semantics where distinct closures reflect binding structure.

Example 3.2. (1) Let X be a discrete nominal set and $t \in X$. Then $\operatorname{supp} t = \emptyset$, so $X_t = \{x \in X \mid \operatorname{supp} x = \emptyset\} = X$ while, $\operatorname{Orb}(t) = \{t\} \subsetneq X_t$.

(2) Let $X = \mathbb{D}^{(2)}$ and $t = (a,b)$. Then $\operatorname{supp} t = \{a,b\}$ and $X_t = \{(a,b),(b,a)\} \subsetneq \operatorname{Orb}(t) = \operatorname{Perm}(\mathbb{D})(a,b) = \mathbb{D}^{(2)}$.

(3) Let $X = \mathbb{D}^3 \cup \mathbb{D}^{(2)}$ and $t = (d_1,d_2)$. Then, $Orb(t) = \mathbb{D}^{(2)}$ and $X_t = \{(d_1,d_2),(d_2,d_1)\} \cup \{(a,b,c) \mid a,b,c \in \{d_1,d_2\}\}$. So in this case $X_t \cap Orb(t) \neq \emptyset$.

Example 3.3. Consider $X = \mathbb{D}^{(2)} \cup \mathbb{D}^{(3)}$.

(1) For $A = \{a,b\}$, we have $\operatorname{supp} A = \{a,b\}$ and $cl(A) = X_{\{a,b\}} = \{(a,b),(b,a)\}$ ($|cl(A)| = 2$).

(2) For $B = \{(a,b,c)\}$, we have $\operatorname{supp} B = \{a,b,c\}$ and $cl(B) = X_{\{a,b,c\}}$ contains all $3! = 6$ permutations of (a,b,c).

Different supports $\{a,b\} \neq \{a,b,c\}$ generate closures of distinct sizes and structures, demonstrating how support determines closure behavior.

The following lemma shows that metric continuity preserves support-classes.

Lemma 3.4. *Let $f : (X,d_X) \to (Y,d_Y)$ be an equivariant metric continuous map. Then, $f(X_t) \subseteq (f(X))_t$ for every $t \in X$.*

Proof. Let $y \in f(X_t)$. Then, we have:

$$\begin{aligned} y \in f(X_t) \quad &\Rightarrow \quad y = f(x), \quad \text{for some } x \in X_t \\ &\Rightarrow \quad \operatorname{supp} x = \operatorname{supp} t \\ &\Rightarrow \quad \operatorname{supp} f(x) = \operatorname{supp} f(t), \quad (f \text{ is metric continuous}) \\ &\Rightarrow \quad y = f(x) \in (f(X))_{f(t)}. \end{aligned}$$

□

Proposition 3.5. *Let X be a nominal set and $A \subseteq X$. The support-classes satisfy:*

(i) (Non-emptiness) *For all $t \in A$, the set X_t is non-empty and finitely supported. Indeed,* $\operatorname{supp} X_t \subseteq \operatorname{supp} t$.
(ii) (*Equality*) $t \in X_s \quad \Rightarrow \quad X_t = X_s$.
(iii) (*Equivalence*) $X_t \cap X_s \neq \emptyset \quad \Rightarrow \quad X_t = X_s$.
(iv) (*Closure realization*) $\overline{A} = \bigcup_{t \in A} X_t$.
(v) (*Equivariance*) *For all $\pi \in \operatorname{Perm}(\mathbb{D})$, we have $\pi X_t = X_{\pi t}$.*
(vi) (*Absorption Property*) *For any $t \in X$, if $X_t \cap \overline{A} \neq \emptyset$, then $X_t \subseteq \overline{A}$.*

Proof. The assertions in parts (i), (ii), (iii), (iv) and (vi) are simple and follow directly from the definitions or previous results. We therefore concentrate on the proof of part (v): We have:

$$\begin{aligned} y \in \pi X_t \quad &\Longleftrightarrow \quad \pi^{-1} y \in X_t \\ &\Longleftrightarrow \quad \operatorname{supp}\left(\pi^{-1} y\right) = \operatorname{supp} t \\ &\Longleftrightarrow \quad \operatorname{supp} y = \operatorname{supp}(\pi t) \\ &\Longleftrightarrow \quad y \in X_{\pi t}. \end{aligned}$$

□

The family of sets $\{X_t\}_{t \in X}$ introduced in Definition 3.1 forms a partition of X according to Proposition 3.5. We denote the equivalence relation corresponding to this partition by

$$\approx_{\mathrm{Supp}} = \left\{ \left(x, x'\right) \in X \times X \mid \operatorname{supp} x = \operatorname{supp} x' \right\}$$

and state the following proposition.

Proposition 3.6. *Let X be a nominal set. Then,*

(i) *The quotient set $X/\approx_{\mathrm{Supp}} = \{\mathsf{X_t} \mid \mathsf{t} \in \mathsf{X}\}$ is a nominal set.*
(ii) *The canonical surjective equivariant map $p : X \to X/\approx_{\mathrm{Supp}}$ is metric continuous.*

Proof. (i) By Proposition 3.5(v), for any $\pi \in \operatorname{Perm}(\mathbb{D})$, we have $\pi X_t = X_{\pi t} \in X/\approx_{\mathrm{Supp}}$. Hence, $X/\approx_{\mathrm{Supp}}$ is a nominal set.

(ii) To prove continuity of p, note that for all $x \in X_t$ and $x' \in X_s$, $d(x, x') = d(t, s)$. Therefore, $d(p(t), p(s)) = d(X_t, X_s) = d(t, s)$ follows that p is metric continuous.

□

Lemma 3.7. *Let X be a nominal set and $A \subseteq X$. Then,*

(i) *if $A \in \mathcal{P}_{\text{fs}}(X)$, then $\operatorname{supp}\overline{A} \subseteq \operatorname{supp} A$ and $\overline{A} \in \mathcal{P}_{\text{fs}}(X)$.*
(ii) *if A is equivariant, then $\overline{A}$ is equivariant.*
(iii) *for each $\pi \in \operatorname{Perm}(\mathbb{D})$, we have $\overline{\pi A} = \pi\overline{A}$.*

Proof. (i) Suppose $a_1, a_2 \notin \operatorname{supp} A$. We show that $(a_1\ a_2)\,\overline{A} = \overline{A}$. Let $x \in \overline{A}$. Then there exists $t \in A$ such that $\operatorname{supp} t = \operatorname{supp} x$. Since $a_1, a_2 \notin \operatorname{supp} A$, it follows that $(a_1\ a_2)\,A = A$ and so $(a_1\ a_2)\,t \in A$. Also, we have $\operatorname{supp}(a_1\ a_2)\,t = (a_1\ a_2)\operatorname{supp} t = (a_1\ a_2)\operatorname{supp} x = \operatorname{supp}(a_1\ a_2)\,x$. Thus $(a_1\ a_2)\,x \in \overline{A}$ and so $(a_1\ a_2)\,\overline{A} = \overline{A}$.

(ii) By Remark 1.6(iii), Since A is equivariant, $\operatorname{supp} A = \emptyset$. Then applying part (i), $\operatorname{supp}\overline{A} \subseteq \operatorname{supp} A = \emptyset$.

(iii) We have:

$$\begin{aligned} t \in \pi\overline{A} &\iff \pi^{-1}t \in \overline{A} \\ &\iff \exists\, s \in A \text{ such that } \operatorname{supp}\left(\pi^{-1}t\right) = \operatorname{supp} s \\ &\iff \exists\, s \in A \text{ such that } \operatorname{supp} t = \operatorname{supp}(\pi s) \\ &\iff t \in \overline{\pi A}. \end{aligned}$$

□

Remark 3.8. Let X be a nominal set and $A \subseteq X$. The operator cl is a closure operator in the sense of [5]; i.e., it satisfies the following following properties:

(i) (Extensivity) $A \subseteq \overline{A}$.
(ii) (Monotonicity) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.
(iii) (Idempotence) $\overline{\left(\overline{A}\right)} = \overline{A}$.

The following result is an immediate consequence of Lemma 3.7:

Corollary 3.9. *For any nominal set X, the closure operator*

$$\text{cl}\colon \mathcal{P}_{\text{fs}}(X) \to \mathcal{P}_{\text{fs}}(X), \quad A \mapsto \overline{A}$$

is a finitely supported map.

Proposition 3.10. *Let $f : X \to Y$ be an equivariant metric continuous map between nominal sets and let $A \subseteq X$. Then, $f\left(\overline{A}\right) \subseteq \left(\overline{f(A)}\right)$.*

Proof. Let $y \in f\left(\overline{A}\right)$. By definition, $\exists x \in \overline{A}$ such that $y = f(x)$. Since $x \in \overline{A}$, there exists $a \in A$ with $d(x, a) = 0$. Now, by continuity of f, the distance inequality yields $d(f(x), f(a)) \leq d(x, a) = 0$. Hence $f(x) \in \overline{f(A)}$, establishing the inclusion. □

Proposition 3.11. *Let X be a nominal set and $A \in \mathcal{P}_{\text{fs}}(X)$. Then,*

(i) *the closure operator* cl *is algebraic; i.e.*

$$\overline{A} = \bigcup \left\{ \overline{B} \mid \ B \text{ is finite }, B \subseteq \ A \right\}.$$

(ii) *if* $\{A_i\}_{i\in I} \subseteq \mathcal{P}_{\text{fs}}(X)$, *then* $\overline{\left(\bigcap_{i\in I} A_i\right)} \subseteq \bigcap_{i\in I} \overline{A_i}$.
(iii) *if* $\{A_i\}_{i\in I} \subseteq \mathcal{P}_{\text{fs}}(X)$, *then* $\overline{\left(\bigcup_{i\in I} A_i\right)} = \bigcup_{i\in I} \overline{A_i}$.

Proof. (i) For any $t \in A$, we have $\overline{\{t\}} = X_t$. Now, applying Proposition 3.5(iii), $\overline{A} = \bigcup_{t\in A} \overline{\{t\}}$. So, cl is algebraic.

(ii) Since $\bigcap_{i\in I} A_i \subseteq A_i$, we get that $\overline{\left(\bigcap_{i\in I} A_i\right)} \subseteq \bigcap_{i\in I} \overline{A_i}$.

(iii) Since $A_i \subseteq \bigcup_{i\in I} A_i$, we have $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcup_{i\in I} A_i}$. Now, let $t \in \overline{\bigcup_{i\in I} A_i}$, Then, there exists $i \in I$ and $s \in A_i$ with $\operatorname{supp} s = \operatorname{supp} t$. So, there exists $s \in A_i$ with $t \in \bigcup_{i\in I} A_i$ which means that $t \in \bigcup_{i\in I} \overline{A_i}$. □

Theorem 3.12. *Let* X *be a nominal set. Then the family* $\overline{\tau} := \left\{ \overline{A} \mid \ A \in \ \mathcal{P}_{\text{fs}}(X) \right\}$ *satisfies the following properties;*

(i) $\overline{\tau}$ *is a nominal set (i.e., an equivariant subset of* $\mathcal{P}_{\text{fs}}(X)$*);*
(ii) $\overline{\tau}$ *is a topology on* X *in the sense of Definition 1.11.*

Proof. (i) By Lemma 3.7(i), we have $\overline{A} \in \mathcal{P}_{\text{fs}}(X)$. Applying Lemma 3.7(iii), for all $\pi \in \operatorname{Perm}(\mathbb{D})$, $\overline{\pi A} = \pi\overline{A}$. Thus, $\pi\overline{A} \in \overline{\tau}$.

(ii) Notice that $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$. Let I be a finite index set and suppose $\left\{\overline{A_i}\right\}_{i\in I} \subseteq \overline{\tau}$. We prove $\overline{\tau}$ is closed under finite intersections. If $\bigcap_{i\in I} \overline{A_i} = \emptyset$, the claim holds trivially. Otherwise, suppose $\bigcap_{i\in I} \overline{A_i} \neq \emptyset$. Then there exists some $k \in I$ such that $\overline{A_k} \cap \bigcap_{i\neq k} \overline{A_i} \neq \emptyset$ and hence there exists $t \in A_k$ with $X_t \cap \bigcap_{i\neq k} \overline{A_i} \neq \emptyset$. Define $B = \left\{ t \in A_k \mid X_t \cap \bigcap_{i\neq k} \overline{A_i} \neq \emptyset \right\}$. By Proposition 3.5(vi), since $X_t \cap \bigcap_{i\neq k} \overline{A_i} \neq \emptyset$, we have

$X_t \subseteq \bigcap_{i\neq k} \overline{A_i}$. Therefore,

$$\begin{aligned}\bigcap_{i\in I} \overline{A_i} &= \overline{A_k} \cap \bigcap_{i\neq k} \overline{A_i} \\ &= \left(\bigcup_{t\in A_k} X_t\right) \cap \bigcap_{i\neq k} \overline{A_i} \\ &= \bigcup_{t\in B} \left(X_t \cap \bigcap_{i\neq k} \overline{A_i}\right) \\ &= \bigcup_{t\in B} \bigcap_{i\neq k} \left(X_t \cap \overline{A_i}\right) \\ &= \bigcup_{t\in B} X_t, \quad \text{by 3.5 (vi)} \\ &= \overline{B}.\end{aligned}$$

Next, we show that B is a finitely supported subset of X and so $\overline{B} \in \overline{\tau}$. Let $t \in B$ and $a, b \notin \left(\bigcup_{i\in I} \operatorname{supp} \overline{A_i}\right)$. Since $(a\ b)\,\overline{A_i} = \overline{A_i}$ for all $i \in I$, it follows that $(a\ b)\,t \in A_k$. Moreover, as $X_t \cap \bigcap_{i\neq k} \overline{A_i} \neq \emptyset$, by Proposition 3.5 we have

$$X_{(a\ b)t} \cap \bigcap_{i\neq k} \overline{A_i} = (a\ b)\, X_t \cap \bigcap_{i\neq k} (a\ b)\, \overline{A_i} \neq \emptyset.$$

so $(a\ b)\,t \in B$, proving that B is finitely supported.

Now, assume $\mathcal{U} \subseteq \overline{\tau}$ is finitely supported. By Proposition 3.11(iii), $\bigcup_{U\in\mathcal{U}} \overline{U} = \overline{\bigcup_{U\in\mathcal{U}} U}$. We show that $\operatorname{supp}(\mathcal{U})$ is a finite support for $\bigcup_{U\in\mathcal{U}} U$, so that $\overline{\tau}$ is closed under finitely supported unions. Take $a, b \notin \operatorname{supp}\mathcal{U}$. Then for every $U \in \mathcal{U}$, $(a\ b)\,U \in \mathcal{U}$. For any $t \in \bigcup_{U\in\mathcal{U}} U$, there exists $U \in \mathcal{U}$ such that $t \in U$. Hence, $(a\ b)\,t \in (a\ b)\,U \in \mathcal{U}$ which implies $(a\ b)\,t \in \bigcup_{U\in\mathcal{U}} U$. □

Example 3.13. (1) Let X be a discrete nominal set (i.e., with trivial action). Then $\operatorname{supp} x = \emptyset$ for all $x \in X$ (Remark 1.6). Thus, $X_x = \{t \in X \mid \operatorname{supp} t = \operatorname{supp} x = \emptyset\} = X$, so $\overline{A} = X$ for any $A \subseteq X$. Hence $\overline{\tau} = \{\emptyset, X\}$ is the non-discrete topology on X.

(2) Let $X = \mathbb{D}$. Then, one can see easily $\overline{\tau}_{\mathbb{D}} = \mathcal{P}_{\mathrm{fs}}(\mathbb{D})$.

Definition 3.14. Suppose $\sim$ is an equivalence relation on a nominal set X and $A \subseteq X$. We say A is saturated by $\sim$, if for every $x \in A$, we have $x/\!\sim \subseteq \mathsf{A}$.

The following lemma shows that elements of the support closure topology are saturated by $\approx_{\mathrm{supp}}$.

Lemma 3.15. *Let X be a nominal set and $A \subseteq X$. Then A is closed with respect to the support closure topology $\overline{\tau}$ if and only if A is saturated by $\approx_{\text{supp}}$.*

Proof. A is closed if and only if $A = \overline{A} = \bigcup_{t \in A} X_t$, which holds if and only if $X_x \subseteq A$ for every $x \in A$. This is precisely the condition that A is saturated by $\approx_{\text{supp}}$. □

Theorem 3.16. *The quotient space $X/\approx_{\text{supp}}$ induced by support equivalence is a T_0 topological space.*

Proof. Let $x/\approx_{\text{supp}} \neq \mathsf{x}'/\approx_{\text{supp}}$. Then, $X_x \neq X_{x'}$. Since the canonical map $p : X \to X/\approx_{\text{supp}}$ is closed, $p(X_x)$ is a closed and contains $p(x) = x/\approx_{\text{supp}}$. Moreover, $p(x') = x'/\approx_{\text{supp}} \notin \mathsf{p}(\mathsf{X}_{\mathsf{x}})$ since $x' \notin X_x$. Thus $p(X_x)$ is a closed neighborhood of $x/\approx_{\text{supp}}$ not containing $x'/\approx_{\text{supp}}$. □

Definition 3.17. Let (X, τ) be a nominal space. Then X is *T_0-support* if for all distinct $x, y \in X$ with $\operatorname{supp} x \neq \operatorname{supp} y$, there exists $U \in \tau$ such that $x \in U$ and $y \notin U$.

Example 3.18. Every nominal metric space (X, τ_d) is T_0-support. Indeed, take $U = B(x, r)$ with $0 < r < 1$. Then $x \in B(x, r)$, but $y \notin B(x, r)$ since

$$d_X(x, y) = |\operatorname{supp} x \,\Delta\, \operatorname{supp} y| \geq 1.$$

Proposition 3.19. *Let X be a nominal set. Then the topology $\overline{\tau}$ is finer than the support metric topology τ_d defined in Definition 2.1. In other words, $\overline{\tau} \subseteq \tau_d$.*

Proof. For any r such that $0 \leq r < 1$, the open ball centered at t with radius r in the metric d is

$$B(t, r) = \{x \in X \mid d(t, x) < r\} = \{x \in X \mid \operatorname{supp} x = \operatorname{supp} t\} = X_t.$$

Therefore, for any finitely supported $A \subseteq X$, we have $\overline{A} = \bigcup_{t \in A} X_t \in \tau_d$, showing that $\overline{\tau} \subseteq \tau_d$. □

Definition 3.20. Let $(X, \overline{\tau})$ be a nominal space and $A \in \mathcal{P}_{\text{fs}}(X)$. We say that A is *compact* if there exist finitely many elements $t_1, \ldots, t_k \in X$ such that $A \subseteq \bigcup_{i=1}^{k} X_{t_i}$.

Theorem 3.21. *Let X be a nominal space. Then, $A \subseteq X$ is compact if and only if A is uniformly finitely supported.*

Proof. Suppose $A \subseteq X_{t_1} \cup \cdots \cup X_{t_k}$ for some $t_1, \ldots, t_k \in X$ and let $C = \bigcup_{i=1}^{k} \operatorname{supp} t_i$. For each $t \in A$, there exists some j such that $\operatorname{supp} t = \operatorname{supp} t_j$. Hence, $\operatorname{supp} t \subseteq C$, which shows that A is uniformly finitely supported.

Conversely, suppose that A is uniformly finitely supported. Then there exists a finite set C of atomic names such that for all $t \in A$, $\operatorname{supp} t \subseteq C$. Consider a decomposition $A \subseteq \bigcup_{i\in I} X_{t_i}$, where the sets X_{t_i} are pairwise disjoint for $i \neq j$. Note that each X_{t_i} either intersects A non-trivially or not at all. Thus, we can write $A \subseteq \bigcup_{j\in J} X_{t_j}$, where $J \subseteq I$ and $X_{t_j} \cap A \neq \emptyset$ for all $j \in J$. Since A is uniformly finitely supported, for each $j \in J$ we have $\operatorname{supp} t_j \subseteq C$. Fix some $k \in J$ and define

$$T = \{d \in \operatorname{supp} t_j \setminus \operatorname{supp} t_k \mid j \in J,\, t_j \neq t_k\} \subseteq C.$$

If J were infinite, then T would also be infinite, contradicting the finiteness of C. Therefore, J is finite and hence A is compact. □

Example 3.22. On $\left(\mathbb{D}^{(2)}, \overline{\tau}\right)$, the set $A = \bigcup_{i=1}^{3} X_{\{(a_i,b_i)\}}$ is compact (uniformly finitely supported by $\{a_1, a_2, a_3, b_1, b_2, b_3\}$). However, the infinite set $\bigcup_n X_{\{(a_n,b_n)\}}$ is non-compact.

Definition 3.23 ([3, Definition 3.10])**.** A nominal topological space X is compact if every open cover C of X has an orbit-finite subcover.

According to Definition 3.23, any orbit-finite nominal set with respect to the nominal topology (Definition 1.11) is compact. The following example shows that orbit-finite nominal sets are *not* compact with respect to the support closure topology $\overline{\tau}$.

Example 3.24. The nominal set $\mathbb{D}$ is not compact with respect to $\overline{\tau}$. Suppose for contradiction there exist $d_1, \ldots, d_n \in \mathbb{D}$ such that $\mathbb{D} \subseteq \bigcup_{i=1}^{n} X_{d_i}$. Since $X_{d_i} = \{d_i\}$ for each i and $\mathbb{D}$ is infinite, choose $d \notin \{d_1, \ldots, d_n\}$. Then $d \in \mathbb{D} \setminus \bigcup_{i=1}^{n} X_{d_i}$, a contradiction.

However, $\mathbb{D}$ is orbit-finite: $\mathbb{D} = \operatorname{Orb}(d)$ is a single orbit. Thus orbit-finiteness does not imply compactness in the support closure topology.

Theorem 3.25. *Let $(X, \overline{\tau})$ be a nominal space. Then the following are equivalent:*

(i) *X is Hausdorff;*
(ii) *the support map $\operatorname{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ is injective;*
(iii) *for each $\overline{A} \in \overline{\tau}$, we have $\overline{A} = A$;*
(iv) *the topology $\overline{\tau}$ is discrete.*

Proof. (i) ⇒ (ii) Suppose $x \neq x' \in X$. Since X is Hausdorff, there exist disjoint finitely supported subsets $\overline{A}, \overline{B}$ of X such that $x \in \overline{A}$ and $x' \in \overline{B}$. Hence, $X_x \cap X_{x'} \subseteq \overline{A} \cap \overline{B} = \emptyset$ which implies that $\operatorname{supp} x \neq \operatorname{supp} x'$.

(ii) ⇒ (iii) By definition,

$$\overline{A} = \{x \in X \mid \exists t \in A,\ \operatorname{supp} x = \operatorname{supp} t\}.$$

Since the support map is injective, $\operatorname{supp} x = \operatorname{supp} t$ implies $x = t$, so

$$\overline{A} = \{x \in X \mid x \in A\} = A.$$

(iii) $\Rightarrow$ (iv) For any $t \in X$, by (iii) we have $X_t = \{t\} \in \overline{\tau}$, which means every singleton is open and thus $\overline{\tau}$ is discrete.

(iv) $\Rightarrow$ (i) If $\overline{\tau}$ is discrete, then for any distinct $x, x' \in X$, the singletons $\{x\}$ and $\{x'\}$ are open and disjoint. Hence, X is Hausdorff. □

4. Continuous Equivariant Maps Between Nominal Spaces and Support-Covered Maps

The support closure topology on nominal sets naturally prompts the study of continuous equivariant maps, so-called topological continuous maps (Definition 4.1). We characterize these in Theorem 4.21 and compare them with metric continuity (Corollary 4.22). We also define and study support-covered maps that address the challenge of ensuring equivariant maps respect finite support constraints across spaces with differing atom dependencies. These maps naturally arise when studying transformations between nominal sets where support sizes may vary, providing a framework for analyzing structure preservation under permutation actions. We also study congruence relations induced by these maps and investigate their continuity and closedness properties.

Definition 4.1. By a *topological continuous map* we mean an equivariant map $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ with $f^{-1}(Y_t) \in \overline{\tau}_X$, for every $t \in Y$.

Example 4.2. Every constant map $f : X \to \mathbb{D}$ defined by $f(x) = a$ is topological continuous. This is because, $f^{-1}(Y_y) = \mathbb{D}$ where $y = a$ and $f^{-1}(Y_y) = \emptyset$ where $y \neq a$.

Proposition 4.3. *Every bijective equivariant map between nominal sets is topological continuous.*

Proof. Let $t \in Y$. Since f is surjective, there exists $x' \in X$ such that $f(x') = t$. Hence, $f^{-1}(Y_t) \neq \emptyset$. Moreover, because f is injective and equivariant, we have $\operatorname{supp} f(x) = \operatorname{supp} x$, for all $x \in X$. Therefore,

$$\begin{aligned} f^{-1}(Y_t) &= \{x \in X \mid f(x) \in Y_t\} \\ &= \{x \in X \mid \operatorname{supp} f(x) = \operatorname{supp} t\} \\ &= \left\{x \in X \mid \operatorname{supp} f(x) = \operatorname{supp} f\left(x'\right)\right\} \\ &= \left\{x \in X \mid \operatorname{supp} x = \operatorname{supp} x'\right\} \\ &= X_{x'} \in \overline{\tau}_X. \end{aligned}$$

Thus, f is topological continuous. □

Lemma 4.4. *If $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ is a topological continuous map, then X_0 and X_+ defined in Remark 2.7 are closed equivariant subsets of X.*

Proof. Suppose f is topological continuous. We first show that

$$X_0 = \{x \in X \mid \operatorname{supp} f(x) = \emptyset\} = \bigcup_{t \in X_0} X_t$$

and therefore $X_0 \in \overline{\tau}_X$.

To prove the non-trivial inclusion, let $x' \in \bigcup_{t \in X_0} X_t$. Then $\operatorname{supp} x' = \operatorname{supp} t$ for some $t \in X_0$. Since f is continuous, we have $\approx_{\mathrm{Supp}} \subseteq \approx_f$, which implies $\operatorname{supp} f(x') = \operatorname{supp} f(t)$. But $\operatorname{supp} f(t) = \emptyset$ by definition of X_0. Thus, $x' \in X_0$, showing X_0 is closed.

By a similar argument, $X_+ \in \overline{\tau}_X$. □

Corollary 4.5. *If $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ is a topological continuous map, then $(X, \overline{\tau}_X)$ is disconnected.*

Remark 4.6. For a given equivariant map $f : X \to Y$ and $y \in Y$, note the following:

(i) If $\operatorname{supp} y = \emptyset$, then $f^{-1}(Y_y) = X_0$. Otherwise, if $\operatorname{supp} y \neq \emptyset$, then $f^{-1}(Y_y) \subseteq X_+$.

(ii) The condition

$$(x \in X_0 \ \Rightarrow \ X_x \subseteq X_0) \quad \Longleftrightarrow \quad (x \in X_+ \ \Rightarrow \ X_x \subseteq X_+)$$

holds.

Proposition 4.7. *Suppose $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ is topological continuous and $x \in X$. Then:*

(i) *If $x \in T$ where $T = X_0$ or X_+, then $X_x \subseteq T$.*

(ii) *We have $X_+ = \{x \in X \setminus \operatorname{Fix}(X) \mid \operatorname{supp} x = \operatorname{supp} f(x)\}$.*

Proof. (i) Let $x \in T$ and $t \in X_x$, so $\operatorname{supp} t = \operatorname{supp} x$. Since f is continuous, $\approx_{\mathrm{Supp}} \subseteq \approx_f$, implying $\operatorname{supp} f(t) = \operatorname{supp} f(x)$.

- If $x \in X_0$, then $\operatorname{supp} f(x) = \emptyset$, so $\operatorname{supp} f(t) = \emptyset$ and $t \in X_0$.
- If $x \in X_+$, then $\operatorname{supp} f(x) \neq \emptyset$, so $\operatorname{supp} f(t) \neq \emptyset$ and $t \in X_+$.

(ii) For $x \in X \setminus \operatorname{Fix}(X)$ with $\operatorname{supp} x = \operatorname{supp} f(x)$, clearly $x \in X_+$. Conversely, for $x \in X_+$, f being equivariant implies $\operatorname{supp} f(x) \subseteq \operatorname{supp} x$ and $\operatorname{supp} f(x) \neq \emptyset$, so $\operatorname{supp} x \neq \emptyset$.

To show $\operatorname{supp} x = \operatorname{supp} f(x)$, use $\approx_{\mathrm{Supp}} \subseteq \approx_f$ to deduce $\operatorname{supp} f(x) = \operatorname{supp} f((a\,a')\,x)$ (as in Proposition 2.8), proving equality.

Hence, $X_+ = \{x \in X \setminus \operatorname{Fix}(X) \mid \operatorname{supp} x = \operatorname{supp} f(x)\}$. □

Definition 4.8. Suppose $f : X \to Y$ is an equivariant map and $y \in Y$. We say that y is a *support-covered element by* f if there exists $x \in X$ such that $\operatorname{supp} f(x) = \operatorname{supp} y$. If every element of Y is support-covered by f, then f is called a *support-covered map.*

Notation 4.9. Let $f : X \to Y$ be an equivariant map between nominal sets. Then, the *support-covered image* of f, denoted by

$$\operatorname{SuppCov}(f) := \{y \in Y \mid \exists x \in X,\ \operatorname{supp} f(x) = \operatorname{supp} y\},$$

is an equivariant subset of Y containing the image $\operatorname{Im}(f)$. With this notation, the equivariant map f is support-covered if and only if $\operatorname{SuppCov}(f) = Y$

Example 4.10.
(1) Every surjective equivariant map is a support-covered map.
(2) Consider the nominal set $Y = (\mathbb{D} \times \{1\}) \cup (\mathbb{D} \times \{2\})$. Define the map $f : \mathbb{D} \to Y, \quad f(x) = (x, 1)$. Then, f is an equivariant map. Moreover, since $\operatorname{supp}(x, i) = \operatorname{supp} x$, for each $x \in \mathbb{D}$ and $i = 1, 2$, it follows that f is a support-covered map, but it is not surjective.

Remark 4.11. Let $f : X \to Y$ be an equivariant map. Then the restricted map $f_s : X \to \operatorname{SuppCov}(f), \quad f_s(x) = f(x)$, is also equivariant. Moreover, for all $y \in \operatorname{SuppCov}(f)$, we have $f_s^{-1}(y) = f^{-1}(y)$.

Lemma 4.12. *Suppose $f : X \to Y$ is an equivariant map. Then f is topological continuous if and only if f_s is topological continuous.*

Proof. We only prove the non-trivial direction. Assume that $f_s : X \to \operatorname{SuppCov}(f)$ is topological continuous and let $y \in Y$. If $f^{-1}(Y_y) = \emptyset$, then clearly $f^{-1}(Y_y) \in \overline{\tau}_X$. Otherwise, suppose $f^{-1}(Y_y) \neq \emptyset$. Then there exists $x \in f^{-1}(Y_y)$ such that $\operatorname{supp} f(x) = \operatorname{supp} y$. Hence, $y \in \operatorname{SuppCov}(f)$. Since f_s is topological continuous, we have $f_s^{-1}(Y_y) \in \overline{\tau}_X$. Note that $f^{-1}(Y_y) = \{x \in X \mid f(x) \in Y_y\} = \{x \in X \mid f_s(x) \in Y_y\} = f_s^{-1}(Y_y)$. Therefore, $f^{-1}(Y_y) \in \overline{\tau}_X$, which implies that f is topological continuous. □

Using Lemma 4.12, we focus on studying continuity for support-covered equivariant maps. But first, we give the following definition.

Definition 4.13. Let $f : X \to Y$ be an equivariant map between nominal sets X and Y. The equivalence relation

$$\approx_f := \left\{(x, x') \in X \times X \mid \operatorname{supp} f(x) = \operatorname{supp} f\left(x'\right)\right\}$$

is a congruence relation on X.

Remark 4.14. Let $f : X \to Y$. Then,

(i) For any $x \in X$, the equivalence class of x under $\approx_f$ is given by

$$\begin{aligned} x/\approx_{\mathsf{f}} &= \left\{x' \in X \mid \operatorname{supp} f\left(x'\right) = \operatorname{supp} f\left(x\right)\right\} \\ &= \left\{x' \in X \mid f\left(x'\right) \in Y_{f(x)}\right\} \\ &= f^{-1}\left(Y_{f(x)}\right). \end{aligned}$$

(ii) For any $\pi \in \operatorname{Perm}(\mathbb{D})$, we have

$$\pi\left(x/\approx_{\mathsf{f}}\right) = \pi f^{-1}\left(Y_{f(x)}\right) = f^{-1}\left(Y_{f(\pi x)}\right) = \left(\pi x\right)/\approx_{\mathsf{f}}.$$

(iii) For any $x \in X$, the set $f^{-1}\left(Y_{f(x)}\right)$ is a finitely supported subset of X and $\operatorname{supp} f^{-1}\left(Y_{f(x)}\right) \subseteq \operatorname{supp} f\left(x\right)$.

Proposition 4.15. *Let $f : X \to Y$ be an equivariant map between nominal sets X and Y. Then,*

(i) *The quotient set $X/\approx_{\mathsf{f}} = \left\{\mathsf{f}^{-1}\left(\mathsf{Y}_{\mathsf{f}(\mathsf{x})}\right) \mid \mathsf{x} \in \mathsf{X}\right\}$ is a nominal set.*
(ii) *There exists a surjective equivariant metric continuous map $q_f : X \to X/\approx_{\mathsf{f}}$.*

Proof. (i) It follows from Remark 4.14 (ii).

(ii) Consider the surjective equivariant map $q_f : X \to X/\approx_{\mathsf{f}}$ defined by $q_f\left(x\right) = f^{-1}\left(Y_{f(x)}\right)$. By Definition 3.1 (ii), $d\left(f^{-1}\left(Y_{f(x)}\right), f^{-1}\left(Y_{f(x')}\right)\right)$ is equal to

$$\min\left\{d\left(t,s\right) \mid t \in f^{-1}\left(Y_{f(x)}\right), s \in f^{-1}\left(Y_{f(x')}\right)\right\}.$$

Now since $\left(x \in f^{-1}\left(Y_{f(x)}\right)\right)$ and $\left(x' \in f^{-1}\left(Y_{f(x')}\right)\right)$, we have

$$\left(f^{-1}\left(Y_{f(x)}\right), f^{-1}\left(Y_{f(x')}\right)\right) \leq \ d\left(x,x'\right),$$

so q_f is metric continuous. □

Proposition 4.16. *Suppose $f : X \to Y$ is an equivariant map between nominal sets.*

(i) *$\approx_{\mathrm{Supp}} \subseteq \approx_f$ if and only if $X_x \subseteq f^{-1}\left(Y_{f(x)}\right)$, for every $x \in X$.*
(ii) *$\approx_f \subseteq \approx_{\mathrm{Supp}}$ if and only if $f^{-1}\left(Y_{f(x)}\right) \subseteq X_x$.*
(iii) *If $\approx_f \subseteq \approx_{\mathrm{Supp}}$ and f is a support-covered map, then $Y_{f(x)} \subseteq f\left(X_x\right)$.*

Proof. (i) ($\Rightarrow$) Assume $\approx_{\mathrm{Supp}} \subseteq \approx_f$. Let $x' \in X_x$. By definition of X_x, this means $\operatorname{supp} x' = \operatorname{supp} x$. Since $\approx_{\mathrm{Supp}} \subseteq \approx_f$, it follows that $(x,x') \in \approx_f$, which implies $\operatorname{supp} f\left(x'\right) = \operatorname{supp} f\left(x\right)$. Hence $x' \in f^{-1}\left(Y_{f(x)}\right)$. Thus, $X_x \subseteq f^{-1}\left(Y_{f(x)}\right)$.

($\Leftarrow$) Assume $X_x \subseteq f^{-1}\left(Y_{f(x)}\right)$, for every $x \in X$. To prove $\approx_{\mathrm{Supp}} \subseteq \approx_f$, let $(x,x') \in \approx_{\mathrm{Supp}}$. This means $\operatorname{supp} x' = \operatorname{supp} x$, so $x' \in X_x$. By our assumption, $x' \in f^{-1}\left(Y_{f(x)}\right)$, which implies $\operatorname{supp} f\left(x'\right) = \operatorname{supp} f\left(x\right)$. Therefore, $(x,x') \in \approx_f$.

(ii) This is proved analogously to (i).

(iii) Let $\approx_f \subseteq \approx_{\mathrm{Supp}}$ and assume f is a support-covered map. Let $t \in Y_{f(x)}$. By definition of $Y_{f(x)}$, we have $\operatorname{supp} t = \operatorname{supp} f(x)$. Since f is a support-covered map, there exists some $x' \in X$ such that $\operatorname{supp} f(x') = \operatorname{supp} t$. Combining these, we get $\operatorname{supp} f(x') = \operatorname{supp} f(x)$. This implies $(x, x') \in \approx_f$. Given $\approx_f \subseteq \approx_{\mathrm{Supp}}$, it follows that $(x, x') \in \approx_{\mathrm{Supp}}$, which means $\operatorname{supp} x' = \operatorname{supp} x$. Since $\operatorname{supp} x' = \operatorname{supp} x$, we have $x' \in X_x$. Finally, since $f(x') \in Y$ and $x' \in X_x$ and we have $\operatorname{supp} f(x') = \operatorname{supp} t$, this means t can be written as $f(x')$ for some $x' \in X_x$. Therefore, $t \in f(X_x)$.

□

Remark 4.17. If $f : X \to Y$ is an equivariant support-covered map, then for any given $t \in Y$, there exists $x \in X$ such that $\operatorname{supp} f(x) = \operatorname{supp} t$. Consequently, $Y_t = Y_{f(x)}$.

Lemma 4.18. *Let $f : X \to Y$ be an equivariant map and let $A \in \mathcal{P}_{\mathrm{fs}}(X)$, $B \in \mathcal{P}_{\mathrm{fs}}(Y)$. Then;*

(i) *If $\approx_{\mathrm{Supp}} \subseteq \approx_f$, then $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.*

(ii) *If f is a support-covered map and $\approx_f \subseteq \approx_{\mathrm{Supp}}$, then*

$$\overline{f(A)} \subseteq f\left(\overline{A}\right).$$

(iii) *If $\approx_{\mathrm{Supp}} \subseteq \approx_f$, then $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right)$.*

(iv) *If f is a support-covered map and $\approx_f \subseteq \approx_{\mathrm{Supp}}$, then*

$$f^{-1}\left(\overline{B}\right) \subseteq \overline{f^{-1}(B)}.$$

Proof. (i) We have $f\left(\overline{A}\right) = f\left(\bigcup_{t \in A} X_t\right) = \bigcup_{t \in A} f(X_t)$. By Proposition 4.16 (iii), $f(X_t) \subseteq Y_{f(t)}$. Thus,

$$f\left(\overline{A}\right) \subseteq \bigcup_{t \in A} Y_{f(t)} = \overline{f(A)}.$$

(ii) Since f is a support-covered map, for any $y \in Y$ there exists $x \in X$ such that $\operatorname{supp} y = \operatorname{supp} f(x)$. Assume $\approx_f \subseteq \approx_{\mathrm{Supp}}$ and let $y \in \overline{f(A)}$. Then there exists $z \in f(A)$ such that $\operatorname{supp} y = \operatorname{supp} z$. Since $z = f(t)$ for some $t \in A$, we have $\operatorname{supp} y = \operatorname{supp} f(x) = \operatorname{supp} f(t)$. By $\approx_f \subseteq \approx_{\mathrm{Supp}}$, $\operatorname{supp} x = \operatorname{supp} t$, which implies $x \in \overline{A}$. Applying f, $f(x) \in f\left(\overline{A}\right)$. Since $\operatorname{supp} y = \operatorname{supp} f(x)$, it follows that $y \in f\left(\overline{A}\right)$. Therefore, $\overline{f(A)} \subseteq f\left(\overline{A}\right)$.

(iii) Let $x \in \overline{f^{-1}(B)}$. Then there exists $t \in f^{-1}(B)$ such that $\operatorname{supp} x = \operatorname{supp} t$. Applying f and using the assumption,

$\operatorname{supp} f(x) = \operatorname{supp} f(t)$. Also, since $t \in f^{-1}(B)$, $f(t) \in B$. Hence, $f(x) \in \overline{B}$, which implies $x \in f^{-1}\left(\overline{B}\right)$. Thus,

$$\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right).$$

(iv) Assume f is support-covered and $\approx_f \subseteq \approx_{\mathrm{Supp}}$. Let $x \in f^{-1}\left(\overline{B}\right)$. Then $f(x) \in \overline{B}$, so there exists $y \in B$ such that $\operatorname{supp} f(x) = \operatorname{supp} y$. Since f is support-covered, there exists $x' \in X$ with $\operatorname{supp} f(x') = \operatorname{supp} y$. By $\approx_f \subseteq \approx_{\mathrm{Supp}}$, $\operatorname{supp} x = \operatorname{supp} x'$, which implies $x \in \overline{\{x'\}} \subseteq \overline{f^{-1}(B)}$. Therefore, $f^{-1}\left(\overline{B}\right) \subseteq \overline{f^{-1}(B)}$. □

Definition 4.19. Let $f : X \to Y$ be an equivariant map between nominal sets. The map f is called *closed* if for every $t \in X$, the image of the orbit X_t under f is a closed set in Y, i.e., $f(X_t) \in \overline{\tau}_Y$.

Theorem 4.20. *Let $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ be an equivariant support-covered map. Then;*

(i) *If $\approx_{\mathrm{Supp}} = \approx_f$, then f is topological continuous and closed.*
(ii) *f is topological continuous if and only if $\approx_{\mathrm{Supp}} \subseteq \approx_f$.*

Proof. (i) Suppose $\approx_{\mathrm{Supp}} = \approx_f$. For a given $t \in Y$, by Remark 4.17 there exists $x \in X$ such that $\operatorname{supp} t = \operatorname{supp} f(x)$ and hence $Y_t = Y_{f(x)}$. Applying Proposition 4.16, we have $f^{-1}\left(Y_{f(x)}\right) = X_x$ and $f(X_x) = Y_{f(x)}$. Therefore, f is both topological continuous and closed.

(ii) Assume f is topological continuous. For all $t \in Y$, by Remark 4.17 there exists $x \in X$ such that $\operatorname{supp} t = \operatorname{supp} f(x)$. Since $x \in f^{-1}\left(Y_{f(x)}\right)$ and $f^{-1}\left(Y_{f(x)}\right)$ is closed in X, it follows that $X_x \subseteq f^{-1}\left(Y_{f(x)}\right)$, which implies $\approx_{\mathrm{Supp}} \subseteq \approx_f$.

To prove the other direction, suppose $\approx_{\mathrm{Supp}} \subseteq \approx_f$. Since f is a support-covered map, by Remark 4.17, it suffices to show that $f^{-1}\left(Y_{f(x)}\right) = \bigcup_{t \in f^{-1}\left(Y_{f(x)}\right)} X_t$. For the non-trivial part, assume $x' \in \bigcup_{t \in f^{-1}\left(Y_{f(x)}\right)} X_t$. Then $\operatorname{supp} x' = \operatorname{supp} t$ for some $t \in f^{-1}\left(Y_{f(x)}\right)$. Since $\approx_{\mathrm{Supp}} \subseteq \approx_f$ and $(x', t) \in \approx_{\mathrm{Supp}}$, it follows that $\operatorname{supp} f(x') = \operatorname{supp} f(t)$. On the other hand, $t \in f^{-1}\left(Y_{f(x)}\right)$ implies $\operatorname{supp} f(t) = \operatorname{supp} f(x)$. Therefore, $\operatorname{supp} f(x) = \operatorname{supp} f(x')$ and hence $x' \in f^{-1}\left(Y_{f(x)}\right)$. □

Theorem 4.21. *Let $f : (X, \overline{\tau}_X) \to (Y, \overline{\tau}_Y)$ be an equivariant map and $x \in X$. Then f is topological continuous if and only if:*

(i) *For $x \in X_+$, $\operatorname{supp} x = \operatorname{supp} f(x)$.*
(ii) *For $x \in X_0$, $X_x \subseteq X_0$.*

Proof. If f is topological continuous, the result follows from Proposition 4.7.

Conversely, assume (i) and (ii). To show $\approx_{\mathrm{Supp}} \subseteq \approx_f$, let $\operatorname{supp} x = \operatorname{supp} x'$, so $x' \in X_x$.

- If $x \in X_0$, then by (ii), $X_x \subseteq X_0$ and $\operatorname{supp} f(x) = \operatorname{supp} f(x') = \emptyset$.
- If $x \in X_+$, then $X_x \subseteq X_+$, so

$$\operatorname{supp} f(x) = \operatorname{supp} x = \operatorname{supp} x' = \operatorname{supp} f\left(x'\right).$$

Thus, $\approx_{\mathrm{Supp}} \subseteq \approx_f$, which implies f is topological continuous. □

Corollary 4.22. *Every metric continuous equivariant map is topological continuous with respect to the support closure topology.*

Proof. We verify conditions (i) and (ii) of Theorem 4.21. Let $f : X \to Y$ be metric continuous. By Corollary 2.9, condition (i) of Theorem 4.21 holds. For $x \in X_0$ and $x' \in X_x$, if $x' \in X_+$, then $|\operatorname{supp} x'| < k$ by Proposition 2.11, contradicting the minimality of $k = \operatorname{Min}\{|\operatorname{supp} x| \mid x \in X_0 \setminus \operatorname{Fix}(X)\}$. Thus, $x' \in X_0$, establishing condition (ii). □

Considering Example 2.6 (3), we show that the converse of Corollary 4.22 does not hold.

Example 4.23 demonstrates that an equivariant map may be topological continuous but not metric continuous, emphasizing the strict distinction between these two notions of continuity.

Example 4.23. Let $X = \mathbb{D}^{(2)} \cup \mathbb{D}^{(3)}$ and $Y = \mathbb{D}^{(2)} \cup \{\theta\}$. Define the map $f : X \to Y$ by $f(d_1, d_2) = (d_2, d_1)$ for every $(d_1, d_2) \in \mathbb{D}^{(2)}$ and $f(d_1, d_2, d_3) = \theta$, for every $(d_1, d_2, d_3) \in \mathbb{D}^{(3)}$. By Example 4.22, f is not metric continuous. While

$$\overline{\tau}_X = \left\{\emptyset, X, \mathbb{D}^{(3)}, \text{finite unions of T}\right\},$$

where $T = \mathcal{P}_2\left(\mathbb{D}^{(2)}\right) \cup \mathcal{P}_4\left(\mathbb{D}^{(2)}\right) \cup \mathcal{P}_6\left(\mathbb{D}^{(2)}\right) \cup \ldots$, in which $\mathcal{P}_n\left(\mathbb{D}^{(2)}\right)$ denotes the collection of all n-element subsets of $\mathbb{D}^{(2)}$. Then, $f^{-1}(Y_\theta) = \mathbb{D}^{(3)} \in \overline{\tau}_X$ and $f^{-1}\left(Y_{(d_1,d_2)}\right) = \{(d_1, d_2), (d_2, d_1)\} \in \overline{\tau}_X$, so f is topological continuous but not metric continuous.

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