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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 3
Pages: 279-304

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2025.2070236.2313



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 3 (2026), 279-304
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2025.2070236.2313

Multidimensional Fuzzy Soft Sets and their Application in Selecting Optimal Transportation Projects under Uncertainty

Shifa Mol A P[1*] and Sunil Jacob John[2]

Abstract. This research presents a new set called multidimensional fuzzy soft set, which is an extension of soft set and multidimensional fuzzy set. The paper explores the fundamental properties of multidimensional fuzzy soft set and introduces operations such as negation, t-norms and t-conorms within this context. Additionally, the study discusses relevant laws, including De Morgan's law as they apply to this new structure. Furthermore, the application of this newly proposed set in the decision-making process such as selecting the most suitable transportation infrastructure project underscores its significance in practical scenarios. A study in comparison with the existing decision making approach utilizing interval valued intuitionistic fuzzy sets highlights the efficacy of the proposed methodology.

1. Introduction

In disciplines like economics, engineering and environmental studies, traditional methods often struggle with uncertainties inherent in real-world problems. While fuzzy set theory [24] has widened the applicability of classical set theory, the challenge remains in assigning precise membership degrees to elements. This limitation has spurred the development of various generalizations. Intuitionistic fuzzy sets [21], introduced by Atanassov in 1999 aimed to address the issue of non-uniqueness in membership assignment by incorporating a degree of non-membership

2020 *Mathematics Subject Classification.* 03E72, 03E75.
Key words and phrases. Multidimensional fuzzy soft set, Multidimensional fuzzy soft negation, Multidimensional fuzzy soft t-norm, Multidimensional fuzzy soft t-conorm, Decision making.
Received: 2025-09-03, Accepted: 2025-12-08.
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alongside traditional membership values as in fuzzy set theory. Subsequent extensions like interval valued fuzzy sets [15, 22, 25, 34] and interval valued intuitionistic fuzzy sets [20] represented the membership degree of elements as intervals of real numbers within the closed interval [0,1]. Later n-dimensional fuzzy sets, fuzzy multisets and hesitant fuzzy sets further refined these concepts, allowing for a range of values to represent linguistic variables. In the context of n-dimensional fuzzy set [6, 41], each element is characterized by precisely n membership values arranged in ascending order. This distinctive attribute leads to their classification as n-dimensional intervals. These structures can be considered as a special case of Yager's fuzzy bags [33], which a decade later came to be referred to as fuzzy multisets in [2, 36]. A fuzzy multiset on the universe $\mathcal{U}$ is described by a function $f : \mathcal{U} \to \mathbb{N}^{[0,1]}$ where $\mathbb{N}^{[0,1]}$ denotes the collection of all maps from $[0, 1]$ to $\mathbb{N}$. Thus, within a fuzzy multiset, an element of $\mathcal{U}$ can appear multiple times, each occurrence being associated with identical or distinct membership values. In hesitant fuzzy sets [40], the membership of an element is expressed as a subset of possible values within the interval $[0, 1]$. The distinction between a hesitant fuzzy set and a fuzzy multiset lies in the fact that fuzzy multisets permit the repetition of individual membership values.

The development further advanced with the introduction of multidimensional fuzzy sets (MDFSs) [1] by Annaxsuel Lima and S. Palmeira, which were presented as a particular case of fuzzy multisets. Using the notation $f(u)$ as f_u, Lima restricted attention to fuzzy multisets with non-empty and finite support, that is, for each $u \in \mathcal{U}$, the set $\{c \in [0, 1] \mid f_u(c) > 0\}$ is non-empty and finite. Moreover, the membership values associated with the elements of $\mathcal{U}$ were arranged in ascending order. However, considering them solely as representations of fuzzy multisets with finite support overlooks the fact that they arise as a natural generalization of n-dimensional fuzzy sets and incorporate an ordering of membership degrees, a feature not explicitly present in fuzzy multisets. Furthermore, existing literature lacks a generally accepted ordering on $\mathbb{N}^{[0,1]}$ that extends the conventional orders used in fuzzy set generalizations represented by fuzzy multisets.

In parallel, Molodtsov's soft set theory [11], emerged in 1999, offered a novel approach to overcome the need for parameterization tools, unlike fuzzy set theory. Maji et al. later expanded on soft sets by introducing fuzzy soft sets [29], which combined the principles of fuzzy sets and soft sets, thereby facilitating solutions to decision-making problems with inherent uncertainty. The subsequent development of various extensions such as intuitionistic fuzzy soft sets [30], generalized fuzzy soft sets [31], picture fuzzy soft sets [42], n-dimensional fuzzy soft sets [35], hesitant

fuzzy soft sets [23], multi fuzzy soft sets [43] further broadened the scope of soft set theory, each aiming to address specific nuances in practical applications.

Various set operators have paved the way for the application of these sets across diverse fields. Key operators such as union, intersection and complement have played significant roles in this regard. The subsequent introduction of t-norms and t-conorms [7] further facilitated their effective manipulation. Initially defined within the [0,1] interval, these operations were subsequently extended to fuzzy sets [8, 17, 32, 38, 39] and also to soft sets by Irfan Ali and Shabir [26]. The evolution continued with the extension of t-norms to $I_n[0,1]$, the set of n dimensional intervals, and to $I_\infty[0,1]$, the union of all n-dimensional intervals across the set of natural numbers, as discussed in [6, 16, 18].

A number of techniques have been developed to address decision-making problems across multiple fields. These methods generally aim at identifying the optimal alternative, ranking a collection of options, or classifying them into specific categories. A wide range of algorithms has been introduced to achieve these goals, as reported in studies such as [3, 10, 12, 14, 21, 27, 28, 44, 45] alongside established approaches including TOPSIS [9] and VIKOR [37]. Such advancements highlight the practical significance and applicability of generalized extensions of classical set theory in real-world scenarios.

While multidimenional fuzzy sets provide a robust mathematical framework for managing uncertainty, they also exhibit limitations similar to those observed in classical fuzzy set theory, as noted in [11]. One of the main challenges lies in the absence of adequate parametrization techniques. To overcome this drawback, we introduce a new structure, namely multidimensional fuzzy soft set (MDFSSs), which integrate the principles of MDFSs and soft sets, thereby extending their representational and analytical potential. This new framework not only accommodates diverse dimensions of attributes but also enhances the clarity and precision in decision-making processes, making it particularly relevant in real-world applications.

The structure of the paper is as follows: Section 2 reviews the fundamental notions that underpin the study. Section 3 introduces multidimensional fuzzy soft sets in detail, along with the associated operations of t-norm, t-conorm and negation, supported by illustrative theorems and examples. It also establishes key related results, including De Morgan's law [4]. Section 4 demonstrates the application of multidimensional fuzzy soft sets in decision-making contexts. Finally, Section 5 summarizes the contributions of the work.

2. Preliminaries

As a preliminary step, we revisit fundamental concepts such as fuzzy sets, soft sets, n dimensional fuzzy set, multidimensional fuzzy set and fuzzy soft set.

Let W represent a starting universe set containing objects and Q denote the set of parameters associated with these objects. Throughout this paper, we assume that W and Q are nonempty unless we specify otherwise. Parameters typically encompass attributes, qualities, or features of the objects.

We use $P(W)$ to represent the power set of W.

Definition 2.1 ([24])**.** A fuzzy set B on W is described by a function $\eta_B : W \to [0,1]$ called fuzzy membership function, which assigns to each element in W a grade of membership ranging between zero and one. i.e., $B = \{(x, \eta_B(x)); x \in W\}$ where $\eta_B(x)$ is the fuzzy membership value of x in W. The set of all fuzzy subsets on W is denoted by $F(W)$.

Definition 2.2 ([41])**.** Let W be a set and n belongs to the set of natural numbers. An n-dimensional fuzzy set (nFS) is defined by the form, $\mathcal{D} = \left\{\left(w, \eta^1_{\mathcal{D}}(w), \ldots \eta^n_{\mathcal{D}}(w)\right) : w \in W\right\}$ where $\eta^i_{\mathcal{D}} : W \to [0,1]$, $i = 1, 2...n$ are the membership functions that satisfy $\eta^1_{\mathcal{D}}(w) \leq \cdots \leq \eta^n_{\mathcal{D}}(w)$.

We use $nFS(W)$ to denote the set of all n-dimensional fuzzy sets on W. Bedregal et al. presented the following set in [5]:

$$\mathcal{D}_n([0,1]) = \{(w_1, \ldots, w_n) \in [0,1]^n \mid w_1 \leq w_2 \leq \cdots \leq w_n\}$$

and introduced a partial order on $\mathcal{D}_n([0,1])$ called the product order as follows:

$$\mathcal{S} \leq_n \mathcal{T} \quad \Leftrightarrow \quad s_j \leq t_j \text{ for } j = 1, \ldots, n$$

where $\mathcal{S} = (s_1, s_2, \ldots, s_n)$ and $\mathcal{T} = (t_1, t_2, \ldots, t_n)$.

Therefore, we obtain $\left(\eta^1_{\mathcal{D}}(w), \ldots, \eta^n_{\mathcal{D}}(w)\right) \in \mathcal{D}_n([0,1])$ indicating that $\mathcal{D}_n([0,1])$ encompasses all n-dimensional membership degrees. The notation $/w/_n$ stands for the element $(w, w, \ldots, w) \in \mathcal{D}_n([0,1])$ and is termed the degenerate element.

Definition 2.3. [1] Let W be an initial set and $q : W \to \mathbb{N}$ be a given function. Define the set $\mathcal{D}$ as follows:

$$\mathcal{D} = \left\{\left(w, \eta^1_{\mathcal{D}}(w), \ldots, \eta^{q(w)}_{\mathcal{D}}(w)\right) \middle| w \in W\right\}.$$

Each $\eta^i_{\mathcal{D}} : W \to [0,1]$ are functions satisfying $\eta^1_{\mathcal{D}}(w) \leq \cdots \leq \eta^{q(w)}_{\mathcal{D}}(w)$ for all $i = 1, \ldots, q(w)$ and for all $w \in W$.

$\eta^i_{\mathcal{D}}(w)$ denotes the i^{th} membership degree of w in the set $\mathcal{D}$ and the $q(w)$-tuple, $\mathcal{D}(w) = \left(\eta^1_{\mathcal{D}}(w), \ldots, \eta^{q(w)}_{\mathcal{D}}(w)\right)$ is the membership degree of

w in $\mathcal{D}$. This set $\mathcal{D}$ is termed as a multidimensional fuzzy set (MDFS) on W. The collection of all multidimensional fuzzy sets on W is denoted by $MDFS(W)$.

$$\mathcal{D}_\infty([0,1]) = \bigcup_{n=1}^{\infty} \mathcal{D}_n([0,1])$$

Hence for any $\mathcal{S} \in \mathcal{D}_\infty([0,1])$, there exists a natural number n with the property that $\mathcal{S} \in \mathcal{D}_n([0,1])$. This unique n is called the cardinality of $\mathcal{S}$, denoted by $|\mathcal{S}|$, in $\mathcal{D}_\infty([0,1])$.

Numerous partial orders would have been developed for $\mathcal{D}_\infty([0,1])$ if the elements had the same dimension. Since this is not the case, we adopt a standard partial order on $\mathcal{D}_\infty([0,1])$ defined as follows:

$\mathcal{S} \leq_\infty \mathcal{T} \Leftrightarrow |\mathcal{S}| = |\mathcal{T}| = n$ and $\mathcal{S} \leq_n \mathcal{T}$.

Definition 2.4 ([11])**.** Let W be a starting universe set and Q be a set of parameters. A soft set is represented by a structure $\langle \Upsilon, K \rangle$, where $K \subseteq Q$ and Υ is a mapping $\Upsilon : K \to P(W)$, the power set of W. Or, a parametrized family of subsets of the universe W is the soft set. Hence $\Upsilon(\alpha)$ can be viewed as the set of α approximate elements of the soft set $\langle \Upsilon, K \rangle$ for each $\alpha \in K$.

Definition 2.5 ([29])**.** A function λ_K which maps elements from Q to $F(W)$ which also satisfy the property $\lambda_K(\alpha) = \phi \; if \; \alpha \notin K$ is called a fuzzy soft set.

The following definitions are taken from [18].
Consider the sets $\overline{1} = \{/1_n : n \in \mathbb{N}\}$, $\overline{0} = \{/0_n : n \in \mathbb{N}\}$. We use $\dot{1}$ and $\dot{0}$ to denote a random element of $\overline{1}$ and $\overline{0}$ respectively .

Definition 2.6 ([18])**.** A function $\mathfrak{C} : \mathcal{D}_\infty([0,1]) \to \mathcal{D}_\infty([0,1])$ is called a multidimensional complement if it adheres to the specified axioms:

C_1 : $|\mathfrak{C}(\mathcal{S})| = |\mathcal{S}|$
C_2 : $\mathfrak{C}(\dot{0}) = \dot{1}$ and $\mathfrak{C}(\dot{1}) = \dot{0}$, for all $\dot{0} \in \overline{0}$ and $\dot{1} \in \overline{1}$
C_3 : If $\mathcal{S} \leq_\infty \mathcal{T}$, then $\mathfrak{C}(\mathcal{T}) \leq_\infty \mathfrak{C}(\mathcal{S})$, for all $\mathcal{S}, \mathcal{T} \in \mathcal{D}_\infty([0,1])$.

Definition 2.7 ([18])**.** A multidimensional fuzzy set $\mathcal{D}$ is called absolute if $\mathcal{D}(w) \in \overline{1}$ for every $w \in W$, and is called null if $\mathcal{D}(w) \in \overline{0}$ for every $w \in W$.

Jomal [18] introduced two functions denoted as $G_1, G_2 : \mathcal{D}_\infty([0,1]) \times \mathcal{D}_\infty([0,1]) \to \mathbb{N}$ by

$$G_1(\mathcal{S},\mathcal{T}) = \begin{cases} |\mathcal{S}| & \text{if } \exists r \in \mathbb{N} : s_i = t_i \\ & \text{for } i = 1,2,\ldots,r-1 \text{ and } s_r < t_r \\ |\mathcal{T}| & \text{if } \exists r \in \mathbb{N} : s_i = t_i \\ & \text{for } i = 1,2,\ldots,r-1 \text{ and } s_r > t_r \\ \min\{|\mathcal{S}|,|\mathcal{T}|\}, & \text{otherwise} \end{cases}$$

$$G_2(\mathcal{S},\mathcal{T}) = \begin{cases} |\mathcal{S}| & \text{if } \exists r \in \mathbb{N} : s_{n+1-i} = t_{m+1-i} \\ & \text{for } i = 1,2,\ldots r-1 \\ & \text{and } s_{n+1-r} > t_{m+1-r} \\ |\mathcal{T}| & \text{if } \exists r \in \mathbb{N} : s_{n+1-i} = t_{m+1-i} \\ & \text{for } i = 1,2,\ldots,r-1 \\ & \text{and } s_{n+1-r} < t_{m+1-r} \\ \min\{|\mathcal{S}|,|\mathcal{T}|\} & \text{otherwise} \end{cases}$$

where $\mathcal{S} = (s_1, s_2, \ldots, s_n)$ and $\mathcal{T} = (t_1, t_2, \ldots, t_m), n, m \in \mathbb{N}$.

Note that by the definition of G_1 and G_2, if $\mathcal{S} = (s_1, s_2, \ldots, s_n)$ and $\mathcal{T} = (t_1, t_2, \ldots, t_m)$, with $s_1 < t_1$ and $s_n < t_m$ then $G_1(\mathcal{S},\mathcal{T}) = |\mathcal{S}| = n$ and $G_2(\mathcal{S},\mathcal{T}) = |\mathcal{T}| = m$.

Definition 2.8 ([18])**.** A function $\mathbb{H} : \mathcal{D}_\infty([0,1]) \times \mathcal{D}_\infty([0,1]) \to \mathcal{D}_\infty([0,1])$ is called multidimensional t norm (MDTN) if it adheres to the specified axioms:

$H1$: $|\mathbb{H}(\mathcal{S},\mathcal{T})| = G_1(\mathcal{S},\mathcal{T})$
$H2$: $\mathbb{H}(\mathcal{S},\mathcal{T}) = \mathbb{H}(\mathcal{T},\mathcal{S})$
$H3$: If $\mathcal{T} \leq_\infty \mathcal{R}$ then $\mathbb{H}(\mathcal{S},\mathcal{T}) \leq_\infty \mathbb{H}(\mathcal{S},\mathcal{R})$, given $|\mathcal{S}| = |\mathcal{T}| = |\mathcal{R}|$
$H4$: $\mathbb{H}\left(\mathcal{S},\dot{1}\right) = \mathcal{S}$, for $\mathcal{S} \notin \bar{1} \setminus \{\dot{1}\}$
$H5$: $\mathbb{H}(\mathcal{S},\mathbb{H}(\mathcal{T},\mathcal{R})) = \mathbb{H}(\mathbb{H}(\mathcal{S},\mathcal{T}),\mathcal{R})$ whenever $|\mathcal{S}| = |\mathcal{T}| = |\mathcal{R}|$ where $\mathcal{S},\mathcal{T},\mathcal{R} \in \mathcal{D}_\infty([0,1])$

Definition 2.9 ([18])**.** A function $\mathbb{J} : \mathcal{D}_\infty([0,1]) \times \mathcal{D}_\infty([0,1]) \to \mathcal{D}_\infty([0,1])$ is called multidimensional t-conorm (MDTCN) if it adheres to the specified axioms:

$J1$: $|\mathbb{J}(\mathcal{S},\mathcal{T})| = G_2(\mathcal{S},\mathcal{T})$
$J2$: $\mathbb{J}(\mathcal{S},\mathcal{T}) = \mathbb{J}(\mathcal{T},\mathcal{S})$
$J3$: If $\mathcal{T} \leq_\infty \mathcal{R}$ then $\mathbb{J}(\mathcal{S},\mathcal{T}) \leq_\infty \mathbb{J}(\mathcal{S},\mathcal{R})$, provided

$$|\mathcal{S}| = |\mathcal{T}| = |\mathcal{R}|$$

$J4$: $\mathbb{J}\left(\mathcal{S}, \dot{0}\right) = \mathcal{S}$, for $\mathcal{S} \notin \bar{0} \setminus \left\{\dot{0}\right\}$
$J5$: $\mathbb{J}(\mathcal{S}, \mathbb{J}(\mathcal{T}, \mathcal{R})) = \mathbb{J}(\mathbb{J}(\mathcal{S}, \mathcal{T}), \mathcal{R})$ whenever $|\mathcal{S}| = |\mathcal{T}| = |\mathcal{R}|$ where $\mathcal{S}, \mathcal{T}, \mathcal{R} \in \mathcal{D}_\infty([0,1])$

3. Multidimensional Fuzzy Soft Set

In this section, we introduce the concept of Multidimensional fuzzy soft sets ($MDFSS$) and explore their various properties across the universal set W.

Definition 3.1. Let W be a starting universe set, Q be a set of parameters. A multidimensional fuzzy soft set ($MDFSS$) over W is represented by a structure $\langle \Upsilon, K \rangle$ where $K \subseteq Q$ and Υ is a mapping $\Upsilon : K \to MDFS(W)$. Hence an MDFSS parametrizes the MDFSs.

The set K is called the domain of the multidimensional fuzzy soft set $\langle \Upsilon, K \rangle$ and the set of all MDFSS over W is denoted by $MDFSS(W)$.

For an $\alpha \in K$, $\Upsilon(\alpha) = \mathcal{D}$ is an MDFS. Hence for $w \in W$, $\mathcal{D}(w) = \Upsilon(\alpha)(w)$ denotes the membership value of w in $\Upsilon(\alpha)$.

Example 3.2. Let $Q = \{a_1, a_2, a_3, a_4, a_5\}$ and $K = \{a_1, a_2, a_3\} \subset Q$ be a set of diseases where a_1: Heart disease, a_2 : Lung disease, a_3 : Kidney disease, a_4 : Liver disease, a_5 : Diabetes. Let $\{w_1, w_2, \ldots, w_7\}$ be a set of patients. If $\{c_1, \ldots, c_{10}\}$ denotes a set of symptoms associated with the disease set K, then the multidimensional fuzzy soft set, $\langle \Upsilon, K \rangle$ represented by

$$\begin{aligned}
\Upsilon(a_1) &= \{(w_1, 0.2_{c_1}, 0.5_{c_3}), (w_3, 0.5_{c_4})\} \\
\Upsilon(a_2) &= \{(w_3, 0.4_{c_5}, 0.6_{c_3}), (w_5, 0.7_{c_2}, 0.8_{c_8}, 0.9_{c_{10}})\} \\
\Upsilon(a_3) &= \{(w_2, 0.6_{c_3}, 0.9_{c_5}, 1.0_{c_2}), (w_5, 0.1_{c_{10}}, 0.3_{c_4}, 0.6_{c_7}, 0.8_{c_5}), \\
&\qquad (w_6, 0.4_{c_2}, 0.5_{c_3}, 0.7_{c_8})\}
\end{aligned}$$

indicates the severity of the symptoms associated with each disease on specific patients.

Definition 3.3. A multidimensional fuzzy soft set $\langle \Upsilon, K \rangle$ is called absolute if $\Upsilon(\alpha)$ is an absolute multidimensional fuzzy set, for all $\alpha \in K$.

An absolute multidimensional fuzzy soft set is represented by $\left\langle \dot{\mathscr{A}}, Q \right\rangle$, and the collection of all absolute multidimensional fuzzy soft sets is represented by $\left\langle \overline{\dot{\mathscr{A}}, Q} \right\rangle$

Definition 3.4. A multidimensional fuzzy soft set $\langle \Upsilon, Q \rangle$ is called null if $\Upsilon(\alpha)$ is a null multidimensional fuzzy set, for all $\alpha \in Q$.

A null multidimensional fuzzy soft set is represented by $\left\langle \dot{\mathscr{N}}, Q \right\rangle$ and the set of all null multidimensional fuzzy soft sets is represented by $\left\langle \overline{\dot{\mathscr{N}}, Q} \right\rangle$

Definition 3.5. Let $\langle \Upsilon, K \rangle$ and $\langle \Psi, L \rangle$ be two $MDFSSs$ over W. Then $\langle \Upsilon, K \rangle$ is said to be a multidimensional fuzzy soft subset of $\langle \Psi, L \rangle$ if the following two conditions are satisfied.

(i) $K \subseteq L$
(ii) $\forall \alpha \in K,\ \Upsilon(\alpha) \subseteq \Psi(\alpha)$

3.1. **Multidimensional Fuzzy Soft Operators.** This section extends the concepts of t-norms, t-conorms and negation to the domain of multidimensional fuzzy soft sets.

Multidimensional Fuzzy Soft Negation. Let $\mathcal{N} : MDFSS(W) \to MDFSS(W)$ be a map with $|\mathcal{N}(\langle \Upsilon, K \rangle (\alpha)(w))| = |\Upsilon(\alpha)(w)|$ for every $\langle \Upsilon, K \rangle \in MDFSS(W)$ and for every $\alpha \in K$. Then $\mathcal{N}$ is called a multidimensional fuzzy soft negation or multi dimensional fuzzy soft complement (MDFSC) if

$N_1 : \mathcal{N}\left(\left\langle \dot{\mathscr{A}}, Q \right\rangle\right) = \left\langle \dot{\mathscr{N}}, Q \right\rangle$ and $\mathcal{N}\left(\left\langle \dot{\mathscr{N}}, Q \right\rangle\right) = \left\langle \dot{\mathscr{A}}, Q \right\rangle$
$N_2 :$ If $\langle \Upsilon, K \rangle \subseteq \langle \Psi, L \rangle$, then $\mathcal{N}(\langle \Psi, L \rangle) \subseteq \mathcal{N}(\langle \Upsilon, K \rangle)$

and is called strict multidimensional fuzzy soft negation if

$N_3 : \mathcal{N}(\mathcal{N}(\langle \Upsilon, K \rangle)) = \langle \Upsilon, K \rangle$

Example 3.6. Let $\langle \Upsilon, K \rangle \in MDFSS(W)$ and

$$\Upsilon(\alpha)(w) = \left(\eta_{\mathcal{D}}^{1}(w), \eta_{\mathcal{D}}^{2}(w), \ldots, \eta_{\mathcal{D}}^{q(w)}(w)\right), \quad \text{for an } \alpha \in K.$$

Define $\mathcal{N}_S(\langle \Upsilon, K \rangle) = \langle \Psi, K \rangle$ where

$$\Psi(\alpha)(w) = \left(1 - \eta_{\mathcal{D}}^{q(w)}(w), 1 - \eta_{\mathcal{D}}^{q(w)-1}(w), \ldots, 1 - \eta_{\mathcal{D}}^{1}(w)\right).$$

Then $\mathcal{N}_S$ is a multidimensional fuzzy soft negation, called standard multidimensional fuzzy soft negation.

Example 3.7. Let $\langle \Upsilon, K \rangle \in MDFSS(W)$ and

$$\Upsilon(\alpha)(w) = \left(\eta_{\mathcal{D}}^{1}(w), \eta_{\mathcal{D}}^{2}(w), \ldots, \eta_{\mathcal{D}}^{q(w)}(w)\right)$$

for an arbitrary $\alpha \in Q$. Define $\mathcal{N}(\langle \Upsilon, K \rangle) = \langle \Psi, K \rangle$ where

$$\Psi(\alpha)(w) = \begin{cases} /1/_{q(w)} & \text{if } \eta_{\mathcal{D}}^{i}(w) = 0 \text{ for all } i = 1, 2, \ldots, q(w) \\ /0/_{q(w)}, & \text{otherwise} \end{cases}$$

Then $\mathcal{N}$ is a multidimensional fuzzy soft negation

Multidimensional Fuzzy Soft t-Norm: Let

$$\mathbb{T} : MDFSS(W) \times MDFSS(W) \to MDFSS(W)$$

be a map such that for every $\langle \Upsilon, K\rangle, \langle \Psi, L\rangle \in MDFSS(W)$ and

$$|\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle)(\alpha)(w)| = G_1(\Upsilon(\alpha)(w), \Psi(\alpha)(w))$$

for every $\alpha \in K \cap L$ and $w \in W$. Then $\mathbb{T}$ is called a multidimensional fuzzy soft t-norm (MDFSTN) if

T_1 : $\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \mathbb{T}(\langle \Psi, L\rangle, \langle \Upsilon, K\rangle)$

T_2 : $\mathbb{T}(\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle), \langle \Theta, M\rangle) = \mathbb{T}(\langle \Upsilon, K\rangle, \mathbb{T}(\langle \Psi, L\rangle, \langle \Theta, M\rangle))$ for every $\langle \Upsilon, K\rangle, \langle \Psi, L\rangle, \langle \Theta, M\rangle$ in $MDFSS(W)$ such that

$$|\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)| = |\Theta(\alpha)(w)|, \quad \alpha \in K \cap L \cap M,\ w \in W$$

T_3 : $\mathbb{T}\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{A}}, Q\right\rangle\right) = \langle \Upsilon, K\rangle$ for every pair $\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{A}}, Q\right\rangle\right)$ with $\Upsilon(\alpha)(w) \notin \overline{1} \setminus \{\dot{\mathscr{A}}(\alpha)(w)\}, \alpha \in K, w \in W$.

T_4 : If $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$, then $\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Theta, M\rangle) \subseteq \mathbb{T}(\langle \Psi, L\rangle, \langle \Theta, M\rangle)$, provided $|\Theta(\alpha)(w)| = |\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)|$ for all $\alpha \in K \cap M$ and for all $w \in W$.

Examples for MDFSTN: For two MDFSSs $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ over the same universe W, let $K \cap L = M$, $\Upsilon(\alpha)(w) = (s_1, s_2, \ldots, s_n)$, $\Psi(\alpha)(w) = (t_1, t_2, \ldots, t_m)$ for an arbitrary $\alpha \in M, w \in W$ and

$$G_1(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) = k.$$

Example 3.8 (Min operator - $\mathbb{T}_{\min}$)**.** The multidimensional fuzzy soft t-norm $\mathbb{T}_{\min}$ is defined as $\mathbb{T}_{\min}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \langle \Theta, M\rangle$, where

$$\Theta(\alpha)(w) = \min(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) = (s_1 \wedge t_1, \ldots, s_k \wedge t_k)$$

with $s_i = t_i = 1$ for all $i > n$ and $j > m$.

For example, let $W = \{s_1, s_2\}, K = \{\alpha_1, \alpha_3, \alpha_5\}, L = \{\alpha_1, \alpha_4, \alpha_5\}$. Let $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ be two MDFSSs defined by

$$\begin{aligned}
\Upsilon(\alpha_1) =& \{(w_1, 0.1, 0.6, 0.7), (w_2, 0.3, 0.4, 0.5, 0.6)\},\\
\Upsilon(\alpha_3) =& \{(w_1, 0.9, 0.9), (w_2, 0.7, 0.8, 0.9)\}\\
\Upsilon(\alpha_5) =& \{(w_1, 0.1, 0.2, 0.3), \{(w_2, 0.3, 0.4, 0.4)\},\\
\Psi(\alpha_1) =& \{w_1, 0.5, 0.5, 0.6, 0.6), (w_2, 0.3, 0.4, 0.4)\},\\
\Psi(\alpha_4) =& \{w_1, 0.8), (w_2, 0.3, 0.3, 0.4)\}\\
\Psi(\alpha_5) =& \{(w_1, 0.5, 0.6), (w_2, 0.3, 0.4, 0.5)
\end{aligned}$$

Then $\mathbb{T}_{\min}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \langle \Theta, M\rangle$ where $M = K \cap L = \{\alpha_1, \alpha_5\}$ and

$$\begin{aligned}
\Theta(\alpha_1) =& \{(w_1, 0.1, 0.5, 0.6), (w_2, 0.3, 0.4, 0.4)\},\\
\Theta(\alpha_5) =& \{(w_1, 0.1, 0.2, 0.3), (w_2, 0.3, 0.4, 0.4)\}
\end{aligned}$$

Example 3.9 (MIN operator - $\mathbb{T}_{MIN}$)**.** The multidimensional fuzzy soft t-norm $\mathbb{T}_{MIN}$ is defined as $\mathbb{T}_{MIN}(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle) = \langle\Theta, M\rangle$, where $\Theta(\alpha)(w) = MIN(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) = (r_1, \ldots, r_k)$ where r_i is the i-th least element in the set $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$. Since s_i and t_j could be same for some i and j, the set $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ could be a multiset.

For example, consider the MDFSS $\langle\Upsilon, K\rangle$ and $\langle\Psi, L\rangle$ mentioned after Example 3.8. Then $MIN(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle) = \langle\Theta, M\rangle$, where $M = K \cap L = \{\alpha_1, \alpha_5\}$

$$\Theta(\alpha_1) = \{w_1, 0.1, 0.5, 0.5), (w_2, 0.3, 0.3, 0.4)\},$$
$$\Theta(\alpha_5) = \{(w_1, 0.1, 0.2, 0.3), (w_2, 0.3, 0.3, 0.4)\}$$

Example 3.10. (**Drastic min operator** $-\ \mathbb{T}_{Drastic}$) : The multidimensional fuzzy soft t-norm $\mathbb{T}_{Drastic}$ is defined as

$$\mathbb{T}_{Drastic}(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle) = \langle\Theta, M\rangle,$$

where

$$\Theta(\alpha)(w) = \begin{cases} \Upsilon(\alpha)(w) & \text{if } \Psi(\alpha)(w) \in \overline{1} \\ \Psi(\alpha)(w) & \text{if } \Upsilon(\alpha)(w) \in \overline{1} \\ /1/_{G_1(\Upsilon(\alpha)(w), \Psi(\alpha)(w))} & \Upsilon(\alpha)(w),\ \Psi(\alpha)(w) \in \overline{1} \\ /0/_{G_1(\Upsilon(\alpha)(w), \Psi(\alpha)(w))} & \text{otherwise} \end{cases}$$

Remark 3.11. Any multidimensional fuzzy t-norm $\mathbb{H}$ can induce a multidimensional fuzzy soft t-norm $\mathbb{T}_{\mathbb{H}}$ given by $\mathbb{T}_{\mathbb{H}}(\langle\Upsilon, K\rangle \langle\Psi, L\rangle) = \langle\Theta, M\rangle$, where $M = K \cap L$ and $\Theta(\alpha) = \mathbb{H}(\Upsilon(\alpha), \Psi(\alpha))$ for $\alpha \in M$.

Multidimensional Fuzzy Soft t-Conorm. Let

$$\mathbb{S} : MDFSS(W) \times MDFSS(W) \to MDFSS(W)$$

be a map such that for every $\langle\Upsilon, K\rangle, \langle\Psi, L\rangle \in MDFSS(W)$ and

$$|\mathbb{S}(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle)(\alpha)(w)| = G_2(\Upsilon(\alpha)(w), \Psi(\alpha)(w))$$

for every $\alpha \in K \cap L$ and $w \in W$. Then $\mathbb{S}$ is called a multidimensional fuzzy soft t-conorm (MDFSTCN) if it satisfies the following conditions:

S_1 : $\mathbb{S}(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle) = \mathbb{S}(\langle\Psi, L\rangle, \langle\Upsilon, K\rangle)$

S_2 : $\mathbb{S}(\mathbb{S}(\langle\Upsilon, K\rangle, \langle\Psi, L\rangle), \langle\Theta, M\rangle) = \mathbb{S}(\langle\Upsilon, K\rangle, \mathbb{S}(\langle\Psi, L\rangle, \langle\Theta, M\rangle))$ for every $\langle\Upsilon, K\rangle, \langle\Psi, L\rangle, \langle\Theta, M\rangle \in MDFSS(W)$ such that

$$|\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)| = |\Theta(\alpha)(w)|, \quad \alpha \in K \cap L \cap M, w \in W.$$

S_3 : $\mathbb{S}\left(\langle\Upsilon, K\rangle, \left\langle\dot{\mathscr{N}}, Q\right\rangle\right) = \langle\Upsilon, K\rangle$, for every pair $\left(\langle\Upsilon, K\rangle, \left\langle\dot{\mathscr{N}}, Q\right\rangle\right)$ with $\Upsilon(\alpha)(w) \notin \overline{0} \setminus \{\dot{\mathscr{N}}(\alpha)(w)\}, \alpha \in K, w \in W$.

S_4 : If $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$, then

$$\mathbb{S}\left(\langle \Upsilon, K\rangle, \langle \Theta, M\rangle\right) \subseteq \mathbb{S}\left(\langle \Psi, L\rangle, \langle \Theta, M\rangle\right),$$

provided $|\Theta(\alpha)(w)| = |\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)|$ for all $\alpha \in K \cap M$ and for all $w \in W$.

Examples for MDFSTCN

For two MDFSS $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ over the same universe W, let $K \cap L = M$, $\Upsilon(\alpha)(w) = (s_1, s_2, \ldots, s_n), \Psi(\alpha)(w) = (t_1, t_2, \ldots, t_m)$ for an arbitrary $\alpha \in M, w \in W$ and $G_2\left((\Upsilon(\alpha)(w), \Psi(\alpha)(w)\right) = k$.

Example 3.12 (Max operator - $\mathbb{S}_{\max}$)**.** The multidimensional fuzzy soft t-conorm $\mathbb{S}_{\max}$ is defined as $\mathbb{S}_{\max}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta, M\rangle$, where $\Theta(\alpha)(w) = \max(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) = (s_{n-(k-1)} \vee t_{m-(k-1)}, \ldots, s_n \vee t_m)$ with $s_{n-i} = t_{m-j} = 0$ for all $i \geq n$ and $j \geq m$.

For example, consider the MDFSS $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ mentioned after Example 3.8.

Then $\mathbb{S}_{\max}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta', M\rangle$, where $M = K \cap L = \{\alpha_1, \alpha_5\}$

$$\Theta'(\alpha_1) = \{(w_1, 0.5, 0.6, 0.7), (w_2, 0.3, 0.4, 0.5, 0.6)\}$$
$$\Theta'(\alpha_5) = \{(w_1, 0.5, 0.6), (w_2, 0.3, 0.4, 0.5)\}$$

Example 3.13 (MAX operator - $\mathbb{S}_{MAX}$)**.** The multidimensional fuzzy soft t-conorm $\mathbb{S}_{MAX}$ is defined as $\mathbb{S}_{MAX}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta, M\rangle$, where $\Theta(\alpha)(w) = MAX(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) = (r_1, \ldots, r_k)$ where r_{k+1-i} is the i-th largest element in the set $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$. Since s_i and t_j could be same for some i and j, the set $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ could be a multiset.

For example, consider the MDFSS $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ mentioned after Example 3.8.

Then $\mathbb{S}_{MAX}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta', M\rangle$, where $M = K \cap L = \{\alpha_1, \alpha_5\}$

$$\begin{aligned}\Theta'(\alpha_1) &= \{w_1, 0.6, 0.6, 0.7), (w_2, 0.4, 0.4, 0.5, 0.6)\}, \Theta'(\alpha_5)\\ &= \{(w_1, 0.5, 0.6), (w_2, 0.4, 0.4, 0.5)\}\end{aligned}$$

Example 3.14. (**Drastic max operator** $-\ \mathbb{S}_{Drastic}$) : The multidimensional fuzzy soft t-conorm $\mathbb{S}_{Drastic}$ is defined as

$$\mathbb{S}_{Drastic}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta, M\rangle,$$

where

$$\Theta(\alpha)(w) = \begin{cases} \Upsilon(\alpha)(w) & \text{if } \Psi(\alpha)(w) \in \overline{0} \\ \Psi(\alpha)(w) & \text{if } \Upsilon(\alpha)(w) \in \overline{0} \\ /0/_{G_2(\Upsilon(\alpha)(w),\Psi(\alpha)(w))} & \Upsilon(\alpha)(w) \in \overline{0},\ \Psi(\alpha)(w) \in \overline{0} \\ /1/_{G_2(\Upsilon(\alpha)(w),\Psi(\alpha)(w))} & \text{otherwise} \end{cases}$$

Remark 3.15. Any multidimensional fuzzy t-conorm $\mathbb{J}$ can induce a multidimensional fuzzy soft t-conorm $\mathbb{S}_{\mathbb{J}}$ given by $\mathbb{S}_{\mathbb{J}}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Theta, M\rangle$, where $M = K \cap L$ and $\Theta(\alpha) = \mathbb{J}(\Upsilon(\alpha), \Psi(\alpha)), \alpha \in M$.

Modified Multidimensional Fuzzy Soft t-Conorm. The introduction of t-norms and t-conorms in classical set theory aimed to refine the notions of intersection and union, respectively. However, within the context of Multidimensional fuzzy soft t-conorms (MDFSTCN), challenges arise in extending the domain of operation $\mathbb{S}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)$ from $K \cap L$ into $K \cup L$, as demonstrated in earlier examples. To address this issue, we propose the introduction of a modified multidimensional fuzzy soft t-conorm.

Let $\langle \Upsilon, K\rangle \in MDFSS(W)$ and $K \subseteq Q$. We define $\widetilde{\langle \Upsilon, K\rangle} = \{\langle \Psi, Q\rangle \mid \Psi(\alpha) = \Upsilon(\alpha)$ for $\alpha \in K$ and $\Psi(\alpha) \in \overline{0}$ for $\alpha \notin K\}$. Hence any multidimensional fuzzy soft set $\langle \Upsilon, K\rangle$ can be identified with a collection of multidimensional fuzzy soft sets $\widetilde{\langle \Upsilon, K\rangle}$.

Now we define the modified multidimensional fuzzy soft t co-norm $\mathbb{S}'$ as follows:

Let $\mathbb{S}' : MDFSS(W) \times MDFSS(W) \to MDFSS(W)$ be a map such that for every $\langle \Upsilon, K\rangle, \langle \Psi, L\rangle \in MDFSS(W)$ and

$$\left|\mathbb{S}'\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)(\alpha)(w)\right| = G_2(\Upsilon(\alpha)(w), \Psi(\alpha)(w))$$

for every $\alpha \in K \cap L$ and $w \in W$. Then $\mathbb{S}'$ is called modified multidimensional fuzzy soft t-norm if

(i) Symmetry : $\mathbb{S}'\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \mathbb{S}'\left(\langle \Psi, L\rangle, \langle \Upsilon, K\rangle\right)$

(ii) Associativity :

$$\mathbb{S}'\left(\mathbb{S}'\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right), \langle \Theta, M\rangle\right) = \mathbb{S}'\left(\langle \Upsilon, K\rangle, \mathbb{S}'\left(\langle \Psi, L\rangle, \langle \Theta, M\rangle\right)\right)$$

for every $\langle \Upsilon, K\rangle, \langle \Psi, L\rangle, \langle \Theta, M\rangle \in MDFSS$ such that

$$|\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)| = |\Theta(\alpha)(w)|, \quad \alpha \in K \cap L \cap M, w \in W.$$

(iii) $\mathbb{S}'\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{N}}, L\right\rangle\right) \in \widetilde{\langle \Upsilon, K\rangle}$, for every pair $\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{N}}, L\right\rangle\right)$ where $\Upsilon(\alpha)(w) \notin \overline{0} \setminus \{\dot{\mathscr{N}}(\alpha)(w)\}$, $\alpha \in K,\ w \in W$ *and* $L \subseteq Q$ provided $\mathbb{S}'\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{N}}, K\right\rangle\right) = \langle \Upsilon, K\rangle$.

(iv) Monotonicity : If $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$, then $\mathbb{S}'(\langle \Upsilon, K\rangle, \langle \Theta, M\rangle) \subseteq \mathbb{S}'(\langle \Psi, L\rangle, \langle \Theta, M\rangle)$, provided

$$|\Theta(\alpha)(w)| = |\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)|$$

for all $\alpha \in K \cap M$ and for all $w \in W$.

Example 3.16. Let $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ be two multidimensional fuzzy soft set over the same universe W. Define $\mathbb{S}'(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \langle \Theta, M\rangle$, where $M = K \cup L$ and

$$\Theta(\alpha)(w) = \begin{cases} \Upsilon(\alpha)(w) & \text{if } \alpha \in K - L \\ \Psi(\alpha)(w) & \text{if } \alpha \in L - K \\ \max(\Upsilon(\alpha)(w), \Psi(\alpha)(w)) & \text{if } \alpha \in K \cap L \end{cases}$$

For example, consider the MDFSS $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ mentioned after Example 3.8.

Then $\mathbb{S}'(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \langle \Theta, M\rangle$, where $M = K \cup L = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$

$$\begin{aligned} \Theta(\alpha_1) &= \{w_1, 0.5, 0.6, 0.7), (w_2, 0.3, 0.4, 0.5, 0.6)\} \\ \Theta(\alpha_3) &= \{(w_1, 0.9, 0.9), (w_2, 0.7, 0.8, 0.9) \\ \Theta(\alpha_4) &= \{(w_1, 0.8), (w_2, 0.3, 0.3, 0.4)\} \\ \Theta(\alpha_5) &= \{(w_1, 0.5, 0.6), (w_2, 0.3, 0.4, 0.5)\} \end{aligned}$$

Remark 3.17. Any multidimensional t-norm $\mathbb{H}$ can generate a modified multidimensional fuzzy soft t-norm $\mathbb{S}_{\mathbb{H}}$ given by $\mathbb{S}_{\mathbb{H}}(\langle \Upsilon, K\rangle \langle \Psi, L\rangle) = \langle \Theta, M\rangle$, where $M = K \cup L$ and

$$\Theta(\alpha) = \begin{cases} \Upsilon(\alpha) & \text{if } \alpha \in K - L \\ \Psi(\alpha) & \text{if } \alpha \in L - K \\ \mathbb{H}(\Upsilon(\alpha), \Psi(\alpha)) & \text{if } \alpha \in K \cap L \end{cases}$$

Definition 3.18. An MDFSTCN, $\mathbb{T}$ and MDFSC, $\mathcal{N}$ are said to obey the law of excluded middle of MDFSS if $\mathbb{T}(\langle \Upsilon, K\rangle, \mathcal{N}(\langle \Upsilon, K\rangle)) \in \left\langle \overline{\dot{\mathcal{N}}, K} \right\rangle$, where

$$\left\langle \overline{\dot{\mathcal{N}}, K} \right\rangle = \{\langle \Upsilon, K\rangle \ \ |\Upsilon(\alpha)(w) \in \overline{0}, \text{for every } \alpha \in K \text{ and } w \in W\}$$

An MDFSTN, $\mathbb{S}$ and MDFSC, $\mathcal{N}$ are said to obey law of contradiction of MDFSS if $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}(\langle \Upsilon, K\rangle) \in \left\langle \overline{\dot{\mathscr{A}}, K} \right\rangle$ where

$$\left\langle \overline{\dot{\mathscr{A}}, K} \right\rangle = \{\langle \Upsilon, K\rangle \ \ |\Upsilon(\alpha)(w) \in \overline{1} \text{for every } \alpha \in K \text{ and } w \in W\}$$

Remark 3.19. If $\mathbb{S}$ is the max operator and $\mathbb{T}$ is the min operator, then $\mathcal{N}_S, \mathbb{T}, \mathbb{S}$ violates the law of contradiction as well as the law of excluded middle.

Proof. Let $\langle \Upsilon, K\rangle$ be an MDFSS with $\Upsilon(\alpha)(w) = (0.2, 0.3, 0.6)$ for some $w \in W$ and for some $\alpha \in K$. Then $\mathcal{N}_S(\langle \Upsilon, K\rangle)(\alpha)(w) = (0.4, 0.7, 0.8)$. Hence

$$\mathbb{T}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) = (0.2, 0.3, 0.6) \notin \overline{0},$$
$$\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) = (0.4, 0.7, 0.8) \notin \overline{1} \qquad \square$$

Remark 3.20. If $\mathbb{T}$ is the $\mathbb{T}_{Drastic}$ operator and $\mathbb{S}$ is the $\mathbb{S}_{Drastic}$ operator, then $\mathbb{T}, \mathbb{S}, \mathcal{N}_S$ satisfy law of contradiction as well as law of excluded middle .

Proof. **Case 1:** If $\langle \Upsilon, K\rangle(\alpha)(w) \in \overline{0}$, then $\mathcal{N}_S(\langle \Upsilon, K\rangle)(\alpha)(w) \in \overline{1}$. So $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) \in \overline{1}$.
Case 2: If $\mathcal{N}_S(\langle \Upsilon, K\rangle)(\alpha)(w) \in \overline{0}$, it implies that $\langle \Upsilon, K\rangle(\alpha)(w) \in \overline{1}$.

Hence again $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) \in \overline{1}$. Now if neither $\langle \Upsilon, K\rangle(\alpha)(w)$ nor $\mathcal{N}_S(\langle \Upsilon, K\rangle)(\alpha)(w)$ belong to $\overline{0}$, then from the way we defined $\mathbb{T}_{Drastic}$, we get $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) \in \overline{1}$. Therefore in every case, we get $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle))(\alpha)(w) \in \overline{1}$, so we can conclude that $\mathbb{S}(\langle \Upsilon, K\rangle, \mathcal{N}_S(\langle \Upsilon, K\rangle)) \in \left\langle \overline{\dot{\varnothing}, K} \right\rangle$. Hence these operators satisfy law of excluded middle.

Similar way proves that they satisfy law of contradiction as well. $\square$

Definition 3.21. An operator $\mathbb{I} : MDFSS(W) \times MDFSS(W) \to MDFSS(W)$ is called an idempotent operator if

$$\mathbb{I}(\langle \Upsilon, K\rangle, \langle \Upsilon, K\rangle) = \langle \Upsilon, K\rangle, \quad \forall \langle \Upsilon, K\rangle \in MDFSS(W).$$

Remark 3.22. Let $\mathbb{T}$ be a MDFSTN , $\mathbb{S}$ be a MDFSTCN and $\mathbb{S}'$ be a modified MDFSTCN which are idempotent. Then for $\langle \Upsilon, K\rangle$, $\langle \Psi, L\rangle \in MDFSS$ with $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$, we have

(i) $\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \langle \Upsilon, K\rangle$
(ii) $\mathbb{S}'(\langle \Upsilon', L\rangle, \langle \Psi, L\rangle) = \langle \Psi, L\rangle$ where

$$\Upsilon'(\alpha)(w) = \begin{cases} \Upsilon(\alpha)(w) & \text{if } \alpha \in K \\ \dot{0} & \text{if } \alpha \in L - K \end{cases}$$

where $|\dot{0}| = |\Psi(\alpha)(w)|$
(iii) $\mathbb{S}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle)$ need not be equal to $\langle \Psi, L\rangle$.

Proof. We have

(i)

$$\langle \Upsilon, K\rangle = \mathbb{T}(\langle \Upsilon, K\rangle, \langle \Upsilon, K\rangle)$$

$$\begin{aligned}&\subseteq \mathbb{T}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)\\ &\subseteq \mathbb{T}(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{A}}, Q\right\rangle\\ &= \langle \Upsilon, K\rangle\end{aligned}$$

where $|\Psi(\alpha)(w)| = |\dot{\mathscr{A}}(\alpha)(w)|$ for every $\alpha \in L, w \in W$. Hence

$$\mathbb{T}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) = \langle \Upsilon, K\rangle.$$

(ii) As $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$, it is already known that $|\Upsilon(\alpha)(w)| = |\Psi(\alpha)(w)|$ for $\alpha \in K$ and $w \in W$. Hence we can choose an MDFSS $\langle \Upsilon', L\rangle$ as follows,

$$\Upsilon'(\alpha) = \begin{cases} \Upsilon(\alpha) & \text{if } \alpha \in K \\ \\ \dot{0} & \text{if } \alpha \in L - K \end{cases}$$

where $|\Upsilon'(\alpha)(w)| = |\Psi(\alpha)(w)|$ for $\alpha \in L$ and $w \in W$.

Then it is clear that $\langle \Upsilon, K\rangle \subseteq \langle \Upsilon', L\rangle \subseteq \langle \Psi, L\rangle$. Hence we can write

$$\begin{aligned}\langle \Psi, L\rangle &= \mathbb{S}'\left(\left\langle \dot{\mathscr{N}}, L\right\rangle, \langle \Psi, L\rangle\right)\\ &\subseteq \mathbb{S}'\left(\left\langle \Upsilon', L\right\rangle, \langle \Psi, L\rangle\right)\\ &\subseteq \mathbb{S}'\left(\langle \Psi, L\rangle, \langle \Psi, L\rangle\right)\\ &= \langle \Psi, L\rangle\end{aligned}$$

Hence $\mathbb{S}'\left(\langle \Upsilon', L\rangle, \langle \Psi, L\rangle\right) = \langle \Psi, L\rangle$.

(iii) If we consider $\mathbb{S}_{\max}$ operator and $Q = \{\alpha_1, \alpha_2, \alpha_3\}$, $K = \{\alpha_1, \alpha_2\}$, $L = Q$, $W = \{w_1, w_2, \ldots, w_{10}\}$ then $\mathbb{S}_{\max}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)$ has $K \cap L = K$ as domain, while domain of $\langle \Psi, L\rangle$ is $L \neq K$. Hence $\mathbb{S}_{\max}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right) \neq \langle \Psi, L\rangle$ here. □

3.2. **De Morgan's Law.** Using multidimensional fuzzy soft t-norm, t-conorm and negation, we can define multidimensional fuzzy soft De Morgan's laws as follows:

Let $\langle \Upsilon, K\rangle$ and $\langle \Psi, L\rangle$ be two MDFSS, $\mathbb{T}$ be a MDFSTN , $\mathbb{S}$ be a MDFSTCN and $\mathcal{N}$ be a MDFSC. Then De Morgan's laws are

$$\mathcal{N}\left(\mathbb{T}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)\right) = \mathbb{S}\left(\mathcal{N}\left(\langle \Upsilon, K\rangle\right), \mathcal{N}\left(\langle \Psi, L\rangle\right)\right) \tag{3.1}$$

$$\mathcal{N}\left(\mathbb{S}\left(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle\right)\right) = \mathbb{T}\left(\mathcal{N}\left(\langle \Upsilon, K\rangle\right), \mathcal{N}\left(\langle \Psi, L\rangle\right)\right) \tag{3.2}$$

Example 3.23. Let $\mathbb{T}$ be the MDFSTN $\mathbb{T}_{\min}$ and $\mathbb{S}$ be the MDFSTCN $\mathbb{S}_{\max}$ and $\mathcal{N}_S$ be the standard multidimensional fuzzy soft negation. Then $\mathbb{T}, \mathbb{S}$ and $\mathcal{N}_{\mathcal{S}}$ satisfy De Morgan's Law.

Proof. Clearly the domains of Left Side and Right Side of equation (3.1) are $K \cap L$.

Let

$$\mathcal{N}_S\left(\mathbb{T}\left(\langle\Upsilon,K\rangle,\langle\Psi,L\rangle\right)\right)=\langle\Theta,K\cap L\rangle$$
$$\mathbb{S}\left(\mathcal{N}_S\left(\langle\Upsilon,K\rangle\right),\mathcal{N}_S\left(\langle\Psi,L\rangle\right)\right)=\langle\Omega,K\cap L\rangle$$

Now let $\alpha\in K\cap L$, $\Upsilon(\alpha)(w)=(s_1,s_2,\ldots,s_n)$, $\Psi(\alpha)(w)=(t_1,t_2,\ldots,t_m)$. Let $G_1\left((\Upsilon(\alpha)(w),\Psi(\alpha)(w))\right)=k$. Then $|\Theta(\alpha)(w)|=k$.

To prove $|\Theta(\alpha)(w)|=|\Omega(\alpha)(w)|$, we consider two cases.

Case 1: $|\Theta(\alpha)(w)|=k\implies$ without loss of generality, there exists $k\in\mathbb{N}$ such that $s_i=t_i, i=1,2,\ldots,k-1$ and $s_k<t_k$. Then $1-s_i=1-t_i$ for $i=1,2,\ldots,k-1$ and $1-s_k=1-t_k$. Hence $G_2\left(\mathcal{N}_S(\Upsilon(\alpha)(w)),\mathcal{N}_S(\Psi(\alpha)(w))\right)=k$.

Case 2: If there doesn't exist an $r\in\mathbb{N}$ such that $s_i=t_i, i=1,2,\ldots,r-1$ and $s_r=t_r$. Then $|\Theta(\alpha)(w)|=|\Omega(\alpha)(w)|=\min\{m,n\}$. Let

$$\begin{aligned}\Theta(\alpha)(w)&=(1-r_k,1-r_{k-1},\ldots,r_1)\\&=(1-\min\{s_k,t_k\},1-\min\{s_{k-1},t_{k-1}\},\ldots,1-\min\{s_1,t_1\})\\&=(\max\{1-s_k,1-t_k\},\max\{1-s_{k-1},1-t_{k-1}\},\ldots,\\&\quad\max\{1-s_1,1-t_1\})\\&=\Omega(\alpha)(w)\end{aligned}$$

Hence left side and right side of equation (3.1) are same. Similarly we can prove equation (3.2). □

Remark 3.24.
(i) Let $\mathbb{T}$ be the MDFSTN induced by the MIN operator and $\mathbb{S}$ be the MDFSTCN induced by the MAX operator and $\mathcal{N}_S$ be the standard multidimensional fuzzy soft negation. Then $\mathbb{T},\mathbb{S}$ and $\mathcal{N}_{\mathcal{S}}$ satisfy De Morgan's Law.
(ii) Let $\mathbb{T}$ be the MDFSTN induced by the Drastic min operator and $\mathbb{S}$ be the MDFSTCN induced by the Drastic max operator and $\mathcal{N}_S$ be the standard multidimensional fuzzy soft negation. Then $\mathbb{T},\mathbb{S}$ and $\mathcal{N}_{\mathcal{S}}$ satisfy De Morgan's Law.

Remark 3.25. If we replace $\mathbb{S}$ in De Morgan's law by the modified multidimensional fuzzy soft t- conorm $\mathbb{S}'$, then the equality need not satisfy.

Proof. For two MDFSS $\langle\Upsilon,K\rangle$ and $\langle\Psi,L\rangle$, consider $\mathbb{S}'\left(\langle\Upsilon,K\rangle,\langle\Psi,L\rangle\right)=\langle\Theta,M\rangle$ where $M=K\cup L$ and

$$\Theta(\alpha)=\begin{cases}\Upsilon(\alpha) & \text{if } \alpha\in K-L\\ \Psi(\alpha) & \text{if } \alpha\in L-K\\ \max(\Upsilon(\alpha),\Psi(\alpha)) & \text{if } \alpha\in K\cap L\end{cases}$$

Then the domain of left side of equation 3.1 is $K \cap L$, whereas domain of right side of equation 3.1 is $K \cup L$. □

Theorem 3.26. *Let $\mathbb{T}$ be an MDFSTN and $\mathcal{N}$ be a strict MDFSC, then $\mathbb{S}$ defined by $\mathbb{S}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Psi, L\rangle))$ is a multidimensional fuzzy soft t-conorm when restricted to n dimensional case.*

Proof. It is trivial to prove S_1. So we prove the other three conditions.
S_2 :

$$\begin{aligned}
&\mathbb{S}(\mathbb{S}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle), \langle \Theta, M\rangle)\\
&\quad = \mathbb{S}(\mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Psi, L\rangle)), \langle \Theta, M\rangle)\\
&\quad = \mathcal{N}(\mathbb{T}(\mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Psi, L\rangle))), \mathcal{N}\langle \Theta, M\rangle))\\
&\quad = \mathcal{N}(\mathbb{T}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Psi, L\rangle, \mathcal{N}\langle \Theta, M\rangle)))\\
&\quad = \mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathbb{T}(\mathcal{N}\langle \Psi, L\rangle, \mathcal{N}\langle \Theta, M\rangle))) \qquad (\text{by } T_2)\\
&\quad = \mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}(\mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Psi, L\rangle, \mathcal{N}\langle \Theta, M\rangle)))))\\
&\quad = \mathbb{S}(\langle \Upsilon, K\rangle, \mathbb{S}(\langle \Psi, L\rangle, \langle \Theta, M\rangle))
\end{aligned}$$

S_3 : By N_1, $\mathcal{N}\left\langle \dot{\mathscr{N}}, Q\right\rangle = \left\langle \dot{\mathscr{A}}, Q\right\rangle$ then

$$\begin{aligned}
\mathbb{T}\left(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\left\langle \dot{\mathscr{N}}, Q\right\rangle\right) &= \mathbb{T}\left(\mathcal{N}\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{A}}, Q\right\rangle\right)\\
&= \mathcal{N}\langle \Upsilon, K\rangle, \qquad (\text{by } T_3)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}\left(\mathbb{T}\left(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\left\langle \dot{\mathscr{N}}, Q\right\rangle\right)\right) &= \mathcal{N}(\mathcal{N}\langle \Upsilon, K\rangle)\\
&= \langle \Upsilon, K\rangle \qquad (\text{by } N_3)
\end{aligned}$$

Hence $\mathbb{S}\left(\langle \Upsilon, K\rangle, \left\langle \dot{\mathscr{N}}, Q\right\rangle\right) = \langle \Upsilon, K\rangle$

S_4 : Let $\langle \Upsilon, K\rangle \subseteq \langle \Psi, L\rangle$ then

$\mathcal{N}\langle \Psi, L\rangle \subseteq \mathcal{N}\langle \Upsilon, K\rangle$, (by N_2)

$\mathbb{T}(\mathcal{N}\langle \Psi, L\rangle, \mathcal{N}\langle \Theta, M\rangle) \subseteq \mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Theta, M\rangle)$, (by T_4)

$\mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Theta, M\rangle)) \subseteq \mathcal{N}(\mathbb{T}(\mathcal{N}\langle \Psi, L\rangle, \mathcal{N}\langle \Theta, M\rangle))$, (by N_2)

$\mathbb{S}(\langle \Upsilon, K\rangle, \langle \Theta, M\rangle) \subseteq \mathbb{S}(\langle \Psi, L\rangle, \langle \Theta, M\rangle)$

Hence $\mathbb{S}$ is a multidimensional fuzzy soft t- conorm when restricted to n dimensional case. □

Theorem 3.27. *Let $\mathbb{S}$ be an MDFSTCN and $\mathcal{N}$ be a strict MDFSC, then $\mathbb{T}$ defined by $\mathbb{T}(\langle \Upsilon, K\rangle, \langle \Psi, L\rangle) = \mathcal{N}(\mathbb{S}(\mathcal{N}\langle \Upsilon, K\rangle, \mathcal{N}\langle \Psi, L\rangle))$ is a multidimensional fuzzy soft t-norm when restricted to n dimensional case.*

Proof. Analogous to proof of theorem 3.26. □

4. Application of MDFSS in Decision Making

The introduction of Multidimensional Fuzzy Sets (MDFS) and Multidimensional Fuzzy Soft Sets (MDFSS) aimed to address challenges associated with confining analysis within fixed dimensional sets. Decision making encountered numerous complexities when varying evaluations across criteria or alternatives were present. For instance, scenarios arose where experts abstained from assessing certain criteria for specific alternatives. In this section, we propose methodologies designed to mitigate these challenges.

First of all, we propose an algorithm for decision-making under uncertainty using Multidimensional Fuzzy Soft Sets (MDFSS). This approach is grounded in the aggregation capability of the multidimensional fuzzy t-norm, which enables the effective integration of information from multiple sources. The algorithm addresses the challenges of decision making when dealing with incomplete and uncertain data. In scenarios such as selecting the best transportation infrastructure project for a city based on multiple criteria evaluated by several decision-makers (DMs), it becomes crucial to handle incomplete evaluations and uncertainties effectively.

Algorithm.

Step1: Represent the given data as an MDFSS $\langle \Upsilon, K\rangle$ over the universal set $W = \{w_1, w_2, \ldots, w_n\}$ and the parameter set

$$K \subseteq Q = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}.$$

Step 2: Calculate the consolidated analysis using an appropriate MDFSTN, say $\mathbb{T}$, as follows:

$$\mathbb{T}\left(\ldots \mathbb{T}\left(\mathbb{T}\left((\Upsilon(\alpha_1)(w_i), \Upsilon(\alpha_2)(w_i)\right), \Upsilon(\alpha_3)(w_i)\right) \ldots \Upsilon(\alpha_m)(w_i)\right),$$

$i = 1, 2, \ldots, n$.

Step 3: Compare the membership values of $w_i, i = 1, 2, \ldots, n$ using an appropriate total order on $\bigcup_{n\in\mathbb{N}}[0,1]^n$.

Step 4: Choose the object with higher value in ordering as the optimal decision.

The following example illustrates the basic concept of the Algorithm outlined above.

Imagine a city planning committee tasked with selecting the most suitable transportation infrastructure project for their city's future development. They have five alternatives (transportation projects) labeled T_1, T_2, T_3, T_4 and T_5. To make an informed decision, they consider seven

key attributes: p_1- Technological feasibility, p_2- Environmental impact, p_3- Socio-political acceptability, p_4 -Economic viability, p_5- Safety, p_6- Accessibility, p_7- Future scalability.

Let $\{E_1, E_2, E_3, E_4\}$ be the set of decision-makers (DMs) or experts including city officials, urban planners, environmentalists and transportation experts who are invited to evaluate the performance of the five transportation alternatives across these attributes. Each DM provides their evaluation values anonymously to avoid bias and influence from others' opinions.

Because of insufficient understanding of certain parameters, uncertainty, or limited expertise in the subject, it becomes challenging to obtain complete information from evaluators, often resulting in dealing with incomplete data.

Table 1, Table 2, Table 3 and Table 4 represents a compiled dataset based on each expert's (DM) perspective, wherein their assessment of each project is transformed into a numerical value ranging from 0 to 1. The symbol * represents an unknown value which occurred due to the incomplete data.

TABLE 1. Incomplete Evaluation of DM E_1

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
T_1	0.9	0.2	0.4	0.5	*	0.8	*
T_2	0.1	1.0	0.4	0.3	*	0.6	*
T_3	0.3	0.2	0.4	0.5	*	0.3	*
T_4	0.3	*	*	0.3	*	*	*
T_5	0.1	0.2	0.9	0.3	*	0.3	*

TABLE 2. Incomplete Evaluation of DM E_2

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
T_1	0.9	*	0.1	0.3	0.2	0.5	0.3
T_2	*	*	*	*	0.9	0.2	*
T_3	0.3	*	0.4	0.2	0.8	0.5	0.6
T_4	0.3	*	0.3	0.7	*	0.6	0.2
T_5	0.1	*	0.8	0.3	0.4	0.2	0.3

Now we can combine the informations provided by a particular expert and represent it using a multidimensional fuzzy set(MDFS). By representing each expert's data as a seperate MDFS and then combining all of them, we arrive at a multidimensional fuzzy soft set $\langle \Upsilon, K \rangle$ as shown in Table 5. Here K is the set of experts and the first column of Table 5 is the MDFS $\Upsilon(E_1)$.

TABLE 3. Incomplete Evaluation of DM E_3

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
T_1	*	0.1	0.3	0.4	0.3	0.6	0.7
T_2	0.1	0.9	0.5	0.3	0.4	0.3	0.3
T_3	*	*	0.3	*	0.4	*	*
T_4	*	0.1	0.2	0.9	0.3	0.4	0.6
T_5	*	*	0.8	*	0.3	0.4	0.2

TABLE 4. Incomplete Evaluation of DM E_4

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
T_1	0.9	*	*	*	0.2	*	0.1
T_2	0.1	1.0	*	0.2	0.5	*	0.3
T_3	0.3	*	0.4	0.3	0.9	*	0.2
T_4	0.2	0.2	0.4	0.8	0.6	*	0.3
T_5	0.1	0.1	0.9	0.3	0.4	0.5	*

That is, $K = \{E_1, E_2, E_3, E_4\}$ and

$$\Upsilon(E_1) = \big\{(T_1, 0.2, 0.4, 0.5, 0.8, 0.9), (T_2, 0.1, 0.3, 0.4, 1.0), (T_3, 0.2, 0.3, 0.3, 0.4, 0.5), (T_4, 0.3, 0.3), (T_5, 0.1, 0.2, 0.3, 0.3, 0.9)\big\}$$

Similarly, other columns of Table 5 represents $\Upsilon(E_i), i = 2, 3, 4$.

To determine which transportation project is more suitable, we apply the MIN operator. The MIN operator is chosen as the appropriate t-norm due to its simplicity and its ability to capture the most precise assessment among all expert evaluations by considering the lowest value provided by the experts. This approach ensures a conservative and reliable decision-making process, reflecting the most cautious perspective in the analysis. That is, for each $i = 1, 2, 3, 4, 5$, we apply

$$MIN\left(MIN\left\{MIN\left(\Upsilon(E_1)(T_i), \Upsilon(E_2)(T_i)\right), \left(\Upsilon(E_3)(T_i)\right\}, \Upsilon(E_4)(T_i)\right)$$

The resulting values are represented in Table 6.

Now we compare the membership values obtained after MIN operation for each T_i by lexicographic ordering. Accordingly, the resulting order is $T_5 < T_2 < T_1 < T_4 < T_3$. Since membership value indicates the degree to which an element is most probable to be included in a set and the obtained lexicographic ordering is based on increasing order of membership values, the ranking order is shown in Table 7.

Hence T_3 is identified as the most suitable transportation infrastructure project for their city's future development.

TABLE 5. MDFSS $\langle \Upsilon, K \rangle$

	E_1	E_2
T_1	(0.2,0.4,0.5,0.8,0.9)	(0.1,0.2,0.3,0.3,0.5,0.9)
T_2	(0.1,0.3,0.4,0.6,1.0)	(0.2,0.9)
T_3	(0.2,0.3,0.3,0.4,0.5)	(0.2,0.3,0.4,0.5,0.6,0.8)
T_4	(0.3,0.3)	(0.2,0.3,0.3,0.6,0.7)
T_5	(0.1,0.2,0.3,0.3,0.9)	(0.1,0.2,0.3,0.3,0.4,0.8)
	E_3	E_4
T_1	(0.1,0.3,0.3,0.4,0.6,0.7)	(0.1,0.2,0.9)
T_2	(0.1,0.3,0.3,0.3,0.4,0.5,0.9)	(0.1,0.2,0.3,0.5,1.0)
T_3	(0.3,0.4)	(0.2,0.3,0.3,0.4,0.9)
T_4	(0.1,0.2,0.3,0.4,0.6,0.9)	(0.2,0.2,0.3,0.4,0.6,0.8)
T_5	(0.2,0.3,0.4,0.8)	(0.1,0.1,0.3,0.4,0.5,0.9)

TABLE 6. MDFS after applying MIN operator

projects	membership value
T_1	(0.1,0.1,0.1,0.2,0.2,0.2)
T_2	(0.1,0.1,0.1,0.2,0.2)
T_3	(0.2,0.2,0.2,0.3,0.3)
T_4	(0.1,0.2,0.2,0.2,0.2,0.3)
T_5	(0.1,0.1,0.1,0.1,0.2,0.2)

TABLE 7. Rank

Projects	Rank
T_1	3
T_2	4
T_3	1
T_4	2
T_5	5

4.1. **Comparison with the Existing Interval-Valued Intuitionistic Fuzzy Set Approach.** In the seminal work [19], Ye delineated a decision-making scenario adapted from [13], wherein an investment entity seeks to allocate funds optimally among four distinct alternatives. These alternatives are identified as follows: B_1, pertaining to a car manufacturing enterprise; B_2, associated with a food production company; B_3, involving a computer technology firm; and B_4, concerning an armaments manufacturer. The decision-making process hinges upon evaluating these alternatives based on three critical attributes (criteria):

P_1, focusing on risk assessment; P_2, emphasizing growth potential; and P_3, scrutinizing environmental ramifications.

The respective attribute weights are encapsulated within the vector $\mathbf{w} = (0.35, 0.25, 0.40)^T$, delineating their relative significance. Ye adopt an interval-valued intuitionistic fuzzy set framework to model the decision data, accommodating both membership and non-membership values for each element. In this framework, the membership of an element is described by two closed intervals within [0,1], where the first interval corresponds to the degree of membership and the second represents the degree of non-membership [20]. However, in the present context, our focus is restricted to membership values, which requires their extraction and representation. Specifically, we extract only the membership interval $\left[\bar{a}, \bar{b}\right]$ from an interval-valued intuitionistic fuzzy number of the form $\left(\left[\bar{a}, \bar{b}\right], \left[\bar{c}, \bar{d}\right]\right)$. This enables us to define an isomorphism f from the set of these extracted elements to the collection of all two-dimensional fuzzy elements, given by $f\left(\left[\bar{a}, \bar{b}\right]\right) = \left(\bar{a}, \bar{b}\right)$.

The obtained membership values are compiled into a tabular format, representing a two-dimensional fuzzy soft set as shown in Table 8, which in essence constitutes a specific instance of the MDFSS. This structured representation facilitates a nuanced examination of the decision-making process, facilitating clearer insights into the optimal investment strategy among the available alternatives.

TABLE 8. Two-dimensional fuzzy soft set

	E_1	E_2	E_3
B_1	(0.4,0.5)	(0.4,0.6)	(0.1,0.3)
B_2	(0.6,0.7)	(0.6,0.7)	(0.4,0.7)
B_3	(0.3,0.6)	(0.5,0.6)	(0.5,0.6)
B_4	(0.7, 0.8)	(0.6,0.7)	(0.3,0.4)

Upon evaluation by team E_i based on criteria P_i, each membership value is multiplied by the corresponding weight assigned to P_i. Subsequently, these weighted values are organized in ascending order, culminating in the structured presentation found in Table 9.

TABLE 9. Weighted two-dimensional fuzzy soft set

	E_1	E_2	E_3
B_1	(0.14,0.175)	(0.1,0.15)	(0.04,0.12)
B_2	(0.21,0.245)	(0.15,0.175)	(0.16,0.28)
B_3	(0.105,0.21)	(0.125,0.15)	(0.2,0.24)
B_4	(0.245, 0.28)	(0.15,0.175)	(0.12,0.16)

Now we apply $MIN(MIN(E_1, E_2), E_3)$ to get the combined membership value of B_i and we get the result shown in Table 10.

TABLE 10. Membership value after applying MIN operation

companies	membership value
B_1	(0.040, 0.100)
B_2	(0.150, 0.160)
B_3	(0.105, 0.125)
B_4	(0.120,0.150)

Upon employing lexicographic ordering to compare the membership values, the sequence $B_1 < B_3 < B_4 < B_2$ emerges, mirroring the findings reported in [13]. This consistency underscores the efficacy of the proposed methodology in addressing decision-making challenges.

5. Conclusion

This research defines multidimensional fuzzy soft sets (MDFSS) and their associated operators such as complement, t-norm and t-conorm. MDFSS extends the concepts of multidimensional fuzzy sets and soft sets. While soft set theory offers a versatile framework for managing datasets categorized by attributes or parameters, thereby facilitating simplified analysis, combining it with multidimensional fuzzy set theory enhances our ability to handle data relationships affected by uncertainty or vagueness. The study includes the proof of several theorems based on newly proposed definitions, including De Morgan's law. Furthermore, a decision-making method for MDFSS is presented and exemplified through a practical application, showcasing its efficacy in addressing problems characterized by uncertainty. This method holds potential for application across various fields and warrants further exploration into areas such as distance measure, similarity measure and entropy within the context of MDFSS. Future research directions also encompass the exploration of algebraic and topological structures associated with MDFSS theory.

Acknowledgment. The first author gratefully acknowledges the University Grants Commission (UGC) for providing financial support through a PhD fellowship.

References

1. A. Lima, E.S. Palmeira, B. Bedregal and H. Bustince, *Multidimensional fuzzy sets*, IEEE Transactions on Fuzzy Systems, 29 (2021), pp. 2195–2208.

2. A. Ramer and C.C. Wang, *Fuzzy multisets*, in: Proc. Asian Fuzzy Systems Symposium, Soft Computing in Intelligent Systems and Information Processing, Kenting, Taiwan, 1996, 429-434.
3. A.R. Roy and P.K. Maji, *A fuzzy soft set theoretic approach to decision making problems*, J. Comput. Appl. Math., 203 (2007), pp. 412-418.
4. B. Bedregal and I. Mezzomo, *Ordinal sums and multiplicative generators of the De Morgan triples*, J. Intell. Fuzzy Syst., 34 (2018), pp. 2159-2170.
5. B. Bedregal, G. Beliakov, H. Bustince, T. Calvo, J. Fernández and R. Mesiar, *A characterization theorem for t-representable n-dimensional triangular norms*, in: Proc. Eurofuse 2011: Workshop on Fuzzy Methods for Knowledge-Based Systems ,Régua, Portugal, 2011, 103-112.
6. B. Bedregal, G. Beliakov, H. Bustince, T. Calvo, R. Mesiar and D. Paternain, *A class of fuzzy multisets with a fixed number of memberships*, Inf. Sci., 189 (2012), pp. 1-17.
7. B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North-Holland, New York, 1983.
8. C. Alsina, E. Trillas and L. Valverde, *On some logical connectives for fuzzy sets theory*, J. Math. Anal. Appl., 93 (1983), pp. 15-26.
9. C. L. Hwang and K. Yoon, *Multiple Attribute Decision Making: Methods and Applications*, Lect. Notes Econ. Math. Syst., 186, Springer-Verlag, New York, 1981.
10. D. Chen, E.C.C. Tsang, D.S. Yeung and X. Wang, *The parameterization reduction of soft sets and its applications*, Comput. Math. Appl., 49 (2005), pp. 5-6.
11. D. Molodtsov, *Soft set theory—first results*, Comput. Math. Appl., 37 (1999), pp. 19-31.
12. E. Herrera-Viedma, F. Chiclana, F. Herrera and S. Alonso, *Group decision-making model with incomplete fuzzy preference relations based on additive consistency*, IEEE Trans. Syst. Man Cybern., Part B (Cybernetics), 37 (2007), pp. 176-189.
13. F. Herrera and E. Herrera-Viedma, *Linguistic decision analysis: steps for solving decision problems under linguistic information*, Fuzzy Sets Syst., 115 (2000), pp. 67-82.
14. F. Meng and X. Chen, *An approach to incomplete multiplicative preference relations and its application in group decision making*, Inf. Sci., 309 (2015), pp.119-137
15. I. Grattan-Guinness, *Fuzzy membership mapped onto intervals and many-valued quantities*, Math. Log. Q., 22 (1976), pp. 149-160.

16. I. Mezzomo, B.C. Bedregal and R.H.S. Reiser, *Natural n-dimensional fuzzy negations for n-dimensional t-norms and t-conorms*, in: Proc. 2017 IEEE international conference on Fuzzy Systems, Naples, Italy, 2017, 1-6.
17. J. Dombi, *A general class of fuzzy operators, the Demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators*, Fuzzy Sets Syst., 8 (1982), pp. 149-163.
18. J. Josen, B. Mathew, S.J. John and J. Vallikavungal, *Integrating Rough Sets and Multidimensional Fuzzy Sets for Approximation Techniques: A Novel Approach*, IEEE Access, (2024).
19. J. Ye, *Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment*, Expert systems with Applications, 36(2009), pp. 6899-6902.
20. K.T. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets Syst., 31 (1989), pp. 343-349.
21. K.T. Atanassov, *Intuitionistic fuzzy sets: theory and applications*, Springer, Heidelberg, 1999.
22. K.U. Jahn, *Intervall-wertige mengen*, Mathematische Nachrichten, 68 (1975), pp. 115-132.
23. K.V. Babitha and S.J. John , *Hesitant fuzzy soft sets*, Journal of New Results in Science, 2 (2013), pp. 98-107.
24. L.A. Zadeh, *Fuzzy sets*, Inf. Control, 8 (1965), pp. 338–353.
25. L.A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning—I*, Inf. Sci., 8 (1975), pp. 199–249.
26. M.I. Ali and M. Shabir, *Logic connectives for soft sets and fuzzy soft sets*, IEEE Trans. Fuzzy Syst., 22 (2013), pp. 1431-1442.
27. M. Weber, *Decision making with incomplete information*, Eur. J. Oper. Res., 28 (1987), pp. 44-57.
28. N. Cagman, S. Enginoglu and F. Citak, *Fuzzy soft set theory and its applications*, Iran. J. Fuzzy Syst., 8 (2011), pp. 137-147.
29. P.K. Maji and R Biswas, *Fuzzy soft sets*, J. Fuzzy Math., 9 (2001), pp. 589-602.
30. P.K. Maji, R. Biswas and A.R. Roy, *Intuitionistic Fuzzy Soft sets*, J. Fuzzy Math., 9 (2001), pp. 677-692.
31. P. Majumdar and S.K. Samanta, *Generalised fuzzy soft sets*, Comput. Math. Appl., 59 (2010), pp. 1425–1432.
32. R.R. Yager, *On a general class of fuzzy connectives*, Fuzzy Sets Syst., 4 (1980), pp. 235-242.
33. R.R. Yager, *On the Theory of Bags*, International Journal of General System, 13 (1986), pp. 23-37.

34. R. Sambuc, *Fonctions and floues: application a l'aide au diagnostic en pathologie thyroidienne*, Ph. D. Thesis, University of Marseille, France, 1975.
35. S.M. AP, S.J. John and A. TM, *Exploring n-Dimensional Fuzzy Soft Sets: A Framework for Multicriteria Decision-Making Problems*, Adv. Fuzzy Syst., 2025 (2025), pp. 1209599
36. S. Miyamoto, *Basic operations of fuzzy multisets*, Journal of Japan Society of Fuzzy Theory and Systems, 8 (1996) pp. 639-645.
37. S. Opricovic, *Multicriteria Optimization of Civil Engineering Systems*, PhD Thesis, Faculty of Civil Engineering, Belgrade, 1998, 302 p.
38. S. Weber, *A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms* , Fuzzy Sets Syst., 11 (1983), pp. 115-134.
39. U. Höhle, *Probabilistic uniformization of fuzzy topologies*, Fuzzy Sets and Systems, 1 (1978), pp. 311-332.
40. V. Torra, *Hesitant fuzzy sets*, Int. J. Intell. Syst., 25 (2010), pp. 529-539.
41. Y. Shang, X. Yuan and E.S. Lee, *The n-dimensional fuzzy sets and zadeh fuzzy sets based on the finite valued fuzzy sets*, Comput. Math. Appl., 60 (2010), pp. 442-463.
42. Y. Yang, C. Liang, S. Ji and T. Liu, *Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making*, J. Intell. Fuzzy Syst., 29, (2015), pp. 1711-1722.
43. Y. Yang, X. Tan and C. Meng, *The multi-fuzzy soft set and its application in decision making*, Appl. Math. Modelling, 37 (2013), pp. 4915-4923.
44. Z. Xu and X. Zhang, *Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information*, Knowledge-Based Systems, 52 (2013), pp. 53–64.
45. Z. Xu, *A procedure for decision making based on incomplete fuzzy preference relation*, Fuzzy Optim. Decis. Mak., 4 (2005), pp. 175–189.

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