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The Optimal Operator Formulation of The Hermite-Hadamard and Fejér Inequalities

Corina Karim[1*], Ekadion Maulana[2], Ratno Bagus Edy Wibowo[3], Abdul Rouf Alghofari[4], Adem Kilicman[5], Nor Hanimah Kamis[6] and Mila Kurniawaty[7]

Abstract. This paper establishes new operator versions of the Hermite-Hadamard and Fejér inequalities for the class of operator (h, m)-convex functions on Hilbert spaces. Operator (h, m)-convex functions generalize operator convex, operator m-convex and operator h-convex functions by including a nonnegative function h and a parameter $m \in (0, 1]$. The results presented not only generalize and improve several known inequalities but also represent best possible generalizations for certain parameter selections such as $h(\kappa) = \kappa$ and $m = 1$. Additionally, Fejér type inequalities are derived by integrating symmetric weights, demanding integrability of the product of functions on operator domains. The approach generalizes integral inequality techniques in a classical manner to operators, offering new perspectives for functional analysis and operator theory.

1. Introduction

Convexity is a concept in Mathematics that has essential roles in the study of applied Mathematics [18]. In this study, one must know what convex set and convex function is. According to [31], a set $K \subseteq \mathbb{R}^n$ is called a convex if for any $u, v \in K$ and $\kappa \in (0, 1)$, then the following statement hold $\kappa u + (1 - \kappa)v \in K$..

Meanwhile, a function $\xi : K \to \mathbb{R}$ is said to be convex if for any $u, v \in K$ and $\kappa \in (0, 1)$, it holds that

$$\xi(\kappa u + (1 - \kappa)v) \leq \kappa\xi(u) + (1 - \kappa)\xi(v). \tag{1.1}$$

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* Corresponding author.

If the inequality sign in (1.1) is reversed, then ξ is called a concave functions. So, when ξ is a convex function, then $-\xi$ is a concave function and vice versa.

Two significant integral inequalities pertaining to convex functions are the Hermite-Hadamard (H-H) inequality and the Fejér inequality. The H-H inequality was firstly introduced by mathematician Jacques Hadamard in 1893 [17]. Let $I = [t, s] \subseteq \mathbb{R}$ be an interval. If $\xi : I \to \mathbb{R}$ is a convex function on I, then the following holds:

$$(b-a)\xi\left(\frac{a+b}{2}\right) \leq \int_a^b \xi(u)\,du \leq (b-s)\left(\frac{\xi(a)+\xi(b)}{2}\right).$$

The Hermite-Hadamard-Fejér inequality or simply called the Fejér inequality was firstly studied by Fejér as a weighted H-H inequality in 1905 [13]. As a generalization of the H-H inequality, the Fejér inequality is constructed by adding a symmetric weight function $\eta : I \to [0, \infty)$ with respect to $\frac{t+s}{2}$, resulting in:

$$\begin{aligned}(b-a)\xi\left(\frac{a+b}{2}\right)\int_a^b \eta(u)\,du &\leq \int_a^b \xi(u)\eta(u)\,du \\ &\leq (b-a)\left(\frac{\xi(a)+\xi(b)}{2}\right)\int_a^b \eta(u)\,du.\end{aligned}$$

Several new classes of convexity have been developed by mathematicians alongside the H-H and Fejér type inequalities. In 1984, the definition of m-convex sets, m-convex functions and the H-H type inequalities for m-convex functions were studied by Toader [36]. Here, $m \in [0, 1]$ and $I = [0, b] \subseteq \mathbb{R}$. According to [21], when ξ is an m-convex function with $m = 0$, one can refer to ξ as a star-shaped function. The Fejér type inequalities for m-convex functions were studied by Bracamonte et al. in 2015 [7]. In 2007, Varošanec [37] studied the definition and properties of a new generalization of convex functions where the domain of the function is still a convex set. It is called the h-convex function. Other classes in convexity generalized by h-convexity include Godunova-Levin functions (see [16, 23]), P-functions (see [30]), s-convex functions (see [19, 35, 38]) and s-Godunova-Levin functions (see [2, 28]). The (H-H) and Fejér type inequalities for h-convex functions were studied by Bombardelli and Varošanec [6]. In 2011, Özdemir et al. [29] introduced the definition of (h, m)-convex functions by combining h-convexity and m-convexity. They also studied the H-H inequality for functions that satisfy the (h, m)-convexity condition. Meanwhile, the Fejér inequalities for (h, m)-convex functions have been studied by various researchers in different ways; for more details, see [1, 20]. This class generalizes other

convexity classes, such as (s,m)-convex functions (see [3, 8, 34]) and (s,m)-Godunova-Levin functions (see [27]).

One of the significant convexity classes is that of operator convex functions, an operator version of the convex functions. The motivation for defining operator convex functions originates from Theorem 1 below. First, we denote that H is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and $\mathbf{1}_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$.

Theorem 1.1 ([14]). *Let $\mathbb{B}(\mathcal{H}) = \{S : \mathcal{H} \to \mathcal{H} | S$ is bounded and linear$\}$ and $\sigma(S)$ is the spectrum of S. Then there exists a unique $*$-isometric isomorphism, known ans the Gelfand Map, $\Lambda : C(\sigma(S)) \to \mathbb{B}(\mathcal{H})$, such that for any $\xi_1, \xi_2 \in C(\sigma(S))$, the following properties hold*

(i) $\Lambda(a\xi + b\eta) = a\Lambda(\xi) + b\Lambda(\eta)$,
(ii) $\Lambda(\xi\eta) = \Lambda(\xi)\Lambda(\eta), \quad \Lambda(\overline{\xi}) = \Lambda(\xi)^*$
(iii) $|\Lambda(\xi)| = |\Lambda(\xi)|$,
(iv) *If* $\xi_0(t) = 1$ *and* $\xi_1(t) = t$, *then* $\Lambda(\xi_0) = \mathbf{1}_{\mathcal{H}}, \quad \Lambda(\xi_1) = \xi_1$ *respectively.*
(v) $\xi_1(u) \geq 0$, *then* $\Lambda(\xi_1)$ *is a positive operator.*

In the class of operator convex functions, the operator is must be self-adjoint and belongs to the family of bounded and linear operators in Hilbert space. Furthermore, the spectra of the operator $\sigma(\cdot)$ must be contained in the domain of the function. According to [5], $\xi : I \to \mathbb{R}$ is said to be operator convex on interval I, if for any self-adjoint operators $T, S \in \mathbb{B}(\mathcal{H})$ whose spectra are contained in I and for any $\kappa \in (0,1)$, the following inequality holds

$$\xi(\kappa T + (1-\kappa)S) \preceq \kappa\xi(T) + (1-\kappa)\xi(S). \tag{1.2}$$

Here, the notation $T \preceq S$ means $\langle Tu, u\rangle \leq \langle Su, u\rangle$ for all $u \in \mathcal{H}$. In 2011, Dragomir [11] studied the H-H type inequalities for operator convex functions.

The definition of operator (h,m)-convex functions is based on the (h,m)-convexity concept. In 2017, Salaş et al. [33] introduced the definition of operator (h,m)-convex functions and studied some H-H type inequalities for this class. This class of convex operator functions is a generalization of several classes, such as operator s-convex functions (see [15, 25]), Godunova-Levin operator functions (see [32]), s-Godunova-Levin operator functions (see [26]), operator h-convex functions (see [9, 10]) and operator m-convex functions ([4]). Since the Fejér type inequalities are generalizations of the H-H type inequalities, we are interested in studying these inequalities. We also find different versions of H-H type inequalities for operator (h,m)-convex functions that are not the same as those in Salaş et al. [33].

2. Preliminaries

In this part, we recall some definitions and theorems that support obtaining our results. We also introduce some notations for convenience in reading this paper.

The domain of m-convex functions is m-convex set. The definition of m-convex sets and m-convex functions can be seen in the following definitions.

Definition 2.1 ([22, 36])**.** Let $L \subseteq \mathbb{R}$ be a linear space. Suppose $K \subseteq L$ is a non-empty set and $m \in [0,1]$. V is called to be an m-convex set if for every $u, v \in K$ and $\kappa \in (0,1)$, it holds that $\kappa u + m(1-\kappa)v \in K$.

From Definition 2.1, ones can easily prove that for $b > 0$, the interval $I_1 = [0, b]$ and $I_2 = [0, \infty)$ is an m-convex set for $m \in [0,1]$.

Definition 2.2 ([28])**.** Let $I = [0,b] \subset \mathbb{R}$, $m \in [0,1]$ and $h : J \supset (0,1) \to [0,\infty)$. A function $\xi : I \to \mathbb{R}$ is said to be an (h,m)-convex function if for every $u, v \in V$ and $\kappa \in (0,1)$, it holds

$$\xi(\kappa u + m(1-\kappa)v) \leq h(\kappa)\xi(u) + mh(1-\kappa)\xi(v).$$

Remark 2.3. Let $\xi : I \to \mathbb{R}$ be an (h,m)-convex function. If $h(\kappa) = \kappa$, then ξ is an m-convex function. Meanwhile, if $m = 1$, then ξ is an h-convex function.

By using Theorem 1.1, we can define $\xi(S) := \Lambda(\xi)$ for every $\xi \in C(\sigma(S))$, also called the continuous functional calculus for self-adjoint operators $S \in \mathbb{B}(\mathcal{H})$. For instance, let $S \in \mathbb{B}(\mathcal{H})$ and $\xi(t) = t^2 + 2$ for $u \in \sigma(T)$, then we have $\xi(S) = S^2 + 2 \cdot 1_{\mathcal{H}}$.

It can be easily observed that, for any self-adjoint operator $S \in \mathbb{B}(\mathcal{H})$ and $\xi \in C(\sigma(S))$, if $\alpha \in \sigma(S)$ and $\xi(\alpha) \geq 0$, then $\xi(S)$ is a positive operator on $\mathcal{H}$. Furthermore, if for each $u \in \sigma(T)$, $\xi(\alpha) \leq \eta(u)$ (with $\xi, \eta \in C(\sigma(S))$), then $\xi(S) \preceq \eta(S)$.

Let $\mathcal{K} \subseteq \mathbb{B}^+(\mathcal{H}) = \{S : \mathcal{H} \to \mathcal{H} \mid S \text{ is positive, bounded and linear}\}$ be an m-convex set. Now we present the definition of operator (h,m)-convex functions. Hence, by using the Definition 2.2, Salaş et al. [33], construct operator (h,m)-convex function's definition as follows.

Definition 2.4 ([33])**.** Let $I = [0,b] \subseteq \mathbb{R}$, $m \in [0,1]$ and $h : J \supseteq [0,1] \to [0,\infty)$. A function $\xi : I \to \mathbb{R}$ is called an operator (h,m)-convex function on I if for any self-adjoint operators $T, S \in K$ with $\sigma(T), \sigma(S) \subseteq I$ and $\kappa \in (0,1)$, it holds

$$\xi(\kappa T + m(1-\kappa)S) \preceq h(\kappa)\xi(T) + mh(1-\kappa)\xi(S).$$

Remark 2.5. Let $\xi : I \to \mathbb{R}$ satisfies the operator (h,m)-convex function condition for operators in K. If $h(\kappa) = \kappa$, then ξ is called an

operator m-convex function. If $m = 1$, then ξ is said to be an operator h-convex function.

Example 2.6. Let $h(\kappa) = 3\kappa$, $m \in (0, 1]$ and $\mathcal{G} = \{T \mid T \in \mathbb{B}^+(\mathcal{H}), T = T^*, \text{and } \sigma(T) = \{\lambda \in \mathbb{R}^+ \mid \lambda \leq 1\}\}$. A function ξ, defined by

$$\xi(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x < 1, \\ \\ \frac{3}{2}x, & 1 \leq x \leq 2, \end{cases}$$

is an operator (h, m)-convex function for operators in $\mathcal{G}$.

The proof of Example 2.6 can be seen in [24].

3. Hermite-Hadamard Type Inequalites

We provide some new types of H-H type inequalities for operator (h, m)-convex functions where the operators are in $\Omega_{m,T,S} := \{\kappa T + m(1-\kappa)S \mid T, S \in \mathcal{K}, \kappa \in [0, 1], m \in (0, 1]\}$. and $\Omega_{1,T,S} := \{\kappa T + (1-\kappa)S \mid \kappa \in [0, 1]\}$.

Theorem 3.1. *Let $m \in (0, 1]$ and $h : J \supseteq (0, 1) \to [0, \infty)$. Suppose that $\xi : I = [0, \infty) \to \mathbb{R}$ is an operator (h, m)-convex function for operators in $\Omega_{1,T,S}$. Then, for any self-adjoint operators $T, S \in \mathcal{K}$, with $T \preceq S$ and $\sigma(T), \sigma(S) \subset I$, it holds that*

(3.1)

$$\int_0^1 \xi(\kappa T + (1-\kappa)S)\, d\kappa \preceq \min\left\{ \left[\xi(T) + m\xi\left(\frac{S}{m}\right)\right] \int_0^1 h(\kappa)\, d\kappa, \left[\xi(S) + m\xi\left(\frac{T}{m}\right)\right] \int_0^1 h(\kappa)\, d\kappa \right\}.$$

Proof. Let $\kappa T + (1-\kappa)S \in \Omega_{1,T,S}$. Then, it holds that

$$\begin{aligned} \xi(\kappa T + (1-\kappa)S) &= \xi\left(\kappa T + m(1-\kappa)\frac{S}{m}\right) \\ &\preceq h(\kappa)\xi(T) + mh(1-\kappa)\xi\left(\frac{S}{m}\right), \end{aligned} \tag{3.2}$$

and similarly, we have

$$\begin{aligned} \xi(\kappa S + (1-\kappa)T) &= \xi\left(\kappa S + m(1-\kappa)\frac{T}{m}\right) \\ &\preceq h(\kappa)\xi(S) + mh(1-\kappa)\xi\left(\frac{T}{m}\right). \end{aligned} \tag{3.3}$$

By integrating the inequality (3.2) with respect to $\kappa \in [0,1]$, we obtain

$$\int_0^1 \xi(\kappa T + (1-\kappa)S)\, d\kappa \preceq \int_0^1 \left[h(\kappa)\xi(T) + mh(1-\kappa)\xi\left(\frac{S}{m}\right)\right] d\kappa$$
$$= \xi(T)\int_0^1 h(\kappa)\, d\kappa + m\xi\left(\frac{S}{m}\right)\int_0^1 h(1-\kappa)\, d\kappa.$$

Since $\int_0^1 h(\kappa)\, d\kappa = \int_0^1 h(1-\kappa)\, d\kappa$, we obtain

$$\int_0^1 \xi(\kappa T + (1-\kappa)S)\, d\kappa \preceq \left[\xi(T) + m\xi\left(\frac{S}{m}\right)\right]\int_0^1 h(\kappa)\, d\kappa.$$

Using the same process for the inequality (3.3), we also get

$$\int_0^1 \xi(\kappa S + (1-\kappa)T)\, d\kappa \preceq \left[\xi(S) + m\xi\left(\frac{T}{m}\right)\right]\int_0^1 h(\kappa)\, d\kappa.$$

Hence, taking the minimum of the two gives the result. □

From the Theorem 3.1, we obtain the corollary below.

Corollary 3.2. *If $h(\kappa) = \kappa$, from Inequality (3.1), we get the H-H type inequality for operator m-convex functions (see [12]).*

For more clear explanation, we might pay attention to this following Example 3.3. We will see how the Theorem 3.1 works.

Example 3.3. Let us consider the family of self-adjoint matrices in $\mathcal{M}_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{C}\right\}$ given by $\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| 0 < a < 1\right\}$.

By Taking $T = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$. Then, using the function ξ as defined in the Example 2.6, we can compute:

$$R_{\text{left}} = \int_0^1 \xi\left(\kappa T + (1-\kappa)S\right)\, d\kappa$$
$$= \int_0^1 \xi\left(\begin{pmatrix} \kappa a + (1-\kappa)b & 0 \\ 0 & 0 \end{pmatrix}\right) d\kappa$$
$$= \frac{1}{2}\begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, substituting T and S into the right-hand side, we have:

$$R_{\text{right}} = \left[\xi(T) + m\xi\left(\frac{S}{m}\right)\right]\int_0^1 h(\kappa)\, d\kappa,$$

where $m = 1$ and $h(\kappa) = 3\kappa$. Thus,

$$R_{\text{right}} = \left[\frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right] \times \int_0^1 3\kappa\, d\kappa$$

$$= \frac{3}{4}\begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}.$$

Ts we can see, $R_{\text{left}} \preceq R_{\text{right}}$ (satisfies the inequality (3.1)), which verifies that our theorem holds.

Theorem 3.4. *Let $m \in (0,1]$ and $h : J \supseteq (0,1) \to [0,\infty)$. Suppose that $\xi : I = [0,\infty) \to \mathbb{R}$ is an operator (h,m)-convex function for operators in $\Omega_{m,T,S}$. Then, for two arbitrary self-adjoint operators $T, S \in \mathcal{K}$ such that $\sigma(T), \sigma(S) \subset I$, we have the following statement:*

(3.4)

$$\begin{aligned} &\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+mS}{2}\right) \\ &\quad \preceq \int_0^1 \xi(\kappa T + m(1-\kappa)S)\,d\kappa + m\int_0^1 \xi\left(\frac{m\kappa S + (1-\kappa)T}{m}\right)d\kappa \\ &\quad \preceq \left[\xi(T) + m\xi(S) + m^2\xi\left(\frac{S}{m}\right) + m\xi\left(\frac{T}{m}\right)\right]\int_0^1 h(\kappa)\,d\kappa. \end{aligned}$$

Proof. Firstly, we prove the left-hand side (LHS) of (3.4). Since ξ is an operator (h,m)-convex function, for $\kappa = \frac{1}{2}$ it holds that

$$\xi\left(\frac{A + m\frac{B}{m}}{2}\right) \preceq h\left(\frac{1}{2}\right)\left[\xi(A) + m\xi\left(\frac{B}{m}\right)\right], \tag{3.5}$$

for every self-adjoint operator $A, B \in \Omega_{m,T,S}$. By taking

$$A = \kappa T + m(1-\kappa)S, \qquad B = m\kappa S + (1-\kappa)T, \tag{3.6}$$

and substituting equation (3.6) into inequality (3.5), we obtain

(3.7)

$$\xi\left(\frac{T+mS}{2}\right) \preceq h\left(\frac{1}{2}\right)\left[\xi(\kappa T + m(1-\kappa)S) + m\xi\left(\frac{m\kappa S + (1-\kappa)T}{m}\right)\right].$$

Integrating (3.7) over $\kappa \in [0,1]$, we have

$$\xi\left(\frac{T+mS}{2}\right) \preceq h\left(\frac{1}{2}\right)\int_0^1 [\xi(\kappa T+m(1-\kappa)S) +m\xi\left(\frac{m\kappa S+(1-\kappa)T}{m}\right)]\,d\kappa$$
$$= h\left(\frac{1}{2}\right)\left[\int_0^1 \xi(\kappa T+m(1-\kappa)S)\,d\kappa +m\int_0^1 \xi\left(\frac{m\kappa S+(1-\kappa)T}{m}\right)\,d\kappa\right].$$

Thus, dividing by $h\left(\frac{1}{2}\right)$, we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+mS}{2}\right) \preceq \int_0^1 \xi(\kappa T+m(1-\kappa)S)\,d\kappa + m\int_0^1 \xi\left(\frac{m\kappa S+(1-\kappa)T}{m}\right)\,d\kappa, \tag{3.8}$$

which proves the left-hand side of (3.4).

Next, we prove the right-hand side (RHS) of (3.4). Since ξ is an operator (h,m)-convex function, we have

$$\int_0^1 \xi(\kappa T+m(1-\kappa)S)\,d\kappa \preceq \int_0^1 [h(\kappa)\xi(T)+mh(1-\kappa)\xi(S)]\,d\kappa. \tag{3.9}$$

Since $\sigma(T),\sigma(S)\subset I$ and by using the linearity of integral, then (3.9) can be rewritten as

$$\int_0^1 \xi(\kappa T+m(1-\kappa)S)\,d\kappa \preceq \xi(T)\int_0^1 h(\kappa)\,d\kappa + m\xi(S)\int_0^1 h(1-\kappa)\,d\kappa.$$

Similarly, we obtain

$$\int_0^1 \xi\left(\frac{m\kappa S+(1-\kappa)T}{m}\right)d\kappa \preceq m\xi\left(\frac{S}{m}\right)\int_0^1 h(\kappa)\,d\kappa + \xi\left(\frac{T}{m}\right)\int_0^1 h(1-\kappa)\,d\kappa.$$

Since

$$\int_0^1 h(\kappa)\,d\kappa = \int_0^1 h(1-\kappa)\,d\kappa,$$

it follows that

$$\int_0^1 \left[\xi(\kappa T+m(1-\kappa)S)+m\xi\left(\frac{m\kappa S+(1-\kappa)T}{m}\right)\right]d\kappa \preceq \left[\xi(T)+m\xi(S)+m^2\xi\left(\frac{S}{m}\right)+m\xi\left(\frac{T}{m}\right)\right]\int_0^1 h(\kappa)\,d\kappa. \tag{3.10}$$

Thus, (3.8) and (3.10) complete the proof. □

Now, we have the following example for Theorem 3.4.

Example 3.5. Let us use the same space as in the previous example. We can obtain:

$$R_{\text{left}} = \frac{1}{6}\begin{pmatrix} a+mb & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_{\text{mid}} = \int_0^1 \frac{1}{2}\left(\kappa T + m(1-\kappa)S\right) d\kappa + m\int_0^1 \frac{1}{2}\left(\frac{m\kappa S + (1-\kappa)T}{m}\right) d\kappa$$
$$= \begin{pmatrix} a+mb & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_{\text{right}} = \frac{3}{2}\begin{pmatrix} a+mb & 0 \\ 0 & 0 \end{pmatrix}.$$

As we can see, it holds that $R_{\text{left}} \preceq R_{\text{mid}} \preceq R_{\text{right}}$.

From the Theorem 3.4, we obtain the following corollary.

Corollary 3.6. (i) *If $m = 1$, from (3.4), we get the H-H type inequalities for operator h-convex functions (see [10]).*

(ii) *If $m = 1$ and $h(\kappa) = \kappa$, from (3.4), we get the H-H type inequalities for operator convex functions (see [11]).*

Theorem 3.7. *Let $I = [0, \infty) \subseteq \mathbb{R}$, $m \in (0, 1]$ and $h : J \supseteq (0, 1) \to [0, \infty)$. Suppose that $\xi : I \to \mathbb{R}$ is an operator (h, m)-convex function for operators in $\Omega_{1,T,S}$. If $T, S \in \mathcal{K}$ are self-adjoint operators with $\sigma(T), \sigma(S) \subset I$, then the following holds:*

(3.11)

$$\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+S}{2}\right)$$
$$\preceq \int_0^1 \left[\xi(\kappa T + (1-\kappa)S) + m\xi\left(\frac{\kappa T + (1-\kappa)S}{m}\right)\right] d\kappa$$
$$\preceq \left[\xi(T) + m\left\{\xi\left(\frac{T}{m}\right) + \xi\left(\frac{S}{m}\right)\right\} + m^2\xi\left(\frac{S}{m^2}\right)\right]\int_0^1 h(\kappa)\, d\kappa.$$

Proof. For the LHS of Inequality (3.11), let us take $A, B \in \Omega_{1,T,S}$. Since ξ is an operator (h, m)-convex function for operators in $\Omega_{1,T,S}$, we have

$$\xi\left(\frac{A+B}{2}\right) = \xi\left(\frac{A + m\frac{B}{m}}{2}\right) \preceq h\left(\frac{1}{2}\right)\left[\xi(A) + m\xi\left(\frac{B}{m}\right)\right],$$

for any $m \in (0, 1]$. By taking

$$A = \kappa T + (1-\kappa)S, \qquad B = \kappa S + (1-\kappa)T,$$

and substituting into the above, we get

(3.12)
$$\xi\left(\frac{T+S}{2}\right) \preceq h\left(\frac{1}{2}\right)\left[\xi(\kappa T+(1-\kappa)S)+m\xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)\right].$$

By integrating (3.12) over $\kappa \in [0,1]$ yields

$$\xi\left(\frac{T+S}{2}\right) \preceq h\left(\frac{1}{2}\right)\left[\int_0^1 \xi(\kappa T+(1-\kappa)S)\,d\kappa + m\int_0^1 \xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)d\kappa\right].$$

Since

$$\int_0^1 \xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)d\kappa = \int_0^1 \xi\left(\frac{\kappa T+(1-\kappa)S}{m}\right)d\kappa,$$

we can write

$$\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+S}{2}\right) \preceq \int_0^1 \xi(\kappa T+(1-\kappa)S)\,d\kappa + m\int_0^1 \xi\left(\frac{\kappa T+(1-\kappa)S}{m}\right)d\kappa.$$

Now, for the RHS, we consider again that $A=\kappa T+(1-\kappa)S$ and $B=\kappa S+(1-\kappa)T$. We have

$$\xi(A)+m\xi\left(\frac{B}{m}\right)=\xi(\kappa T+(1-\kappa)S)+m\xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right).$$

By using the assumption that ξ is operator (h,m)-convex, we have

(3.13)
$$\begin{aligned}\xi(\kappa T+(1-\kappa)S)+m\xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)&\\ \preceq h(\kappa)\xi(T)+mh(1-\kappa)\xi\left(\frac{S}{m}\right)&\\ +m\left[mh(\kappa)\xi\left(\frac{S}{m^2}\right)+h(1-\kappa)\xi\left(\frac{T}{m}\right)\right].&\end{aligned}$$

Next, by integrating (3.13) over $\kappa \in [0,1]$ gives

$$\begin{aligned}(3.14)\quad &\int_0^1 \left[\xi(\kappa T+(1-\kappa)S)+m\xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)\right]d\kappa\\ &\preceq \xi(T)\int_0^1 h(\kappa)\,d\kappa + m\left[\xi\left(\frac{T}{m}\right)+\xi\left(\frac{S}{m}\right)\right]\int_0^1 h(1-\kappa)\,d\kappa\\ &\quad + m^2\xi\left(\frac{S}{m^2}\right)\int_0^1 h(\kappa)\,d\kappa.\end{aligned}$$

Since $\int_0^1 h(\kappa)\,d\kappa = \int_0^1 h(1-\kappa)\,d\kappa$, then (3.14) can be simplified to

$$\begin{aligned}&\int_0^1 \left[\xi(\kappa T+(1-\kappa)S)+m\xi\left(\frac{\kappa S+(1-\kappa)T}{m}\right)\right]d\kappa\\ &\preceq \left[\xi(T)+m\left\{\xi\left(\frac{T}{m}\right)+\xi\left(\frac{S}{m}\right)\right\}+m^2\xi\left(\frac{S}{m^2}\right)\right]\int_0^1 h(\kappa)\,d\kappa.\ \square\end{aligned}$$

4. Fejér Type Inequalities

The concepts of Fejér type inequalities are similar to the H-H type inequalities. However, there is an addition of weight functions (denoted as η). Thus, for these inequalities, the product of the functions ξ and η must be integrable on their domain, that is, $\xi\eta \in L^1$.

We present the Fejér type inequalities for operator (h,m)-convex functions in the following Theorem 4.1 up to Theorem 4.5.

Theorem 4.1. *Let $m \in (0,1]$ and $h : J \supseteq (0,1) \to [0,\infty)$. Suppose that $\xi : I = [0,\infty) \to \mathbb{R}$ is an operator (h,m)-convex function and $\eta : I \to [0,\infty)$ such that*

$$\eta(\kappa t+(1-\kappa)s) = \eta(\kappa s+(1-\kappa)t) \quad \text{for all} \quad \kappa \in [0,1] \quad \text{and} \quad t,s \in I.$$

Then, for any self-adjoint operators $T, S \in K$ with $\sigma(T), \sigma(S) \subset I$, the following inequality holds:

$$\begin{aligned}(4.1)\quad &\int_0^1 \xi\eta(\kappa T+(1-\kappa)S)\,d\kappa \preceq \left[\xi(T)+m\xi\left(\frac{S}{m}\right)\right]\\ &\qquad\qquad \times \int_0^1 \eta(\kappa T+(1-\kappa)S)h(\kappa)\,d\kappa.\end{aligned}$$

Proof. Since ξ is an operator (h,m)-convex function, we have

$$\xi\eta(\kappa T+(1-\kappa)S)=\xi\left(\kappa T+m(1-\kappa)\frac{S}{m}\right)\eta(\kappa T+(1-\kappa)S) \tag{4.2}$$
$$\preceq\left[h(\kappa)\xi(T)+mh(1-\kappa)\xi\left(\frac{S}{m}\right)\right]\times\eta(\kappa T+(1-\kappa)S).$$

Let us integrate (4.2) over $\kappa\in[0,1]$, then it yields

$$\int_0^1\xi\eta(\kappa T+(1-\kappa)S)\,d\kappa\preceq\xi(T)\int_0^1 h(\kappa)\eta(\kappa T+(1-\kappa)S)\,d\kappa+m\xi\left(\frac{S}{m}\right)\times\int_0^1 h(1-\kappa)\eta(\kappa T+(1-\kappa)S)\,d\kappa. \tag{4.3}$$

By using the assumption that $\eta(\kappa T+(1-\kappa)S)=\eta(\kappa S+(1-\kappa)T)$, we can rewrite (4.3) as

$$\int_0^1\xi\eta(\kappa T+(1-\kappa)S)\,d\kappa\preceq\left[\xi(T)+m\xi\left(\frac{S}{m}\right)\right]\times\int_0^1\eta(\kappa T+(1-\kappa)S)h(\kappa)\,d\kappa.$$

Thus, the proof is complete. □

Theorem 4.2. *Let $m\in[0,1]$ and $h:J\supseteq(0,1)\to[0,\infty)$. Suppose that $\xi:I=[0,\infty)\to\mathbb{R}$ is an operator (h,m)-convex function and $\eta:I\to[0,\infty)$ such that $\eta(\kappa t+m(1-\kappa)s)=\eta(m\kappa s+(1-\kappa)t)$ for all $\kappa\in[0,1]$ and $a,b\in I$ with $t\in\sigma(T)$ and $s\in\sigma(S)$. Then, for any self-adjoint operators $T,S\in\mathcal{K}$ with $\sigma(T),\sigma(S)\subset I$, the following inequality holds:*

$$\int_0^1\xi\eta(\kappa T+m(1-\kappa)S)\,d\kappa\preceq[\xi(T)+m\xi(S)]\times\int_0^1 h(\kappa)\eta(\kappa T+m(1-\kappa)S)\,d\kappa. \tag{4.4}$$

Proof. Since ξ is an operator (h,m)-convex function, we have

$$\xi(\kappa T+m(1-\kappa)S)\preceq h(\kappa)\xi(T)+mh(1-\kappa)\xi(S). \tag{4.5}$$

Multiplying both sides by $\eta(\kappa T+m(1-\kappa)S)$ and integrating over $\kappa\in[0,1]$, we get

$$\int_0^1\xi\eta(\kappa T+m(1-\kappa)S)\,d\kappa\preceq\int_0^1[h(\kappa)\xi(T)+mh(1-\kappa)\xi(S)]\times\eta(\kappa T+m(1-\kappa)S)\,d\kappa. \tag{4.6}$$

Expanding and rearranging (4.6), it can be written as

$$\int_0^1 \xi\eta(\kappa T + m(1-\kappa)S)\,d\kappa \preceq \xi(T)\int_0^1 h(\kappa)\eta(\kappa T + m(1-\kappa)S)\,d\kappa + m\xi(S) \times \int_0^1 h(1-\kappa)\eta(m\kappa S + (1-\kappa)T)\,d\kappa.$$

By using the symmetry assumption

$$\eta(\kappa T + m(1-\kappa)S) = \eta(m\kappa S + (1-\kappa)T),$$

we conclude

$$\int_0^1 \xi\eta(\kappa T + m(1-\kappa)S)\,d\kappa \preceq [\xi(T) + m\xi(S)] \times \int_0^1 h(\kappa)\eta(\kappa T + m(1-\kappa)S)\,d\kappa. \quad \square$$

We can see that if the weight function has a constant value, it satisfies symmetrical condition. Thus, we have the following Corollary .

Corollary 4.3. *If we take the weight function $\eta(\cdot) = 1$ in Theorem 4.2, we get a variation of the H-H type inequalities when the function ξ is an operator (h, m)-convex function (see [33]).*

Theorem 4.4. *Let $m \in (0, 1]$ and $h : J \supseteq (0, 1) \to [0, \infty)$. Suppose that $\xi : I = [0, \infty) \to \mathbb{R}$ is an operator (h, m)-convex function for operators in $\Omega_{1,T,S}$ and $g : I \to [0, \infty)$ such that $\eta(\kappa t + (1-\kappa)s) = \eta(\kappa s + (1-\kappa)t)$ for $\kappa \in [0, 1]$ and $t, s \in I$ with $t \in \sigma(T)$ and $s \in \sigma(S)$. Then, for any self-adjoint operators $T, S \in \mathcal{K}$ where $\sigma(T), \sigma(S) \subset I$, the following statement holds:*

(4.7)

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+S}{2}\right)\int_0^1 \eta(\kappa T + (1-\kappa)S)\,d\kappa \\
&\preceq \int_0^1 \left[\xi(\kappa T + (1-\kappa)S) + m\xi\left(\frac{\kappa T + (1-\kappa)S}{m}\right)\right]\eta(\kappa T + (1-\kappa)S)\,d\kappa \\
&\preceq \left[\xi(T) + m\left(\xi\left(\frac{S}{m}\right) + \xi\left(\frac{T}{m}\right)\right) + m^2\xi\left(\frac{S}{m^2}\right)\right] \\
&\quad \times \int_0^1 h(\kappa)\eta(\kappa T + (1-\kappa)S)\,d\kappa.
\end{aligned}$$

Proof. Let us multiply inequality (3.12) by $\eta(\kappa T + (1-\kappa)S)$ and integrating it over $\kappa \in [0, 1]$, we can obtain

$$\frac{\int_0^1 \eta(\kappa T + (1-\kappa S))}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+S}{2}\right) \tag{4.8}$$

$$\preceq \int_0^1 \xi(\kappa T + (1-\kappa)S)\eta(\kappa T + (1-\kappa)S)\, d\kappa + \int_0^1 m\xi\left(\frac{\kappa T + (1-\kappa)S}{m}\right)\eta(\kappa T + (1-\kappa)S)\, d\kappa.$$

By using assumption that $\eta(\kappa T + (1-\kappa)S) = \eta(\kappa S + (1-\kappa)T)$, we have

$$\int_0^1 \xi\left(\frac{\kappa S + (1-\kappa)T}{m}\right)\eta(\kappa S + (1-\kappa)T)\, d\kappa = \int_0^1 \xi\left(\frac{\kappa S + (1-\kappa)T}{m}\right)\eta(\kappa T + (1-\kappa)S)\, d\kappa.$$

Then, from (4.8) we have

$$\frac{\int_0^1 \eta(\kappa T + (1-\kappa)S)}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+S}{2}\right) \preceq \int_0^1 \xi\eta(\kappa T + (1-\kappa)S)\, d\kappa + m\int_0^1 \xi\left(\frac{\kappa S + (1-\kappa)T}{m}\right)\eta(\kappa T + (1-\kappa)S)\, d\kappa.$$

So, we obtain the LHS of (4.7). By using the same way as in Theorem 3.7, since ξ is an operator (h,m)-convex function, we have

$$\xi(\kappa T + (1-\kappa)S) \preceq h(\kappa)\xi(T) + mh(1-\kappa)\xi\left(\frac{S}{m}\right).$$

By multiplying it with $\eta(\kappa T + (1-\kappa)S)$ and integrating it over $\kappa \in [0,1]$, we can obtain the RHS of (4.7). □

Theorem 4.5. *Let $m \in (0,1]$ and $h : J \supseteq (0,1) \to [0,\infty)$. Suppose $\xi : I = [0,\infty) \to \mathbb{R}$ is an operator (h,m)-convex function for operators in $\Omega_{m,T,S}$ and let $\eta : I \to [0,\infty)$ be such that*

$$\eta(\kappa t + m(1-\kappa)s) = \eta(m\kappa s + (1-\kappa)t), \quad \text{for all } \kappa \in [0,1],$$

where $t, s \in I$ with $t \in \sigma(T), s \in \sigma(S)$. Then, for two arbitrary self-adjoint operators $T, S \in \mathcal{K}$ with $\sigma(T), \sigma(S) \subseteq I$, the following inequality holds:

$$\frac{\int_0^1 \eta(\kappa T + m(1-\kappa)S)\, d\kappa}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+mS}{2}\right) \preceq \int_0^1 \left[\xi(\kappa T + (1-\kappa)S) + m\xi\left(\frac{\kappa T + m(1-\kappa)S}{m}\right)\right] \times \eta(T + (1-\kappa)S)\, d\kappa \tag{4.9}$$

$$\preceq \left[\xi(T) + m\left\{\xi\left(\frac{S}{m}\right) + \xi\left(\frac{T}{m}\right)\right\} + m^2\xi\left(\frac{S}{m}\right)\right] \times \int_0^1 h(\kappa)\eta(\kappa T + m(1-\kappa)S)\, d\kappa.$$

Proof. By multiplying (3.7) by $\eta(\kappa T + (1-\kappa)S)$ and integrating it over $\kappa \in [0,1]$, we obtain

$$\begin{aligned}
&\frac{\int_0^1 \eta(\kappa T + m(1-\kappa)S)\, d\kappa}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+mS}{2}\right) \\
&\preceq \int_0^1 \xi(\kappa T + (1-\kappa)S)\eta(\kappa T + m(1-\kappa)S)\, d\kappa \\
&\quad + \int_0^1 m\xi\left(\frac{\kappa T + m(1-\kappa)S}{m}\right)\eta(\kappa T + m(1-\kappa)S)\, d\kappa.
\end{aligned}$$

By using the assumption that $\eta(\kappa T + m(1-\kappa)S) = \eta(m\kappa S + (1-\kappa)T)$, then we have

$$\begin{aligned}
&\frac{\int_0^1 \eta(\kappa T + m(1-\kappa)S)\, d\kappa}{h\left(\frac{1}{2}\right)}\xi\left(\frac{T+mS}{2}\right) \\
&\preceq \int_0^1 \xi(\kappa T + (1-\kappa)S)\eta(\kappa T + m(1-\kappa)S)\, d\kappa \\
&\quad + \int_0^1 m\xi\left(\frac{\kappa T + m(1-\kappa)S}{m}\right)\eta(m\kappa S + (1-\kappa)T)\, d\kappa.
\end{aligned}$$

Next, in a similar manner, since ξ is an operator (h,m)-convex function, we have

$$\xi(\kappa T + m(1-\kappa)S) \preceq h(\kappa)\xi(T) + mh(1-\kappa)\xi(S).$$

By multiplying it with $\eta(\kappa T + m(1-\kappa)S)$ and using the same way as in Theorem 3.4. □

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[1] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran, Malang, 65145, Indonesia.
Email address: co_mathub@ub.ac.id

[2] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran, Malang, 65145, Indonesia.
Email address: rm.ekadion.m@gmail.com

[3] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran, Malang, 65145, Indonesia.
Email address: rbagus@ub.ac.id

[4] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran, Malang, 65145, Indonesia.
Email address: abdul_rouf@ub.ac.id

[5] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Jl. Ilmu 1/1, Shah Alam, 40450, Malaysia.
Email address: kilicman@uitm.edu.my

[6] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Jl. Ilmu 1/1, Shah Alam, 40450, Malaysia.
Email address: norha480@uitm.edu.my

[7] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran, Malang, 65145, Indonesia.
Email address: mila_n12@ub.ac.id