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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 3
Pages: 189-205

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2026.2071604.2330



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 3 (2026), 189-205
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2026.2071604.2330

On Conformable Fractional Schweitzer-Type Inequality

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Abstract. Schweitzer's inequality is a classical result in mathematical analysis that provides bounds for the product of integrals involving positive, increasing, and concave functions; it plays a significant role in the study of integral inequalities and has inspired various generalizations in modern fractional calculus. In this paper, we establish an extension of the classical Schweitzer inequality in the framework of left conformable fractional integrals. Building upon recent works based on Riemann-Liouville fractional integrals, we generalize these results to the framework of conformable fractional integrals. Finally, we provide some examples showcasing how our result refines and improves upon previous results.

1. Introduction

Integral inequalities play a central role in mathematical analysis, providing essential tools for the estimation and comparison of integrals involving functions with specific properties. These inequalities are widely used in various fields such as optimization theory, probability, statistics, differential equations, and applied sciences. Among the classical results, the Schweitzer inequality stands out due to its elegant structure and far-reaching applications.

In 1914, Schweitzer [9] established the following remarkable result: if $\mathscr{G}$ and $\frac{1}{\mathscr{G}}$ are integrable functions on $[\eta, \rho]$ satisfying $0 < \mathfrak{q} \leq \mathscr{G}(\mathfrak{l}) \leq \mathfrak{p}$,

2020 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. Schweitzer inequality, Concave functions, Cauchy–Schwarz inequality, Riemann-Liouville fractional integrals, Conformable fractional integrals.
Received: 2025-09-13, Accepted: 2026-01-31.
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then

$$\left(\frac{1}{\rho-\eta}\mathfrak{u}_{\eta}^{\rho}\mathscr{G}\left(\mathfrak{l}\right)d\mathfrak{l}\right)\left(\frac{1}{\rho-\eta}\int_{\eta}^{\rho}\frac{1}{\mathscr{G}\left(\mathfrak{l}\right)}d\mathfrak{l}\right)\leq\frac{\left(\mathfrak{q}+\mathfrak{p}\right)^{2}}{4\mathfrak{q}\mathfrak{p}}. \tag{1.1}$$

This inequality provides a bound on the product of the average values of a positive function and its reciprocal, and has been extensively studied and generalized over the decades. In the same work, Schweitzer also derived a discrete counterpart:

$$\left(\frac{1}{m}\sum_{k=1}^{m}\sigma_{k}\right)\left(\frac{1}{m}\sum_{k=1}^{m}\frac{1}{\sigma_{k}}\right)\leq\frac{\left(\mathfrak{q}+\mathfrak{p}\right)^{2}}{4\mathfrak{q}\mathfrak{p}},$$

where $0<\mathfrak{q}\leq\sigma_{k}\leq\mathfrak{p}$, $k=1,\ldots,m$. This discrete version was later extended by Kantorovich [6], who provided refined bounds for weighted sums:

$$1\leq\left(\frac{1}{\sum_{k=1}^{m}\mu_{k}}\sum_{k=1}^{m}\mu_{k}\sigma_{k}\right)\left(\frac{1}{\sum_{k=1}^{m}\mu_{k}}\sum_{k=1}^{m}\frac{\mu_{k}}{\sigma_{k}}\right)\leq\frac{\left(\mathfrak{q}+\mathfrak{p}\right)^{2}}{4\mathfrak{q}\mathfrak{p}}. \tag{1.2}$$

Further generalizations have been obtained in continuous settings. For example, Wilkins [11] showed that for a monotone concave function $\mathscr{G}$ on $(0,1)$ satisfying $0<\mathfrak{q}\leq\mathscr{G}\left(\mathfrak{l}\right)\leq\mathfrak{p}$, one has

$$1\leq\left(\int_{0}^{1}\mathscr{G}\left(\mathfrak{l}\right)d\mathfrak{l}\right)\left(\int_{0}^{1}\frac{1}{\mathscr{G}\left(\mathfrak{l}\right)}d\mathfrak{l}\right)\leq\frac{\left(\frac{\mathfrak{p}(\ln\mathfrak{p}-\ln\mathfrak{q})}{\mathfrak{p}-\mathfrak{q}}-\frac{\mathfrak{q}+\mathfrak{p}}{2\mathfrak{p}}\right)^{2}}{2\left(\mathfrak{p}-\mathfrak{q}\right)\left(\frac{\ln\mathfrak{p}-\ln\mathfrak{q}}{\mathfrak{p}-\mathfrak{q}}-\frac{1}{\mathfrak{p}}\right)}.$$

Brenner and Alzer [3] contributed yet another refinement under the assumption that $\mathscr{G}$ is continuous and concave on $[\eta,\rho]$, proving that

$$1\leq\frac{1}{\left(\rho-\eta\right)^{2}}\left(\int_{\eta}^{\rho}\mathscr{G}\left(\mathfrak{l}\right)d\mathfrak{l}\right)\left(\int_{\eta}^{\rho}\frac{1}{\mathscr{G}\left(\mathfrak{l}\right)}d\mathfrak{l}\right)\leq1+\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\sqrt{\mathscr{G}\left(\rho\right)\mathscr{G}\left(\eta\right)}}. \tag{1.3}$$

These results illustrate the deep interplay between integral estimates and functional properties, such as concavity or boundedness. Their relevance extends well beyond pure mathematics, influencing disciplines like economics, physics, and engineering.

With the emergence of fractional calculus , which generalizes differentiation and integration to non-integer orders, there has been growing interest in extending classical inequalities to this broader framework. Fractional calculus provides more accurate models for systems with memory effects, long-range dependence, and anomalous diffusion processes, making it a powerful tool in modern science and engineering.

One of the most commonly used operators in fractional calculus is the left Riemann–Liouville fractional integral of order $\alpha > 0$, defined for a function $\mathscr{G}$ on $[\eta, \rho]$ as

$$\mathscr{J}_{\eta^+}^{\alpha}\mathscr{G}(\ell) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{\ell}(\ell - \mathfrak{l})^{\alpha-1}\mathscr{G}(\mathfrak{l})d\mathfrak{l}, \quad \ell > \eta, \tag{1.4}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-\mathfrak{l}}\mathfrak{l}^{\alpha-1}d\mathfrak{l}$, is the Gamma function and $\mathscr{J}_{\eta^+}^{0}\mathscr{G}(\ell) = \mathscr{G}(\ell)$.

Recently, Abdeljawad et al. [1] proposed a fractional analogue of the Schweitzer inequality using this operator. Specifically, they proved the following theorem:

Theorem 1.1. *Let $\mathscr{G} : [\eta, \rho] \to \mathbb{R}^+$ be continuously differentiable increasing and concave function, then for $\alpha > 0$ we have*

$$\begin{aligned} 1 \leq & \frac{\Gamma(\alpha+1)^2}{(\rho-\eta)^{2\alpha}}\mathscr{J}_{\eta^+}^{\alpha}(\mathscr{G})(\rho)\,\mathscr{J}_{\eta^+}^{\alpha}(\mathfrak{g}\circ\mathscr{G})(\rho) \\ & \leq \begin{cases} \varrho(\alpha,\eta,\rho), & \text{if } 0 < \alpha \leq 1, \\ \varsigma(\alpha,\eta,\rho), & \text{if } 1 < \alpha \leq 2, \\ \upsilon(\alpha,\eta,\rho), & \text{if } \alpha > 2, \end{cases} \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} &\varrho(\alpha,\eta,\rho) \\ &= 1 + \left(\frac{\alpha(1-\alpha)\rho^{\alpha}}{(\rho-\eta)^{\alpha}} - \frac{(2-\alpha)\alpha^2\rho^{\alpha-1}}{(\alpha+1)(\rho-\eta)^{\alpha-1}}\right)\ln\mathscr{G}(\rho) \\ &\quad + \left(\frac{\alpha^2\rho^{\alpha-2}(\rho-(\alpha-1)\eta)}{(\alpha+1)(\rho-\eta)^{\alpha-1}} - \frac{\alpha\rho^{\alpha-1}(\rho-\alpha\eta)}{(\rho-\eta)^{\alpha}}\right)\ln\mathscr{G}(\eta) \\ &\quad + \rho^{\alpha-1}\left(\frac{(2-\alpha)\alpha}{(\rho-\eta)^{\alpha}} + \frac{(1-\alpha)\alpha^2}{\rho(\alpha+1)(\rho-\eta)^{\alpha-1}}\right)(\rho-\eta)\ln\mathscr{G}\left(\frac{\eta+\rho}{2}\right), \end{aligned}$$

and

$$\varsigma(\alpha,\eta,\rho) = 1 + \frac{(\alpha+1)\alpha\rho^{\alpha-1} - \alpha^2\rho^{\alpha-2}(\rho-\eta)}{(\alpha+1)(\rho-\eta)^{\alpha-1}}\ln\mathscr{G}(\eta)\,\mathscr{G}\left(\frac{\eta+\rho}{2}\right)$$

and

$$\begin{aligned} &\upsilon(\alpha,\eta,\rho) \\ &= 1 + \frac{\alpha^2\rho^{\alpha-1}((3-\alpha)\rho + (1-\alpha)\eta)}{2(\alpha+1)\rho(\rho-\eta)^{\alpha-1}}\ln\frac{\mathscr{G}(\eta)}{\mathscr{G}(\rho)} + \frac{\alpha\rho^{\alpha-1}}{(\rho-\eta)^{\alpha-1}}\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\eta)}, \end{aligned}$$

with $\mathfrak{g}(\mathfrak{l}) = \frac{1}{\mathfrak{l}}$.

This result highlights how the classical Schweitzer inequality can be generalized to the fractional setting, offering new insights into the behavior of products of integrals in the presence of memory effects.

Motivated by the above developments, we aim in this paper to establish a new extension of the Schweitzer-type inequality , now formulated in terms of the left conformable fractional integral, a recent alternative to the Riemann-Liouville approach that avoids singular kernels and allows for more localized modeling.

We show that, under similar assumptions on $\mathscr{G}$, our formulation leads to bounds analogous to those in Theorem 1.1, but adapted to the conformable framework. Moreover, we demonstrate that these bounds reduce to known results when the fractional order and the conformable parameter approache unity, thereby recovering the classical case.

Finally, although an analytical comparison between our result when fixing the conformable parameter $\beta = 1$ and that obtained by Abdeljawad et al. in Theorem 1.1 proves challenging, we provide a few illustrative examples in which our bound clearly refines the earlier result—at least for the specific functions considered.

2. Preliminaries

In this section, we recall some basic definitions from conformable fractional calculus, focusing on the left conformable fractional integral, and we also introduce the beta function, which plays a key role in the theory of fractional operators.

Definition 2.1 ([5])**.** Let $\mathscr{G} \in L[\eta, \rho]$, where $L[\eta, \rho]$ denotes the space of Lebesgue integrable functions on the interval $[\eta, \rho]$. Then, the left conformable fractional integral of order $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$ and $\beta > 0$

$$ {}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}(\ell) = \tfrac{1}{\Gamma(\alpha)} \int_{\eta}^{\ell} \left(\tfrac{(\ell-\eta)^{\beta} - (\mathfrak{l}-\eta)^{\beta}}{\beta} \right)^{\alpha-1} (\mathfrak{l}-\eta)^{\beta-1} \mathscr{G}(\mathfrak{l}) d\mathfrak{l}, \quad \eta < \ell \leq \rho. $$

Several studies on integral inequalities have been conducted using this operator; see [2, 4, 8, 10, 12].

Definition 2.2 ([7])**.** For any complex numbers and nonpositive integers $\mathfrak{v}_1, \mathfrak{v}_2$ such that $Re(\mathfrak{v}_1) > 0$ and $Re(\mathfrak{v}_2) > 0$. The Beta function is defined by

$$ \begin{aligned} B(\mathfrak{v}_1, \mathfrak{v}_2) &= \int_0^1 \mathfrak{l}^{\mathfrak{v}_1 - 1} (1-\mathfrak{l})^{\mathfrak{v}_2 - 1} \, d\mathfrak{l} \\ &= \frac{\Gamma(\mathfrak{v}_1)\Gamma(\mathfrak{v}_2)}{\Gamma(\mathfrak{v}_1 + \mathfrak{v}_2)}, \end{aligned} $$

where $\Gamma(.)$ is the Euler gamma function.

3. Main Results

In this section, we first establish a generalization of Schweitzer's inequality given by (1.1). We then introduce a key lemma that will serve as a tool for deriving the analogues of inequalities (1.3) and (1.5) provided by Brener and Alzer in [3], and Abdeljawad et al. in [1], respectively.

Theorem 3.1. *Let $\mathscr{G} : [\eta, \rho] \to \mathbb{R}^+$ be continuously differentiable such that $0 < \mathfrak{q} \leq \mathscr{G}(\mathfrak{l}) \leq \mathfrak{p}$. If $\mathscr{G}, \frac{1}{\mathscr{G}} \in L[\eta, \rho]$, then for $\alpha, \beta > 0$ we have*

$$1 \leq \frac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}}\left[\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\right] \leq \frac{(\mathfrak{p}+\mathfrak{q})^2}{4\mathfrak{q}\mathfrak{p}}, \tag{3.1}$$

where $\mathfrak{g}(\mathfrak{l}) = \frac{1}{\mathfrak{l}}$.

Proof. First, we demonstrate the inequality's left side in (3.1), from differentiability and positivity of $\mathscr{G}$, we have

(3.2)

$$\begin{aligned}
&\left(\tfrac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)\right)(\rho)\left(\tfrac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)\right)(\rho)\\
&= \left(\tfrac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}} \times \tfrac{1}{\Gamma(\alpha)}\int_{\eta}^{\rho}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\mathscr{G}(\mathfrak{l})\,d\mathfrak{l}\right)\\
&\quad\times\left(\tfrac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}} \times \tfrac{1}{\Gamma(\alpha)}\int_{\eta}^{\rho}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\frac{1}{\mathscr{G}(\mathfrak{l})}d\mathfrak{k}\right)\\
&= \left(\tfrac{\alpha\beta^{\alpha}}{(\rho-\eta)^{\alpha\beta}}\right)^2\left(\int_{\eta}^{\rho}\left(\left(\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\right)^{\frac{1}{2}}\sqrt{\mathscr{G}(\mathfrak{l})}\right)^2 d\mathfrak{l}\right)\\
&\quad\times\left(\int_{\eta}^{\rho}\left(\left(\left(\tfrac{(\rho-\eta)^{\beta}+(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\right)^{\frac{1}{2}}\tfrac{1}{\sqrt{\mathscr{G}(\mathfrak{l})}}\right)^2 d\mathfrak{l}\right).
\end{aligned}$$

Applying the Cauchy–Schwarz inequality to (3.2) yields

$$\begin{aligned}
&\left(\tfrac{\alpha\beta^{\alpha}}{(\rho-\eta)^{\alpha\beta}}\right)^2\left(\int_{\eta}^{\rho}\left(\left(\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\right)^{\frac{1}{2}}\sqrt{\mathscr{G}(\mathfrak{l})}\right)^2 d\mathfrak{l}\right)\\
&\quad\times\left(\int_{\eta}^{\rho}\left(\left(\left(\tfrac{(\rho-\eta)^{\beta}+(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\right)^{\frac{1}{2}}\tfrac{1}{\sqrt{\mathscr{G}(\mathfrak{l})}}\right)^2 d\mathfrak{l}\right)\\
&\geq \left(\tfrac{\alpha\beta^{\alpha}}{(\rho-\eta)^{\alpha\beta}}\right)^2\left(\int_{\eta}^{\rho}\left(\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\right)d\mathfrak{l}\right)^2
\end{aligned} \tag{3.3}$$

$$= \left(\frac{\alpha\beta^{\alpha}}{(\rho-\eta)^{\alpha\beta}}\right)^{2} \left(\frac{1}{\alpha}\left(\frac{(\rho-\eta)^{\beta}}{\beta}\right)^{\alpha}\right)^{2}$$
$$= 1.$$

This proves the inequality on the left side of (3.1). Now let us demonstrate the right-sided one.

Clearly from the hypothesis $0 < \mathfrak{q} \leq \mathscr{G}(\mathfrak{l}) \leq \mathfrak{p}$, we have

$$0 < \frac{1}{\mathfrak{p}} \leq \frac{1}{\mathscr{G}(\mathfrak{l})} \leq \frac{1}{\mathfrak{q}}.$$

So,

$$\mathscr{G}(\mathfrak{l}) - \mathfrak{q} \geq 0, \qquad \frac{1}{\mathfrak{p}} - \frac{1}{\mathscr{G}(\mathfrak{l})} \leq 0. \tag{3.4}$$

Obviously from (3.4), we have

$$(\mathscr{G}(\mathfrak{l}) - \mathfrak{q})\left(\frac{1}{\mathfrak{p}} - \frac{1}{\mathscr{G}(\mathfrak{l})}\right) = \frac{\mathscr{G}(\mathfrak{l})}{\mathfrak{p}} - 1 - \frac{\mathfrak{q}}{\mathfrak{p}} + \frac{\mathfrak{q}}{\mathscr{G}(\mathfrak{l})} \leq 0,$$

which gives

$$\frac{\mathscr{G}(\mathfrak{l})}{\mathfrak{p}} + \frac{\mathfrak{q}}{\mathscr{G}(\mathfrak{l})} \leq \frac{\mathfrak{p}+\mathfrak{q}}{\mathfrak{p}}. \tag{3.5}$$

Multiplying both sides of (3.5) by $\frac{1}{\Gamma(\alpha)}\left(\frac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\sqrt{\frac{\mathfrak{p}}{\mathfrak{q}}}$, then integrating the result over $[\eta, \rho]$, we get

$$\frac{1}{\sqrt{\mathfrak{q}\mathfrak{p}}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho) + \sqrt{\mathfrak{q}\mathfrak{p}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho) \leq \frac{(\rho-\eta)^{\alpha\beta}}{\beta^{\alpha}\Gamma(\alpha+1)}\frac{\mathfrak{p}+\mathfrak{q}}{\sqrt{\mathfrak{q}\mathfrak{p}}}. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned} 0 &\leq \left(\frac{1}{\sqrt{\mathfrak{q}\mathfrak{p}}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho) - \sqrt{\mathfrak{q}\mathfrak{p}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\right)^{2} \\ &= \left(\frac{1}{\sqrt{\mathfrak{q}\mathfrak{p}}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho)\right)^{2} + \left(\sqrt{\mathfrak{q}\mathfrak{p}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\right)^{2} \\ &\quad - 2\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho). \end{aligned} \tag{3.7}$$

By adding $4\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)$ to both sides of (3.7), we obtain

$$\begin{aligned} &4\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathcal{H})\right)(\rho) \\ &\quad \leq \left(\frac{1}{\sqrt{\mathfrak{q}\mathfrak{p}}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}\mathscr{G}\right)(\rho) + \sqrt{\mathfrak{q}\mathfrak{p}}\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\right)^{2}. \end{aligned} \tag{3.8}$$

Using (3.6) into (3.8) yields

$$4\left({}^{\alpha}\mathscr{I}_{\eta^+}^{\beta}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}_{\eta^+}^{\beta}(\mathfrak{g}\circ\mathcal{H})\right)(\rho)\leq\frac{(\rho-\eta)^{2\alpha\beta}}{\beta^{2\alpha}\left(\Gamma(\alpha+1)\right)^{2}}\frac{(\mathfrak{p}+\mathfrak{q})^{2}}{\mathfrak{q}\mathfrak{p}}. \tag{3.9}$$

So, from (3.9), we have

$$\left(\frac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}}\left({}^{\alpha}\mathscr{I}_{\eta^+}^{\beta}\mathscr{G}\right)\right)(\rho)\left(\frac{\beta^{\alpha}\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha\beta}}\left({}^{\alpha}\mathscr{I}_{\eta^+}^{\beta}(\mathfrak{g}\circ\mathscr{G})\right)\right)(\rho)$$
$$\leq\frac{(\mathfrak{p}+\mathfrak{q})^{2}}{4\mathfrak{q}\mathfrak{p}}.$$

It follows from inequalities (3.3) and (3.9) that inequality (3.1) of Theorem 3.1 holds □

Corollary 3.2. *In Theorem 3.1, if we take $\beta=1$, we obtain*

$$1\leq\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha}}\left(\mathscr{I}_{\eta^+}^{\alpha}\mathscr{G}\right)\right)(\rho)\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha}}\left(\mathscr{I}_{\eta^+}^{\alpha}(\mathfrak{g}\circ\mathscr{G})\right)\right)(\rho)\leq\frac{(\mathfrak{p}+\mathfrak{q})^{2}}{4\mathfrak{q}\mathfrak{p}}.$$

Corollary 3.3. *In Theorem 3.1, if we take $\beta=\alpha=1$, we obtain*

$$1\leq\left(\frac{1}{\rho-\eta}\int_{\eta}^{\rho}\mathscr{G}(\mathfrak{l})d\mathfrak{l}\right)\left(\frac{1}{\rho-\eta}\int_{\eta}^{\rho}\frac{1}{\mathscr{G}(\mathfrak{l})}d\mathfrak{l}\right)\leq\frac{(\mathfrak{p}+\mathfrak{q})^{2}}{4\mathfrak{q}\mathfrak{p}},$$

which is the same result as (1.1) *established by Schweitzer [9].*

Lemma 3.4. *Let $\alpha,\beta>0$ and $\mathfrak{f}:[0,1]\to\mathbb{R}$ defined for $\beta\alpha>1$ by*

$$\mathfrak{f}(z)=(1-z)^{\alpha-1}z^{\frac{\beta-1}{\beta}},$$

and let $\mathfrak{h}:[0,1]\to\mathbb{R}$ defined for $\beta\alpha\leq1$ by

$$\mathfrak{h}(z)=\begin{cases}(1-z)^{\frac{\beta\alpha-1}{\beta}}, & \text{if } 0\leq z\leq\frac{1}{2},\\ z^{\frac{\beta\alpha-1}{\beta}}, & \text{if } \frac{1}{2}\leq z\leq1.\end{cases}$$

Then, we have

$$\sup_{0\leq z\leq1}\mathfrak{f}(z)=\left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}},$$

and

$$\sup_{0\leq z\leq1}\mathfrak{h}(z)=\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}.$$

Proof. Clearly we have

$$\mathfrak{f}'(z)=\begin{cases}\left(\frac{\beta-1}{\beta}-\frac{\beta\alpha-1}{\beta}z\right)(1-z)^{\alpha-2}\,z^{-\frac{1}{\beta}}, & \text{if } \beta>1,\\ \\ (\alpha-1)(1-z)^{\alpha-2}, & \text{if } \beta=1.\end{cases}$$

The sign of $\mathfrak{f}'(z)$ depends on the sign of $\frac{\beta-1}{\beta}-\frac{\beta\alpha-1}{\beta}z$, because $(1-z)^{\alpha-2}\,z^{-\frac{1}{\beta}}$ is positive on $[0,1]$. Since $\beta\alpha>1$, then

$$\sup_{0\leq z\leq 1}\mathfrak{f}(z)=\left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}.$$

Obviously,

$$\mathfrak{h}'(z)=\begin{cases}-\frac{\beta\alpha-1}{\beta}(1-z)^{\frac{\beta\alpha-1}{\beta}-1}, & \text{if } 0\leq z\leq\frac{1}{2},\\ \\ \frac{\beta\alpha-1}{\beta}z^{\frac{\beta\alpha-1}{\beta}-1}, & \text{if } \frac{1}{2}\leq z\leq 1.\end{cases}$$

Since $\beta\alpha\leq 1$ and $\beta>1>0$, $\mathfrak{h}$ is increasing on $\left[0,\frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2},1\right]$. Consequently, we have $\sup\limits_{0\leq z\leq 1}\mathfrak{h}(z)=\mathfrak{h}\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}$. □

Theorem 3.5. *Let $\mathscr{G}:[\eta,\rho]\to\mathbb{R}^+$ be a continuously differentiable, increasing, and concave function; then for $\alpha,\beta>0$, we have*

(3.10)

$$1\leq\left(\frac{\Gamma(\alpha+1)^{\alpha}}{(\rho-\eta)^{\alpha}}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left(\frac{\Gamma(\alpha+1)^{\alpha}}{(\rho-\eta)^{\alpha}}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)$$

$$\leq\begin{cases}\frac{(\rho-\eta)^{2\alpha(\beta-1)}}{\beta^{2\alpha}}\left(1+\left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\alpha\beta(\rho-\eta)^{1-\beta}\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\\ +\left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\frac{(\rho-\eta)^{(2\alpha-1)(\beta-1)}}{\beta^{2(\alpha-1)+1}}\alpha^2\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\frac{1}{\beta}+1,\alpha\right)\\ \quad \textit{if}\quad \beta\alpha>1 \textit{ and } \beta\neq 1,\\ \\ \frac{(\rho-\eta)^{2\alpha(\beta-1)}}{\beta^{2\alpha}}\left(1+\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}(\rho-\eta)^{1-\beta}\alpha\beta\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\\ +\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\frac{(\rho-\eta)^{(2\alpha-1)(\beta-1)}}{\beta^{2\alpha-1}}\alpha^2\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\frac{1}{\beta}+1,\alpha\right)\\ \quad \textit{if}\quad \beta\alpha\leq 1\end{cases}$$

with $\mathfrak{g}(\mathfrak{l}) = \frac{1}{t}$.

Proof. The left side inequality in (3.10) is the same as in Theorem 3.1. It follows from (3.2) and (3.3). For the second inequality of (3.10). From differentiability of $\mathscr{G}$ we have for all $\mathfrak{l}, \mathfrak{k} \in [\eta, \rho]$

$$\mathscr{G}(\mathfrak{l}) \leq \mathscr{G}(\mathfrak{k}) + (\mathfrak{l} - \mathfrak{k})\mathscr{G}'(\mathfrak{k}). \tag{3.11}$$

Since $\mathscr{G}$ is positive, (3.3) gives

$$\frac{\mathscr{G}(\mathfrak{l})}{\mathscr{G}(\mathfrak{k})} \leq 1 + (\mathfrak{l} - \mathfrak{k})\frac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}. \tag{3.12}$$

Multiplying both sides of (3.12) by

$$\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{k}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{k}-\eta)^{\beta-1},$$

we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{k}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{k}-\eta)^{\beta-1}\frac{\mathscr{G}(\mathfrak{l})}{\mathscr{G}(\mathfrak{k})}\\
&\quad\leq \frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{k}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{k}-\eta)^{\beta-1}\\
&\qquad+\frac{1}{\Gamma(\alpha)}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\\
&\qquad\times\frac{1}{\Gamma(\alpha)}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{k}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{k}-\eta)^{\beta-1}(\mathfrak{l}-\mathfrak{k})\frac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}.
\end{aligned} \tag{3.13}$$

Integrating both sides of (3.13) with respect to $\mathfrak{k}$ over $[\eta, \rho]$ yields

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\mathscr{G}(\mathfrak{l})\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\\
&\quad\leq \frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\frac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^\alpha}\\
&\qquad+\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\\
&\qquad\times\int_\eta^\rho\left(\frac{(\rho-\eta)^\beta-(\mathfrak{k}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{k}-\eta)^{\beta-1}(\mathfrak{l}-\mathfrak{k})\frac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}d\mathfrak{k}\\
&\quad=\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\frac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^\alpha}\\
&\qquad+\frac{(\rho-\eta)^{\beta(\alpha-1)}}{\beta^{\alpha-1}}\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^\beta-(\mathfrak{l}-\eta)^\beta}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\\
&\qquad\times\int_\eta^\rho\left(1-\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^\beta\right)^{\alpha-1}\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta-1}(\mathfrak{l}-\mathfrak{k})\frac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}d\mathfrak{k}.
\end{aligned} \tag{3.14}$$

Assume that $\beta\alpha > 1$, since $\beta > 0$ and $\eta \leq \mathfrak{k} \leq \rho$, then $\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta} \in [0,1]$, we observe that

$$\left(1-\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta}\right)^{\alpha-1}\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta-1} = \mathfrak{f}\left(\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta}\right) \leq \left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}. \tag{3.15}$$

From (3.14) and (3.15), we have

(3.16)

$$\begin{aligned}
&\tfrac{1}{\Gamma(\alpha)}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\mathscr{G}(\mathfrak{l})\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\\
&\quad\leq \tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\\
&\qquad+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}\\
&\qquad\times(\mathfrak{l}-\eta)^{\beta-1}\int_{\eta}^{\rho}(\mathfrak{l}-\mathfrak{k})\tfrac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}d\mathfrak{k}\\
&\quad=\tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\\
&\qquad+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}\\
&\qquad\times(\mathfrak{l}-\eta)^{\beta-1}\left((\mathfrak{l}-\rho)\ln\mathscr{G}(\rho)-(\mathfrak{l}-\eta)\ln\mathscr{G}(\eta)+\int_{\eta}^{\rho}\ln\mathscr{G}(\mathfrak{k})\,d\mathfrak{k}\right).
\end{aligned}$$

Since $\ln\mathscr{G}(\mathfrak{k})$ is concave, we have

$$\int_{\eta}^{\rho}\ln\mathscr{G}(\mathfrak{k})\,d\mathfrak{k} \leq (\rho-\eta)\ln\mathscr{G}\left(\frac{\eta+\rho}{2}\right). \tag{3.17}$$

Combining (3.16) and (3.17), we get

(3.18)

$$\begin{aligned}
&\tfrac{1}{\Gamma(\alpha)}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\mathscr{G}(\mathfrak{l})\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^{+}}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\\
&\quad\leq \tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\\
&\qquad+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^{2}}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\\
&\qquad\times\left(((\mathfrak{l}-\eta)-(\rho-\eta))\ln\mathscr{G}(\rho)-(\mathfrak{l}-\eta)\ln\mathscr{G}(\eta)+(\rho-\eta)\ln\mathscr{G}\left(\tfrac{\eta+\rho}{2}\right)\right)\\
&\quad=\tfrac{1}{\Gamma(\alpha)^{2}}\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\left(1+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\alpha\beta(\rho-\eta)^{1-\beta}\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)
\end{aligned}$$

$$\times \left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}$$
$$\times \tfrac{1}{\Gamma(\alpha)^2}\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta}.$$

Now, integrating both sides of (3.18) with respect to $\mathfrak{l}$ over $[\eta,\rho]$ we deduce

(3.19)

$$\begin{aligned}
&\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathcal{H})\right)(\rho)\\
&\leq \tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}\Gamma(\alpha)^2}\left(1+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\alpha\beta\,(\rho-\eta)^{1-\beta}\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\\
&\quad\times\int_{\eta}^{\rho}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\,d\mathfrak{l}\\
&\quad+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^2}\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}\\
&\quad\times\int_{\eta}^{\rho}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta}\,d\mathfrak{l}\\
&=\left(\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}\Gamma(\alpha)}\right)^2\left(1+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\alpha\beta\,(\rho-\eta)^{1-\beta}\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\\
&\quad+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\tfrac{(\rho-\eta)^{2\beta\alpha-\beta+1}}{\beta^{2(\alpha-1)+1}}\tfrac{1}{\Gamma(\alpha)^2}\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\tfrac{1}{\beta}+1,\alpha\right),
\end{aligned}$$

where we have used

(3.20)

$$\begin{aligned}
\int_{\eta}^{\rho}\left(1-\left(\tfrac{\mathfrak{l}-\eta}{\rho-\eta}\right)^{\beta}\right)^{\alpha-1}\left(\tfrac{\mathfrak{l}-\eta}{\rho-\eta}\right)^{\beta}d\mathfrak{l}&=(\rho-\eta)\int_0^1\left(1-t^{\beta}\right)^{\alpha-1}t^{\beta}dt\\
&=\tfrac{\rho-\eta}{\beta}\int_0^1(1-l)^{\alpha-1}\,l^{\frac{1}{\beta}}dl\\
&=\tfrac{\rho-\eta}{\beta}B\left(\tfrac{1}{\beta}+1,\alpha\right).
\end{aligned}$$

Multiplying both sides of (3.20) by $\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha}}\right)^2$, we get

(3.21)

$$\begin{aligned}
&\left(\tfrac{\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha}}\,{}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left(\tfrac{\Gamma(\alpha+1)}{(\rho-\eta)^{\alpha}}\,{}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\\
&\leq\tfrac{(\rho-\eta)^{2\alpha(\beta-1)}}{\beta^{2\alpha}}\left(1+\left(\tfrac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\tfrac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\alpha\beta\,(\rho-\eta)^{1-\beta}\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)
\end{aligned}$$

$$+\left(\frac{\beta\alpha-\beta}{\beta\alpha-1}\right)^{\alpha-1}\left(\frac{\beta-1}{\beta\alpha-1}\right)^{\frac{\beta-1}{\beta}}\frac{(\rho-\eta)^{(2\alpha-1)(\beta-1)}}{\beta^{2(\alpha-1)+1}}\alpha^2\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\frac{1}{\beta}+1,\alpha\right).$$

Assume that $\beta\alpha\leq 1$, since $\beta>0$ and $\eta\leq\mathfrak{k}\leq\rho$, then $\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta}\in[0,1]$, we observe that

(3.22)
$$\left(1-\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta}\right)^{\alpha-1}\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta-1}\leq\mathfrak{h}\left(\left(\frac{\mathfrak{k}-\eta}{\rho-\eta}\right)^{\beta}\right)\leq\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}.$$

Combining (3.14) and (3.22), then using (3.17), we get

(3.23)
$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\left(\frac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\mathscr{G}(\mathfrak{l})\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)\\
&\leq\frac{1}{\Gamma(\alpha)^2}\left(\frac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\frac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\\
&\quad+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\frac{(\rho-\eta)^{\beta(\alpha-1)}}{\beta^{\alpha-1}\Gamma(\alpha)^2}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\int_{\eta}^{\rho}(\mathfrak{l}-\mathfrak{k})\tfrac{\mathscr{G}'(\mathfrak{k})}{\mathscr{G}(\mathfrak{k})}d\mathfrak{k}\\
&\leq\tfrac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}\Gamma(\alpha)^2}\left(1+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\tfrac{\alpha\beta}{(\rho-\eta)^{\beta-1}}\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}\\
&\quad+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^2}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta}\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}.
\end{aligned}$$

By integrating both sides of (3.23) with respect to $\mathfrak{l}$ over $[\eta,\rho]$, then using (3.22) it yields

(3.24)
$$\begin{aligned}
&\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left({}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathcal{H})\right)(\rho)\\
&\leq\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{(\rho-\eta)^{\beta\alpha}}{\alpha\beta^{\alpha}}\left(1+\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}(\rho-\eta)^{1-\beta}\alpha\beta\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)\\
&\quad\times\int_{\eta}^{\rho}\left(\frac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta-1}d\mathfrak{l}\\
&\quad+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\tfrac{(\rho-\eta)^{\beta\alpha-\beta}}{\beta^{\alpha-1}}\tfrac{1}{\Gamma(\alpha)^2}\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}\int_{\eta}^{\rho}\left(\tfrac{(\rho-\eta)^{\beta}-(\mathfrak{l}-\eta)^{\beta}}{\beta}\right)^{\alpha-1}(\mathfrak{l}-\eta)^{\beta}d\mathfrak{l}\\
&=\tfrac{1}{\Gamma(\alpha+1)^2}\tfrac{(\rho-\eta)^{2\beta\alpha}}{\beta^{2\alpha}}\left(1+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}(\rho-\eta)^{1-\beta}\alpha\beta\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)
\end{aligned}$$

$$+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\frac{(\rho-\eta)^{2\beta\alpha-\beta}}{\beta^{2\alpha-2}}\left(\frac{1}{\Gamma(\alpha)}\right)^2\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}\int_\eta^\rho\left(1-\left(\tfrac{\mathfrak{l}-\eta}{\rho-\eta}\right)^\beta\right)^{\alpha-1}\left(\tfrac{\mathfrak{l}-\eta}{\rho-\eta}\right)^\beta d\mathfrak{l}$$

$$=\frac{1}{\Gamma(\alpha+1)^2}\frac{(\rho-\eta)^{2\beta\alpha}}{\beta^{2\alpha}}\left(1+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}(\rho-\eta)^{1-\beta}\alpha\beta\ln\tfrac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)$$

$$+\left(\tfrac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\frac{(\rho-\eta)^{2\beta\alpha-\beta+1}}{\beta^{2\alpha-1}}\frac{1}{\Gamma(\alpha)^2}\ln\tfrac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\tfrac{1}{\beta}+1,\alpha\right),$$

where we have used (3.20). Multiplying both sides of (3.24) by $\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^\alpha}\right)^2$, we get

$$\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^\alpha}\,{}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}\mathscr{G}\right)(\rho)\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^\alpha}\,{}^{\alpha}\mathscr{I}^{\beta}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)(\rho)$$

$$\leq\frac{(\rho-\eta)^{2\alpha(\beta-1)}}{\beta^{2\alpha}}\left(1+\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}(\rho-\eta)^{1-\beta}\alpha\beta\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}\right)$$

$$+\left(\frac{1}{2}\right)^{\frac{\beta\alpha-1}{\beta}}\frac{(\rho-\eta)^{(2\alpha-1)(\beta-1)}}{\beta^{2\alpha-1}}\alpha^2\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}B\left(\frac{1}{\beta}+1,\alpha\right).$$

The proof is completed. □

Corollary 3.6. *In Theorem 3.5, if we take $\beta=1$, we obtain*

$$1\leq\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^\alpha}\left(\mathscr{I}^{\alpha}_{\eta^+}\mathscr{G}\right)\right)(\rho)\left(\frac{\Gamma(\alpha+1)}{(\rho-\eta)^\alpha}\left(\mathscr{I}^{\alpha}_{\eta^+}(\mathfrak{g}\circ\mathscr{G})\right)\right)(\rho)$$

$$\leq 1+\left(\frac{1}{2}\right)^{\alpha-1}\alpha\ln\frac{\mathscr{G}\left(\frac{\eta+\rho}{2}\right)}{\mathscr{G}(\rho)}+\left(\frac{1}{2}\right)^{\alpha-1}\frac{\alpha}{\alpha+1}\ln\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}. \tag{3.25}$$

Corollary 3.7. *In Theorem 3.5, if we take $\beta=\alpha=1$, we obtain*

$$1\leq\left(\frac{1}{\rho-\eta}\int_\eta^\rho\mathscr{G}(\mathfrak{l})d\mathfrak{l}\right)\left(\frac{1}{\rho-\eta}\int_\eta^\rho\frac{1}{\mathscr{G}(\mathfrak{l})}d\mathfrak{l}\right)$$

$$\leq 1+\ln\mathscr{G}\left(\frac{\eta+\rho}{2}\right)-\ln\sqrt{\mathscr{G}(\rho)\ln\mathscr{G}(\eta)},$$

which is the same result as Theorem 4.1 from [3].

Remark 3.8 (Comparison of the Upper Bounds)**.** We compare the upper bound derived in Theorem 3.1 (denoted as $\mathcal{B}_{Kant}$) with the refined upper bound derived in Theorem 3.5 (denoted as $\mathcal{B}_{Conc}$).

- The bound $\mathcal{B}_{Kant}=\frac{(\mathfrak{p}+\mathfrak{q})^2}{4\mathfrak{p}\mathfrak{q}}$ relies solely on the codomain $[\mathfrak{q},\mathfrak{p}]$. It corresponds to the Kantorovich constant, which is the supremum attained when the function is a step function taking values $\mathfrak{q}$ and $\mathfrak{p}$ with equal probability.

- The bound $\mathcal{B}_{Conc}$ in (3.10) exploits the geometric property that $\mathscr{G}$ is increasing and concave. Such functions cannot exhibit the extreme discontinuous variations of step functions. Consequently, $\mathcal{B}_{Conc}$ provides a strictly sharper estimate.

Analytically, we demonstrate that $\mathcal{B}_{Conc} \leq \mathcal{B}_{Kant}$ by examining their asymptotic behavior with respect to the ratio $\kappa = \mathfrak{p}/\mathfrak{q} \geq 1$.

Justification of Sharpness. Let $\kappa = \frac{\mathfrak{p}}{\mathfrak{q}}$. The Kantorovich bound from Theorem 3.1 can be rewritten as:

$$\mathcal{B}_{Kant}(\kappa) = \frac{(\kappa+1)^2}{4\kappa} = \frac{1}{4}\left(\kappa + 2 + \frac{1}{\kappa}\right). \tag{3.26}$$

For large deviations ($\kappa \gg 1$), $\mathcal{B}_{Kant}$ scales linearly with κ:

$$\mathcal{B}_{Kant}(\kappa) \sim \frac{\kappa}{4}. \tag{3.27}$$

In contrast, the bound $\mathcal{B}_{Conc}$ in Theorem 3.5 depends on terms involving $\ln\left(\frac{\mathscr{G}(\rho)}{\mathscr{G}(\eta)}\right)$. Since $\mathfrak{q} \leq \mathscr{G} \leq \mathfrak{p}$, this logarithmic term is bounded by $\ln(\kappa)$. The structure of (3.10) is of the form:

$$\mathcal{B}_{Conc}(\kappa) \approx C_1 + C_2 \ln(\kappa), \tag{3.28}$$

where C_1, C_2 are constants depending on α, β and the interval length. Using the fundamental inequality $\ln(\kappa) < \kappa - 1$ for $\kappa > 1$, and specifically the fact that logarithmic growth is asymptotically negligible compared to linear growth ($\ln \kappa \ll \kappa$), it follows that:

$$\mathcal{B}_{Conc} < \mathcal{B}_{Kant}, \quad \text{for } \mathfrak{p} \neq \mathfrak{q}. \tag{3.29}$$

Thus, Theorem 3.5 yields a sharper inequality by utilizing the concavity of $\mathscr{G}$ to dampen the estimation error from a linear factor (in $\mathcal{B}_{Kant}$) to a logarithmic factor.

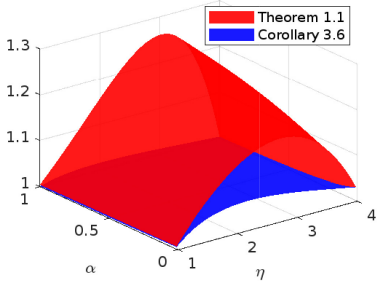
4. Numerical Comparison: Corollary 3.6 vs. Theorem 1.1

In this section, to compare the result obtained in Corollary 3.6 with that of Theorem 1.1, we provide several numerical examples demonstrating that our result offers a sharper bound.

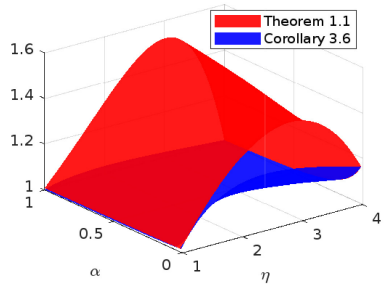
Example 4.1. Let us consider the following three functions: $\mathscr{G}(\mathfrak{l}) = \sqrt{\mathfrak{l}}$, $\mathscr{G}(\mathfrak{l}) = 1 - e^{-\mathfrak{l}}$, and $\mathscr{G}(\mathfrak{l}) = \ln(\mathfrak{l} + 2)$, defined on the interval $[\eta, \eta + 1]$ where $\eta \geq 1$. These functions satisfy the conditions of Theorem 1.1 and Corollary 3.6. The right-hand sides derived from Theorem 1.1 and Corollary 3.6 for each of these functions are displayed in Figure 1(A), Figure 1(B), and Figure 1(C), respectively, for $\alpha \in [0.1, 1]$ and $\eta \in [1, 4]$.

Based on these graphical representations, we confirm that, for the considered functions, the result provided in Corollary 3.6 offers a sharper

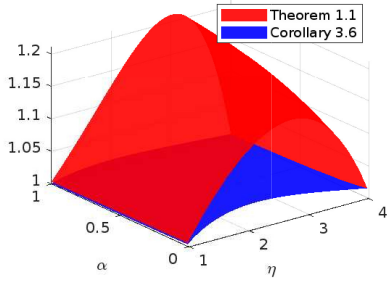
bound than that obtained from Theorem 1.1 by Abdeljawad et al. in [1, Theorem 2.3].



(A) Result for $\mathscr{G}(\mathfrak{l}) = \sqrt{\mathfrak{l}}$



(B) Result for $\mathscr{G}(\mathfrak{l}) = 1 - e^{-\mathfrak{l}}$



(C) Result for $\mathscr{G}(\mathfrak{l}) = \ln(\mathfrak{l} + 1)$

FIGURE 1. Comparison of bounds from Theorem 1.1 and Corollary 3.6 for different functions.

5. Conclusion

In this work, we have successfully extended the Schweitzer inequality to the context of right conformable fractional integrals, offering a novel perspective in the study of integral inequalities within local fractional calculus. The derived bounds explicitly depend on fractional order α and the conformable parameter β and recover known results in the limit case $\alpha = \beta = 1$. Our findings demonstrate a notable improvement over existing formulations in terms of both simplicity and sharpness. This new inequality can serve as a fundamental tool in the study of other classes of fractional inequalities and may find applications in various fields such as optimization, signal theory, and mathematical physics. As future perspectives, it would be interesting to explore extensions to multi-dimensional or stochastic settings.

Conflict of interest. The authors state that they do not have any conflicts of interest.

Data availability. Not applicable

Funding. No funding was received for this work.

Acknowledgment. T. Abdeljawad would like to thank Prince Sultan University for the support through TAS research lab.

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