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Composition Operators on Some Semi-Hilbert Spaces

Yousef Estaremi[1] and Mohammed S. Al Ghafri[2*]

Abstract. In this paper, we study composition and weighted composition operators on semi-Hilbert spaces induced by positive multiplication operators on $L^2(\mu)$. Within the framework of A-adjoint operators, we characterize the conditions under which a composition operator C_φ is A-selfadjoint, A-normal, A-quasinormal, or A-unitary, where $A = M_u$ denotes a positive multiplication operator. In addition, we provide necessary and sufficient conditions for C_φ to be an A-isometry or an A-partial isometry. Analogous results are also obtained when A is a positive composition operator. Several examples are included to illustrate the applicability of the main results.

1. Introduction and Preliminaries

The study of composition operators C_φ and multiplication operators M_u in function spaces, has led to significant advances in characterizing their adjoints, isometric properties, and spectral behavior [10, 14, 18, 19]. In particular, the self-adjointness of weighted composition operators $W = M_u C_\varphi$ in $L^2(\mu)$ and their unitary equivalences have been explored in recent literature [8]. A fundamental aspect of operator theory is the study of adjoint operators. The concept of A-adjoint operators, which generalizes classical adjoints, is crucial in defining an alternative inner product structure. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})^+$ is the cone of positive semidefinite operators, i.e.

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) : \langle A\xi, \xi \rangle \geq 0 \text{ for all } \xi \in \mathcal{H}\}.$$

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For any operator T in $\mathcal{B}(\mathcal{H})$, its range and null space are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, while its adjoint is represented as T^*. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, the orthogonal projection onto $\mathcal{M}$ is written as $P_{\mathcal{M}}$. A subspace $\mathcal{M}$ is said to be invariant under T if $T\mathcal{M} \subset \mathcal{M}$. Moreover, $\mathcal{M}$ is called a reducing subspace for T when both $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp$ remain invariant under T. The set of complex numbers is denoted by $\mathbb{C}$, and the complex conjugate of a number λ is expressed as $\overline{\lambda}$. Additionally, the closure of $\mathcal{R}(T)$ is represented as $\overline{\mathcal{R}(T)}$.

Let $A \in \mathcal{B}(\mathcal{H})^+$ be a positive operator. The semi-inner product induced by A is given by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. Then $\langle \cdot, \cdot \rangle_A$ is a semi-inner product on $\mathcal{H}$. It follows that

$$\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2} = \langle A\xi, \xi \rangle^{1/2} = \langle A^{1/2}\xi, A^{1/2}\xi \rangle^{1/2} = \|A^{1/2}\xi\|.$$

It follows that $\|\xi\|_A = 0$ if and only if $\xi \in \mathcal{N}(A)$. Hence, $\| \cdot \|_A$ is a norm if and only if A is injective. Moreover, this induces a seminorm on a certain subset of $\mathcal{B}(\mathcal{H})$, which contains all $T \in \mathcal{B}(\mathcal{H})$ for which there exists a constant $c > 0$ such that $\|T\xi\|_A \leq c\|\xi\|_A$ for all $\xi \in \mathcal{H}$. For such operators, we have

$$\|T\|_A = \sup_{\xi \in \mathcal{R}(A), \xi \neq 0} \frac{\|T\xi\|_A}{\|\xi\|_A} < \infty.$$

For $\mathcal{L} \subseteq \mathcal{H}$, we denote by $\mathcal{L}^{\perp_A} = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle_A = 0 \text{ for all } \eta \in \mathcal{L}\}$. It is easily seen that $\mathcal{L}^{\perp_A} = (A\mathcal{L})^\perp = A^{-1}(\mathcal{L}^\perp)$. Moreover, since $A(A^{-1}(\mathcal{L})) = \mathcal{L} \cap \mathcal{R}(A)$, then we have $(\mathcal{L}^{\perp_A})^{\perp_A} = \mathcal{L}^{\perp_A} \cap \mathcal{R}(A)^\perp$.

Within this framework, the concept of an A-adjoint becomes central. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to admit an A-adjoint if there exists $W \in \mathcal{B}(\mathcal{H})$ such that $\langle W\xi, \eta \rangle_A = \langle \xi, T\eta \rangle_A$ for all $\xi, \eta \in \mathcal{H}$. Hence T has an A-adjoint if and only if there exists $W \in \mathcal{B}(\mathcal{H})$ such that $AW = T^*A$. Therefore, T has an A-adjoint if and only if the equation $AX = T^*A$ has a solution. The following theorem due to Douglas [9] will be used in the sequel.

Theorem 1.1 (Douglas' Theorem)**.** *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.
(ii) *There exists $\lambda > 0$ such that $B^*B \leq \lambda AA^*$.*
(iii) *There exists $C \in \mathcal{B}(\mathcal{H})$ such that $AC = B$.*

If any of these conditions holds, then there exists a unique operator $W \in \mathcal{B}(\mathcal{H})$ such that $AW = B$, $\mathcal{R}(W) \subseteq \overline{\mathcal{R}(A^*)}$, and $\mathcal{N}(W) = \mathcal{N}(B)$. The operator W is called the reduced solution of $AX = B$.

From the above observations, it follows that $T \in \mathcal{B}(\mathcal{H})$ admits an A-adjoint if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. In this paper, $\mathcal{B}_A(\mathcal{H})$ denotes

the set of all operators $T \in \mathcal{B}(\mathcal{H})$ that admit an A-adjoint, that is, $\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}$. The set $\mathcal{B}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ that is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover,

$$\overline{\mathcal{B}_A(\mathcal{H})} \subseteq \left\{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \overline{\mathcal{R}(A)}\right\}.$$

By Theorem 1.1, we obtain

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \exists c > 0 \text{ such that } \|T\xi\|_A \leq c\|\xi\|_A \forall \xi \in \mathcal{H}\}.$$

To fix notation, let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, and let $\mathcal{H} = L^2(\mu)$ be the Hilbert space of square-integrable complex-valued functions on X. Let $u : X \to \mathbb{C}$ be a measurable function. The multiplication operator M_u acts on $L^2(\mu)$ by $(M_u f)(x) = u(x)f(x)$, $f \in L^2(\mu)$. If $u \geq 0$ almost everywhere, then M_u is a positive operator. We define the modified inner product on $\mathcal{H}$ as

$$\langle f, g\rangle_A = \langle Af, g\rangle = \int_X u(x)f(x)\overline{g(x)}\, d\mu(x),$$

which induces the norm

$$\|f\|_A := \|A^{1/2}f\| = \left(\int_X u(x)|f(x)|^2\, d\mu(x)\right)^{1/2}.$$

Now, consider a measurable transformation $\varphi : X \to X$ that is non-singular, meaning that the pushforward measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. Define the Radon-Nikodym derivative:

$$h_\varphi = \frac{d(\mu \circ \varphi^{-1})}{d\mu}, \tag{1.1}$$

and let $E(f) = E(f \mid \varphi^{-1}(\mathcal{F}))$ denote the conditional expectation of f with respect to the σ-algebra $\varphi^{-1}(\mathcal{F})$.

The composition operator associated with φ is defined as $(C_\varphi f)(x) = f(\varphi(x))$ for all $f \in L^2(\mu)$. The adjoint of C_φ satisfies the following properties:

(i) Adjoint formula: $C_\varphi^* f = h_\varphi E(f) \circ \varphi^{-1}$.
(ii) Product of the operator and its adjoint: $C_\varphi^* C_\varphi f = h_\varphi \cdot f$.
(iii) Composition with the adjoint operator: $C_\varphi C_\varphi^* f = (h_\varphi \circ \varphi)Ef$.
(iv) Absolute value of the operator: $|C_\varphi| f = \sqrt{h_\varphi} \cdot f$.

Now, let us define the weighted composition operator W by $(Wf)(x) = u(x)f(\varphi(x))$ for all $f \in L^2(\mu)$. This operator combines both multiplication and composition, and it satisfies the following properties:

(i) Adjoint formula: $W^* f = h_\varphi E(u \cdot f) \circ \varphi^{-1}$.
(ii) Composition of the operator and its adjoint: $W^*Wf = h_\varphi E(u^2) \circ \varphi^{-1} \cdot f$, and $WW^*f = u \cdot (h_\varphi \circ \varphi)E(uf)$.
(iii) Absolute value of the operator: $|W|f = \sqrt{h_\varphi E(u^2) \circ \varphi^{-1}} \cdot f$.

Although a significant body of work has addressed boundedness, compactness, and spectral properties of such operators (see [1, 2, 8, 12, 17]), relatively few studies have examined their behavior within the semi-Hilbert space framework.

For any operator $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution to the equation $AT^\sharp = T^*A$ is uniquely given by $T^\sharp = A^\dagger T^*A$, where $A^\dagger$ denotes the Moore–Penrose inverse of A [5, 9].

Our focus lies on generalized notions such as A-selfadjointness ($T = T^\sharp$), A-normality ($TT^\sharp = T^\sharp T$), A-quasinormality ($TT^\sharp T = T^\sharp T^2$), and A-unitarity. These concepts extend classical operator theory and provide a deeper understanding of weighted dynamics under modified inner product structures. For more comprehensive studies on these properties, we refer the reader to [3, 4, 6, 13, 15, 16]. Recently, in [11] the authors have investigated some measure-theoretic characterizations for composition operators in some operator classes on $L^2(\Sigma)$-semi-Hilbertian spaces with respect to positive multiplication operators.

In this study, we establish a comprehensive framework for analyzing composition operators and their weighted variants in semi-Hilbertian settings induced by both multiplication and composition operators. By leveraging Douglas' range inclusion theorem, conditional expectations, and the Moore–Penrose inverse, we derive necessary and sufficient conditions for various operator-theoretic properties. The results not only generalize classical notions such as self-adjointness, normality, and isometry, but also offer new insights into the structural interplay between operators in weighted settings. Concrete examples are provided to demonstrate the effectiveness and sharpness of the developed criteria.

2. Main Results

In this section, we establish characterizations for when a composition operator C_φ on $L^2(\mu)$ becomes A-selfadjoint, A-normal, A-quasinormal, A-isometric, or A-unitary, where $A = M_u$ is a positive multiplication operator. We also present conditions under which C_φ acts as an A-partial isometry. These results are obtained using the notion of A-adjoint operators and the Douglas factorization theorem. Analogous results are also given when A is a positive composition operator. Each theorem is followed by proof and interpretation within the semi-Hilbert space context.

Theorem 2.1. *Let M_u be a positive multiplication operator, C_φ a composition operator, and define the weighted composition operator $W = M_u C_\varphi$ on $L^2(\mu)$. Then C_φ is M_u-selfadjoint on $L^2(\mu)$ if and only if $M_u C_\varphi = C_\varphi^* M_u$, if and only if W is selfadjoint (i.e., $W^* = W$).*

Proof. Recall that C_φ is M_u-selfadjoint if and only if $\langle C_\varphi f, g\rangle_{M_u} = \langle f, C_\varphi g\rangle_{M_u}$ for all $f, g \in L^2(\mu)$, which is equivalent to $\langle M_u C_\varphi f, g\rangle = \langle M_u f, C_\varphi g\rangle$ for all $f, g \in L^2(\mu)$. This, in turn, is equivalent to $\langle Wf, g\rangle = \langle M_u C_\varphi f, g\rangle = \langle C_\varphi^* M_u f, g\rangle = \langle W^* f, g\rangle$. Hence, C_φ is M_u-selfadjoint on $L^2(\mu)$ if and only if W is selfadjoint. □

We now recall several equivalent conditions for the selfadjointness of a weighted composition operator W on the Hilbert space $L^2(\mu)$.

Theorem 2.2 ([7])**.** *The weighted composition operator $W = M_u C_\varphi$ is selfadjoint on $L^2(\mu)$ if and only if the restriction $\varphi_{S_J} := \varphi|_{S_J}$ is periodic of period 2, and $J := h_\varphi \cdot E(u) \circ \varphi^{-1} = u$, where $S_J = \{x \in X : J(x) \neq 0\}$.*

Combining the previous two theorems, we get the following corollary.

Corollary 2.3. *If M_u is a positive multiplication operator, C_φ is a composition operator and $W = M_u C_\varphi$ the related weighted composition operator on $L^2(\mu)$, then C_φ is M_u-selfadjoint on $L^2(\mu)$ if and only if φ_{S_J} is periodic of period 2 and $J := h_\varphi E(u) \circ \varphi^{-1} = u$.*

Let $T \in \mathcal{B}_A(\mathcal{H})$. Then there exists a unique operator associated with T, referred to as the A-adjoint, which solves the equation $AX = T^*A$ in the reduced sense. This operator, denoted by $T^\sharp$, is given by the formula $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is the Moore–Penrose inverse of A.

We recall that the operator $T \in \mathcal{B}_A(\mathcal{H})$ is called A-normal if $TT^\sharp = T^\sharp T$ and T is A-quasi-normal if $TT^\sharp T = T^\sharp T^2$. Also, a bounded operator T on the Hilbert space $\mathcal{H}$ is called hyponormal if $T^*T \geq TT^*$. If $T \in \mathcal{B}_A(\mathcal{H})$, then T is called A-hyponormal if $T^*AT \geq TAT^*$. We now investigate normality and quasi-normality of the composition operator C_φ with respect to the multiplication operator M_u on the Hilbert space $L^2(\mu)$.

Theorem 2.4. *Let M_u be a positive multiplication operator and C_φ a bounded composition operator on the Hilbert space $L^2(\mu)$. Then the following statements hold:*

(a) *The operator C_φ is M_u-normal if and only if*

$$\left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \cdot u = \frac{1}{u}\chi_{S(u)} \cdot J,$$

where $\chi_{S(u)}$ denotes the characteristic function of the support $S(u) = \{x \in X : u(x) \neq 0\}$, and $J = h_\varphi \cdot E(u) \circ \varphi^{-1}$.

(b) *The operator C_φ is M_u-quasi-normal if and only if*

$$\frac{1}{u}\chi_{S(u)} \cdot J \cdot h = \left(\frac{1}{u}\chi_{S(u)} \cdot J\right) \circ \varphi^{-1} \cdot h,$$

where $J = h_\varphi \cdot E(u) \circ \varphi^{-1}$.

Proof. (a) As we know, $M_u^\dagger = M_{\frac{1}{u}\chi_{S(u)}}$ and $C_\varphi^* = M_h C_{\varphi^{-1}} E$. Therefore,

$$C_\varphi^\sharp = M_u^\dagger C_\varphi^* M_u = M_{\frac{1}{u}\chi_{S(u)}} M_h C_{\varphi^{-1}} E M_u.$$

By definition, C_φ is M_u-normal if and only if

$$C_\varphi^\sharp C_\varphi = C_\varphi C_\varphi^\sharp.$$

For every $f \in L^2(\mu)$, we compute:

$$\begin{aligned} C_\varphi^\sharp C_\varphi f &= M_{\frac{1}{u}\chi_{S(u)}} M_h C_{\varphi^{-1}} E M_u C_\varphi f \\ &= \frac{1}{u}\chi_{S(u)} J \cdot f, \end{aligned}$$

$$\begin{aligned} C_\varphi C_\varphi^\sharp f &= C_\varphi M_{\frac{1}{u}\chi_{S(u)}} M_h C_{\varphi^{-1}} E M_u f \\ &= \left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \cdot E(uf). \end{aligned}$$

Hence, $C_\varphi^\sharp C_\varphi f = C_\varphi C_\varphi^\sharp f$ if and only if

$$\frac{1}{u}\chi_{S(u)} J \cdot f = \left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \cdot E(uf).$$

Equivalently, we obtain

$$\begin{aligned} \int_X \frac{1}{u}\chi_{S(u)} J \cdot f \, d\mu &= \int_X \left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \, E(uf) \, d\mu \\ &= \int_X \left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \, uf \, d\mu, \end{aligned}$$

for all $f \in L^2(\mu)$. Therefore, C_φ is M_u-normal if and only if

$$\left(\frac{1}{u}\chi_{S(u)} h\right) \circ \varphi \cdot u = \frac{1}{u}\chi_{S(u)} \cdot J.$$

(b) By definition, C_φ is M_u-quasi-normal if and only if

$$C_\varphi^\sharp C_{\varphi^2} = C_\varphi C_\varphi^\sharp C_\varphi.$$

For every $f \in L^2(\mu)$, we compute:

$$\begin{aligned} C_\varphi^\sharp C_{\varphi^2} f &= M_{\frac{1}{u}\chi_{S(u)}} M_h C_{\varphi^{-1}} E M_u C_{\varphi^2} f \\ &= \frac{1}{u}\chi_{S(u)} J \cdot f \circ \varphi, \end{aligned}$$

$$\begin{aligned} C_\varphi C_\varphi^\sharp C_\varphi f &= C_\varphi M_{\frac{1}{u}\chi_{S(u)}} M_h C_{\varphi^{-1}} E M_u C_\varphi f \\ &= \left(\frac{1}{u}\chi_{S(u)} \cdot J\right) \circ \varphi \cdot f \circ \varphi. \end{aligned}$$

Thus, $C_\varphi^\sharp C_{\varphi^2} f = C_\varphi C_\varphi^\sharp C_\varphi f$ if and only if

$$\frac{1}{u}\chi_{S(u)} J \cdot f \circ \varphi = \left(\frac{1}{u}\chi_{S(u)} \cdot J\right) \circ \varphi \cdot f \circ \varphi.$$

Equivalently, we obtain

$$\begin{aligned}\int_X \left(\frac{1}{u}\chi_{S(u)} J\right) \circ \varphi^{-1} \cdot h \cdot f \, d\mu &= \int_X \frac{1}{u}\chi_{S(u)} J \cdot f \circ \varphi \, d\mu \\ &= \int_X \left(\frac{1}{u}\chi_{S(u)} \cdot J\right) \circ \varphi \cdot f \circ \varphi \, d\mu \\ &= \int_X \frac{1}{u}\chi_{S(u)} \cdot J \cdot h \cdot f \, d\mu,\end{aligned}$$

for all $f \in L^2(\mu)$. Therefore, C_φ is M_u-quasi-normal if and only if $\frac{1}{u}\chi_{S(u)} J \cdot h = \left(\frac{1}{u}\chi_{S(u)} \cdot J\right) \circ \varphi^{-1} \cdot h$. □

We also recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called an A-isometry if $\|Tx\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. Hence, T is an A-isometry if and only if $\langle T^*ATx, x\rangle = \langle ATx, Tx\rangle = \|Tx\|_A^2 = \|x\|_A^2 = \langle Ax, x\rangle$ for all $x \in \mathcal{H}$. Therefore, $T \in \mathcal{B}(\mathcal{H})$ is an A-isometry if and only if $T^*AT = A$.

Moreover, T is called A-unitary if both T and T^* are A-isometries. Thus, T is A-unitary if and only if $T^*AT = TAT^* = A$.

In the following theorem, we investigate the conditions under which the composition operator C_φ is an isometry with respect to the multiplication operator M_u on the Hilbert space $L^2(\mu)$.

Theorem 2.5. *Let M_u be a positive multiplication operator and C_φ a bounded composition operator on the Hilbert space $L^2(\mu)$. Then C_φ is an M_u-isometry if and only if $u = J = h_\varphi \cdot E(u) \circ \varphi^{-1}$.*

Proof. The operator C_φ is an M_u-isometry if and only if $\|C_\varphi f\|_{M_u} = \|f\|_{M_u}$ for all $f \in L^2(\mu)$, which is equivalent to $\|C_\varphi f\|_{M_u}^2 = \|f\|_{M_u}^2$. Using the M_u-inner product, this condition becomes $\langle C_\varphi^* M_u C_\varphi f, f\rangle = \langle M_u f, f\rangle$. It is known that $C_\varphi^* M_u C_\varphi f = h_\varphi \cdot E(u) \circ \varphi^{-1} \cdot f = Jf$. Therefore, C_φ is M_u-isometric if and only if $Jf = uf$ for all $f \in L^2(\mu)$, that is, $u = J = h_\varphi \cdot E(u) \circ \varphi^{-1}$. □

The operator $T \in \mathcal{B}(\mathcal{H})$ is called an A-partial isometry if $\|Tx\|_A = \|x\|_A$ for all $x \in \mathcal{N}_A(T)^{\perp_A}$. In the following, we aim to determine conditions under which the composition operator C_φ is an M_u-partial isometry on the Hilbert space $L^2(\mu)$. As a first step, we compute the space $\mathcal{N}_{M_u}(C_\varphi)^{\perp_{M_u}}$. In the next lemma, we characterize this orthogonal complement as a subspace of $L^2(\mu)$.

Lemma 2.6. *Let M_u be a positive multiplication operator and C_φ a bounded composition operator on the Hilbert space $L^2(\mu)$. Then*

$$\mathcal{N}_{M_u}(C_\varphi)^{\perp_{M_u}} = \big(uL^2(X \setminus S)\big)^{\perp}, \tag{2.1}$$

where $J = h_\varphi \cdot E(u) \circ \varphi^{-1}$ and $S = S(J) = \{x \in X : J(x) \neq 0\}$.

Proof. A function $f \in L^2(\mu)$ belongs to $\mathcal{N}(u^{1/2}C_\varphi)$ if and only if

$$\begin{aligned} \|u^{1/2}C_\varphi f\|^2 &= \int_X u \cdot |f \circ \varphi|^2 \, d\mu \\ &= \int_X h_\varphi \cdot E(u) \circ \varphi^{-1} \cdot |f|^2 \, d\mu \\ &= \int_X J|f|^2 \, d\mu = 0. \end{aligned}$$

Thus, $f \in \mathcal{N}(u^{1/2}C_\varphi)$ if and only if $\|Jf\| = 0$, which implies $f \in L^2(X \setminus S)$. Hence, $\mathcal{N}_{M_u}(C_\varphi) = \mathcal{N}(u^{1/2}C_\varphi) = L^2(X \setminus S)$.

Now, the M_u-orthogonal complement is given by:

$$\begin{aligned} \mathcal{N}_{M_u}(C_\varphi)^{\perp_{M_u}} &= L^2(X \setminus S)^{\perp_{M_u}} \\ &= \big\{f \in L^2(\mu) : \langle M_u g, f\rangle = 0 \text{ for all } g \in L^2(X \setminus S)\big\} \\ &= \left\{f \in L^2(\mu) : \int_X u\, g\, \overline{f}\, d\mu = 0 \text{ for all } g \in L^2(X \setminus S)\right\}. \end{aligned}$$

This space is exactly the orthogonal complement (in the usual inner product) of $uL^2(X \setminus S)$, which gives (2.1). □

Now we are ready to provide necessary and sufficient conditions under which the composition operator C_φ is an M_u-partial isometry on the Hilbert space $L^2(\mu)$.

Theorem 2.7. *Let M_u be a positive multiplication operator and C_φ a bounded composition operator on the Hilbert space $L^2(\mu)$. Then C_φ is an M_u-partial isometry on $L^2(\mu)$ if and only if $\int_X (J-u)|f|^2 \, d\mu = 0$ for all $f \in \big(uL^2(X \setminus S)\big)^{\perp}$, where $J = h_\varphi \cdot E(u) \circ \varphi^{-1}$ and $S = \{x \in X : J(x) \neq 0\}$.*

Proof. By Lemma 2.6, we know that $\mathcal{N}_{M_u}(C_\varphi)^{\perp_{M_u}} = \big(uL^2(X \setminus S)\big)^{\perp}$. Therefore, C_φ is an M_u-partial isometry if and only if $\|C_\varphi f\|_{M_u} = \|f\|_{M_u}$ for all $f \in \big(uL^2(X \setminus S)\big)^{\perp}$. This is equivalent to $\|u^{1/2} f \circ \varphi\| = \|u^{1/2} f\|$ for all $f \in \big(uL^2(X \setminus S)\big)^{\perp}$. Squaring both sides gives $\int_X u \cdot |f \circ \varphi|^2 \, d\mu = \int_X u \cdot |f|^2 \, d\mu$. Using the change-of-variable identity associated with the adjoint of the composition operator, we get $\int_X u \cdot |f \circ \varphi|^2 \, d\mu = \int_X J \cdot |f|^2 \, d\mu$. Thus, $\int_X (J-u)|f|^2 \, d\mu = 0$ for all $f \in \big(uL^2(X \setminus S)\big)^{\perp}$. □

In the next theorem, we investigate conditions under which the composition operator C_φ is M_u-unitary on the Hilbert space $L^2(\mu)$. Specifically, we provide necessary and sufficient conditions for C_φ to be M_u-unitary.

Theorem 2.8. *Let M_u be a positive multiplication operator and C_φ be a bounded composition operator on the Hilbert space $L^2(\mu)$. Then C_φ is M_u-unitary if and only if*

$$u = J = h_\varphi E(u) \circ \varphi^{-1} = (u \cdot h_\varphi) \circ \varphi.$$

Proof. By definition, C_φ is M_u-unitary if and only if $C_\varphi^* M_u C_\varphi = M_u$ and $C_\varphi M_u C_\varphi^* = M_u$. For every $f \in L^2(\mu)$, we compute

$$\langle C_\varphi^* M_u C_\varphi f, f\rangle = \langle h_\varphi E(u) \circ \varphi^{-1} f, f\rangle = \langle M_u f, f\rangle,$$

and

$$\langle C_\varphi M_u C_\varphi^* f, f\rangle = \langle (u \circ \varphi) \cdot (h_\varphi \circ \varphi) \cdot E(f), f\rangle = \langle M_u f, f\rangle.$$

Therefore, C_φ is M_u-unitary if and only if $u = h_\varphi E(u) \circ \varphi^{-1}$ and $(u \cdot h_\varphi) \circ \varphi \cdot E(f) = u \cdot f$ for all $f \in L^2(\mu)$. To simplify the second condition, integrate both sides of $(u \cdot h_\varphi) \circ \varphi \cdot E(f) = u \cdot f$ and use the fact that E is a conditional expectation operator. It follows that the above holds for all $f \in L^2(\mu)$ if and only if $(u \cdot h_\varphi) \circ \varphi = u$. Thus, the result follows. □

By Theorem 2.8, if C_φ is M_u-unitary, then

$$\begin{aligned} R(M_u) &= R\left(M_{(u\circ\varphi)\cdot(h_\varphi\circ\varphi)} E\right) \\ &= R\left(E M_{(u\circ\varphi)\cdot(h_\varphi\circ\varphi)}\right) \subseteq L^2(\varphi^{-1}(\Sigma)) \\ &= R(C_\varphi). \end{aligned}$$

Hence, by Theorem 1.1, there exists an operator $C \in \mathcal{B}(L^2(\mu))$ such that $M_u C = C_\varphi$.

Theorem 2.9. *Let C_φ be a composition operator on $L^2(\mu)$, with C_ψ positive. Then C_φ is C_ψ-selfadjoint if and only if the restricted map $\varphi_{S_J} := \varphi|_{S_J}$ is periodic of period two, and $J := h_\varphi \cdot E^\varphi(\sqrt{h_\psi}) \circ \varphi^{-1} = \sqrt{h_\psi}$, where $S_J = \{x \in X : J(x) \neq 0\}$, and $E^\varphi(\cdot) := E(\cdot \mid \varphi^{-1}(\mathcal{F}))$ denotes the conditional expectation with respect to the σ-algebra $\varphi^{-1}(\mathcal{F})$.*

Proof. Since C_ψ is a positive composition operator, it follows that $C_\psi = |C_\psi| = M_{\sqrt{h_\psi}}$, and $C_\psi^{1/2} = M_{h_\psi^{1/4}}$. The operator C_φ is C_ψ-selfadjoint if and only if $C_\psi C_\varphi = C_{\varphi\circ\psi} = C_\varphi^* C_\psi$. Since $C_\psi = M_{\sqrt{h_\psi}}$, this condition becomes $M_{\sqrt{h_\psi}} C_\varphi = C_\varphi^* M_{\sqrt{h_\psi}} = (M_{\sqrt{h_\psi}} C_\varphi)^*$. Define the weighted composition operator $W_\psi := M_{\sqrt{h_\psi}} C_\varphi$. Then C_φ is C_ψ-selfadjoint if and only if $W_\psi = W_\psi^*$, that is, W_ψ is selfadjoint. Therefore, by Theorem 2.2,

the operator W_ψ is selfadjoint if and only if $\varphi_{S_J} = \varphi|_{S_J}$ is periodic of period two and $J := h_\varphi \cdot E^\varphi(\sqrt{h_\psi}) \circ \varphi^{-1} = \sqrt{h_\psi}$. This completes the proof. □

Theorem 2.10. *Let C_ψ be a positive composition operator and C_φ a bounded composition operator on the Hilbert space $L^2(\mu)$. Let $J := h_\varphi \cdot E^\varphi(\sqrt{h_\psi}) \circ \varphi^{-1}$, and $S := \{x \in X : J(x) \neq 0\}$. Then the following statements hold:*

(a) *The operator C_φ is a C_ψ-isometry if and only if*

$$\sqrt{h_\psi} = J = h_\varphi \cdot E^\varphi(\sqrt{h_\psi}) \circ \varphi^{-1}.$$

(b) *The operator C_φ is a C_ψ-partial isometry on $L^2(\mu)$ if and only if*

$$\int_X (J - \sqrt{h_\psi})|f|^2 \, d\mu = 0, \quad \text{for all } f \in \left(\sqrt{h_\psi} \cdot L^2(X \setminus S)\right)^\perp.$$

(c) *The operator C_φ is C_ψ-unitary if and only if*

$$\sqrt{h_\psi} = J = h_\varphi \cdot E^\varphi(\sqrt{h_\psi}) \circ \varphi^{-1} \quad \text{and} \quad \left(\sqrt{h_\psi} \cdot h_\varphi\right) \circ \varphi = \sqrt{h_\psi},$$

or equivalently,

$$\left(\sqrt{h_\psi} \cdot h_\varphi\right) \circ \varphi \cdot E^\varphi(f) = \sqrt{h_\psi} \cdot f, \quad \text{for all } f \in L^2(\mu).$$

Proof. The proof follows directly from the structure of the operator $C_\psi = M_{\sqrt{h_\psi}}$, combined with the arguments used in Theorem 2.9. Applying the operator-theoretic characterizations from Theorems 2.5, 2.7, and 2.8 to the weighted composition operator $W_\psi = M_{\sqrt{h_\psi}} C_\varphi$, we obtain the stated equivalences. □

Remark 2.11. If C_φ is C_ψ-unitary, then by Theorem 2.10, item (c), we have

$$R(C_\psi) = R\left(M_{(\sqrt{h_\psi} \cdot h_\varphi) \circ \varphi} \cdot E^\varphi\right) = R\left(E^\varphi M_{(\sqrt{h_\psi} \cdot h_\varphi) \circ \varphi}\right).$$

Therefore,

$$L^2(\psi^{-1}(\Sigma)) = R(C_\psi) \subseteq R(C_\varphi) = L^2(\varphi^{-1}(\Sigma)).$$

This implies that

$$\psi^{-1}(\Sigma) \subseteq \varphi^{-1}(\Sigma).$$

As a direct consequence of this necessary condition, we obtain the following:

Corollary 2.12. *Let C_φ and C_ψ be composition operators on the Hilbert space $L^2(\mu)$, with C_ψ positive. If $\psi^{-1}(\Sigma) \nsubseteq \varphi^{-1}(\Sigma)$, then C_φ cannot be C_ψ-unitary.*

3. Examples

In this section, we present two concrete examples of non-singular transformations on the interval $[0,1]$. These examples illustrate how the properties of M_u-normality, M_u-isometry, and M_u-unitarity of composition operators depend on the structure of the transformation and the choice of the weight function u.

Example 3.1. Let $X=[0,1]$, let $d\mu=dx$, and let Σ be the Lebesgue σ-algebra. Define the non-singular transformation $\varphi: X\to X$ by

$$\varphi(x)=\begin{cases}2x, & \text{if } x\in[0,\frac{1}{2}],\\ \\ 2x-1, & \text{if } x\in(\frac{1}{2},1].\end{cases}$$

Note that h_φ, the Radon–Nikodym derivative defined in (1.1), satisfies $h_\varphi(x)=1$.

For each $0\le a<b\le 1$ and $f\in L^2(\Sigma)$, we have

$$\begin{aligned}\int_{\varphi^{-1}(a,b)} f(x)\,dx &= \int_{a/2}^{b/2} f(x)\,dx+\int_{(a+1)/2}^{(b+1)/2} f(x)\,dx\\ &= \int_a^b \frac{1}{2}\left\{f\left(\frac{x}{2}\right)+f\left(\frac{1+x}{2}\right)\right\}dx.\end{aligned}$$

Hence, $(E(f)\circ\varphi^{-1})(x)=\frac{1}{2}\left\{f\left(\frac{x}{2}\right)+f\left(\frac{1+x}{2}\right)\right\}$. It follows that

$$E(f)(x)=\begin{cases}\frac{1}{2}\left\{f(x)+f\left(\frac{1+2x}{2}\right)\right\}, & x\in[0,\frac{1}{2}],\\ \\ \frac{1}{2}\left\{f\left(\frac{2x-1}{2}\right)+f(x)\right\}, & x\in(\frac{1}{2},1].\end{cases}$$

Let $u: X\to\mathbb{R}$ be a positive, Σ-measurable function such that $S(u)=X$. By Theorem 2.4, the operator C_φ is M_u-normal if and only if

$$\frac{u(x)}{u(2x)}\chi_{[0,1/2]}(x)+\frac{u(x)}{u(2x-1)}\chi_{(1/2,1]}(x)=\frac{u(x/2)+u((1+x)/2)}{2u(x)},$$

for all $x\in[0,1]$. In particular, this identity implies that $u(0)=u(1/2)$. Therefore, for any measurable function u such that $u(0)\neq u(1/2)$, the operator C_φ is not M_u-normal. For example, if $u(x)=e^x$, then C_φ is not M_u-normal.

By Theorem 2.5, C_φ is M_u-isometric if and only if

$$u(x)=\frac{u(x/2)+u((x+1)/2)}{2},\quad \text{for all } x\in[0,1].$$

Moreover, by Theorem 2.8, the operator C_φ is M_u-unitary if and only if

$$u(x)=\frac{u(x/2)+u((x+1)/2)}{2}$$

$$= u(2x)\chi_{[0,1/2]}(x) + u(2x-1)\chi_{(1/2,1]}(x).$$

Example 3.2. Let $X = [0,1]$, let $d\mu = dx$, and let Σ be the Lebesgue σ-algebra. Define the non-singular transformation $\varphi : X \to X$ by

$$\varphi(x) = \begin{cases} 1-2x, & x \in [0, \frac{1}{2}], \\ \\ 2x-1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Note that h_φ, the Radon–Nikodym derivative defined in (1.1), satisfies $h_\varphi(x) = 1$.

For each $0 \le a < b \le 1$ and $f \in L^2(\Sigma)$, we have

$$\begin{aligned} \int_{\varphi^{-1}(a,b)} f(x)\,dx &= \int_{\frac{1-b}{2}}^{\frac{1-a}{2}} f(x)\,dx + \int_{\frac{a+1}{2}}^{\frac{b+1}{2}} f(x)\,dx \\ &= \int_a^b \frac{1}{2}\left\{f\left(\tfrac{1-x}{2}\right) + f\left(\tfrac{1+x}{2}\right)\right\} dx. \end{aligned}$$

Hence,

$$(E(f)\circ\varphi^{-1})(x) = \frac{1}{2}\left\{f\left(\tfrac{1-x}{2}\right) + f\left(\tfrac{1+x}{2}\right)\right\}.$$

It follows that

$$E(f)(x) = \begin{cases} \frac{1}{2}\left\{f(x) + f(1-x)\right\}, & x \in [0, \frac{1}{2}], \\ \\ \frac{1}{2}\left\{f(1-x) + f(x)\right\}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Let $u : X \to \mathbb{R}$ be a positive Σ-measurable function. Then, by Theorems 2.4, 2.5 and 2.8, we have:

- The operator C_φ is M_u-normal on the Hilbert space $L^2(\Sigma)$ if and only if
$$\begin{aligned} &\frac{u(x)}{u(1-2x)}\chi_{[0,\frac{1}{2}]}(x) + \frac{u(x)}{u(2x-1)}\chi_{(\frac{1}{2},1]}(x) \\ &\quad = \frac{u\left(\frac{1-x}{2}\right) + u\left(\frac{1+x}{2}\right)}{2u(x)}, \quad \text{for all } x \in [0,1]. \end{aligned}$$
- The operator C_φ is M_u-isometric if and only if
$$u(x) = \frac{u\left(\frac{1-x}{2}\right) + u\left(\frac{1+x}{2}\right)}{2}, \quad \text{for all } x \in [0,1].$$
- The operator C_φ is M_u-unitary if and only if
$$\begin{aligned} u(x) &= \frac{u\left(\frac{1-x}{2}\right) + u\left(\frac{1+x}{2}\right)}{2} \\ &= u(1-2x)\chi_{[0,1/2]}(x) + u(2x-1)\chi_{(1/2,1]}(x), \end{aligned}$$
for all $x \in [0,1]$.

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