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Operational Techniques for ${}_pR_q$ Function via Integral Transforms and Fractional Calculus

Belete Debalkie[1] and D.L. Suthar[2*]

Abstract. This paper presents a comprehensive analysis of various integral transforms applied to the recently introduced ${}_pR_q$ function, a generalization of the classical Mittag-Leffler-type functions. Due to their extended structure, these functions are significant in addressing complex problems across science, engineering, and technology. The study systematically investigates and derives the integral transforms of the ${}_pR_q$ function, including the Laplace, Euler-Beta, Mellin, Mohand, Aboodh, SEE, Sadik, Whittaker, and K-transforms. In addition, the work establishes key operational properties such as n^{th} order fractional derivative formulas and recurrence relations. Lastly, we solve a fractional differential equations involving ${}_pR_q$ functions as applications and develop its graphical representations.

1. Introduction

The process of recasting a function typically a time domain signal or spatial distribution into a new function defined in a different conceptual domain is known as an integral transform. Using a carefully selected kernel function as a template to extract, highlight, or reinterpret specific aspects of the original function, this transformation is accomplished through an organized integration process. Novel integral transformations and extensions of classical ones have been developed recently as a result of advancements in generalized special functions and fractional calculus. Transforms pertaining the generalized Mittag-Leffler functions, ${}_pR_q$ function and their variations are of special interest since they have

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been used to explain intricate phenomena like viscoelastic behavior and anomalous diffusion. Several integral transforms are frequently used in fractional calculus (FC), particularly in fractional order differential equations, much as they are in classical calculus and differential equations of integer order. Fractional differential equations (FDEs) have long been important to every aspect of practical mathematics, and since the introduction of the computer, their importance has only increased. It is also well recognized that scientific computing is attempting to understand and manage our natural environment in order to develop new technological processes in the domains of engineering, science, and many other areas. As a result of the investigation and analysis of FDEs that emerged in applications, many methods for solving FDEs were established here. Many researchers used the integral transform to solve the FDEs. Laplace's well-known research (1749–1827) provided the foundation for the integral transform that now bears his name.

Luchko et al.[8] studied fractional calculus (FC) frequently utilizes the Fourier and Laplace integral transforms to tackle a variety of challenges. In this article, the authors describe how the Mellin integral transform works in FC. They note that integral transforms of the Mellin convolution type can be used to define the FC operators, the Cos- and Sin-Fourier transforms, and the Laplace integral transform. Additionally, a particular example of each of FC's special functions is the Fox H-function, which is an inverse Mellin transform of a quotient of given Gamma function products. This article offers an example discussion of several well-known and some innovative applications of the Mellin integral transform to various FC problems.

Parmar et al.[12] presented extensive generalizations of the extended Gauss hypergeometric function, the extended Gammafunction, and the extended beta function in a collaborative publication with Srivastava and Chopra. In this work, the authors used the extended beta function to expand the generalized Mittag-Leffler function. They then methodically examine a number of the extended Mittag-Leffler function's properties, for instance the beta- Laplace, Euler transform transform, Mellin transform, and some fundamental properties. Additionally, specific characteristics of the derivatives and Riemann-Liouville fractional integrals corresponding to the extended Mittag-Leffler function are examined.The authors also highlight a few intriguing exceptional examples of our major findings.

Zayed et al.[22] studied on the fractional Fourier transform, which is a variation of the Fourier transform and has several uses in signal processing and optics.The author used two different methods, one algebraic

and one analytic, to extend the fractional Fourier transform to multiple spaces of generalized functions. For the fractional Fourier transform to be used algebraically, a new convolution process must be invented. When two functions are convolutional, this technique makes the transform of the convolution almost equal to the product of their transforms. In recent years, a lot of research articles have focused on the fractional Fourier transform, which is a variation of the Fourier transform and has several uses in signal processing and optics. In this paper, the authors used two different methods, one algebraic and one analytic, to extend the fractional Fourier transform to multiple spaces of generalized functions. For the fractional Fourier transform to be used algebraically, a new convolution process must be invented. When two functions are convolutional, this technique makes the transform of the convolution almost equal to the product of their transforms.

Srivastava et al.[16] created several formulas using integral transforms, fractional derivatives, and certain types of extended generalized hypergeometric functions.The authors primary findings are demonstrated to have numerous intriguing corollaries and ramifications. The authors also presented and examined the features of a probability density function that incorporates the extended generalized hypergeometric function. Their results can be readily extended to derive the appropriate features of some of the probability density functions and the classical probability distributions that go along with them.

Suthar et al.[17] introduced generalized fractional integral operators with the I function as kernel and established an image formula for the ${}_pR_q$ function, whose properties are also examined. Connections between the ${}_pR_q$ function and standard fractional integrals (Weyl, Riemann–Liouville, Erdélyi–Kober, and Saigo) as well as classical integral transforms (Euler and Laplace) are established, and due to the generality of the I function and ${}_pR_q$ function, many known special-function results are recovered as particular cases.

Debalkie and Suthar[4] studied image formulas for the generalized ${}_pR_q(\alpha, \beta; z)$ function are derived using generalized fractional operators, revealing structural properties of fractional calculus and yielding several special cases related to generalized hypergeometric and Mittag–Leffler functions, thereby contributing to further developments and applications in mathematical analysis. Debalkie and Suthar[3] presented fractional integral and derivative formulas involving the generalized ${}_pR_q$ function are established via Saigo operators and expressed in terms of generalized Wright hypergeometric functions, including results for Weyl,

Erdélyi–Kober, Saigo, and Riemann–Liouville types, along with composition formulas derived using certain integral transforms, highlighting their theoretical significance and potential applications.

The current study is motivated by the increasing significance of generalized Mittag–Leffler-type functions in fractional calculus and their broad relevance in simulating intricate phenomena that arise in science, industry, and engineering. A systematic and comprehensive approach of integral transforms related to the recently published ${}_pR_q$ function has remained largely unexplored despite the quick growth of fractional operators and integral transforms.

The framework of the paper is arranged as follows. The brief introduction provided in Section 1. Section 2, we outline the definitions of certain integral transforms and a few special functions relevant to fractional calculus (FC). Different integral transform of ${}_pR_q$ function are presented in Section 3. Section 4 examines certain properties of the ${}_pR_q$ function, including n^{th} order fractional derivatives, recurrence relations, and more. Section 5 presents application on the solution of fractional differential equations in Caputo sense involving ${}_pR_q$ functions. Finally, we discussed the conclusion and future study.

The following well-known facts are required for our inquiry of integral transforms of the ${}_pR_q$ function.

2. Preliminaries

In this section, presents a brief summary of certain chosen integral transforms, describing their fundamental meanings and theoretical implications.

(i) **Beta Transform** ([15])
The definition of the function $\Psi(w)$ beta (Euler) transform is

$$\{\Psi(w); a, b\} = \int_0^1 w^{a-1}(1-w)^{b-1}\Psi(w)dw, \quad \Re(a) > 0, \Re(b) > 0. \tag{2.1}$$

(ii) **Laplace Transform** ([10])
The Laplace transform of the function $\omega(z)$ is defined by

$$\Omega(s) = \{\omega(z); s\} = \int_0^\infty e^{-sz}\omega(z)\,dz, \quad \Re(s) > 0. \tag{2.2}$$

(iii) **K-Transform** ([10])
The following integral equation defines the K-transform of function $f(x)$,

$$R_\nu\{f(x); \ell\} = \int_0^\infty (\ell x)^{\frac{1}{2}} K_\nu(\ell x) f(x)\,dx, \tag{2.3}$$

for $K_\nu(z)$ is a third-type modified Bessel function and ℓ is a complex parameter.

(iv) **Mohand Transform** ([11])
This is represented by the operator M and transform is defined as follows for $z \geq 0$:

$$M\{f(z);p\} = p^2 \int_0^\infty e^{-pz} f(z)\, dz, \tag{2.4}$$

(v) **Varma Transform** ([10])
The following integral equation defines the Varma transform of function $f(\ell)$,

$$V(f,k,m;s) = \int_0^\infty (s\ell)^{m-\frac{1}{2}} e^{\left(-\frac{1}{2}s\ell\right)} W_{k,v}(s\ell) f(\ell)\, d\ell, \quad \Re(s) > 0, \tag{2.5}$$

and the Whittaker function described by ([10]) is represented by $W_{k,v}(z)$.

(vi) **Hankel Transform** ([10])
The definition of the Hankel transform of function $f(z)$ is

$$H_\nu\{f(x);\ell\} = \int_0^\infty (\ell x)^{\frac{1}{2}} J_\nu(\ell x) f(x) dx, \quad \ell > 0, \tag{2.6}$$

where $J_\nu(z)$ denotes a first-kind Bessel function.

(vii) **Sumudu Transform** ([21])
The definition of the Sumudu transform is

$$G(u) = [\ell(r)] = \int_0^\infty e^{-r} \ell(ur) dr, \quad u \in (-\varrho_1, \varrho_2), \tag{2.7}$$

across the collection of functions

$$A = \left\{ \ell(r) \in \mathbb{R} \;\middle|\; \exists M, \varrho_1, \varrho_2 > 0,\ |\ell(r)| < M e^{\frac{|r|}{j}},\ \text{if} \right.$$
$$\left. \text{if}\ \ r \in (-1)^j \times [0,\infty), j = 1,2 \right\}.$$

(viii) **Fractional Fourier Transform** ([9])
The definition of the fractional Fourier transform of order $\wp$, where $0 < \wp \leq 1$, is

$$\hat{u}_\wp(k) = \Im_\wp[u](k) = \int_{\mathbb{R}} e^{ik^{1/\wp} z} u(z) dz, \tag{2.8}$$

where the function u is a member of the Lizorkin space of functions. When $\wp = 1$ is chosen, (2.8) reduces to the standard Fourier transform provided by

$$\hat{u}(k) = \Im[u](k) = \int_{\mathbb{R}} e^{ikz} u(z) dz. \tag{2.9}$$

(ix) **Sadik Transform** ([14])
The definition of the Sadik transform, represented by S_a, is the following:

$$S_a\{h(\varrho)\} = \frac{1}{v^{\Im}} \int_0^\infty e^{-v\varrho} h(\varrho)\, d\varrho, \quad \text{for } v \in \mathbb{C},\ \Im \in \mathbb{R}. \tag{2.10}$$

The ${}_pR_q(\wp, \Im; z)$ function was introduced by Desai and Shukla [5], who also provided an interpretation by

$$\begin{aligned} {}_pR_q\left(\wp, \Im; z\right) &= {}_pR_q\left[\begin{array}{c} e_p \\ \zeta_q \end{array}\middle| \wp, \Im; z\right] \\ &= {}_pR_q\left[\begin{array}{c} e_1, e_2, \ldots, e_p \\ \zeta_1, \zeta_2, \ldots, \zeta_q \end{array}\middle| \wp, \Im; z\right] \\ &= \sum_{\Upsilon=0}^{\infty} \frac{1}{\Gamma(\wp\Upsilon + \Im)} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_n} \frac{z^\Upsilon}{\Upsilon!}, \end{aligned} \tag{2.11}$$

where $\wp, \Im \in \mathbb{C}$, $\Re(\wp), \Re(\Im), \Re(e_i), \Re(\zeta_j) > 0$ for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, and $p, q \in \mathbb{Z}^+ \cup \{0\}$. e_p is a multi-parameter symbol in this case, meaning that $e_p = (e_1, \ldots, e_p)$ and $(e_p)_\Upsilon = (e_1)_\Upsilon \cdots (e_p)_\Upsilon$, where $(e)_\Upsilon$ is the Pochhammer symbol that is established by

$$(e)_\Upsilon = \frac{\Gamma(e + \Upsilon)}{\Gamma(e)} = \begin{cases} 1 & (\Upsilon = 0,\ e \in \mathbb{C} \setminus \{0\}), \\ e(e+1)\cdots(e+\Upsilon-1) & (\Upsilon \in \mathbb{N},\ e \in \mathbb{C}). \end{cases}$$

Series (2.11) is defined when $\zeta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$. In the instance that any of the numerator parameters e_i $(i = 1, 2, \ldots, p)$ are zero or negative, the series ends in a polynomial in z. The following are updated convergence conditions for the ${}_pR_q\left(\wp, \Im; z\right)$ function:

(i) Any finite value of z will result in the series to converge if $\Re(\wp) \geq p - q$.
(ii) Given the case that $\Re(\wp) = p - q - 1$, the series diverges for $|z| > 1$ and converges for $|z| < 1$.
(iii) According to the parameters, the series may converge conditionally when $\Re(\wp) = p - q - 1$ and $|z| = 1$. If $\wp = p - q - 1$, then $\Re\left(k + \sum_{j=1}^{q} \zeta_j - \sum_{i=1}^{p} e_i\right) > 0$ indicates that the series in (2.11) is absolutely convergent on the circle $|z| = 1$.

When $e_1 = 1$, $\Im = 1$, and $\wp = 1$ are present, (2.11) reduces to

$${}_pR_q\left(\begin{array}{c} 1, e_2, \ldots, e_p \\ \zeta_1, \zeta_2, \ldots, \zeta_q \end{array}\middle| 1, 1, z\right) = \sum_{\Upsilon=0}^{\infty} \frac{1}{\Gamma(\Upsilon + 1)} \frac{(1)_\Upsilon (e_2)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon (\zeta_2)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{z^\Upsilon}{\Upsilon!}$$

$$= {}_{p-1}F_q\left(\begin{array}{c} e_2, \ldots, e_p \\ \zeta_1, \zeta_2, \ldots, \zeta_q \end{array} \middle| z \right),$$

where the generalized hypergeometric (GH) function is denoted by ${}_{p-1}F_q(z)$ [13]. When $p = 1$, $q = 0$, and $e_1 = 1$, (2.11) yields

$$\begin{aligned} {}_pR_q(1|\wp, \Im; z) &= \sum_{\Upsilon=0}^{\infty} \frac{(1)_\Upsilon}{\Gamma(\wp\Upsilon + \Im)} \frac{z^\Upsilon}{\Upsilon!} \\ &= E_{\wp,\Im}(z), \end{aligned}$$

where the generalized Mittag-Leffler (M-L) function is denoted by $E_{\wp,\Im}(z)$ [18].

Here, our goal is to determine the integeral transform and certain properties of the ${}_pR_q$ function. The generalized Wright function is used to express the primary formulas found here. For $z, a_i, b_j \in \mathbb{C}$, and $A_i, B_j \in \Re$ $(A_i, B_j \neq 0;\ i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, ${}_p\psi_q(z)$ is provided by the series

$$\begin{aligned} {}_p\psi_q(z) &= {}_p\psi_q\left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{array} \middle| z \right] \\ &= \sum_{\Upsilon=0}^{\infty} \frac{\Gamma(a_1 + A_1\Upsilon) \ldots \Gamma(a_p + A_p\Upsilon) z^\Upsilon}{\Gamma(b_1 + B_1\Upsilon) \ldots \Gamma(b_q + B_q\Upsilon)\Upsilon!}, \end{aligned} \tag{2.12}$$

where $\Gamma(z)$ is the Euler gamma function, often known as the generalized Wright function, which was first proposed by Wright [19]. For all values of the argument z, a number of asymptotic expansion theorems for ${}_p\psi_q(z)$ are exist, provided that:

$$1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0. \tag{2.13}$$

We refer to the work (for example, see [19, 20]) for a thorough examination of the different generalization, properties, and applications of the generalized Wright function and Wright function.

To obtain the generalized hypergeometric function(GHF) ${}_pF_q$, equation (2.12) reduces to $A_i = B_i = 1$, ([6]).

$${}_pF_q\left[\begin{array}{c} e_1, \ldots, e_p \\ \zeta_1, \ldots, \zeta_q \end{array} \middle| z \right] = \frac{\Gamma(\zeta_1) \ldots \Gamma(\zeta_q)}{\Gamma(e_1) \ldots \Gamma(e_p)} {}_p\psi_q\left[\begin{array}{c} (e_i, 1)_{1,p} \\ (\zeta_j, 1)_{1,q} \end{array} \middle| z \right]. \tag{2.14}$$

By adding a regularization factor, Chaudhry and Zubair [2] in 1994 and Chaudhry *et al.* [1] in 1997 expanded the domain of beta and gamma functions to the whole complex plane:

$$\Gamma_b(v) = \int_0^\infty t^{v-1} e^{-bt} dt, \quad (\Re(b) \geq 0),$$

$$B_b(w,m)=\int_0^1 (1-t)^{m-1}t^{w-1}e^{-bt(1-t)}dt,\quad (\Re(b)\geq 0). \tag{2.15}$$

Note that

$$B_b(w,m)\leq e^{4b}B(w,m),\quad \text{where } \Re(w)>0,\Re(m)>0,\Re(b)\geq 0.$$

We used the following identity to define the generalized version of these functions:

$$\frac{\Gamma(w)\Gamma(m)}{\Gamma(w+m)}=B(w,m),\quad \Re(w),\Re(m)>0.$$

We used the generalized versions of these functions using the equality.

$$\frac{(\wp)_\nu}{(\gamma)_\nu}=\frac{B(\wp+\nu,\gamma-\wp)}{B(\wp,\gamma-\wp)},\quad \Re(\gamma)>\Re(\wp)>0,$$

and substitute $B_b(x,y)$, its generalization for $B(x,y)$ in the numerator.

3. Integral Transforms of ${}_pR_q$ Function

A number of integral transforms are then described for the ${}_pR_q$ function using a number of theorems, including the Laplace transform, Sumudu transform, beta transform, Varma transform, Mohand transform, K-transform, Hankel transform, fractional Fourier transform and sadik transform.

Theorem 3.1 (Laplace Transform)**.** *The following formula holds if* $\wp,\Im,a,\ s\in\mathbb{C},\ \Re(\wp)>0,\ \Re(\Im)>0,\ \Re(s)>0,$ *and* $|as^{-\wp}|<1$ *are satisfied:*

(3.1)

$$L\left(z^{\Im-1}{}_pR_q\left(\wp,\Im;az^{\wp}\right);s\right)=\frac{1}{s^{\Im}}\,{}_{p+1}F_q\left(1,e_1,\ldots,e_p;\zeta_1,\ldots,\zeta_q;\frac{a}{s^{\wp}}\right).$$

Proof. Applying equation (2.2), we get

$$L\left(z^{\Im-1}{}_pR_q\left(\wp,\Im;az^{\wp}\right);s\right)=\int_0^\infty z^{\Im-1}e^{-sz}{}_pR_q\left(\wp,\Im;az^{\wp}\right)\,dz.$$

By utilizing definition of ${}_pR_q$, we obtain

$$=\int_0^\infty z^{\Im-1}e^{-sz}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{a^\Upsilon z^{\wp\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,dz.$$

Changing the order of the summation and integral, which gives us

$$=\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{a^\Upsilon}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\int_0^\infty e^{-sz}z^{\wp\Upsilon+\Im-1}\,dz.$$

By changing $sz = t$, we obtain

$$= \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{a^\Upsilon}{\Gamma(\wp\Upsilon + \Im)\Upsilon!} \frac{1}{s^{\wp\Upsilon+\Im}} \int_0^\infty e^{-t} t^{\wp\Upsilon+\Im-1}\, dt.$$

Applying the gamma function definition, we arrive at

$$= \frac{1}{s^\Im} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{1}{\Upsilon!} \left(\frac{a}{s^\wp}\right)^\Upsilon$$

By interpreting in terms of the GHF defined by (2.14), we reached the required result (3.1). □

Theorem 3.2 (Beta Transform)**.** *If the condition* $\wp$, $\Im$, ϱ, σ, $b \in \mathbb{C}$, $\Re(\wp) > 0$, $\Re(\Im) > 0$, $\Re(\varrho) > 0$, $\Re(\sigma) > 0$, *and* $\nu > 0, \tau > 0$. *Then the following formula holds:*

(3.2)

$$B\left(z^{\Im-1}{}_pR_q\left(\wp, \Im; bz^\nu(t-z)^\tau\right); \varrho, \sigma\right)$$
$$= t^{\varrho+\sigma-1} \left(\frac{\prod_{j=1}^{q} \Gamma(\zeta_j)}{\prod_{j=1}^{p} \Gamma(e_j)} \right)$$
$$\times {}_{p+2}\psi_{q+2}\left[\begin{matrix} (e_1, 1), \ldots, (e_p, 1), (\varrho, \nu), (\sigma, \tau) \\ (\zeta_1, 1), \ldots, (\zeta_q, 1), (\Im, \wp), (\varrho+\sigma, \nu+\tau) \end{matrix} \middle| bt^{\nu+\tau} \right].$$

Proof. Applying the definitions (2.1) to the provided integral yields

$$B\left(z^{\Im-1}{}_pR_q\left(\wp, \Im; bz^\nu(t-z)^\tau\right); \varrho, \sigma\right)$$
$$= \int_0^t z^{\varrho-1}(t-z)^{\sigma-1}{}_pR_q\left(\wp, \Im; bz^\nu(t-z)^\tau\right)\, dz.$$

Utilizing definition (2.11), we obtain

$$= \int_0^t (t-z)^{\sigma-1} z^{\varrho-1} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{b^\Upsilon z^{\nu\Upsilon}(t-z)^{\tau\Upsilon}}{\Gamma(\wp\Upsilon + \Im)\Upsilon!}\, dz.$$

Changing the order of the summation and integral, it gives us

$$= \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{b^\Upsilon}{\Gamma(\wp\Upsilon + \Im)\Upsilon!} \int_0^t z^{\nu\Upsilon+\varrho-1}(t-z)^{\tau\Upsilon+\sigma-1}\, dz.$$

By changing $z = st$, we obtain

$$= t^{\varrho+\sigma-1} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{b^\Upsilon t^{(\nu+\tau)\Upsilon}}{\Gamma(\wp\Upsilon + \Im)\Upsilon!} \int_0^1 s^{\nu\Upsilon+\varrho-1}(1-s)^{\tau\Upsilon+\sigma-1}\, ds.$$

Using the properties of beta and gamma function, we have

$$= t^{\varrho+\sigma-1} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{\Gamma(\nu\Upsilon + \varrho)\Gamma(\tau\Upsilon + \sigma) b t^{(\nu+\tau)\Upsilon}}{\Gamma(\wp\Upsilon + \Im)\Gamma((\nu+\tau)\Upsilon + \varrho + \sigma)\Upsilon!}.$$

By expressing it in terms of the generalized Wright function, ${}_p\psi_q$, as provided by (2.12), we finally obtain (3.2). □

Theorem 3.3. *(Mohand Transform) The following formula holds if the conditions* $\wp$, $\Im$, a, $p \in \mathbb{C}$, $\Re(\wp) > 0$, $\Re(\Im) > 0$, $\Re(p) > 0$, *and* $|ap^{-\wp}| < 1$ *are satisfied:*

(3.3)

$$M\left(z^{\Im-1}{}_pR_q\left(\wp, \Im; az^\wp\right); p\right) = p^{2-\Im}\, {}_pF_q\left(1, e_1, \ldots, e_p; \zeta_1, \ldots, \zeta_q; \frac{a}{p^\wp}\right).$$

Proof. Applying Mohand Transform, we have

$$M\left(z^{\Im-1}{}_pR_q\left(\wp, \Im; az^\wp\right); p\right) = p^2 \int_0^\infty z^{\Im-1} e^{-pz}\, {}_pR_q\left(\wp, \Im; az^\wp\right)\, dz.$$

Using the definition of (2.11), we obtain

$$= p^2 \int_0^\infty z^{\Im-1} e^{-pz} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{a^\Upsilon z^{\wp\Upsilon}}{\Gamma(\wp\Upsilon + \Im)\Upsilon!}\, dz.$$

By changing the order of integral and summation, we get

$$= p^2 \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{a^\Upsilon}{\Gamma(\wp\Upsilon + \Im)\Upsilon!} \int_0^\infty e^{-pz} z^{\wp\Upsilon+\Im-1}\, dz.$$

By changing $pz = t$, we obtain

$$= p^2 \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{a^\Upsilon}{\Gamma(\wp\Upsilon + \Im)\Upsilon!} \frac{1}{p^{\wp\Upsilon+\Im}} \int_0^\infty e^{-t} t^{\wp\Upsilon+\Im-1}\, dt.$$

Using the gamma function definition, we arrive at

$$= p^{2-\Im} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{1}{\Upsilon!} \left(\frac{a}{p^\wp}\right)^\Upsilon.$$

Finally, by interpreting in terms of the generalized hypergeometric function, we get the desired result (3.3). □

Theorem 3.4 (Varma Transform)**.** *The following formula holds if the condition* $\wp,\Im,\varrho,\nu,\lambda,a,b\in\mathbb{C},\sigma>0$ *and* $\Re(\wp)>0,\Re(\Im)>0,\Re(\varrho)>0,\Re(\nu)>0,\Re(\lambda)>0,\Re(a)>0$ *are satisfied:*

(3.4)

$$
\begin{aligned}
&V\left({}_pR_q,k,m;s\right)\\
&\quad=\int_0^\infty z^{\varrho-1}\exp\left(-\frac{1}{2}az\right)W_{\Upsilon,\nu}(az)\,{}_pR_q(\wp,\Im;bz^\sigma)\,dz\\
&\quad=a^{-\varrho}\left(\frac{\prod_{j=1}^{q}\Gamma(\zeta_j)}{\prod_{j=1}^{p}\Gamma(e_j)}\right)\\
&\qquad\times{}_{p+2}\psi_{q+2}\left[\begin{matrix}(e_1,1),\ldots,(e_p,1),\left(\varrho+\nu+\frac{1}{2},\sigma\right),\left(\varrho-\nu+\frac{1}{2},\sigma\right)\\(\zeta_1,1),\ldots,(\zeta_q,1),(\Im,\wp),(1-\lambda+\varrho,\sigma)\end{matrix}\;\middle|\;\frac{b}{a^\sigma}\right].
\end{aligned}
$$

Proof. Applying the definition of the ${}_pR_q$ function (2.11) to the provided integral yields

$$
\begin{aligned}
&\int_0^\infty z^{\varrho-1}\exp\left(-\frac{1}{2}az\right)W_{\lambda,\nu}(az)\,{}_pR_q(\wp,\Im;bz^\sigma)\,dz\\
&\quad=\int_0^\infty z^{\varrho-1}\exp\left(-\frac{1}{2}az\right)W_{\lambda,\nu}(az)\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{b^\Upsilon z^{\sigma\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,dz.
\end{aligned}
$$

By changing the order of the summation and integral, it enables us

$$
\begin{aligned}
&=\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{b^\Upsilon}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\\
&\quad\times\int_0^\infty z^{\sigma\Upsilon+\varrho-1}\exp\left(-\frac{1}{2}az\right)W_{\lambda,\nu}(az)\,dz.
\end{aligned}
$$

Now, by utilizing Mathai and Saxena's integral [10], we obtain

(3.5)
$$
\int_0^\infty z^{\varrho-1}\exp\left(-\frac{1}{2}az\right)W_{\Upsilon,\nu}(az)\,dz=a^{-\varrho}\frac{\Gamma(\varrho+\nu+\frac{1}{2})\Gamma(\varrho-\nu+\frac{1}{2})}{\Gamma(1-\Upsilon+\varrho)}.
$$

By applying equation above (3.5), we acquire

$$
=a^{-\varrho}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{\Gamma\left(\sigma\Upsilon+\varrho+\nu+\frac{1}{2}\right)\Gamma\left(\sigma\Upsilon+\varrho-\nu+\frac{1}{2}\right)}{\Gamma(\wp\Upsilon+\Im)\Gamma(1-\lambda+\varrho+\sigma\Upsilon)\Upsilon!}\left(\frac{b}{a^\sigma}\right)^\Upsilon.
$$

By interpreting the Wright function (2.12), we reached the desired result (3.4). □

Theorem 3.5 (K-Transform)**.** *If* $\wp, \Im, \varrho, \nu, a, b \in \mathbb{C}, \sigma > 0$ *and* $\Re(\wp) > 0, \Re(\Im) > 0, \Re(\varrho) > 0, \Re(\nu) > 0$*, thus, the following formula is true:*

$$\begin{aligned}(3.6)\quad &\int_0^\infty z^{\varrho-1} K_\nu(az)\, {}_pR_q(\wp, \Im; bz^\sigma)\, dz \\ &= \frac{1}{4}\left(\frac{2}{a}\right)^{\varrho} {}_{p+1}\psi_{q+1}\left[\begin{array}{c}(e_1,1),\ldots,(e_p,1),\left(\frac{\varrho\pm\nu}{2},\frac{\sigma}{2}\right)\\(\zeta_1,1),\ldots,(\zeta_q,1),(\Im,\wp)\end{array}\middle| b\left(\frac{2}{a}\right)^{\sigma}\right].\end{aligned}$$

Proof. Applying the ${}_pR_q$ function definition (2.11) to the given integral yields

$$\begin{aligned}&\int_0^\infty z^{\varrho-1} K_\nu(az)\, {}_pR_q(\wp, \Im; bz^\sigma)\, dz \\ &= \int_0^\infty z^{\varrho-1} K_\nu(az) \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{b^\Upsilon z^{\sigma\Upsilon}}{\Gamma(\wp\Upsilon + \Im)\Upsilon!}\, dz.\end{aligned}$$

Changing the order of the integral and summation, which gives us

$$= \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon} \frac{b^\Upsilon}{\Gamma(\wp\Upsilon + \Im)} \int_0^\infty z^{\sigma\Upsilon+\varrho-1} K_\nu(az)\, dz.$$

By utilizing Mathai and Saxena's integral [10], gives

$$(3.7)\qquad \int_0^\infty z^{\varrho-1} K_\nu(az)\, dz = 2^{\varrho-2} a^{-\varrho} \Gamma\left(\frac{\varrho \pm \nu}{2}\right).$$

By applying equation above (3.7), we acquire

$$= \frac{1}{4}\left(\frac{2}{a}\right)^{\varrho} \sum_{\Upsilon=0}^{\infty} \frac{(e_1)_\Upsilon \cdots (e_p)_\Upsilon \Gamma\left(\frac{\sigma\Upsilon}{2} + \left(\frac{\varrho\pm\nu}{2}\right)\right)(1)_\Upsilon}{(\zeta_1)_\Upsilon \cdots (\zeta_q)_\Upsilon \Gamma(\wp\Upsilon + \Im)\,\Upsilon!} \left(b\left(\frac{2}{a}\right)^{\sigma}\right)^{\Upsilon}.$$

By using the definition of the Wright function (2.12), we obtain (3.6). □

Theorem 3.6 (Hankel Transform)**.** *The following formula holds if the condition* $\wp, \Im, \varrho, \nu, a, b \in \mathbb{C}, \sigma > 0$ *and* $\Re(\wp) > 0, \Re(\Im) > 0, \Re(\varrho) > 0$ *are satisfied:*

(3.8)

$$\begin{aligned}&H_v\left({}_pR_q(\wp, \Im; bz^\sigma); a\right) \\ &= \int_0^\infty z^{\varrho-1} J_\nu(az)\, {}_pR_q(\wp, \Im; bz^\sigma)\, dz \\ &= \frac{1}{2}\left(\frac{2}{a}\right)^{\varrho} \left(\frac{\prod_{j=1}^{q} \Gamma(\zeta_j)}{\prod_{j=1}^{p} \Gamma(e_j)}\right)\end{aligned}$$

$$\times {}_{p+2}\psi_{q+2}\left[\begin{array}{c}(e_1,1),\ldots,(e_p,1),\left(\frac{\varrho+\nu}{2},\frac{\sigma}{2}\right)\\(\zeta_1,1),\ldots,(\zeta_q,1),(\Im,\wp),\left(\frac{2+\nu-\varrho}{2},-\frac{\sigma}{2}\right)\end{array}\middle| b\left(\frac{2}{a}\right)^{\sigma}\right].$$

Proof. Applying the ${}_pR_q$ function definition (2.11) to the provided integral yields

$$\int_0^\infty z^{\varrho-1}J_\nu(az)\,{}_pR_q(\wp,\Im;bz^\sigma)\,dz$$
$$=\int_0^\infty z^{\varrho-1}J_\nu(az)\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{b^\Upsilon z^{\sigma\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,dz.$$

By interchanging the integral and summation order, we obtain

$$=\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{b^\Upsilon}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\int_0^\infty z^{\sigma\Upsilon+\varrho-1}J_\nu(az)\,dz.$$

Using Mathai and Saxena's integral [10], yields

$$\int_0^\infty z^{\varrho-1}J_\nu(az)\,dz=2^{\varrho-1}a^{-\varrho}\frac{\Gamma\left(\frac{\varrho+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-\varrho}{2}\right)},\tag{3.9}$$

applying equation above (3.9), we acquire

$$=\frac{2^{\varrho-1}}{a^\varrho}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon\Gamma\left(\frac{\sigma\Upsilon}{2}+\frac{\varrho+\nu}{2}\right)}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon\Gamma(\wp\Upsilon+\Im)\Gamma\left(\frac{2+\nu-\varrho}{2}-\frac{\sigma\Upsilon}{2}\right)\Upsilon!}\left(b\left(\frac{2}{a}\right)^{\sigma}\right)^{\Upsilon}.$$

By applying the wright function definition, we obtain the result (3.8). □

Theorem 3.7. *(Sumudu Transform) The following formula holds if* $\wp,\Im\in\mathbb{C},\Re(\wp)>0,\Re(\Im)>0$, *and* $|u^\wp|<1$ *are satisfied:*

$$S\left({}_pR_q(\wp,\Im;(uz)^\sigma)\right)=u^{\Im-1}\,{}_{p+1}F_q\left(1,e_1,\ldots,e_p;\zeta_1,\ldots,\zeta_q;u^\wp\right).\tag{3.10}$$

Proof. According to (2.7) and (2.11), we get

$$S\left({}_pR_q(\wp,\Im;(uz)^\sigma);a\right)$$
$$=\int_0^\infty(uz)^{\Im-1}e^{-z}\,{}_pR_q(\wp,\Im;(uz)^\sigma)\,dz$$
$$=\int_0^\infty(uz)^{\Im-1}e^{-z}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{u^{\wp\Upsilon}z^{\wp\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,dz.$$

Changing the order of the integral and summation, which gives us

$$=u^{\Im-1}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{u^{\wp\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\int_0^\infty e^{-z}z^{\wp\Upsilon+\Im-1}\,dz.$$

Using the gamma function definition, we arrive at

$$= u^{\Im-1}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{u^{\wp\Upsilon}}{\Upsilon!}.$$

Applying definition (2.14), we require the result (3.10). □

Theorem 3.8. *(Fractional Fourier Transform) Assume that* $\sigma, \Im \in \mathbb{C}, \Re(\sigma) > 0$, $\Re(\Im) > 0, z < 0$,*and* $\left|\left(\frac{-a}{i\omega^{\frac{1}{\wp}}}\right)^{\sigma}\right| < 1$ *are satisfied, then the following formula holds :*

(3.11)

$$\Im_\wp[{}_pR_q](\omega) = (-1)^{\Im-1}(i\omega\wp)^{-\Im}{}_pF_q\left(e,\ldots,e_p;\zeta_1,\ldots,\zeta_q;\left(\frac{-a}{i\omega^{\frac{1}{\wp}}}\right)^{\sigma}\right).$$

Proof. Applying the ${}_pR_q$ function definition (2.11) to the provided integral yields

$$\begin{aligned}\Im_\wp[{}_pR_q](\omega) &= \int_{\mathbb{R}} z^{\Im-1}e^{i\omega^{\frac{1}{\wp}}z}{}_pR_q(\wp,\Im;(az)^\sigma)dz\\ &= \int_{\mathbb{R}} z^{\Im-1}e^{i\omega^{\frac{1}{\wp}}z}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{a^{\sigma\Upsilon}z^{\sigma\Upsilon}}{\Gamma(\sigma\Upsilon+\Im)\Upsilon!}dz.\end{aligned}$$

Changing the order of the integral and summation, which is gives us

$$= \sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{(a^\sigma)^\Upsilon}{\Gamma(\sigma\Upsilon+\Im)\Upsilon!}\int_{-\infty}^{0}e^{i\omega^{\frac{1}{\wp}}z}z^{\sigma\Upsilon+\Im-1}\,dz.$$

By altering the variable $i\omega^{\frac{1}{\wp}}z = -h$, we obtain

$$= \sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\frac{(a^\sigma)^\Upsilon}{\Gamma(\sigma\Upsilon+\Im)\Upsilon!}\frac{(-1)^{\sigma\Upsilon+\Im-1}}{(i\omega^{\frac{1}{\wp}})^{\sigma\Upsilon+\Im}}\int_0^{\infty}e^{-h}h^{\sigma\Upsilon+\Im-1}\,dh.$$

Using the gamma function definition, we arrive at

$$= \frac{(-1)^{\Im-1}}{(i\omega^{\frac{1}{\wp}})^{\Im}}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon\Upsilon!}\frac{(-1)^{\sigma\Upsilon}a^{\sigma\Upsilon}}{(i\omega^{\frac{1}{\wp}})^{\sigma\Upsilon}}.$$

By utilizing definition (2.14), we obtain the desired result (3.11). □

Theorem 3.9. *(Sadik Transform) The following formula holds if the condition* $\wp, \Im, a, s \in \mathbb{C}, \Re(\wp) > 0, \Re(\Im) > 0$, $\Re(s) > 0$, *and* $|a\nu^{-\wp}| < 1$ *are fulfilled:*

(3.12)

$$S_a\left(z^{\Im-1}{}_pR_q\left(\wp,\Im;az^\wp\right)\right) = \frac{1}{\nu^{2\Im}}\,{}_pF_q\left(e_1,\ldots,e_p;\zeta_1,\ldots,\zeta_q;\frac{a}{\nu^\wp}\right).$$

Proof. Applying definition (2.10), we acquire

$$Sa\left(z^{\Im-1}{}_pR_q\left(\wp,\Im;az^{\wp}\right);p\right)=\frac{1}{\nu^{\Im}}\int_0^{\infty}z^{\Im-1}e^{-\nu z}{}_pR_q\left(\wp,\Im;az^{\wp}\right)\,dz.$$

Utilizing ${}_pR_q$ definition, we get

$$=\frac{1}{\nu^{\Im}}\int_0^{\infty}z^{\Im-1}e^{-\nu z}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{a^\Upsilon z^{\wp\Upsilon}}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,dz.$$

By interchanging integral and summation, we obtain the following

$$=\frac{1}{\nu^{\Im}}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{a^\Upsilon}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\int_0^{\infty}e^{-\nu z}z^{\wp\Upsilon+\Im-1}\,dz.$$

By changing $\nu z=t$, we obtain

$$=\frac{1}{\nu^{\Im}}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{a^\Upsilon}{\Gamma(\wp\Upsilon+\Im)\Upsilon!}\,\frac{1}{\nu^{\wp\Upsilon+\Im}}\int_0^{\infty}e^{-t}t^{\wp\Upsilon+\Im-1}\,dt.$$

Using the Gamma function definition, we arrive at

$$=\frac{1}{\nu^{2\Im}}\sum_{\Upsilon=0}^{\infty}\frac{(e_1)_\Upsilon\cdots(e_p)_\Upsilon}{(\zeta_1)_\Upsilon\cdots(\zeta_q)_\Upsilon}\,\frac{1}{\Upsilon!}\left(\frac{a}{\nu^{\wp}}\right)^{\Upsilon}.$$

By using the definition (2.14), the required result (3.12) is obtained. □

4. Properties of ${}_pR_q$ Function

Inspired by the previous study, let us introduce the ${}_pR_q$ function:

(4.1)

$${}_pR_q\left(\wp,\Im;z,b\right)={}_pR_q\left[\begin{array}{c}e_1,e_2,\ldots,e_p\\ \zeta_1,\zeta_2,\ldots,\zeta_q\end{array}\middle|\wp,\Im;z,b\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\wp n+\Im)}\,\frac{(e_2)_n\cdots(e_p)_n}{(\zeta_2)_n\cdots(\zeta_q)_n}\,\frac{B_b(e_1+k,\zeta_1-e_1)}{B(e_1,\zeta_1-e_1)}\,\frac{z^n}{n!},$$

where $\wp,\Im\in\mathbb{C}$, $\Re(\wp),\Re(\Im),\Re(e_i),\Re(\zeta_j)>0,\Re(b)>0$, $\Re(\zeta_1)>\Re(e_1)>0$ for all $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$, and $p,q\in\mathbb{Z}^+\cup\{0\}$. We begin our investigation by deriving certain integral representations of the ${}_pR_q$ function.

Theorem 4.1. *Assume that* $\wp,\Im\in\mathbb{C}$, $\Re(\wp),\Re(\Im),\Re(e_i),\Re(\zeta_j)>0$, $\Re(b)>0,\Re(\zeta_1)>\Re(e_1)>0$ *for all* $i=1,2,\ldots,p$ *and* $j=1,2,\ldots,q$, *and* $p,q\in\mathbb{Z}^+\cup\{0\}$. *Then the following are the integral representations of the* ${}_pR_q$ *function:*

${}_pR_q(\wp,\Im;z,b)$

$$= \frac{1}{B(e_1, \zeta_1 - e_1)} \times \int_0^1 \left[t^{e_1-1}(1-t)^{\zeta_1-e_1-1} e^{\frac{-b}{t(1-t)}} {}_{p-1}R_{q-1} \left[\begin{array}{c} e_2, \ldots, e_p \\ \zeta_2, \ldots, \zeta_q \end{array} \middle| \wp, \Im; zt \right] \right] dt$$

$$= \frac{1}{B(e_1, \zeta_1 - e_1)} \times \int_0^\infty \left[{}^{e_1-1}(1+u)^{-\zeta_1} e^{-b\frac{(u+1)^2}{u}} {}_{p-1}R_{q-1} \left[\begin{array}{c} e_2, \ldots, e_p \\ \zeta_2, \ldots, \zeta_q \end{array} \middle| \wp, \Im; \frac{zu}{u+1} \right] \right] du$$

$$= \frac{2}{B(e_1, \zeta_1 - e_1)} \int_0^{\pi/2} (\sin\varpi)^{2a_1-1} (\cos\varpi)^{2c_1-2a_1-1} e^{-\frac{b}{\sin^2\varpi\cos^2\varpi}} \times {}_{p-1}R_{q-1} \left[\begin{array}{c} e_2, \ldots, e_p \\ \zeta_2, \ldots, \zeta_q \end{array} \middle| \wp, \Im; z\sin^2\varpi) \right] d\varpi.$$

Proof. Utilizing the right-hand side of (2.15) in (4.1) and interchanging the integration and summation order, we obtain:

$${}_pR_q(\wp, \Im; zt, b)$$

$$= \frac{1}{B(e_1, \zeta_1 - e_1)} \times \int_0^1 \left[t^{e_1-1}(1-t)^{\zeta_1-e_1-1} e^{\frac{-b}{t(1-t)}} \sum_{n=0}^\infty \frac{1}{\Gamma(\wp n + \Im)} \frac{(e_2)_n \cdots (e_p)_n}{(\zeta_2)_n \cdots (\zeta_q)_n} \frac{(zt)^n}{n!} \right] dt$$

$$= \frac{1}{B(e_1, \zeta_1 - e_1)} \times \int_0^1 \left[t^{e_1-1}(1-t)^{\zeta_1-e_1-1} e^{\frac{-b}{t(1-t)}} {}_{p-1}R_{q-1}(e_2, \ldots, e_p; \zeta_2, \ldots, \zeta_q; zt) \right] dt,$$

since the first equality is satisfied.

To prove the second equality, we use the transformation $t = \frac{u}{1+u}$, implies $dt = \frac{1}{(1+u)^2} du$. Again, when $t \to 0 \implies u \to 0$ and when $t \to 1 \implies u \to \infty$.

Following some simplification, we acquire the following:

$${}_pR_q(\wp, \Im; zt; b)$$

$$= \frac{1}{B(e_1, \zeta_1 - e_1)} \times \int_0^\infty \left[(1+u)^{-\zeta_1} u^{e_1-1} e^{-b\frac{(u+1)^2}{u}} {}_{p-1}R_{q-1} \left(e_2, \ldots, e_p; \zeta_2, \ldots, \zeta_q; \frac{uz}{u+1} \right) \right] du.$$

To prove the third equality by transformations, $t = sin^2\varpi$, implies $dt = 2sin\varpi cos\varpi$. Again, when $t \to 0 \implies \varpi \to 0$ and when $t \to 1 \implies$

$\varpi \to \frac{\pi}{2}$.
After some simplification, we obtain:

$$
\begin{aligned}
{}_pR_q(\wp,\Im;zt) = {} & \frac{1}{B(e_1,\zeta_1-e_1)} \\
& \times \left[\int_0^{\pi/2} (\sin\varpi)^{2e_1-1}(\cos\varpi)^{2\zeta_1-2e_1-1} e^{-\frac{b}{\sin^2\varpi\cos^2\varpi}}\right. \\
& \left. \times {}_{p-1}R_{q-1}(e_2,\ldots,e_p;\zeta_2,\ldots,\zeta_q;z\sin^2\varpi)d\varpi\right]. \qquad \square
\end{aligned}
$$

Theorem 4.2. *Let $\wp,\Im \in \mathbb{C}$, $\Re(\wp),\Re(\Im),\Re(e_i),\Re(\zeta_j) > 0, \Re(b) > 0, \Re(\zeta_1) > \Re(e_1) > 0$ for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$, and $p,q \in \mathbb{Z}^+ \cup \{0\}$. Then the following recurrence relation ${}_pR_q$ function is hold:*

$$
\begin{aligned}
{}_pR_q(\wp,\Im;zt,b) = {} & \frac{\zeta_1-e_1}{\zeta_1}{}_pR_q\left[\begin{array}{c} e_1,e_2,\ldots,e_p \\ \zeta_1+1,\zeta_2,\ldots,\zeta_q \end{array}\middle|\wp,\Im;zt,b\right] \\
& + \frac{e_1}{\zeta_1}{}_pR_q\left[\begin{array}{c} e_1+1,e_2,\ldots,e_p \\ \zeta_1,\zeta_2,\ldots,\zeta_q \end{array}\middle|\wp,\Im;zt,b\right].
\end{aligned}
$$

Proof. Applying Theorem (4.1), we obtain

$$
\begin{aligned}
& {}_pR_q(\wp,\Im;zt,b) \\
& = \frac{1}{B(e_1,\zeta_1-e_1)}\int_0^1 t^{e_1-1}(1-t)^{\zeta_1-e_1-1}(1-t+t)e^{-\frac{b}{t(1-t)}} \\
& \quad \times {}_{p-1}R_{q-1}\left[\begin{array}{c} e_2,\ldots,e_p \\ \zeta_2,\ldots,\zeta_q \end{array}\middle|\wp,\Im;zt\right]dt \\
& = \frac{1}{B(e_1,\zeta_1-e_1)} \\
& \quad \times \left[\int_0^1 t^{e_1-1}(1-t)^{\zeta_1-e_1}e^{-\frac{b}{t(1-t)}}{}_{p-1}R_{q-1}\left[\begin{array}{c} e_2,\ldots,e_p \\ \zeta_2,\ldots,\zeta_q \end{array}\middle|\wp,\Im;zt\right]dt\right. \\
& \quad \left. + \int_0^1 t^{e_1}(1-t)^{\zeta_1-e_1-1}e^{-\frac{b}{t(1-t)}}{}_{p-1}R_{q-1}\left[\begin{array}{c} e_2,\ldots,e_p \\ \zeta_2,\ldots,\zeta_q \end{array}\middle|\wp,\Im;zt\right]dt\right].
\end{aligned}
$$

The result is obtained by using the gamma function property and multiplying the terms by $\frac{B(e_1,\zeta_1+1-e_1)}{B(e_1,\zeta_1+1-e_1)}$ and $\frac{B(e_1+1,\zeta_1-e_1)}{B(e_1+1,\zeta_1-e_1)}$ respectively. $\square$

Corollary 4.3. *Assume that $\Im \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\Im),\Re(e_i),\Re(\zeta_j) > 0, \Re(b) > 0, \Re(\zeta_1) > \Re(e_1) > 0$ for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$, and $p,q \in \mathbb{Z}^+ \cup \{0\}$. The then the following transformation formula for*

$\wp = 1$ *is holds:*

$$
\begin{aligned}
&{}_pR_q\left[\begin{array}{c} e_1, e_2, \ldots, e_p \\ \zeta_1, \zeta_2, \ldots, \zeta_q \end{array}\middle| 1, \Im; z, b\right] \\
&\quad = \frac{\Gamma(\gamma)}{\Gamma(\Im)}\ {}_{p+1}R_{q+1}\left[\begin{array}{c} e_1, e_2, \ldots, e_p, \gamma \\ \zeta_1, \zeta_2, \ldots, \zeta_q, \Im \end{array}\middle| 1, \gamma; z, b\right].
\end{aligned}
$$

Theorem 4.4. *If* $\wp, \Im \in \mathbb{C}$, $\Re(\wp), \Re(\Im), \Re(e_i), \Re(\zeta_j) > 0, \Re(b) > 0, \Re(\zeta_1) > \Re(e_1) > 0$ *for all* $i = 1, 2, \ldots, p$ *and* $j = 1, 2, \ldots, q$, *and* $p, q \in \mathbb{Z}^+ \cup \{0\}$. *Then the* n^{th} *derivative formula of* ${}_pR_q$ *function is as follows:*

(4.2)

$$
\begin{aligned}
&\frac{d^n}{dz^n}\left\{{}_pR_q(\wp, \Im; z, b)\right\} \\
&\quad = n!\frac{(e_1)_n \cdots (e_p)_n}{(\zeta_1)_n \cdots (\zeta_q)_n} \\
&\qquad \times {}_{p+1}R_{q+1}\left[\begin{array}{c} n+1, e_1+n, e_2+n, \ldots, e_p+n \\ 1, \zeta_1+n, \zeta_2+n, \ldots, \zeta_q+n \end{array}\middle| \wp, \Im+\wp n; z, b\right].
\end{aligned}
$$

Proof. Using the standard differentiation formula for power functions, we have

$$
\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^\lambda\right) = \frac{\Gamma(1+\lambda)}{\Gamma(\lambda-n+1)}z^{\lambda-n},
$$

Using the above formula, interchanging the n^{th} derivative and summation in the equation of (4.1) yields the following result.

$$
\begin{aligned}
&\frac{d^n}{dz^n}\left\{{}_pR_q(\wp, \Im; z, b)\right\} \\
&\quad = \sum_{k=n}^{\infty} \frac{(e_2)_k \cdots (e_p)_k}{(\zeta_2)_k \cdots (\zeta_q)_k} \frac{B_b(e_1+k, \zeta_1-e_1)}{B(e_1, \zeta_1-e_1)} \frac{\Gamma(k+1)}{\Gamma(\wp n+\Im)} \frac{z^{k-n}}{k!}.
\end{aligned}
$$

Substituting $k \to k+n$ and use $(e)_{k+n} = (e)_n(e+n)_k$, we obtain:

$$
\begin{aligned}
&= \frac{(e_2)_n \cdots (e_p)_n}{(\zeta_2)_n \cdots (\zeta_q)_n} \sum_{k=0}^{\infty} \frac{(e_2+n)_k \cdots (e_p+n)_k}{(\zeta_2+n)_k \cdots (\zeta_q+n)_k} \frac{B_b(e_1+k+n, \zeta_1-e_1)}{B(e_1, \zeta_1-e_1)} \\
&\quad \times \frac{\Gamma(k+n+1)}{\Gamma(k+1)} \frac{z^k}{\Gamma(\wp k+\Im+\wp n)k!}.
\end{aligned}
$$

Multiply by $\frac{\Gamma(n+1)B(e_1+n, \zeta_1-e_1)}{\Gamma(n+1)B(e_1+n, \zeta_1-e_1)}$, and after some simplification, we get

$$
\frac{d^n}{dz^n}\left\{{}_pR_q(\wp, \Im; z)\right\}
$$

$$= n!\frac{(e_1)_n\cdots(e_p)_n}{(\zeta_1)_n\cdots(\zeta_q)_n}\sum_{k=0}^{\infty}\frac{(e_2+n)_k\cdots(e_p+n)_k}{(\zeta_2+n)_k\cdots(\zeta_q+n)_k}\frac{(n+1)_k}{(1)_k}$$
$$\times\frac{B_b(e_1+n+k,\zeta_1-e_1)}{B(e_1+n,\zeta_1-e_1)}\frac{z^k}{\Gamma(\wp k+\Im+\wp n)k!}.$$

Applying definition (4.1), we get the desired result (4.2). □

Corollary 4.5. *Suppose that* $\Re(e_i),\Re(\zeta_j)>0,\Re(b)>0,\Re(\zeta_1)>\Re(e_1)>0$ *for all* $i=1,2,\ldots,p$ *and* $j=1,2,\ldots,q$, *and* $p,q\in\mathbb{Z}^+\cup\{0\}$. *If* $\wp=\Im=1$ *in equation (4.2), then we get the following special case formula:*

$$\frac{d^n}{dz^n}\left\{{}_pR_q(1,1;z,b)\right\}$$
$$=\frac{(e_1)_n\cdots(e_p)_n}{(\zeta_1)_n\cdots(\zeta_q)_n}\times{}_{p+1}R_{q+1}\left[\begin{array}{c}n+1,e_1+n,e_2+n,\ldots,e_p+n\\1,\zeta_1+n,\zeta_2+n,\ldots,\zeta_q+n\end{array}\middle|1,1;z,b\right].$$

Corollary 4.6. *Assume that* $\wp,\Im\in\mathbb{C}$, $\Re(\wp),\Re(\Im),\Re(e_i),\Re(\zeta_j)>0,\Re(b)>0,\Re(\zeta_1)>\Re(e_1)>0$ *for all* $i=1,2,\ldots,p$ *and* $j=1,2,\ldots,q$, *and* $p,q\in\mathbb{Z}^+\cup\{0\}$. *Then the derivative formula that follows is valid:*

$$\frac{d^n}{dz^n}\left\{z^{\Im-1}{}_pR_q(\wp,\Im;\lambda z^{\wp},b)\right\}=z^{\Im-n-1}{}_pR_q(\wp,\Im-n;\lambda z^{\wp},b).$$

5. Application on the Solution of Fractional Differential Equations in Caputo Seense Involving ${}_pR_q$ Functions

Fractional differential equations (FDEs) in the Caputo sense are more flexible than their conventional integer-order counterparts and have become effective instruments for simulating complicated events with memory and hereditary characteristics. Generalized gamma, beta, and hypergeometric functions are examples of recent extensions of special functions that have improved the analytical framework for resolving these FDEs. This study uses the integral representations and transform features of extended special functions and Caputo derivatives to study exact solutions to FDEs involving these functions. One of the most practical mathematical methods is the integral transform method, which utilizes very useful strategies that are frequently employed to derive the analytical solution of fractional differential equations.

Definition 5.1 ([7])**.** Let $\Im>0$, $m=\lceil\Im\rceil$ and $\varrho_0\in\mathbb{R}$. For a function $f\in A^m[\varrho_0,\varrho]$, i.e., such that $f^{(m-1)}$ is absolutely continuous, the Caputo derivative is defined as:

$${}^CD^{\Im}_{\varrho_0}f(\varrho)=\frac{1}{\Gamma(m-\Im)}\int_{\varrho_0}^{\varrho}(\varrho-\tau)^{m-\Im-1}f^{(m)}(\tau)\,d\tau,\quad \varrho>\varrho_0,$$

where D^m and $f^{(m)}$ denote integer-order derivatives.

Lemma 5.2 ([7]). *The Laplace transform of the Caputo's derivative of a function $f(\varrho)$ is*

(5.1)

$$\mathcal{L}\{f^{(\wp)}(\varrho)\} = \Im^{\wp}\mathcal{L}\{f(\varrho)\} - \Im^{\wp-1}f(0) - \Im^{\wp-2}f'(0) - \cdots - \Im^{\wp-(n+1)}f^{(n)}(0)$$

for $n < \wp \leq n+1$, $n \geq 1$.

Example 5.3. Let $1 < \Re(\wp) \leq 2$, $\Re(\sigma) > 0$ and $e_1, \ldots, e_p, f_1, \ldots, f_q \neq 0, -1, -2, \ldots$. Then, the fractional differential equation

$$^{\mathbb{C}}\mathscr{D}_{\varrho}^{\wp}\{f(\ell)\} = {}_pR_q\left[\begin{array}{c} e_1, e_2, \ldots, e_p \\ f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \wp\ell\right], \tag{5.2}$$

with initial conditions

$$f(0) = f'(0) = 0,$$

has its solution given by

$$f(\ell) = \frac{\ell^{\wp}}{\Gamma(1+\wp)}\ {}_{p+1}R_{q+1}\left[\begin{array}{c} 1, e_1, e_2, \ldots, e_p \\ 1+\wp, f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \wp\ell\right].$$

Proof. Applying the Laplace transform to both sides of the equation (5.2),

$$\mathcal{L}\left\{^{\mathbb{C}}\mathscr{D}_{\varrho}^{\wp}f(\ell); \Im\right\} = \mathcal{L}\left\{{}_pR_q\left[\begin{array}{c} e_1, e_2, \ldots, e_p \\ f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \wp\ell\right]; \Im\right\},$$

and then by the help of equation (5.1), we acquire

$$\Im^{\wp}F(\Im) - \Im^{\wp-1}f(0) - \Im^{\wp-2}f'(0) = \frac{1}{\Im}\ {}_{p+1}R_q\left[\begin{array}{c} e_1, e_2, \ldots, e_p, 1 \\ f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \frac{\wp}{\Im}\right].$$

Using the initial conditions, we obtain

$$F(\Im) = \frac{1}{\Im^{\wp+3}}\ {}_{p+1}R_q\left[\begin{array}{c} 1, e_1, e_2, \ldots, e_p \\ f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \frac{\wp}{\Im}\right].$$

By utilizing the inverse Laplace transform on both sides of the final equation and doing the required computations, we are able to get the solution function as follows:

$$f(\ell) = \frac{\ell^{\wp}}{\Gamma(1+\wp)}{}_{p+1}R_{q+1}\left[\begin{array}{c} 1, e_1, e_2, \ldots, e_p \\ 1+\wp, f_1, f_2, \ldots, f_q \end{array}\middle|\sigma, \psi; \wp\ell\right].$$

We are presented approximate graph of the solution of $f(\ell)$ of example 5.3 below.

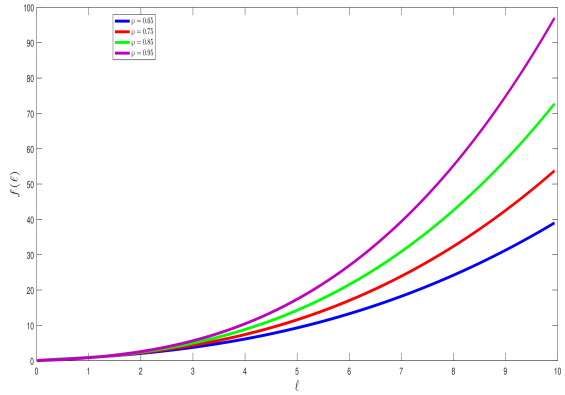


FIGURE 1. The behavior of the approximate 2D graph of the solution of $f(\ell)$ for $\wp = 0.95, 0.85, 0.75$ and $\wp = 0.65$, where $p = q = e_1 = f_1 = \sigma = \psi = 1, x = y = 5$.

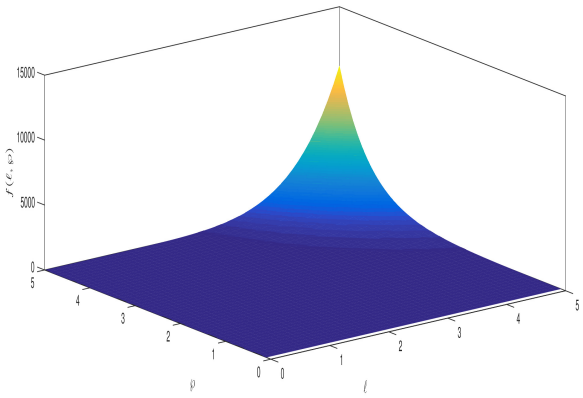


FIGURE 2. The behavior of the approximate 3D graph of the solution a function $f(\ell, \wp)$, where $p = q = e_1 = f_1 = \sigma = \psi = 1, x = y = 5$.

The fractional order has a significant impact on the solutions of the fractional differential equation. The sensitivity of the solutions to fractional order changes is demonstrated in Figure 1, where changing the fractional order results in different growth patterns, including rapid or divergent behavior for larger values of $\wp$. Fractional differential equation solutions, which are extremely sensitive to the fractional order, can be extended from 2D curves to 3D surfaces, as seen in Figure 2. Surface

shape and development behavior are greatly impacted by order variations, yielding results that range from smooth to rapidly diverging.

6. Conclusion and Future Study

In conclusion, this paper provides a comprehensive analysis of the operational characteristics and integral transforms of the generalized ${}_pR_q$ function, including the Laplace, Euler-Beta, Mellin, Mohand, Aboodh, SEE, Sadik, Whittaker, and K-transforms. In addition to improving theoretical knowledge of these extended Mittag-Leffler-type functions, the study offers useful analytical tools for solving challenging problems in applied mathematics, science, and engineering by obtaining n^{th} order fractional derivative formula, recurrence relations and others.
Building on these results, developing computational algorithms and numerical implementations of the derived transformations and fractional operators for the generalized ${}_pR_q$ function is an interesting area for future research. By incorporating these analytical methods into simulation frameworks, practical applications in anomalous diffusion modeling, signal processing, and fractional-order control systems could be investigated. Additionally, this method would be much more useful in developing interdisciplinary fields like computational biology and quantitative finance if it were extended to multivariate and matrix-variate settings or combined with machine learning techniques for solving high-dimensional fractional PDEs.

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References

1. M.A. Chaudhry, A. Qadir, M. Rafique and S.M. Zubair, *Extension of Euler's* Γ *function*, J. Comput. Appl. Math., 78 (1) (1997), pp. 19–32.
2. M.A. Chaudhry and S.M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math., 55 (1) (1994), pp. 99–124.
3. B. Debalkie and D.L. Suthar, *Composition formula for Saigo fractional calculus operator on* ${}_pR_q$ *function*, Int. J. of Mathematics and Mathematical Sciences (accepted).
4. B. Debalkie and D.L. Suthar, *Generalized fractional integral and derivative formulas for* ${}_pR_q$ *function*, Int. J. of Applied Nonlinear Science (accepted).

5. R. Desai, and A.K.Shukla, *Some results on function* ${}_pR_q(\mu, \Im; z)$, J. Math. Anal. Appl., 448 (1) (2017), pp. 187–197.
6. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of integral transforms. Vol. 1*, McGraw-Hill, New York–Toronto–London, 1953.
7. R. Garrappa, E. Kaslik and M. Popolizio, *Evaluation of fractional integrals and derivatives of elementary functions: Overview and tutorial*, Mathematics, 7 (5) (2019), 407.
8. Y. Luchko and V. Kiryakova, *The Mellin integral transform in fractional calculus*, Fract. Calc. Appl. Anal., 16 (2) (2013), pp. 405–430.
9. Y.F. Luchko, H. Matrínez and Trujillo J.J., *Fractional Fourier transform and some of its applications*, Fract. Calc. Appl. Anal., 11 (4) (2008), pp. 457–470.
10. A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-function theory and applications*, Springer, New York, 2010.
11. M. Mohand and A. Mahgoub, *The new integral transform Mohand transform*, Advances in Theoretical and Applied Mathematics, 12 (2) (2017), pp. 113–120.
12. R.K. Parmar, *A class of extended Mittag–Leffler functions and their properties related to integral transforms and fractional calculus*, Mathematics, 3 (4) (2015), pp. 1069–1082.
13. E.D. Rainville, *Special functions*, The Macmillan Company, New York, 1960, xii+365 pp.
14. L.S. Sadikali, *Introducing a new integral transform, Sadik transform*, American International Journal of Research in Science, Technology, Engineering and Mathematics, 22 (1) (2018), pp. 100–102.
15. I.N. Sneddon, *The Use of Integral Transform*, Tata Mc Graw-Hill, New Delhi, 1979.
16. H.M. Srivastava, R. Agarwal and S. Jain, *Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions*, Math. Methods Appl. Sci., 40 (1) (2017), pp. 255–273.
17. D.L. Suthar, N. Abeye, B. Shimelis and F. Gidaf, *Generalized fractional integral operators pertaining to the* ${}_pR_q(\lambda, \vartheta; x)$*-function*, Research in Mathematics, 12 (1) (2025), 2505280.
18. A. Wiman, *Über den Fundamentalsatz in der Teorie der Funktionen* $E_a(x)$, Acta Math., New York, 1905, pp. 191–201.
19. E.M. Wright, *The Asymptotic expansion of the generalized Bessel Function*, Proc. London Math. Soc., 38 (2) (1935), pp. 257-270.
20. E.M. Wright, *The asymptotic expansion of integral functions defined by Taylor series*, Philos. Trans. Roy. Soc. London Ser. A, 238,

1940, pp. 423–451.
21. G.K. Watugala, *Sumudu transform, a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Tech., 24 (1) (1993), pp. 35–43.
22. A.I. Zayed, *Fractional Fourier transform of generalized functions*, Integral Transform. Spec. Funct., 7 (3–4) (1998), pp. 299–312.

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