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A Generalization of Suzuki and Ćirić Fixed point Theorems

Deepak Khantwal[1*] and Rajendra Pant[2]

Abstract. This paper aims to present a generalization of the main results of Suzuki [Proc. Amer. Math. Soc. 136(5):1861–1869, 2008] and Ćirić [Proc. Amer. Math. Soc. 45: 267–273, 1974]. We introduce a new class of mappings that contains several existing classes of contraction mappings as special cases. We then establish some fixed point results for this new class of mappings. We also provide illustrative examples to support our theoretical findings. Moreover, we discuss an application to delay differential equations.

1. Introduction

The classical Banach contraction principle (BCP) is one of the most fundamental and widely used results in metric fixed point theory. It plays a crucial role in various branches of analysis, including metric theory, fractal theory, integration theory, and many others. Over the years, this principle has been generalized and extended by numerous mathematicians. Notable generalizations of the BCP include those by Kannan [11], Bianchini [29], Chatterjee [3], Hardy and Rogers [7], among others (see also [1, 9, 10, 23, 24, 30, 33]).

In 1974, Ćirić [4] introduced a significant generalization of the BCP, now known as Ćirić's quasi-contraction. This concept is considered as one of the most general forms of contraction, as highlighted by Rhoades in [26]. Later, in 2017, Kumam et al. [16] introduced the notion of generalized quasi-contractions (GQC) and established fixed point results for such mappings. Several important developments in this direction can be found in [2, 5, 6, 8, 12, 16–18, 20, 25, 27, 32].

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On the other hand, in 2007, Suzuki [28] proposed another notable generalization of the BCP, which, unlike Ćirić's quasi-contractions and their extensions, applies only to certain pairs of elements in the underlying space rather than the entire space. This class of mappings, now known as Suzuki contraction mappings, has since been extensively studied and generalized (see [13–15, 18, 19, 31]). In [21], Pant et al. introduced the concept of semi-quasi contractions (SQC), which contains both quasi-contractions and certain Suzuki-type contractions. Subsequently, Pant [18] further extended this idea to a more general class, termed generalized quasi-contractions, though this class still does not fully contain the Suzuki's contractions.

In this paper, we establish fixed point results for a broader class of contraction mappings that unifies and extends the classes of generalized quasi-contractions, Suzuki contractions, quasi contractions, and many others. We refer to this new class as generalized Suzuki-Ćirić quasi-contractions. Moreover, we provide illustrative examples to support our results and discuss their application to delay differential equations.

2. Preliminaries

Throughout this paper, let Ω be a nonempty set, $\mathbb{N}$ the set of positive integers, and (Ω, ρ) a metric space. A mapping Γ is said to be asymptotically regular on Ω if for each $\omega \in \Omega$, $\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) \to 0$ as $n \to \infty$. Consider $\Gamma\colon \Omega \to \Omega$ as a self-mapping. For any $\omega \in \Omega$ and $n \in \mathbb{N}$, define

$$O_\Gamma(\omega, n) = \{\omega, \Gamma\omega, \dots, \Gamma^n\omega\} \quad \text{and} \quad O_\Gamma(\omega, \infty) = \{\omega, \Gamma\omega, \dots, \Gamma^n\omega, \dots\}.$$

The set $O_\Gamma(\omega, \infty)$ is referred to as the orbit of Γ at ω, and Ω is said to be a Γ-orbitally complete metric space or Γ-orbitally CMS if every Cauchy sequence in $O_\Gamma(\omega, \infty)$ converges in Ω. It is worth noting that every complete metric space (CMS) is Γ-orbitally complete for any self-map $\Gamma\colon \Omega \to \Omega$.

Ćirić [4] introduced a broad class of contraction mappings known as quasi-contractions.

Definition 2.1. A mapping $\Gamma\colon \Omega \to \Omega$ is called a Ćirić quasi-contraction (or simply a quasi-contraction, QC) if there exists $\lambda \in [0, 1)$ such that for all $\omega, \vartheta \in \Omega$,

$$\rho(\Gamma\omega, \Gamma\vartheta) \leq \lambda Q_\Gamma^\rho(\omega, \vartheta),$$

where

$$Q_\Gamma^\rho(\omega, \vartheta) = \max\{\rho(\omega, \vartheta), \rho(\omega, \Gamma\omega), \rho(\vartheta, \Gamma\vartheta), \rho(\omega, \Gamma\vartheta), \rho(\vartheta, \Gamma\omega)\}.$$

Kumam et al. [16] proposed a more general class of quasi-contractions, called generalized quasi-contractions (GQC).

Definition 2.2. A mapping $\Gamma\colon \Omega \to \Omega$ is called a GQC if there exists $\lambda \in [0,1)$ such that for all $\omega, \vartheta \in \Omega$,

$$\rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda G_{\Gamma}^{\rho}\left(\omega, \vartheta\right),$$

where

$$G_{\Gamma}^{\rho}\left(\omega, \vartheta\right) = \max\left\{ \begin{array}{c} \rho\left(\omega, \vartheta\right), \rho\left(\omega, \Gamma\omega\right), \rho\left(\vartheta, \Gamma\vartheta\right), \rho\left(\omega, \Gamma\vartheta\right), \rho\left(\vartheta, \Gamma\omega\right), \\ \rho\left(\Gamma^2\omega, \omega\right), \rho\left(\Gamma^2\omega, \Gamma\omega\right), \rho\left(\Gamma^2\omega, \vartheta\right), \rho\left(\Gamma^2\omega, \Gamma\vartheta\right) \end{array} \right\}.$$

In 2008, Suzuki [28] established the following contraction mapping.

Definition 2.3. A self-mapping Γ on Ω is called a Suzuki's contraction if there exists $\lambda \in [0,1)$ such that

$$\zeta(\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \Rightarrow \quad \rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda\rho\left(\omega, \vartheta\right), \tag{2.1}$$

for all $\omega, \vartheta \in \Omega$, where $\zeta\colon [0,1) \to \left(\frac{1}{2}, 1\right]$ is a non-decreasing mapping defined as

$$\zeta(\lambda) = \begin{cases} 1, & \text{if } 0 \leq \lambda \leq \frac{\sqrt{5}-1}{2}, \\ (1-\lambda)\lambda^{-2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq \lambda \leq \frac{1}{\sqrt{2}}, \\ (1+\lambda)^{-1}, & \text{if } \frac{1}{\sqrt{2}} \leq \lambda < 1. \end{cases}$$

In [21], authors proposed the notion of a semi-quasi contractions (SQC).

Definition 2.4. A mapping $\Gamma\colon \Omega \to \Omega$ is called a SQC if there exists a number $\lambda \in [0,1)$ such that for all $\omega, \vartheta \in \Omega$

$$(1-\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \Rightarrow \quad \rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda Q_{\Gamma}^{\rho}\left(\omega, \vartheta\right),$$

where $Q_{\Gamma}^{\rho}\left(\omega, \vartheta\right)$ is defined in Definition 2.1.

Pant [18] later extended the concept of SQC to generalized semi-quasi contractions (GSQC).

Definition 2.5. A mapping $\Gamma\colon \Omega \to \Omega$ is called a GSQC if there exists a number $\lambda \in [0,1)$ such that for all $\omega, \vartheta \in \Omega$

$$(1-\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \Rightarrow \quad \rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda G_{\Gamma}^{\rho}\left(\omega, \vartheta\right),$$

where $G_{\Gamma}^{\rho}\left(\omega, \vartheta\right)$ is defined in the Definition 2.2.

Theorem 2.6. *Let Γ be a QC, GQC, SQC, or GSQC mapping on Ω, where Ω is a Γ-orbitally CMS. Then, the following hold:*

(i) *Γ has a unique fixed point $\upsilon \in \Omega$.*

(ii) *The sequence $\{\Gamma^n\omega\}$ converges to υ for all $\omega \in \Omega$, i.e.,*

$$\lim_{n\to\infty} \Gamma^n\omega = \upsilon.$$

(iii) *The following error estimate holds for all* $\omega \in \Omega$*:*

$$\rho\left(\Gamma^n\omega, \upsilon\right) \leq \frac{\lambda^n}{1-\lambda}\rho\left(\omega, \Gamma\omega\right).$$

Lemma 2.7 ([22])**.** *Let* (Ω, ρ) *be a metric space and* $\{\omega_n\}$ *a sequence in* Ω *which is not Cauchy, and suppose that*

$$\lim_{n\to\infty} \rho\left(\omega_n, \omega_{n+1}\right) = 0.$$

Then there exist $\varepsilon > 0$ *and two subsequences* $\{\omega_{n_k}\}$ *and* $\{\omega_{m_k}\}$ *of* $\{\omega_n\}$ *such that* $n_k > m_k > k$ *and the following sequences tends to* ε^+ *when* $k \to \infty$*,*

$$\begin{array}{lll} \rho\left(\omega_{m_k+1}, \omega_{n_k+1}\right), & \rho\left(\omega_{m_k}, \omega_{n_k}\right), & \rho\left(\omega_{m_k}, \omega_{n_k+1}\right), \\ \rho\left(\omega_{m_k-1}, \omega_{n_k}\right), & \rho\left(\omega_{m_k-1}, \omega_{n_k+1}\right). & \end{array}$$

3. Generalized Suzuki-Ćirić quasi contractions

We now extend the concept of QC mappings to a more general class of generalized Suzuki-Ćirić quasi contractions, defined as follows:

Definition 3.1. A mapping $\Gamma : \Omega \to \Omega$ is said to be a generalized Suzuki-Ćirić quasi contraction (GSCQC) if there exists $\lambda \in [0, 1)$ and $\zeta(\lambda)$ defined in the Definition 2.3 such that for all $\omega, \vartheta \in \Omega$,

$$\zeta(\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \Rightarrow \quad \rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda G_\Gamma^\rho\left(\omega, \vartheta\right) \tag{3.1}$$

where $G_\Gamma^\rho\left(\omega, \vartheta\right)$ is defined in the Definition 2.2.

Remark 3.2. All GCQC mappings and GSQC mappings are GSCQC mappings.

If we take $G_\Gamma^\rho\left(\omega, \vartheta\right) = Q_\Gamma^\rho\left(\omega, \vartheta\right)$ for all $\omega, \vartheta \in \Omega$ then we get the following definition.

Definition 3.3. A mapping $\Gamma : \Omega \to \Omega$ is said to be a Suzuki-Ćirić quasi contraction (SCQC) if there exists $\lambda \in [0, 1)$ such that for all $\omega, \vartheta \in \Omega$,

$$\zeta(\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \Rightarrow \quad \rho\left(\Gamma\omega, \Gamma\vartheta\right) \leq \lambda Q_\Gamma^\rho\left(\omega, \vartheta\right),$$

where $\zeta(\lambda)$ and $Q_\Gamma^\rho\left(\omega, \vartheta\right)$ are defined in the Definition 2.3 and 2.1, respectively.

The following examples demonstrate that the class of GSCQCs is distinct from the class of QCs, SQCs, GQCs, and GSQCs.

Example 3.4. Let $\Omega = \{(0,0), (4,0), (0,4), (4,5), (5,4)\}$ be a metric space endowed with the metric d defined by

$$\rho\left(\left(\omega_1, \omega_2\right), \left(\vartheta_1, \vartheta_2\right)\right) = \left|\omega_1 - \vartheta_1\right| + \left|\omega_2 - \vartheta_2\right|.$$

Then, (Ω, ρ) is a CMS. Let $\Gamma : \Omega \to \Omega$ be defined by

$$\begin{array}{lll} \Gamma(0,0) = (0,0), & \Gamma(4,0) = (0,0), & \Gamma(0,4) = (0,0), \\ \Gamma(4,5) = (4,0), & \Gamma(5,4) = (0,4). & \end{array}$$

For each pair $(\omega, \vartheta) \notin \{(4,5),(5,4)\}$, the inequality

$$\rho(\Gamma\omega, \Gamma\vartheta) \leq \frac{4}{5}\rho(\vartheta, \Gamma\vartheta) = \frac{4}{5}G_{\Gamma}^{\rho}(\omega, \vartheta)$$

implies that Γ satisfies the GSCQC condition.

Next, we verify that Γ does not satisfy the conditions of GQC and GSQC mappings. For $\lambda \in [0.6, 1)$ and $\omega = (4,5)$, $\vartheta = (5,4)$, we have

$$(1-\lambda)\rho(\omega, \Gamma\omega) = (1-\lambda)5 \leq 2 = \rho(\omega, \vartheta),$$

also,

$$\rho(\Gamma\omega, \Gamma\vartheta) = 8, \qquad G_{\Gamma}^{\rho}(\omega, \vartheta) = 8, \qquad \rho(\Gamma\omega, \Gamma\vartheta) > \lambda G_{\Gamma}^{\rho}(\omega, \vartheta).$$

If $\lambda \in [0, 0.6)$ then for $\omega = (0,4)$ and $\vartheta = (4,5)$, we obtain

$$(1-\lambda)\rho(\omega, \Gamma\omega) \leq \rho(\omega, \vartheta), \qquad \rho(\Gamma\omega, \Gamma\vartheta) = 4 > 5r = 5G_{\Gamma}^{\rho}(\omega, \vartheta).$$

Thus, Γ is neither GQC nor GSQC.

Example 3.5. Let $\Omega = \{0, 1, 3, 4\}$ be a metric space equipped with Euclidean metric ρ and Γ be a self-mapping on Ω defined as

$$\Gamma\omega = \begin{cases} 1, & \text{if } \omega \neq 4, \\ 0, & \text{if } \omega = 4. \end{cases}$$

Then Γ is a GSCQC mapping, but Γ does not satisfy the condition (2.1). Since

$$\rho(\Gamma\omega, \Gamma 4) = 1 \leq \frac{1}{4}\rho(4, \Gamma 4) = \frac{1}{4}G_{\Gamma}^{\rho}(\omega, 4),$$

for all $\omega \in \Omega$, Γ is a GSCQC mapping. But, since

$$\zeta(\lambda)\rho(1, \Gamma 1) = 0 \leq \rho(1,4), \qquad \text{and} \qquad \rho(\Gamma 1, \Gamma 4) = 1 = \rho(1,4)$$

for every $\lambda \in [0, 1)$. Thus Γ is not a Suzuki contraction mapping.

4. Main Results

Theorem 4.1. *Let $\Gamma : \Omega \to \Omega$ be an asymptotically regular GSCQC and Ω a Γ-orbitally CMS. Then, the following assertions hold:*

(i) *Γ has a unique fixed point $\upsilon \in \Omega$;*
(ii) *$\lim_{n\to\infty} \Gamma^n\omega = \upsilon$ for all $\omega \in \Omega$;*

Proof. (i) Let $\omega \in \Omega$ be arbitrary. Since Γ is an asymptotically regular GSCQC mapping, so,

$$\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) \to 0 \text{ as } n \to \infty. \tag{4.1}$$

We aim to show that the sequence $\{\omega_n\} = \{\Gamma^n\omega\}$ is Cauchy. It follows from Lemma 2.7 that there exist $\varepsilon > 0$ and two subsequences $\{\omega_{n_k}\}$ and $\{\omega_{m_k}\}$ of $\{\omega_n\}$ such that

$$\begin{aligned}
\lim_{k\to\infty} \rho\left(\omega_{n_k+1}, \omega_{m_k+1}\right) &= \lim_{k\to\infty} \rho\left(\omega_{n_k}, \omega_{m_k}\right) \\
&= \lim_{k\to\infty} \rho\left(\omega_{m_k}, \omega_{n_k+1}\right) \\
&= \lim_{k\to\infty} \rho\left(\omega_{m_k-1}, \omega_{n_k}\right) \\
&= \lim_{k\to\infty} \rho\left(\omega_{m_k-1}, \omega_{n_k+1}\right) \\
&= \varepsilon^+.
\end{aligned}$$

Since $\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) \to 0$, therefore $\zeta(\lambda)\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) \leq \rho\left(\omega_{n_k}, \omega_{m_k}\right)$ for sufficiently large n and from (3.1), we have

$$\rho\left(\omega_{n_k+1}, \omega_{m_k+1}\right) \leq \lambda G_\Gamma^\rho\left(\omega_{n_k}, \omega_{m_k}\right).$$

Making $n \to \infty$, we get $\varepsilon^+ \leq \lambda\varepsilon^+$, a contradiction. Hence the sequence $\{\Gamma^n\omega\}$ is Cauchy in Ω. By the Γ-orbital completeness of Ω, there exists $\upsilon \in \Omega$ such that

$$\lim_{n\to\infty} \Gamma^n\omega = \upsilon.$$

Now, we prove that υ satisfies the inequality

$$\rho\left(\Gamma\vartheta, \upsilon\right) \leq \lambda\rho\left(\vartheta, \upsilon\right), \quad \text{for all } \vartheta \in \Omega \setminus \{\upsilon\}.$$

For $\vartheta \in \Omega \setminus \{\upsilon\}$, we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\rho\left(\Gamma^n\omega, \upsilon\right) < \frac{1}{3}\rho\left(\vartheta, \upsilon\right).$$

Then,

$$\begin{aligned}
\zeta(\lambda)\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) &\leq \rho\left(\Gamma^n\omega, \upsilon\right) + \rho\left(\Gamma^{n+1}\omega, \upsilon\right) \\
&< \frac{2}{3}\rho\left(\vartheta, \upsilon\right) \\
&= \rho\left(\vartheta, \upsilon\right) - \frac{1}{3}\rho\left(\vartheta, \upsilon\right) \\
&\leq \rho\left(\vartheta, \upsilon\right) - \rho\left(\Gamma^n\omega, \upsilon\right) \\
&\leq \rho\left(\Gamma^n\omega, \vartheta\right).
\end{aligned}$$

Hence, by the condition (3.1),

$$\rho\left(\Gamma^{n+1}\omega, \Gamma\vartheta\right) \leq \lambda \max\left\{\begin{array}{l} \rho\left(\Gamma^n\omega, \vartheta\right), \rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right), \rho\left(\Gamma^{n+2}\omega, \Gamma\vartheta\right), \\ \rho\left(\Gamma^n\omega, \Gamma\vartheta\right), \rho\left(\vartheta, \Gamma^{n+1}\omega\right), \rho\left(\Gamma^{n+2}\omega, \Gamma^n\omega\right), \\ \rho\left(\Gamma^{n+2}\omega, \Gamma^{n+1}\omega\right), \rho\left(\Gamma^{n+2}\omega, \vartheta\right), \rho\left(\vartheta, \Gamma\vartheta\right) \end{array}\right\}.$$

Letting $n \to \infty$, and using continuity of ρ, we obtain

$$\rho(\upsilon, \Gamma\vartheta) \leq \lambda \max\{\rho(\upsilon, \vartheta), \rho(\vartheta, \Gamma\vartheta)\}, \quad \text{for all } \vartheta \in \Omega \setminus \{\upsilon\}. \tag{4.2}$$

Taking $\vartheta = \Gamma\upsilon$ in (4.2), we obtain

$$\rho\left(\upsilon, \Gamma^2\upsilon\right) \leq \lambda \max\left\{\rho(\upsilon, \Gamma\upsilon), \rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right)\right\}. \tag{4.3}$$

From the condition (3.1),

$$\rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right) \leq \lambda \max\left\{\rho(\upsilon, \Gamma\upsilon), \rho\left(\upsilon, \Gamma^2\upsilon\right)\right\}.$$

Suppose the maximum above is $\rho\left(\upsilon, \Gamma^2\upsilon\right)$, then $\rho(\upsilon, \Gamma\upsilon) \leq \rho\left(\upsilon, \Gamma^2\upsilon\right)$, contradicting (4.3). Hence,

$$\rho\left(\upsilon, \Gamma^2\upsilon\right) \leq \lambda\rho(\upsilon, \Gamma\upsilon), \qquad \rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right) \leq \lambda\rho(\upsilon, \Gamma\upsilon).$$

Now, taking $\vartheta = \Gamma^2\upsilon$ in (4.2), we obtain

$$\rho\left(\upsilon, \Gamma^3\upsilon\right) \leq \lambda \max\left\{\rho\left(\upsilon, \Gamma^2\upsilon\right), \rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right)\right\}. \tag{4.4}$$

Also,

$$\begin{aligned}
\rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right) &\leq \lambda \max\left\{\rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right), \rho\left(\Gamma\upsilon, \Gamma^3\upsilon\right)\right\} \\
&\leq \lambda \max\left\{\lambda\rho(\upsilon, \Gamma\upsilon), \lambda\rho\left(\upsilon, \Gamma^2\upsilon\right), \lambda\rho\left(\upsilon, \Gamma^3\upsilon\right)\right\} \\
&= \lambda^2 \max\left\{\rho(\upsilon, \Gamma\upsilon), \rho\left(\upsilon, \Gamma^3\upsilon\right)\right\}.
\end{aligned}$$

Assuming $\max\left\{\rho(\upsilon, \Gamma\upsilon), \rho\left(\upsilon, \Gamma^3\upsilon\right)\right\} = \rho\left(\upsilon, \Gamma^3\upsilon\right)$, contradict to (4.4), hence

$$\rho\left(\upsilon, \Gamma^3\upsilon\right) \leq \lambda^2\rho(\upsilon, \Gamma\upsilon), \qquad \rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right) \leq \lambda^2\rho(\upsilon, \Gamma\upsilon).$$

We now analyze two cases:

Case 1: When $0 \leq \lambda < 2^{-1/2}$. In this case, $\zeta(\lambda)\lambda^2 + \lambda - 1 \leq 0$ and $2r^2 < 1$. Here, we consider two subcases:

Subcase (I): Let $\rho\left(\Gamma^2\upsilon, \upsilon\right) < \zeta(\lambda)\rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right)$. Then

$$\begin{aligned}
\rho(\upsilon, \Gamma\upsilon) &\leq \rho\left(\upsilon, \Gamma^2\upsilon\right) + \rho\left(\Gamma^2\upsilon, \Gamma\upsilon\right) \\
&< \zeta(\lambda)\rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right) + \rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right) \\
&\leq \zeta(\lambda)\lambda^2\rho(\upsilon, \Gamma\upsilon) + \lambda\rho(\upsilon, \Gamma\upsilon) \\
&< \rho(\upsilon, \Gamma\upsilon),
\end{aligned}$$

yields a contradiction.

Subcase (II): Let $\rho\left(\Gamma^2\upsilon, \upsilon\right) \geq \zeta(\lambda)\rho\left(\Gamma^2\upsilon, \Gamma^3\upsilon\right)$. Then, by triangle inequality, we get

$$\begin{aligned}
\rho(\upsilon, \Gamma\upsilon) &\leq \rho\left(\upsilon, \Gamma^3\upsilon\right) + \rho\left(\Gamma^3\upsilon, \Gamma\upsilon\right) \\
&\leq \lambda^2\rho(\upsilon, \Gamma\upsilon) + \lambda\rho\left(\Gamma^2\upsilon, \upsilon\right) \\
&\leq \lambda^2\rho(\upsilon, \Gamma\upsilon) + \lambda^2\rho(\Gamma\upsilon, \upsilon) \\
&= 2\lambda^2\rho(\upsilon, \Gamma\upsilon)
\end{aligned}$$

$$< \rho\left(\upsilon, \Gamma\upsilon\right),$$

a contradiction.

Case 2: When $2^{-1/2} \leq \lambda < 1$. Then, any $\omega, \vartheta \in \Omega$, either

$$\zeta(\lambda)\rho\left(\omega, \Gamma\omega\right) \leq \rho\left(\omega, \vartheta\right) \quad \text{or} \quad \zeta(\lambda)\rho\left(\Gamma\omega, \Gamma^2\omega\right) \leq \rho\left(\Gamma\omega, \vartheta\right).$$

If both inequalities fail, we get

$$\begin{aligned}
\rho\left(\omega, \Gamma\omega\right) &\leq \rho\left(\omega, \vartheta\right) + \rho\left(\Gamma\omega, \vartheta\right) \\
&< \zeta(\lambda)\left(\rho\left(\omega, \Gamma\omega\right) + \rho\left(\Gamma\omega, \Gamma^2\omega\right)\right) \\
&\leq \zeta(\lambda)(1+\lambda)\rho\left(\omega, \Gamma\omega\right) \\
&< \rho\left(\omega, \Gamma\omega\right),
\end{aligned}$$

a contradiction. Hence, for the sequence $\{\omega_n\}$, either

$$\zeta(\lambda)\rho\left(\omega_{2n}, \omega_{2n+1}\right) \leq \rho\left(\omega_{2n}, \upsilon\right) \quad \text{or} \quad \zeta(\lambda)\rho\left(\omega_{2n+1}, \omega_{2n+2}\right) \leq \rho\left(\omega_{2n+1}, \upsilon\right),$$

and therefore,

$$\rho\left(\omega_{2n+1}, \Gamma\upsilon\right) \leq \lambda\rho\left(\omega_{2n}, \upsilon\right) \quad \text{or} \quad \rho\left(\omega_{2n+2}, \Gamma\upsilon\right) \leq \lambda\rho\left(\omega_{2n+1}, \upsilon\right).$$

Since $\Gamma^n\omega \to \upsilon$, there exists a subsequence converging to $\Gamma\upsilon$, hence $\Gamma\upsilon = \upsilon$. Thus, in all cases, $\Gamma\upsilon = \upsilon$, i.e., υ is a fixed point of Γ.

(ii) To prove uniqueness, suppose υ, w are fixed points of Γ. Then,

$$\begin{aligned}
\rho\left(\upsilon, w\right) &= \rho\left(\Gamma\upsilon, \Gamma w\right) \\
&\leq \lambda \max\left\{\rho(\Gamma\upsilon, w), \rho\left(\Gamma\upsilon, \Gamma^2\upsilon\right), \rho(w, \Gamma w), \rho(\Gamma\upsilon, \Gamma w), \rho\left(w, \Gamma^2\upsilon\right)\right\} \\
&= \lambda\rho\left(\upsilon, w\right).
\end{aligned}$$

Since $0 \leq \lambda < 1$, we get $\rho\left(\upsilon, w\right) = 0$, hence $\upsilon = w$. So, the fixed point is unique. □

Remark 4.2. It is worth noting that the classes of mappings QC, GQC, SQC, and GSQC are all asymptotically regular.

Here we revisit Examples 3.4 and 3.5 to verify whether Theorem 4.1 is applicable to these examples.

Remark 4.3. In Example 3.4, it is not difficult to verify that (Ω, ρ) is a CMS, and it has been demonstrated that Γ is a GSCQC mapping therein. Also, it is easy to see Γ is an asymptotically regular on Ω. Therefore, by Theorem 4.1, the mapping T has a unique fixed point at $\omega = (0, 0)$.

Similarly, in Example 3.5, (Ω, ρ) is a Euclidean metric space, which is well known to be complete. Furthermore, the mapping Γ in Example 3.5 is a GSCQC and an asymptotically regular. Hence, by Theorem 4.1, the mapping T has a unique fixed point at $\omega = 1$.

Corollary 4.4. *Let Γ be an asymptotically regular and SCQC mapping on Ω and Ω, a Γ-orbitally CMS. Then, the following assertions hold:*

(i) *Γ has a unique fixed point $\upsilon \in \omega$.*
(ii) *$\lim_{n\to\infty} \Gamma^n\omega = \upsilon$ for all $\omega \in \Omega$.*

Now, we present the following fixed point results without assuming the underlying mapping to be asymptotically regular.

Theorem 4.5. *Let (Ω, ρ) be a CMS and $\Gamma : \Omega \to \Omega$ be a mapping satisfying*

$$\zeta(\lambda)\rho(\omega, \Gamma\omega) \leq \rho(\omega, \vartheta) \quad \Rightarrow \quad \rho(\Gamma\omega, \Gamma\vartheta) \leq \lambda H_\Gamma^\rho(\omega, \vartheta), \tag{4.5}$$

for all $\omega, \vartheta \in \Omega$, where $\zeta(\lambda), r$ are same as in the Theorem 4.1 and

$$H_\Gamma^\rho(\omega, \vartheta) = \max\left\{ \begin{array}{c} \rho(\omega, \vartheta), \rho(\omega, \Gamma\omega), \rho(\vartheta, \Gamma\vartheta), \rho(\Gamma\omega, \vartheta), \\ \rho\left(\Gamma^2\omega, \Gamma\omega\right), \rho\left(\Gamma^2\omega, \vartheta\right), \rho\left(\Gamma^2\omega, \Gamma\vartheta\right) \end{array} \right\}.$$

Then Γ has a unique fixed point.

Proof. It is evident that for each $\omega \in \Omega$, $\zeta(\lambda)\rho(\omega, \Gamma\omega) \leq \rho(\omega, \Gamma\omega)$. From (4.5), we have

$$\rho\left(\Gamma\omega, \Gamma^2\omega\right) \leq \lambda\rho(\omega, \Gamma\omega). \tag{4.6}$$

It follows that $\left\{\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right)\right\}$ is monotonic decreasing sequence of non-negative real numbers which converges to some $c \geq 0$. If $c > 0$ then from the condition (4.5), we get a contradiction. Hence $\rho\left(\Gamma^n\omega, \Gamma^{n+1}\omega\right) \to 0$ implies Γ is an asymptotically regular mapping. Now, following the proof of Theorem 4.1, we reach to the conclusion. □

Corollary 4.6. *Let $\Gamma : \Omega \to \Omega$. If Ω is a Γ-orbitally CMS and there exists $k \in \mathbb{N}$ such that the following condition holds*

$$\zeta(\lambda)\rho\left(\omega, \Gamma^k\omega\right) \leq \rho(\omega, \vartheta) \quad \Rightarrow \quad \rho\left(\Gamma^k\omega, \Gamma^k\vartheta\right) \leq \lambda H_{\Gamma^k}^\rho(\omega, \vartheta).$$

for some $\lambda \in [0, 1)$ and all $\omega, \vartheta \in \Omega$. Then, the following statements hold:

(i) *Γ has a unique fixed point $\upsilon \in \Omega$.*
(ii) *$\lim_{n\to\infty} \Gamma^n\omega = \upsilon$ for all $\omega \in \Omega$.*

Proof. The proof followed easily from Corollary 3.2 in [16]. □

5. Applications to Delay Differential Equations

A general delay differential equation (DDE) is of the form:

$$\omega'(t) = f(t, \omega(t), \omega(t - \tau_1), \omega(t - \tau_2), \ldots, \omega(t - \tau_n)),$$

where the parameters $\tau_1, \tau_2, \ldots, \tau_n$ represent time delays. A fundamental example of a DDE is the delay logistic equation:

$$\omega'(t) = \lambda\omega(t)\left(1 - \omega\left(t - \frac{\tau}{K}\right)\right),$$

where λ denotes the growth rate, K is the carrying capacity, and τ is the delay factor.

In this section, we focus on the following DDE:

$$\omega'(t) = f\left(t, \omega(t), \omega(t-\tau)\right), \quad t \in [t_0, b], \tag{5.1}$$

with the initial condition:

$$\omega(t) = \phi(t), \quad t \in [t_0 - \tau, t_0]. \tag{5.2}$$

Let $\Omega = L^\infty[t_0 - \tau, b]$ equipped with the essentially supremum norm, i.e

$$\begin{aligned}|\omega(t)|_\infty &= \operatorname{esssup}\{|\omega(t)| : t \in [t_0 - \tau, t_0]\} \\ &= \inf\{M \,:\, |\omega(t)| \le M, \quad \text{for almost all } t \in [t_0 - \tau, t_0]\} \text{ for } \omega \in \Omega.\end{aligned}$$

This makes (Ω, ρ) a CMS. Assume that the DDE ((5.1)-(5.2)) satisfies the following conditions:

(C1) $t_0, b \in \mathbb{R}$ with $\tau > 0$,
(C2) the function f is continuous on $[t_0, b] \times \Omega^2$,
(C3) the initial function ϕ is bounded and integrable on $[t_0 - \tau, b]$.

Now, finding the solution of the problem (5.1) is equivalent to find a fixed point of the operator $\Gamma : L^\infty[t_0 - \tau, b] \to L^\infty[t_0 - \tau, b]$ is given by

$$\Gamma(\omega(t)) = \begin{cases} \phi(t), & \text{if } t \in [t_0 - \tau, t_0], \\ \phi(t_0) + \int_{t_0}^{t} f\left(s, \omega(s), \omega(s-\tau)\right) ds, & \text{if } t \in [t_0, b]. \end{cases} \tag{5.3}$$

Additionally, suppose that:

(C4) there exists a constant $\lambda > 0$ such that for all $t \in \mathbb{R}$ and $\omega(t), \vartheta(t) \in L^\infty[t_0, b]$,

$$\zeta(\lambda)\,|\omega(t) - \Gamma\omega(t)| \le |\omega(t) - \vartheta(t)|$$

implies

$$|f\left(t, \omega(t), \omega(t-\tau)\right) - f\left(t, \vartheta(t), \vartheta(t-\tau)\right)| \le \lambda H_\Gamma^\rho\left(\omega(t), \vartheta(t)\right)$$

where the operator Γ is defined as in (5.3), and ζ, $H_\Gamma^\rho(\omega, \vartheta)$ are same as in Theorem 4.5,

(C5) $\lambda(b - t_0) < 1$.

Theorem 5.1. *If conditions* $(C1)-(C5)$ *hold, then the DDE* (5.1)–(5.2) *has the following properties:*

(a) *there exists a unique solution* $\bar{\omega} \in L^\infty[t_0 - \tau, b]$,

(b) *the sequence* $\{\omega_n\}$*, defined iteratively as*

$$\omega_{n+1}(s) = \Gamma(\omega_n(s)), \quad \text{for } n = 0, 1, 2, \ldots$$

converges to $\bar{\omega}$*, where* Γ *is defined in* (5.3)*.*

Proof. We note that Γ is a self mapping on Ω. For any $\omega, \vartheta \in \Omega$ with $\zeta(\lambda)|\omega(t) - \Gamma\omega(t)| \leq |\omega(t) - \vartheta(t)|$, we have

$$\begin{aligned}
\rho(\Gamma\omega, \Gamma\vartheta) &= \int_{t_0}^{t} |f(s, \omega(s), \omega(s-\tau)) - f(s, \vartheta(s), \vartheta(s-\tau))|\, ds \\
&\leq \lambda \int_{t_0}^{t} H_{\Gamma}^{\rho}(\omega(t), \vartheta(t))\, ds \\
&\leq \lambda H_{\Gamma}^{\rho}(\omega(t), \vartheta(t)) \int_{t_0}^{t} ds \\
&\leq \lambda(b - t_0) H_{\Gamma}^{\rho}(\omega(t), \vartheta(t)).
\end{aligned}$$

Thus Γ is a GSCQC mapping and the Theorem 4.5 guarantees that Γ has a unique fixed point in Ω, proving (a). The iterative process $\omega_{n+1} = \Gamma(\omega_n)$ converges to this fixed point, establishing (b). □

Example 5.2. Consider the following DDE

$$\omega'(t) = k\omega(t), \quad t \geq 0,\ k \in [0, 1), \tag{5.4}$$

with the initial condition:

$$\omega(t) = \phi(t), \quad t \in [0 - \tau, 0],$$

where $\tau > 0$ is a fixed delay and the initial function $\phi(t)$ is given on $[-\tau, 0]$ as

$$\phi(t) = \begin{cases} 0, & t \in [-\tau, -\tau/2), \\ 1, & t \in [-\tau/2, 0]. \end{cases}$$

Then all the hypothesis of Theorem 5.1 hold for the equation (5.4) and the DDE (5.4) has a unique solution in $L^{\infty}[-\tau, \tau]$. Note that, we have $f(s, \omega(t), \omega(t-\tau)) = k\omega(t)$, $t_0 = 0$ and $b = \tau > 0$. To find the solution of (5.4), we evaluate $\omega(t)$ for $t \geq 0$. From (5.4), we have

$$\omega(t) = Ae^t,$$

where $t \geq 0$ and A is constant. Applying initial condition at $t = 0$, $\omega(0) = \phi(0) = 1$, we get $A = 1$. Thus, the solution of equation (5.4) in $L^{\infty}[-\tau, \tau]$ is

$$\omega(t) = \begin{cases} e^t, & 0 \leq t \leq \tau, \\ \phi(t), & -\tau \leq t \leq 0. \end{cases}$$

Conclusion

In this paper, we have introduced a new class of GSCQC mappings that unifies and extends several well-known classes of contractions, including GQCs, Suzuki contractions, GSQCs, and others. We have established existence results for these mappings. Several illustrative examples are provided to highlight the advantages of our results over existing findings in the literature. These examples demonstrate that the GSCQC condition is more general than some recent extensions of Ćirić quasi-contractions. In addition, we have presented an application to delay differential equations. The results obtained here open new directions for further research. One can extend these results to nonlinear contractive mappings or in more general setting of metric spaces such as semimetric space, b-metric space (1993), partial metric space (1994) etc.

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