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Intuitionistic-Neutrosophic Fuzzy Normed Linear Spaces: Theory and Applications

Farnaz Yaqub Azari[1] and Ildar Sadeqi[2*]

Abstract. We introduce intuitionistic-neutrosophic fuzzy normed linear spaces ($INFNLS$), which unifies and extends both intuitionistic and neutrosophic fuzzy normed linear spaces. We introduce its topological structure and prove that every $INFNL$ space is a Hausdorff topological vector space and first-countable. This structure provides a foundational framework for various applications, including the modeling of systems with multi-perspective uncertainty.

1. Introduction

Classical normed linear spaces provide the foundation of functional analysis but cannot adequately handle the imprecise, incomplete, or contradictory information arising in decision theory, artificial intelligence and complex engineering systems. To address this, Katsaras [10] introduced fuzzy normed linear spaces, where the norm is given by a membership function $\mu : X \times \mathbb{R}^+ \to [0, 1]$. This was extended via Atanassov's intuitionistic fuzzy sets [3] to intuitionistic fuzzy normed linear spaces, which employ both a membership μ and a non-membership ν function to capture uncertainty. Formally introduced in 2004 [18], intuitionistic fuzzy normed spaces were subsequently studied for completeness [11, 19], operator boundedness [11] and 2-normed variants [15, 17]. Recent work has established further topological properties [2] and decomposition theorems linking type-2 intuitionistic fuzzy norms to Felbin-type fuzzy norm

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pairs, together with investigations of boundedness, completeness, compactness and two forms of continuity [5].

In a parallel direction, applying Smarandache's neutrosophic logic [21] to probability theory led to neutrosophic normed linear spaces in 2020 [12], where each vector is assigned a triplet (T, I, F) of truth, indeterminacy and falsity, allowing explicit quantification of contradictory or neutral evidence. Subsequent work has addressed convergence, operator theory, topological characteristics, neutrosophic soft normed spaces [4, 16], inner-product spaces [1] and decision-making applications [9].Topological structures on neutrosophic sets have also been significantly developed, including pairwise neutrosophic b-continuous functions, neutrosophic simply b-open sets and topologies on rough pentapartitioned neutrosophic sets [6, 7, 22]. Neutrosophic methods have also been successfully applied in decision-making, including aggregation-based outranking and multi-criteria group decision-making [8, 20].

Despite these advances, a critical limitation remains: existing frameworks employ either the intuitionistic dual structure (membership/non-membership) or the neutrosophic triplet (T, I, F), but not both simultaneously. The former models hesitation between acceptance and rejection, while the latter separately quantifies truth, indeterminacy and falsity. Yet many practical scenarios — including multi-agent systems, medical diagnosis and engineering design — exhibit layered uncertainty: hesitation exists between membership and non-membership, while each of these components itself contains degrees of truth, indeterminacy and falsity. This nested uncertainty structure remains unaddressed, creating a significant gap between theory and application.

To fill this gap, we introduce Intuitionistic–Neutrosophic Fuzzy Normed Linear Spaces ($INFNLS$), a structure that unifies and extends both intuitionistic and neutrosophic fuzzy normed linear spaces. Our framework employs two neutrosophic triplets: (T_μ, I_μ, F_μ) for the membership component and (T_ν, I_ν, F_ν) for the non-membership component. Using two triplets provides a comprehensive representation of uncertainty, where each traditional intuitionistic component is itself subject to neutrosophic analysis. This allows simultaneous treatment of two uncertainty types and enables the transfer of analytical techniques between the two settings. This is not just a generalization; it helps analyze real-world problems with uncertainty.

In this paper, we first establish the basic mathematical notions related to $INFNLS$ and prove that every $INFNLS$ is a Hausdorff topological vector space. Moreover, we show that every classical normed space can be embedded into an $INFNLS$. These results confirm that the new

definition of $INFNLS$ extends the classical theory by incorporating uncertainty modeling capabilities.

The remainder of this paper is organized as follows. Section 2 reviews essential definitions from intuitionistic and neutrosophic fuzzy normed linear spaces. In Section 3, we introduce the axiomatic definition of $INFNLS$, provide a concrete exampleand establish basic properties. Subsection 3.2 studies convergence and completeness and finally in Subsection 3.3 we investigate the topological structure induced by an $INFNLS$.

2. Preliminaries

In this section, we recall the essential definitions and notations used throughout the paper.

Definition 2.1 ([13])**.** A mapping $* : [0,1]^2 \to [0,1]$ is called a continuous triangular norm (t-norm) if for all $a, b, c, d \in [0,1]$:

(i) $a * b = b * a$ (commutativity),
(ii) $(a * b) * c = a * (b * c)$ (associativity),
(iii) $a * 1 = a$ (boundary condition),
(iv) If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$ (monotonicity),
(v) $*$ is continuous in both variables simultaneously.

Similarly, a mapping $\diamond : [0,1]^2 \to [0,1]$ is called a continuous triangular conorm (t-conorm) if it satisfies the dual conditions with $a \diamond 0 = a$ as the boundary condition.

Example 2.2. The following are important continuous t-norms and their dual$t$-conorms:

. Standard: $a *_M b = \min(a, b)$, $\quad a \diamond_M b = \max(a, b)$
. Algebraic: $a *_P b = ab$, $\quad a \diamond_P b = a + b - ab$
. Łukasiewicz: $a *_L b = \max(0, a + b - 1)$, $\quad a \diamond_L b = \min(a + b, 1)$

These satisfy the ordering: $*_L \leq *_P \leq *_M$ and $\diamond_M \leq \diamond_P \leq \diamond_L$.

Definition 2.3 ([19])**.** Let X be a linear space over $\mathbb{R}$ (or $\mathbb{C}$) and let $*$, $\diamond$ be a continuous t-norm and, t-conorm, respectively. A pair (μ, ν) of mappings $\mu, \nu : X \times \mathbb{R}^+ \to [0,1]$ is called an intuitionistic fuzzy norm on X if for all $x, y \in X$, $t, s > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$:

(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$ and $\nu(x,t) < 1$ for $x \neq 0$,
(iii) $\mu(x,t) = 1$ and $\nu(x,t) = 0$ if and only if $x = 0$,
(iv) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ and $\nu(\alpha x, t) = \nu(x, t/|\alpha|)$,
(v) $\mu(x + y, t + s) \geq \mu(x,t) * \mu(y,s)$ and $\nu(x + y, t + s) \leq \nu(x,t) \diamond \nu(y,s)$,
(vi) $\lim_{t\to\infty} \mu(x,t) = 1$, $\lim_{t\to\infty} \nu(x,t) = 0$, $\lim_{t\to 0} \mu(x,t) = 0$, $\lim_{t\to 0} \nu(x,t) = 1$,

(vii) $\mu(x,\cdot)$ and $\nu(x,\cdot)$ are continuous on $\mathbb{R}^+$.

The quintuple $(X,\mu,\nu,*,\diamond)$ is called an intuitionistic fuzzy normed linear space $(IFNLS)$.

Definition 2.4 ([12])**.** Let X be a linear space and $*$, $\diamond$ are continuous t-norm and t-conorm, respectively. A mapping $N : X \times \mathbb{R}^+ \to [0,1]^3$, written as $N(x,t) = (T(x,t), I(x,t), F(x,t))$, is called a neutrosophic fuzzy norm on X if for all $x, y \in X$, $t, s > 0$ and $\alpha \neq 0$:

(i) $T(x,t) = 1$, $I(x,t) = 0$, $F(x,t) = 0$ if and only if $x = 0$;
(ii) $T(x,t) > 0$, $I(x,t) < 1$, $F(x,t) < 1$ for $x \neq 0$;
(iii) $T(\alpha x, t) = T(x, t/|\alpha|)$, $I(\alpha x, t) = I(x, t/|\alpha|)$, $F(\alpha x, t) = F(x, t/|\alpha|)$;
(iv) $T(x+y, t+s) \geq T(x,t) * T(y,s)$, $I(x+y, t+s) \leq I(x,t) \diamond I(y,s)$, $F(x+y, t+s) \leq F(x,t) \diamond F(y,s)$;
(v) $\lim_{t\to\infty} T(x,t) = 1$, $\lim_{t\to\infty} I(x,t) = 0$, $\lim_{t\to\infty} F(x,t) = 0$;
(vi) $\lim_{t\to 0} T(x,t) = 0$, $\lim_{t\to 0} I(x,t) = 1$, $\lim_{t\to 0} F(x,t) = 1$;
(vii) $T(x,\cdot)$, $I(x,\cdot)$ and $F(x,\cdot)$ are continuous on $\mathbb{R}^+$.

The quadruple $(X, N, *, \diamond)$ is called a neutrosophic fuzzy normed linear space $(NFNLS)$.

Definition 2.5 ([14])**.** Let X be a linear space over $\mathbb{R}$. A mapping $I : X \times (0,\infty) \to [0,1]$ is called a degree of indeterminacy (uncertainty measure) if it satisfies the following axioms:

(i) $I(0,t) = 1$ for every $t > 0$;
(ii) For every $x \in X$ and $t > 0$, we have

$$I(x,t) + \bar{I}(x,t) = 1,$$

where $\bar{I}(x,t) := 1 - I(x,t)$ denotes the degree of determinacy (the complement of indeterminacy). In neutrosophic logic, this duality relation is often expressed as $I(x,t) = 1 - T(x,t) - F(x,t)$ when truth and falsity are present.

(iii) For any countable sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X and any $t > 0$,

$$I\left(\bigcup_{n=1}^{\infty} A_n, t\right) \leq \sum_{n=1}^{\infty} I(A_n, t).$$

The value $I(x,t)$ quantifies the ambiguity, uncertainty, neutrality, or conflicting information associated with x at parameter t. A value $I(x,t)$-$= 0$ represents complete determinacy (no ambiguity), while $I(x,t) =$

1 represents total indeterminacy. Unlike fuzzy sets which only capture membership degrees, indeterminacy represents situations where evidence is neither supportive nor contradictory, capturing the unknown or neutral aspect of information.

3. Main Results

3.1. **Intuitionistic–Neutrosophic Fuzzy Normed Linear Spaces.**

Definition 3.1. Let X be a linear space over $\mathbb{R}$ and let $*, \diamond$ be a continuous t-norm and t-conorm, respectively. A mapping $N : X \times \mathbb{R}^+ \to [0,1]^6$, written as

$$N(x,t) = \big(T_\mu(x,t), I_\mu(x,t), F_\mu(x,t),\ T_\nu(x,t), I_\nu(x,t), F_\nu(x,t)\big),$$

is called an intuitionistic–neutrosophic fuzzy norm if it satisfies:

(IN1) (a) $T_\mu(0,t) = 1$, $I_\mu(0,t) = F_\mu(0,t) = 0$ and $T_\nu(0,t) = 0$ for all $t > 0$;

(b) If $T_\mu(x,t) = 1$, $I_\mu(x,t) = F_\mu(x,t) = 0$ and $T_\nu(x,t) = 0$ for some $t > 0$, then $x = 0$.

(IN2) For all $x \in X$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $t > 0$, $N(\alpha x, t) = N(x, t/|\alpha|)$, in other words:

$$\begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} (\alpha x, t) = \begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} (x, t/|\alpha|).$$

(IN3) For all $x, y \in X$ and $t, s > 0$:

$$\begin{aligned} T_\mu(x+y, t+s) &\geq T_\mu(x,t) * T_\mu(y,s), \\ I_\mu(x+y, t+s) &\leq I_\mu(x,t) \diamond I_\mu(y,s), \\ F_\mu(x+y, t+s) &\leq F_\mu(x,t) \diamond F_\mu(y,s), \\ T_\nu(x+y, t+s) &\leq T_\nu(x,t) \diamond T_\nu(y,s), \\ I_\nu(x+y, t+s) &\geq I_\nu(x,t) * I_\nu(y,s), \\ F_\nu(x+y, t+s) &\geq F_\nu(x,t) * F_\nu(y,s). \end{aligned}$$

(IN4) For all $x \in X$:

$$\lim_{t\to\infty} \begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} (x,t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(IN5) All components are continuous on $(0, \infty)$.

The quadruple $(X, N, *, \diamond)$ is called an intuitionistic–neutrosophic fuzzy normed linear space ($INFNLS$).

Remark 3.2. In the rest of this paper, unless otherwise stated, we assume the standard t-norm and t-conorm, i.e., $* = \min$ and $\diamond = \max$. Consequently, the inequalities in (IN3) simplify to:

$$\begin{aligned}
T_\mu(x+y, t+s) &\geq \min\{T_\mu(x,t), T_\mu(y,s)\}, \\
I_\mu(x+y, t+s) &\leq \max\{I_\mu(x,t), I_\mu(y,s)\}, \\
F_\mu(x+y, t+s) &\leq \max\{F_\mu(x,t), F_\mu(y,s)\}, \\
T_\nu(x+y, t+s) &\leq \max\{T_\nu(x,t), T_\nu(y,s)\}, \\
I_\nu(x+y, t+s) &\geq \min\{I_\nu(x,t), I_\nu(y,s)\}, \\
F_\nu(x+y, t+s) &\geq \min\{F_\nu(x,t), F_\nu(y,s)\}.
\end{aligned}$$

This choice is made for simplicity and aligns with common practice in fuzzy functional analysis, while preserving the generality of the definition.

Remark 3.3. It is easy to see that any intuitionistic-neutrosophic fuzzy normed linear space reduces to an intuitionistic fuzzy normed space ($IFNLS$) if $I_\mu = I_\nu = F_\mu = F_\nu \equiv 0$ and $T_\mu(x,t) + T_\nu(x,t) \leq 1$ for all x, t. Also, it reduces to a neutrosophic fuzzy normed space ($NFNLS$) if $(T_\nu, I_\nu, F_\nu) \equiv (0, 1, 1)$. These reductions are used in Theorem 3.5 and in Theorem 3.27. Note that these are special cases, not additional axioms like those listed in Remark 3.4.

Remark 3.4. The following conditions are not part of the basic definition but may be assumed when needed for specific theorems.

(IN6) (Strong zero characterization)

$$T_\mu(x,t) = 1 \quad \text{if and only if} \quad x = 0.$$

(IN7) (Consistency implications)

$$\begin{aligned}
T_\mu(x,t) = 1 \quad &\Rightarrow \quad I_\mu(x,t) = F_\mu(x,t) = 0, \\
T_\nu(x,t) = 0 \quad &\Rightarrow \quad I_\nu(x,t) = F_\nu(x,t) = 1.
\end{aligned}$$

(IN8) (Intuitionistic consistency)

$$T_\mu(x,t) + T_\nu(x,t) \leq 1, \quad I_\mu(x,t) + I_\nu(x,t) \geq 1,$$

$$F_\mu(x,t) + F_\nu(x,t) \geq 1 \quad T_\mu(x,t) + F_\mu(x,t) \leq 1.$$

(IN9) (Normality) For every $x \in X$, there exists $t_x > 0$ such that $T_\mu(x,t) = 1$, $I_\mu(x,t) = F_\mu(x,t) = 0$, $T_\nu(x,t) = 0$ and $I_\nu(x,t) = F_\nu(x,t) = 1$ for all $t \geq t_x$.

Theorem 3.5 exhibits canonical reductions from the intuitionistic–neutrosophic setting to the classical intuitionistic and neutrosophic settings.

Theorem 3.5. *Let (X, N) be an intuitionistic-neutrosophic fuzzy normed linear space satisfying (IN8). We have*

a) *Define $\mu(x,t) = T_\mu(x,t)$ and $\nu(x,t) = F_\mu(x,t)$, then $(X, \mu, \nu, \min, \max)$ is an intuitionistic fuzzy normed linear space in the sense of Definition 2.3.*

b) *Define*

$$\mathcal{T}(x,t) = T_\mu(x,t), \qquad \mathcal{I}(x,t) = I_\mu(x,t), \qquad \mathcal{F}(x,t) = F_\mu(x,t),$$

then $(X, (\mathcal{T}, \mathcal{I}, \mathcal{F}), \min, \max)$ is a neutrosophic fuzzy normed linear space in the sense of Definition 2.4.

Proof. We verify that the conditions in (a) and (b) satisfy all axioms of $IFNLS$ and $NFNLS$, respectively.

a) Let $\mu(x,t) = T_\mu(x,t)$ and $\nu(x,t) = F_\mu(x,t)$. We verify the conditions of $IFNLS$ as follows:

(i) $\mu(x,t)+\nu(x,t) = T_\mu(x,t)+F_\mu(x,t) \leq 1$ does hold by (IN8).

(ii) For $x \neq 0$, the strict inequalities $\mu(x,t) > 0$ and $\nu(x,t) < 1$ follow from (IN4) and (IN5). Indeed, if $\mu(x,t) = 0$ for some t, then by (IN5) it would be identically zero, contradicting $\lim\limits_{t\to\infty} \mu(x,t) = 1$ from (IN4). A similar argument gives $\nu(x,t) < 1$.

(iii) If $x = 0$, then (IN1-a) gives $T_\mu(0,t) = 1$, $I_\mu(0,t) = F_\mu(0,t)$-$= 0$ and $T_\nu(0,t) = 0$ for every $t > 0$; consequently $\mu(0,t) = 1$ and $\nu(0,t) = 0$ for all $t > 0$. Conversely, assume $\mu(x,t) = 1$ and $\nu(x,t) = 0$ for all $t > 0$. Then $T_\mu(x,t) = 1$ and $F_\mu(x,t) = 0$. From (IN8) we have $T_\mu(x,t) + T_\nu(x,t) \leq 1$, so $T_\nu(x,t) \leq 1 - T_\mu(x,t) = 0$, so $T_\nu(x,t) = 0$. Hence, $T_\mu(x,t) = 1$, $I_\mu(x,t) = F_\mu(x,t) = 0$ and $T_\nu(x,t) = 0$ for all $t > 0$ and (IN1-b) implies $x = 0$.

(iv) Homogeneity follows immediately from (IN2).

(v) The triangle inequalities are inherited from (IN3):

$$\begin{aligned}\mu(x+y,t+s) &= T_\mu(x+y,t+s)\\ &\geq \min\{T_\mu(x,t), T_\mu(y,s)\}\\ &= \min\{\mu(x,t), \mu(y,s)\},\end{aligned}$$

and

$$\begin{aligned}\nu(x+y,t+s) &= F_\mu(x+y,t+s)\\ &\leq \max\{F_\mu(x,t), F_\mu(y,s)\}\end{aligned}$$

$$= \max\{\nu(x,t), \nu(y,s)\}.$$

(vi) $\lim_{t\to\infty} \mu(x,t) = 1$, $\lim_{t\to\infty} \nu(x,t) = 0$, $\lim_{t\to 0} \mu(x,t) = 0$ and - $\lim_{t\to 0} \nu(x,t) = 1$ follow directly from (IN4).

(vii) Continuity of $\mu(x,\cdot)$ and $\nu(x,\cdot)$ is a simple result of (IN5). Thus, $(X, \mu, \nu, \min, \max)$ satisfies all requirements of Definition 2.3.

b) Let $\mathcal{T}(x,t) = T_\mu(x,t)$, $\mathcal{I}(x,t) = I_\mu(x,t)$ and $\mathcal{F}(x,t) = F_\mu(x,t)$. We verify the following needed conditions:

(i) For $x = 0$, (IN1-a) yields $\mathcal{T}(0,t) = 1$, $\mathcal{I}(0,t) = 0$, $\mathcal{F}(0,t) = 0$ for all $t > 0$. Conversely, if $\mathcal{T}(x,t) = 1$, $\mathcal{I}(x,t) = 0$ and $\mathcal{F}(x,t) = 0$ for some $t > 0$, then by (IN1-b) and (IN8) (which gives $T_\nu(x,t) = 0$) we obtain $x = 0$.

(ii) For $x \neq 0$, the strict inequalities $\mathcal{T}(x,t) > 0$, $\mathcal{I}(x,t) < 1$ and $\mathcal{F}(x,t) < 1$ hold for every $t > 0$. Indeed, (IN4) gives $\lim_{t\to\infty} \mathcal{T}(x,t) = 1$, $\lim_{t\to\infty} \mathcal{I}(x,t) = 0$ and $\lim_{t\to\infty} \mathcal{F}(x,t) = 0$. If $\mathcal{T}(x,t_0) = 0$ for some $t_0 > 0$, then continuity (IN5) would force $\mathcal{T}(x,t) = 0$ for all $t > 0$, contradicting the limit 1 as $t \to \infty$. The same argument applied to $1 - \mathcal{I}$ and $1 - \mathcal{F}$ yields $\mathcal{I}(x,t) < 1$ and $\mathcal{F}(x,t) < 1$.

(iii) Homogeneity is resulted directly from (IN2).

(iv) The triangle inequalities are exactly the first three inequalities of (IN3):

$$\begin{aligned}
\mathcal{T}(x+y, t+s) &\geq \min\{\mathcal{T}(x,t), \mathcal{T}(y,s)\},\\
\mathcal{I}(x+y, t+s) &\leq \max\{\mathcal{I}(x,t), \mathcal{I}(y,s)\},\\
\mathcal{F}(x+y, t+s) &\leq \max\{\mathcal{F}(x,t), \mathcal{F}(y,s)\}.
\end{aligned}$$

(v) The limits required are exactly those stated in (IN4): $\lim_{t\to\infty} \mathcal{T}(x,t) = 1$, $\lim_{t\to\infty} \mathcal{I}(x,t) = 0$, $\lim_{t\to\infty} \mathcal{F}(x,t) = 0$ and $\lim_{t\to 0} \mathcal{T}(x,t) = 0$, $\lim_{t\to 0} \mathcal{I}(x,t) = 1$, $\lim_{t\to 0} \mathcal{F}(x,t) = 1$.

(vi) Continuity of $\mathcal{T}(x,\cdot)$, $\mathcal{I}(x,\cdot)$ and $\mathcal{F}(x,\cdot)$ follows from (IN5). Therefore, $(X, (\mathcal{T}, \mathcal{I}, \mathcal{F}), \min, \max)$ is an $NFNLS$ in the sense of Definition 2.4. □

This example demonstrates that condition (IN8) is essential for Theorem 3.5; without it, the natural candidates obtained from the six components do not necessarily form intuitionistic or neutrosophic fuzzy normed spaces.

Example 3.6. Let $X = \mathbb{R}$ and define an intuitionistic–neutrosophic fuzzy normed linear space (X, N) by

$$T_\mu(x,t) = \tfrac{t}{t+|x|}, \quad I_\mu(x,t) = 0.9, \quad F_\mu(x,t) = \tfrac{t}{t+|x|},$$

$$T_\nu(x,t) = 0.5, \qquad I_\nu(x,t) = 0.3, \quad F_\nu(x,t) = 0.4$$

for all $x \in X$ and all $t > 0$. One can verify that axioms (IN1)–(IN7) are satisfied, but (IN8) fails because

$$T_\mu(x,t) + F_\mu(x,t) = \frac{2t}{t+|x|}$$

exceeds 1 for sufficiently large t (e.g., $t = 10|x|$ gives $20/11 > 1$). Consequently, the sum condition $T_\mu + F_\mu \leq 1$ in (IN8) doesn't hold.

Now examine the two reductions proposed in Theorem 3.5:

(a) Set $\mu(x,t) = T_\mu(x,t)$ and $\nu(x,t) = F_\mu(x,t)$. Then

$$\mu(x,t) + \nu(x,t) = \frac{2t}{t+|x|},$$

which is not ≤ 1 for all $t > 0$. Hence $(X, \mu, \nu, \min, \max)$ fails to satisfy the first axiom of an intuitionistic fuzzy normed space (Definition 2.3). Therefore, it is not an $IFNLS$.

(b) Define $\mathcal{T} = T_\mu$, $\mathcal{I} = I_\mu$, $\mathcal{F} = F_\mu$. In a neutrosophic fuzzy normed space one requires $\lim_{t\to 0} I(x,t) = 1$ for $x \neq 0$ (see Definition 2.4). However, here $I_\mu(x,t) = 0.9$ for all t, so

$$\lim_{t\to 0} I_\mu(x,t) = 0.9 \neq 1.$$

Thus $(X, (\mathcal{T}, \mathcal{I}, \mathcal{F}), \min, \max)$ is not a valid $NFNLS$.

Remark 3.7. Theorem 3.5 provides canonical reductions from any $INFNLS$ satisfying (IN8) to an $IFNLS$ and to an $NFNLS$. The converse does not hold, because an $INFNL$ is not determined by a pair consisting of an $IFNLS$ and an $NFNLS$. The axioms (IN3) and (IN4) impose an asymmetric relation between the positive and negative triples that is not captured by the separate structures. Thus, $INFNLS$ is not merely a product of $IFNLS$ and $NFNLS$, but a new structure with interdependent components.

Example 3.8. Let $X = \mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|$. For $x \in X$ and $t > 0$, define

$$N(x,t) = (T_\mu(x,t), I_\mu(x,t), F_\mu(x,t),\ T_\nu(x,t), I_\nu(x,t), F_\nu(x,t))$$

where

$$T_\mu(x,t) = \frac{t}{t+\|x\|}, \qquad I_\mu(x,t) = \frac{\|x\|}{2(t+\|x\|)},$$

$$F_\mu(x,t)=\frac{\|x\|}{t+\|x\|+1},\qquad T_\nu(x,t)=\frac{\|x\|}{t+\|x\|},$$

$$\begin{aligned}I_\nu(x,t)&=1-\frac{\|x\|}{2(t+\|x\|)}\\&=\frac{2t+\|x\|}{2(t+\|x\|)},\\F_\nu(x,t)&=1-\frac{\|x\|}{t+\|x\|+1}\\&=\frac{t+1}{t+\|x\|+1}.\end{aligned}$$

We verify that N satisfies all axioms of Definition 3.1.

(IN1)

(i) For $x=0$:

$$\begin{pmatrix}T_\mu\\I_\mu\\F_\mu\\T_\nu\\I_\nu\\F_\nu\end{pmatrix}(0,t)=\begin{pmatrix}1\\0\\0\\0\\1\\1\end{pmatrix}.$$

(ii) Suppose $T_\mu(x,t)=1$, $I_\mu(x,t)=F_\mu(x,t)=0$ and $T_\nu(x,t)=0$ for some $t>0$. From $T_\mu(x,t)=1$ we have

$$\frac{t}{t+\|x\|}=1\quad\Rightarrow\quad\|x\|=0.$$

Also, for $I_\mu(x,t)=F_\mu(x,t)=0$ and $T_\nu(x,t)=0$, we have $x=0$.

(IN2) For $\alpha\neq 0$,

$$\begin{aligned}T_\mu(\alpha x,t)&=\frac{t}{t+\|\alpha x\|}\\&=\frac{t/|\alpha|}{t/|\alpha|+\|x\|}\\&=T_\mu\left(x,\frac{t}{|\alpha|}\right).\end{aligned}$$

The verification for I_μ, F_μ, T_ν and F_ν is identical. For I_ν we have:

$$\begin{aligned}I_\nu(\alpha x,t)&=1-\frac{\|\alpha x\|}{2(t+\|\alpha x\|)}\\&=1-\frac{\|x\|}{2(t/|\alpha|+\|x\|)}\end{aligned}$$

$$= I_\nu\left(x, \frac{t}{|\alpha|}\right).$$

(IN3) We verify the triangle inequality for T_μ; the other five are similar. For $x, y \in \mathbb{R}^n$ and $t, s > 0$,

$$\begin{aligned} T_\mu(x+y, t+s) &= \frac{t+s}{t+s+\|x+y\|} \\ &\geq \frac{t+s}{t+s+\|x\|+\|y\|}. \end{aligned}$$

Assume without loss of generality that $\frac{t}{t+\|x\|} \leq \frac{s}{s+\|y\|}$. Then

$$\frac{t}{t+\|x\|} \leq \frac{s}{s+\|y\|} \quad \iff \quad t\|y\| \leq s\|x\|.$$

Now we have:

$$\begin{aligned} \frac{t+s}{t+s+\|x\|+\|y\|} - \frac{t}{t+\|x\|} &= \frac{(t+s)(t+\|x\|) - t(t+s+\|x\|+\|y\|)}{(t+s+\|x\|+\|y\|)(t+\|x\|)} \\ &= \frac{s\|x\| - t\|y\|}{(t+s+\|x\|+\|y\|)(t+\|x\|)} \\ &\geq 0, \end{aligned}$$

because $s\|x\| - t\|y\| \geq 0$. Hence, $T_\mu(x+y, t+s) \geq \min\{T_\mu(x,t), T_\mu(y,s)\}$.
(IN4) The limit conditions follow directly:

$$\lim_{t\to\infty} T_\mu(x,t) = 1, \quad \lim_{t\to\infty} T_\nu(x,t) = 0$$

$$\lim_{t\to\infty} I_\mu(x,t) = 0, \quad \lim_{t\to\infty} I_\nu(x,t) = 1$$

$$\lim_{t\to\infty} F_\mu(x,t) = 0, \quad \lim_{t\to\infty} F_\nu(x,t) = 1$$

(IN5) For a fixed $x \in X$, each component of $N(x,t)$—as a function of the single variable t—is a rational function of t. Moreover, for $t > 0$ the denominators of these rational expressions are strictly positive (they are of the form $t + \|x\|$, $t + \|x\| + 1$, or constant multiples of them). Consequently, each component is continuous in t on the interval $(0, \infty)$, which verifies axiom (IN5).

Therefore, $(\mathbb{R}^n, N)$ is an $INFNLS$. Moreover, this example satisfies the strong consistency equality

$$T_\mu(x,t) + T_\nu(x,t) = 1, \quad I_\mu(x,t) + I_\nu(x,t) = 1, \quad F_\mu(x,t) + F_\nu(x,t) = 1,$$

which are stronger than the inequalities typically assumed in intuitionistic fuzzy structures.
In particular:

(IN6)

$$\begin{aligned} T_\mu(x,t) = 1 \quad &\iff \quad \frac{t}{t+\|x\|} = 1 \\ &\iff \quad \|x\| = 0 \\ &\iff \quad x = 0 \end{aligned}$$

(IN7) If $T_\mu(x,t) = 1$, then by (IN6) we have $x = 0$; substituting $x = 0$ in the definitions gives $I_\mu(0,t) = F_\mu(0,t) = 0$.
If $T_\nu(x,t) = 0$, then $\frac{\|x\|}{t+\|x\|} = 0$, which forces $\|x\| = 0$, i.e., $x = 0$; consequently $I_\nu(0,t) = F_\nu(0,t) = 1$.

(IN8) Direct computation shows $T_\mu + T_\nu = 1$, $I_\mu + I_\nu = 1$, $F_\mu + F_\nu = 1$ and $T_\mu + F_\mu \leq 1$.

Condition (IN9) are satisfied as well (the components are even differentiable in t). Therefore, this example is an $INFNLS$ of types (IN6)-(IN9).

Definition 3.9. For a fixed $x \in X$, consider the functions $f_x : (0,\infty) \to [0,1]$ given by $f_x(t) = T_\mu(x,t)$, $f_x(t) = I_\mu(x,t)$, etc. We say that

a) f_x is non-decreasing if $t_1 < t_2$ implies $f_x(t_1) \leq f_x(t_2)$;
b) f_x is non-increasing if $t_1 < t_2$ implies $f_x(t_1) \geq f_x(t_2)$.

Strictly increasing/decreasing are defined analogously with strict inequalities.

Proposition 3.10. *Let (X,N) be an $INFNLS$. Then for each $x \in X$:*

a) *$T_\mu(x,\cdot)$, $I_\nu(x,\cdot)$, $F_\nu(x,\cdot)$ are non-decreasing,*
b) *$I_\mu(x,\cdot)$, $F_\mu(x,\cdot)$, $T_\nu(x,\cdot)$ are non-increasing.*

Proof. For $0 < t_1 < t_2$, set $s = t_2 - t_1 > 0$. From (IN3) with $y = 0$ and using (IN1)(a), we have:

$$\begin{aligned} T_\mu(x, t_1 + s) &\geq \min\{T_\mu(x,t_1), T_\mu(0,s)\} \\ &= \min\{T_\mu(x,t_1), 1\} \\ &= T_\mu(x,t_1). \end{aligned}$$

Since $t_1 + s = t_2$, we obtain $T_\mu(x,t_2) \geq T_\mu(x,t_1)$. Hence, $T_\mu(x,\cdot)$ is non-decreasing.

For T_ν, using the corresponding inequality in (IN3):

$$\begin{aligned} T_\nu(x, t_1 + s) &\leq \max\{T_\nu(x,t_1), T_\nu(0,s)\} \\ &= \max\{T_\nu(x,t_1), 0\} \\ &= T_\nu(x,t_1), \end{aligned}$$

so $T_\nu(x,t_2) \leq T_\nu(x,t_1)$. Thus, $T_\nu(x,\cdot)$ is non-increasing. The remaining four cases are proved similarly:

- For I_μ and F_μ, use $I_\mu(x+y,t+s) \leq \max\{I_\mu(x,t), I_\mu(y,s)\}$ and $F_\mu(x+y,t+s) \leq \max\{F_\mu(x,t), F_\mu(y,s)\}$ with $y=0$ and (IN1)(a) to obtain non-increasing behavior.
- For I_ν and F_ν, use $I_\nu(x+y,t+s) \geq \min\{I_\nu(x,t), I_\nu(y,s)\}$ and $F_\nu(x+y,t+s) \geq \min\{F_\nu(x,t), F_\nu(y,s)\}$ with $y=0$ and (IN1)(a) to obtain non-decreasing behavior.

This completes the proof. □

3.2. **Convergence and Completeness in INFNLS.**

Definition 3.11. Let (X,N) be an intuitionistic–neutrosophic fuzzy normed linear space. A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ (denoted $x_n \to x$) if for every $t > 0$,

$$\lim_{t\to\infty} \begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} (x_n - x, t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Definition 3.12. Let (X,N) be an $INFNLS$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $t > 0$,

$$\lim_{t\to\infty} \begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} (x_m - x_n, t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Theorem 3.13. *Let (X,N) be an $INFNLS$ satisfying (IN6) and (IN7), then the limit point is unique.*

Proof. Assume $x_n \to x$ and $x_n \to y$ in the sense of Definition 3.11. Fix an arbitrary $t > 0$ and write $t = \frac{t}{2} + \frac{t}{2}$.
By applying (IN3) for T_μ we have

$$\begin{aligned} T_\mu(x-y,t) &= T_\mu\left((x_n - y) - (x_n - x), t\right) \\ &\geq \min\left\{T_\mu\left(x_n - y, \tfrac{t}{2}\right), T_\mu\left(x_n - x, \tfrac{t}{2}\right)\right\}. \end{aligned}$$

Letting $n \to \infty$ and using the convergence $x_n \to x$ and $x_n \to y$, we obtain

$$\lim_{t\to\infty} T_\mu\left(x_n - x, \tfrac{t}{2}\right) = 1, \qquad \lim_{t\to\infty} T_\mu\left(x_n - y, \tfrac{t}{2}\right) = 1.$$

So $T_\mu(x-y,t) \geq 1$. Since T_μ takes values in $[0,1]$, we have $T_\mu(x-y,t) = 1$ for every $t > 0$.

Similarly, applying (IN3) for T_ν yields

$$T_\nu(x-y,t) \leq \max\left\{T_\nu\left(x_n-y,\tfrac{t}{2}\right), T_\nu\left(x_n-x,\tfrac{t}{2}\right)\right\}.$$

Again letting $n \to \infty$, by using

$$\lim_{t\to\infty} T_\nu\left(x_n-x,\tfrac{t}{2}\right)=0, \qquad \lim_{t\to\infty} T_\nu\left(x_n-y,\tfrac{t}{2}\right)=0,$$

we get $T_\nu(x-y,t) \leq 0$, hence, $T_\nu(x-y,t)=0$ for every $t>0$.

Now by using (IN7), we have $I_\mu(x-y,t) = F_\mu(x-y,t) = 0$ and $I_\nu(x-y,t) = F_\nu(x-y,t) = 1$.

Finally, by using (IN6), we get $x-y=0$, which means the limit is unique. □

The following two examples show that conditions (IN6) and (IN7) are essential in Theorem 3.13; without them, the theorem fails to hold.

Example 3.14. Let $X=\mathbb{R}$ and define an intuitionistic–neutrosophic fuzzy normed linear space (X,N) by

$$\begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix}(x,t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

for all $x \in X$ and all $t>0$. One checks easily that axioms (IN1)–(IN5) and (IN7) are satisfied, but (IN6) fails because $T_\mu(x,t)=1$ for every x, even when $x \neq 0$.

Consider the sequence $x_n = n$. For any $t>0$,

$$\begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix}(x_n-0,t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Hence, $x_n \to 0$ in the sense of Definition 3.11. The same computation with $x_n - 1$ gives $x_n \to 1$. Thus, the limit is not unique. Therefore, condition (IN6) is essential for Theorem 3.13.

Example 3.15. Let $X = \mathbb{R}$ and define an $INFNLS$ by

$$\begin{pmatrix} T_\mu \\ I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix}(x,t) = \begin{pmatrix} \frac{t}{t+|x|} \\ 0.5 \\ 0.5 \\ \frac{|x|}{t+|x|} \\ 0.5 \\ 0.5 \end{pmatrix}.$$

All axioms (IN1)–(IN6) are satisfied, but (IN7) fails because $T_\mu(0,t) = 1$ does not imply $I_\mu(0,t) = F_\mu(0,t) = 0$ (they are 0.5).

Consider the sequence $x_n = 1/n$. For any fixed $t > 0$,

$$T_\mu(x_n, t) = \frac{t}{t + 1/n} \to 1,$$

but $I_\mu(x_n,t) = 0.5$ and $F_\mu(x_n,t) = 0.5$ do not tend to 0. Hence, x_n doesn't converge to 0 in the sense of Definition 3.11. In fact, no sequence can converge unless $I_\mu, F_\mu \to 0$ and $I_\nu, F_\nu \to 1$, which is precisely what (IN7) guarantees when $T_\mu \to 1$ or $T_\nu \to 0$. Thus, (IN7) is necessary for a consistent notion of convergence and for the proof of Theorem 3.13.

Lemma 3.16. *Let (X, N) be an $INFNLS$ satisfying (IN6) and let $\dim X < \infty$. For any norm $\|\cdot\|$ on X there exists a constant $C > 0$ such that for every $x \in X$,*

$$T_\mu(x,1) > \tfrac{1}{2} \quad \textit{and} \quad T_\nu(x,1) < \tfrac{1}{2} \quad \Longrightarrow \quad \|x\| < C.$$

Proof. Assume, for contradiction, that no such C exists. Then there exists a sequence $\{y_k\}$ in X with

$$T_\mu(y_k,1) > \tfrac{1}{2}, \quad T_\nu(y_k,1) < \tfrac{1}{2}, \quad \text{and} \quad \|y_k\| \to \infty.$$

Define, $z_k := y_k/\|y_k\|$, then $\|z_k\| = 1$ for all k. Since the unit sphere $\{z \in X : \|z\| = 1\}$ is compact (due to finite-dimensionality), there exists a subsequence $\{z_{k_j}\}$ converging to some z with $\|z\| = 1$; in particular $z \neq 0$.

By (IN2) and the hypothesis on y_k,

$$T_\mu(z_{k_j}, 1/\|y_{k_j}\|) = T_\mu(y_{k_j}, 1) > \tfrac{1}{2}.$$

As $j \to \infty$, we have $z_{k_j} \to z$ in norm and $1/\|y_{k_j}\| \to 0$. For any fixed $\delta > 0$, eventually $1/\|y_{k_j}\| < \delta$, so by monotonicity of $T_\mu(z_{k_j}, \cdot)$ (Proposition 3.10), we have $T_\mu(z_{k_j}, 1/\|y_{k_j}\|) \leq T_\mu(z_{k_j}, \delta)$.

Taking lim sup as $j \to \infty$ and using the continuity of $T_\mu(\cdot, \delta)$ (which follows from the definition of the topology and is consistent with (IN5)), we obtain

$$\limsup_{j\to\infty} T_\mu(z_{k_j}, 1/\|y_{k_j}\|) \leq T_\mu(z, \delta).$$

Since this holds for every $\delta > 0$, we can let $\delta \to 0^+$ and apply (IN4) to conclude

$$\limsup_{j\to\infty} T_\mu(z_{k_j}, 1/\|y_{k_j}\|) \leq \lim_{\delta\to 0^+} T_\mu(z, \delta) = 0.$$

Hence, $\lim_{j\to\infty} T_\mu(z_{k_j}, 1/\|y_{k_j}\|) = 0$, which contradicts the fact that

$$T_\mu(z_{k_j}, 1/\|y_{k_j}\|) > \tfrac{1}{2}, \quad \forall j.$$

Therefore, our assumption was false and such a constant C must exist. A similar contradiction can be derived from the condition on T_ν. □

Theorem 3.17. *Let (X, N) be an intuitionistic– neutrosophic fuzzy normed linear space satisfying (IN6) and $\dim X = n < \infty$, then (X, N) is complete; i.e., every Cauchy sequence in X converges to some element of X.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of X. For $x = \sum_{i=1}^{n} \alpha_i e_i$ define the classical maximum norm $\|x\|_{\max} := \max_{1\leq i\leq n} |\alpha_i|$.

First, suppose $x^{(m)} \to x$ in $\|\cdot\|_{\max}$. Write $x^{(m)} - x = \sum_{i=1}^{n} \beta_i^{(m)} e_i$ with $\beta_i^{(m)} \to 0$ as $m \to \infty$. For any fixed $t > 0$, choose m_0 large enough so that $\left|\beta_i^{(m)}\right| < \frac{t}{n}$ for all i and all $m \geq m_0$. Then by repeated application of (IN3) for T_μ and using (IN2), we have:

$$\begin{aligned} T_\mu(x^{(m)} - x, t) &= T_\mu\left(\sum_{i=1}^{n} \beta_i^{(m)} e_i,\ t\right) \\ &\geq \min_{1\leq i\leq n} T_\mu\left(\beta_i^{(m)} e_i,\ \frac{t}{n}\right) \\ &= \min_{1\leq i\leq n} T_\mu\left(e_i,\ \frac{t}{n\left|\beta_i^{(m)}\right|}\right), \quad (\text{if } \beta_i^{(m)} \neq 0). \end{aligned}$$

Since $\left|\beta_i^{(m)}\right| \to 0$, the arguments $\frac{t}{n\left|\beta_i^{(m)}\right|}$ tend to $+\infty$ and by (IN4), each $T_\mu(e_i, \cdot)$ tends to 1. Hence, $T_\mu(x^{(m)} - x, t) \to 1$ as $m \to \infty$. A similar argument (using the reversed inequalities for $I_\mu, F_\mu, T_\nu, I_\nu, F_\nu$) shows that all six required limits hold. Thus, $x^{(m)} \to x$ in (X, N).
Now assume $\{x^{(m)}\}$ is Cauchy in (X, N). By Lemma 3.16, there exists a constant $C > 0$ such that for any $x \in X$,

$$T_\mu(x, 1) > \tfrac{1}{2} \text{ and } T_\nu(x, 1) < \tfrac{1}{2} \quad \Longrightarrow \quad \|x\|_{\max} < C.$$

Since $\left\{x^{(m)}\right\}$ is Cauchy in (X, N), for $t = 1$ there exists M such that for all $m, p \geq M$ we have $T_\mu(x^{(m)} - x^{(p)}, 1) > \frac{1}{2}$ and $T_\nu(x^{(m)} - x^{(p)}, 1) < \frac{1}{2}$.

Taking $p = M$ and applying the lemma to $x = x^{(m)} - x^{(M)}$, we obtain $\|x^{(m)} - x^{(M)}\|_{\max} < C$ for all $m \geq M$. Hence, the whole sequence $\{x^{(m)}\}$ is bounded in $\|\cdot\|_{\max}$.

By the Bolzano–Weierstrass theorem in finite-dimensional spaces, the bounded sequence $x^{(m)}$ has a convergent subsequence. Let $x^{(m_k)} \to x$ in $\|\cdot\|_{\max}$. By the first part of the proof, the subsequence $\{x^{(m_k)}\}$ also converges to x in the $INFNLS$ sense. Since $\{x^{(m)}\}$ is Cauchy in (X, N) and has a subsequence converging to x, by using (IN3) exactly as in the proof of Theorem 3.13, shows that the whole sequence $\{x^{(m)}\}$ converges to x in (X, N). Therefore, every Cauchy sequence in (X, N) converges, i.e., (X, N) is complete. □

Theorem 3.18. *Let (X, N) be an $INFNLS$. Then we have:*

a) *The mapping $(x, y) \mapsto x + y$ from $X \times X$ to X is continuous.*
b) *The mapping $(\alpha, x) \mapsto \alpha x$ from $\mathbb{K} \times X$ to X is continuous.*

Proof. a) Let $(x_n, y_n) \to (x, y)$ in $X \times X$. Fix $t > 0$ and write $t = t/2 + t/2$. From (IN3) we have

$$T_\mu((x_n + y_n) - (x + y), t) \geq \min\{T_\mu(x_n - x, t/2),\ T_\mu(y_n - y, t/2)\}.$$

Since $x_n \to x$ and $y_n \to y$, hence, $\lim_{n\to\infty} T_\mu((x_n+y_n)-(x+y), t) = 1$. Similar estimates using the inequalities in (IN3) give

$$\lim_{n\to\infty} \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \\ I_\nu \\ F_\nu \end{pmatrix} ((x_n + y_n) - (x + y), t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, $x_n + y_n \to x + y$ and addition is continuous.

b) Let $(\alpha_n, x_n) \to (\alpha, x)$ in $\mathbb{K} \times X$; i.e., $\alpha_n \to \alpha$ in $\mathbb{K}$ and $x_n \to x$ in (X, N). We treat two cases. **Case 1**: Let $\alpha \neq 0$. Since $\alpha_n \to \alpha \neq 0$, there exists N_0 such that $|\alpha_n| \geq |\alpha|/2 > 0$ for all $n \geq N_0$. We know $\alpha_n x_n - \alpha x = \alpha_n(x_n - x) + (\alpha_n - \alpha)x$.

For a fixed $t > 0$ and by using (IN3), we have

$$T_\mu(\alpha_n x_n - \alpha x, t) \geq \min\left\{T_\mu\big(\alpha_n(x_n - x), \tfrac{t}{2}\big),\ T_\mu\big((\alpha_n - \alpha)x, \tfrac{t}{2}\big)\right\}.$$

By (IN2),

$$T_\mu\big(\alpha_n(x_n - x), \tfrac{t}{2}\big) = T_\mu\Big(x_n - x, \tfrac{t}{2|\alpha_n|}\Big),$$

and

$$_\mu\big((\alpha_n - \alpha)x, \tfrac{t}{2}\big) = T_\mu\Big(x, \tfrac{t}{2|\alpha_n - \alpha|}\Big).$$

For the first term, by $x_n \to x$, boundedness of $\{t/(2|\alpha_n|)\}$ and continuity of $T_\mu(x,\cdot)$ (IN5), the convergence definition implies

$$\lim_{n\to\infty} T_\mu\Big(x_n - x, \tfrac{t}{2|\alpha_n|}\Big) = 1.$$

For the second term, since $|\alpha_n - \alpha| \to 0$, we have $t/(2|\alpha_n - \alpha|) \to \infty$; hence, (IN4) yields $\lim_{n\to\infty} T_\mu\Big(x, \frac{t}{2|\alpha_n-\alpha|}\Big) = 1$. Consequently, $\lim_{n\to\infty} T_\mu(\alpha_n x_n - \alpha x, t) = 1$.

Case 2: Let $\alpha = 0$. Now $\alpha_n \to 0$ and we must show $\alpha_n x_n \to 0$ in (X, N). Because $x_n \to x$, the sequence $\{x_n\}$ is bounded in the $INFNLS$ sense; in particular, for any fixed $t > 0$ the quantities $T_\mu(x_n, t)$ are bounded away from 0 for large n. Using (IN2),

$$T_\mu(\alpha_n x_n, t) = T_\mu\Big(x_n, \tfrac{t}{|\alpha_n|}\Big).$$

Since $|\alpha_n| \to 0$, the argument $t/|\alpha_n|$ tends to $+\infty$ and by (IN4) we have $T_\mu(x_n, t/|\alpha_n|) \to 1$; moreover, because $\{x_n\}$ is bounded, this convergence is uniform in n (i.e., for every $\varepsilon > 0$ there exists N such that $T_\mu(x_n, t/|\alpha_n|) > 1 - \varepsilon$ for all $n \geq N$ and all $t > 0$). Thus, $T_\mu(\alpha_n x_n, t) \to 1$.

In both cases, analogous estimates for $I_\mu, F_\mu, T_\nu, I_\nu, F_\nu$ (employing the respective inequalities in (IN3)) give the remaining limits required for convergence in (X, N). Hence, $\alpha_n x_n \to \alpha x$ and scalar multiplication is continuous. □

Theorem 3.19. *Let X be a finite-dimensional linear space ($\dim X = n < \infty$) and let N_1, N_2 be two intuitionistic–neutrosophic fuzzy norms on X with* max *norm, that both satisfy conditions (IN6) and (IN7). Then the topologies τ_1, τ_2 induced by N_1 and N_2 coincide; moreover, they coincide with the usual Euclidean topology on X.*

Proof. Fix a basis $\{e_1, \ldots, e_n\}$ of X and let $\|\cdot\|$ be the corresponding maximum norm (which induces the Euclidean topology). Denote by $\tau_{\|\cdot\|}$ the topology of $\|\cdot\|$. From the argument in Theorem 3.17, convergence in $\|\cdot\|_{\max}$ implies convergence in any $INFNLS$ structure on X (because the estimates used there depend only on the basis and the properties (IN2)–(IN5), which are common to all $INFNLS$). Hence, every sequence that converges in $\|.\|$ also converges in (X, N_1) and (X, N_2).

Assume $x_k \to x$ in (X, N_1). By the definition of convergence, there exists an index K such that for all $k \geq K$,

$$T_\mu^{(1)}(x_k - x, 1) > \tfrac{1}{2} \quad \text{and} \quad T_\nu^{(1)}(x_k - x, 1) < \tfrac{1}{2},$$

where the notion (1) refers to the components of N_1. Applying Lemma 3.16 to the $INFNLS$ (X, N_1), we obtain a constant $C > 0$ such that

$\|x_k - x\| < C$ for all $k \geq K$. Thus, $\{x_k\}$ is bounded in $\|\cdot\|$. Now, let $x_k \to x$ in (X, N_1), where $\{x_k\}$ is bounded in $\|\cdot\|$. Since $\dim X < \infty$, the closed unit ball of $\|\cdot\|$ is compact; therefore, every subsequence $\{x_{k_j}\}$ of $\{x_k\}$ possesses a further subsequence $\left\{x_{k_{j_\ell}}\right\}$ that converges in $\|\cdot\|$ to some limit y, where this subsequence also converges to y in (X, N_1). By Theorem 3.13, we must have $y = x$.

Consequently, every subsequence of $\{x_k\}$ has a further subsequence converging to x in $\|\cdot\|$; a standard result in metric spaces (or first countable spaces) implies that the whole sequence $\{x_k\}$ converges to x in $\|\cdot\|$. Hence, $x_k \to x$ in $\|\cdot\|$. This shows that convergence in (X, N_1) implies convergence in $\|\cdot\|$, consequently, the topology induced by N_1 is contained in the topology induced by $\|\cdot\|$. So we obtain $\tau_1 = \tau_{\|\cdot\|}$. Repeating the same argument for N_2 gives $\tau_2 = \tau_{\|\cdot\|}$. Therefore, $\tau_1 = \tau_2 = \tau_{\|\cdot\|}$, which is precisely the Euclidean topology on X. □

3.3. **Topological Structure.**

Definition 3.20. Let (X, N) be an $INFNLS$. For $x_0 \in X$, $r \in (0, 1)$ and $t > 0$, the (r, t)-neighbourhood of x_0 is defined as

$$U(x_0, r, t) = \left\{ \begin{array}{c} \qquad T_\mu(x - x_0, t) > 1 - r,\ I_\mu(x - x_0, t) < r, \\ x \in X : F_\mu(x - x_0, t) < r, T_\nu(x - x_0, t) < r, \\ \qquad I_\nu(x - x_0, t) > 1 - r,\ F_\nu(x - x_0, t) > 1 - r \end{array} \right\}.$$

Remark 3.21. This definition generalizes the neighbourhood concepts in both intuitionistic and neutrosophic fuzzy normed spaces:

- For an $IFNLS$, only the conditions on T_μ and T_ν remain (with $I_\mu = I_\nu = F_\mu = F_\nu \equiv 0$).
- For an $NFNLS$, we have $(T_\nu, I_\nu, F_\nu) \equiv (0, 1, 1)$, so only the first three inequalities matter.

Definition 3.22. For a fixed $x_0 \in X$, define

$$\mathcal{B}_{x_0} = \{U(x_0, r, t) : r \in (0, 1),\ t > 0\}.$$

Lemma 3.23. *For any $INFNLS$ (X, N), the family $\mathcal{B}_{x_0}$ satisfies:*

a) *For each $U \in \mathcal{B}_{x_0}$, $x_0 \in U$.*
b) *If $U_1, U_2 \in \mathcal{B}_{x_0}$, then there exists $U_3 \in \mathcal{B}_{x_0}$ such that $U_3 \subseteq U_1 \cap U_2$.*
c) *For each $U \in \mathcal{B}_{x_0}$ and $y \in U$, there exists $V \in \mathcal{B}_y$ such that $V \subseteq U$.*

Proof. Let (X, N) be an $INFNLS$ and fix $x_0 \in X$.

a) Let $U = U(x_0, r, t) \in \mathcal{B}_{x_0}$ with $r \in (0,1)$, $t > 0$. Since $x_0 - x_0 = 0$, from (IN1) we have:

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(0,t) = 1 > 1 - r, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(0,t) = 0 < r.$$

Hence, $x_0 \in U(x_0, r, t)$.

b) Let $U_1 = U(x_0, r_1, t_1)$ and $U_2 = U(x_0, r_2, t_2)$. Choose $r = \min\{r_1, r_2\} > 0$ and $t = \min\{t_1, t_2\} > 0$ and set $U_3 = U(x_0, r, t)$. We show $U_3 \subseteq U_1 \cap U_2$. Take $x \in U_3$, then by definition:

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(x - x_0, t) > 1 - r, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(x - x_0, t) < r.$$

Since $t \leq t_1$ and $T_\mu(x - x_0, \cdot)$ is non-decreasing (which follows from (IN3) with $y = 0$), we have

$$\begin{aligned} T_\mu(x - x_0, t_1) &\geq T_\mu(x - x_0, t) \\ &> 1 - r \\ &\geq 1 - r_1. \end{aligned}$$

Similarly, $I_\mu(x - x_0, \cdot)$ and $F_\mu(x - x_0, \cdot)$ are non-increasing, so

$$\begin{aligned} I_\mu(x - x_0, t_1) &\leq I_\mu(x - x_0, t) \\ &< r \\ &\leq r_1, \end{aligned}$$

and

$$\begin{aligned} F_\mu(x - x_0, t_1) &\leq F_\mu(x - x_0, t) \\ &< r \\ &\leq r_1. \end{aligned}$$

For the ν-components, $T_\nu(x - x_0, \cdot)$ is non-increasing, while I_ν and F_ν are non-decreasing. Hence,

$$\begin{aligned} T_\nu(x - x_0, t_1) &\leq T_\nu(x - x_0, t) \\ &< r \\ &\leq r_1, \end{aligned}$$

and

$$\begin{aligned} I_\nu(x - x_0, t_1) &\geq I_\nu(x - x_0, t) \\ &> 1 - r \\ &\geq 1 - r_1, \end{aligned}$$

also

$$\begin{aligned} F_\nu(x - x_0, t_1) &\geq F_\nu(x - x_0, t) \\ &> 1 - r \\ &\geq 1 - r_1. \end{aligned}$$

Thus, $x \in U(x_0, r_1, t_1) = U_1$. The same argument with t_2, r_2 gives $x \in U_2$. Therefore, $x \in U_1 \cap U_2$ and consequently $U_3 \subseteq U_1 \cap U_2$.

c) Let $U = U(x_0, r, t) \in \mathcal{B}_{x_0}$ and $y \in U$. We need to find $\rho > 0$ and $s > 0$ such that $V = U(y, \rho, s) \subseteq U$. Since $y \in U(x_0, r, t)$, we have

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(y - x_0, t) > 1 - r, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(y - x_0, t) < r.$$

By continuity of the six components (IN5), there exists $\delta > 0$ such that for all $s \in (t - \delta, t + \delta)$,

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(y - x_0, s) > 1 - r, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(y - x_0, s) < r.$$

Fix such an s with $0 < s < t$. Now choose $\rho > 0$ small enough so that

$$\rho < \min \begin{Bmatrix} 1 - T_\mu(y - x_0, s),\ I_\mu(y - x_0, s), \\ F_\mu(y - x_0, s),\ T_\nu(y - x_0, s), \\ 1 - I_\nu(y - x_0, s),\ 1 - F_\nu(y - x_0, s). \end{Bmatrix}.$$

Note that each term inside the min is positive because of the inequalities above.

Take $V = U(y, \rho, t - s)$. We show $V \subseteq U$. Let $z \in V$, i.e.,

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(z - y, t - s) > 1 - \rho, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(z - y, t - s) < \rho.$$

Write $z - x_0 = (z - y) + (y - x_0)$ and apply (IN3) with $t = (t - s) + s$:

$$\begin{aligned} T_\mu(z - x_0, t) &\geq \min\{T_\mu(z - y, t - s),\ T_\mu(y - x_0, s)\} \\ &> \min\{1 - \rho,\ T_\mu(y - x_0, s)\} \\ &\geq 1 - r, \end{aligned}$$

because, $1 - \rho > T_\mu(y - x_0, s)$ (by choice of ρ) and $T_\mu(y - x_0, s) > 1 - r$.

Similarly,

$$\begin{aligned} I_\mu(z-x_0,t) &\leq \max\{I_\mu(z-y,t-s),\ I_\mu(y-x_0,s)\} \\ &< \max\{\rho,\ I_\mu(y-x_0,s)\} \\ &\leq r, \end{aligned}$$

since $\rho < I_\mu(y-x_0,s)$ and $I_\mu(y-x_0,s) < r$. The same reasoning yields

$$\begin{pmatrix} I_\nu \\ F_\nu \end{pmatrix}(z-x_0,t) > 1-r, \qquad \begin{pmatrix} F_\mu \\ T_\nu \end{pmatrix}(z-x_0,t) < r.$$

Therefore, $z \in U(x_0,r,t) = U$ and $V \subseteq U$, which proves (c). □

Theorem 3.24. *Let (X,N) be an $INFNLS$ that satisfies conditions (IN6) and (IN7). The collection*

$$\tau_N = \{A \subseteq X : \forall x \in A,\ \exists U \in \mathcal{B}_x \text{ such that } U \subseteq A\}$$

is a Hausdorff topology on X. Moreover:

a) *τ_N is first-countable.*
b) *The algebraic operations are continuous with respect to τ_N.*
c) *Sequence convergence in (X,N) coincides with topological convergence in (X,τ_N).*

Proof. We verify the axioms of a topology.

(i) $\emptyset \in \tau_N$ vacuously and $X \in \tau_N$ because for any $x \in X$ we can take any $U \in \mathcal{B}_x$, which is a subset of X.
(ii) Let $\{A_i\}_{i\in I} \subseteq \tau_N$ and put $A = \bigcup_{i\in I} A_i$. If $x \in A$, then $x \in A_i$ for some i. Since $A_i \in \tau_N$, there exists $U \in \mathcal{B}_x$ with $U \subseteq A_i \subseteq A$. Hence, $A \in \tau_N$.
(iii) Let $A_1, A_2 \in \tau_N$ and $x \in A_1 \cap A_2$. There exist $U_1, U_2 \in \mathcal{B}_x$ with $U_1 \subseteq A_1$, $U_2 \subseteq A_2$. By Lemma 3.23, there exists $U_3 \in \mathcal{B}_x$ such that $U_3 \subseteq U_1 \cap U_2 \subseteq A_1 \cap A_2$. Thus, $A_1 \cap A_2 \in \tau_N$.

Therefore, τ_N is a topology on X.

Hausdorff property: Let $x \neq y$ be two distinct points of X. We construct disjoint neighbourhoods of x and y.

Since $x - y \neq 0$, condition (IN6) implies $T_\mu(x-y,1) < 1$. Choose a number r with $0 < r < \frac{1}{2}(1 - T_\mu(x-y,1))$; note that $1 - T_\mu(x-y,1) > 0$. Set $t = 1$ and consider the neighbourhoods

$$U_x = U(x,r,1), \qquad U_y = U(y,r,1).$$

We claim $U_x \cap U_y = \emptyset$. Suppose, on the contrary, that there exists $z \in U_x \cap U_y$, then by definition,

$$T_\mu(z-x,1) > 1-r, \qquad T_\mu(z-y,1) > 1-r.$$

Applying (IN3) with $s = t = 1/2$, we obtain

$$\begin{aligned} T_\mu(x-y,1) &\geq \min\{T_\mu(x-z,1/2),\ T_\mu(z-y,1/2)\} \\ &\geq \min\{T_\mu(z-x,1),\ T_\mu(z-y,1)\} \quad \text{(by monotonicity)} \\ &> \min\{1-r,\ 1-r\} \\ &= 1-r. \end{aligned}$$

Thus, $T_\mu(x-y,1) > 1-r$. But from the choice of r we have

$$\begin{aligned} 1-r &> 1-\frac{1}{2}\left(1-T_\mu(x-y,1)\right) \\ &= \frac{1}{2}+\frac{1}{2}T_\mu(x-y,1) \\ &> T_\mu(x-y,1), \end{aligned}$$

which contradicts the inequality. Hence, $U_x \cap U_y = \emptyset$ and τ_N is Hausdorff.

a) For each $x \in X$ consider the countable family

$$\left\{U\left(x,\frac{1}{n},\frac{1}{n}\right) : n \in \mathbb{N},\ n \geq 2\right\} \subseteq \mathcal{B}_x.$$

We show it is a neighbourhood basis at x. Let $U = U(x,r,t) \in \mathcal{B}_x$. Choose n large enough so that $\frac{1}{n} < \min\{r,t\}$. Then $U\left(x,\frac{1}{n},\frac{1}{n}\right) \subseteq U(x,r,t)$ because for any $y \in U\left(x,\frac{1}{n},\frac{1}{n}\right)$,

$$\begin{aligned} T_\mu(y-x,t) &\geq T_\mu\left(y-x,\frac{1}{n}\right) \\ &> 1-\frac{1}{n} \\ &> 1-r, \end{aligned}$$

and

$$\begin{aligned} I_\mu(y-x,t) &\leq I_\mu\left(y-x,\frac{1}{n}\right) \\ &< \frac{1}{n} \\ &< r, \end{aligned}$$

and similarly for the other components. Hence, τ_N is first - countable.

b) Continuity of addition and scalar multiplication was already proved in Theorem 3.18. That proof used only the convergence of sequences, which is equivalent to topological convergence by part (iii) below. Consequently, the operations are continuous with respect to τ_N.

(c) Let $\{x_n\}$ be a sequence in X and $x \in X$. Recall from the definition of $U(x,r,t)$ that $x_n \in U(x,r,t)$ iff

$$\begin{pmatrix} T_\mu \\ I_\nu \\ F_\nu \end{pmatrix}(x_n - x, t) > 1 - r, \qquad \begin{pmatrix} I_\mu \\ F_\mu \\ T_\nu \end{pmatrix}(x_n - x, t) < r. \tag{3.1}$$

($\Rightarrow$) Assume $x_n \to x$ in (X,N), i.e., the six limits in Definition 3.11 hold. For any $U = U(x,r,t) \in \mathcal{B}_x$, those limits yield an index n_0 such that (3.1) holds for all $n \geq n_0$. Hence $x_n \in U$ for $n \geq n_0$, proving $x_n \to x$ in (X, τ_N).

($\Leftarrow$) Assume $x_n \to x$ in (X, τ_N). Fix $t > 0$ and $r \in (0,1)$. Since $U(x,r,t)$ is a τ_N-neighborhood of x, there exists n_0 such that $x_n \in U(x,r,t)$ for all $n \geq n_0$, i.e., (3.1) holds. As r was arbitrary, letting $r \to 0$ yields precisely the six limits in Definition 3.11. □

Definition 3.25. Let (X, τ_N) be the topological space induced by an $INFNLS$. For a fixed $t > 0$, consider a function $f : X \to [0,1]$, e.g., $f(x) = T_\mu(x,t)$.

(i) f is called upper semi-continuous at x_0 if for every $\alpha > f(x_0)$ there exists a neighbourhood U of x_0 such that $f(x) < \alpha$ for all $x \in U$.
(ii) f is called lower semi-continuous at x_0 if for every $\alpha < f(x_0)$ there exists a neighbourhood U of x_0 such that $f(x) > \alpha$ for all $x \in U$.

Proposition 3.26. *Assume (X,N) satisfies (IN8) and let τ_N be the topology induced by N. Then for each fixed $t > 0$:*

a) *$x \mapsto T_\mu(x,t)$ is upper semi-continuous on (X, τ_N).*
b) *$x \mapsto T_\nu(x,t)$ is lower semi-continuous on (X, τ_N).*
c) *$x \mapsto I_\mu(x,t)$ and $x \mapsto F_\mu(x,t)$ are lower semi-continuous.*
d) *$x \mapsto I_\nu(x,t)$ and $x \mapsto F_\nu(x,t)$ are upper semi-continuous.*

Proof. We prove only (a) and (c); the other statements follow by analogous arguments or by using the duality relations (IN8).

a) We prove that $T_\mu(\cdot, t)$ is upper semi-continuous, i.e., we show that the set $A = \{x \in X : T_\mu(x,t) < \alpha\}$ is open in τ_N.
If $\alpha \geq 1$, then $A = X$ (since $T_\mu(x,t) \leq 1$), which is open. Assume $\alpha < 1$ and take $x_0 \in A$, i.e., $T_\mu(x_0,t) < \alpha$. Choose $r > 0$ such that $T_\mu(x_0,t) < \alpha - r$. Consider the neighbourhood $U = U(x_0,r,t)$. For any $y \in U$ we have, by definition,

$$T_\mu(y - x_0, t) > 1 - r, \qquad T_\nu(y - x_0, t) < r.$$

Using (IN8) ($T_\mu + T_\nu \leq 1$) we obtain

$$T_\mu(y,t) \leq 1 - T_\nu(y,t).$$

Now apply (IN3) for T_ν with $s = t/2$:

$$T_\nu(y,t) \leq \max\left\{T_\nu(y - x_0, t/2),\ T_\nu(x_0, t/2)\right\}.$$

Since $T_\nu(\cdot,\cdot)$ is non-increasing in the second variable, $T_\nu(y - x_0, t/2) \leq T_\nu(y - x_0, t) < r$. Moreover, from (IN8) and the choice of r,

$$\begin{aligned} T_\nu(x_0, t/2) &\leq 1 - T_\mu(x_0, t/2) \\ &\leq 1 - T_\mu(x_0, t) \\ &< 1 - (\alpha - r). \end{aligned}$$

Consequently, as $1 - \alpha > 0$

$$\begin{aligned} T_\nu(y,t) &\leq \max\left\{r,\ 1 - \alpha + r\right\} \\ &= 1 - \alpha + r. \end{aligned}$$

Hence,

$$\begin{aligned} T_\mu(y,t) &\leq 1 - T_\nu(y,t) \\ &\leq 1 - (1 - \alpha + r) \\ &= \alpha - r \\ &< \alpha. \end{aligned}$$

Thus, $y \in A$, showing that $U \subseteq A$. Therefore, A is open and $T_\mu(\cdot, t)$ is upper semi-continuous.

c) To prove lower semi-continuity of $I_\mu(\cdot, t)$, let $\alpha \in \mathbb{R}$ and consider $B = \{x \in X : I_\mu(x,t) > \alpha\}$. If $\alpha < 0$, then $B = X$ (since $I_\mu \geq 0$), which is open. Assume $\alpha \geq 0$ and take $x_0 \in B$, i.e., $I_\mu(x_0, t) > \alpha$. Choose $r > 0$ with $I_\mu(x_0, t) > \alpha + r$. Set $U = U(x_0, r, t)$. For $y \in U$ we have $I_\nu(y - x_0, t) > 1 - r$. Applying (IN3) for I_ν with $s = t/2$ gives

$$I_\nu(y,t) \geq \min\left\{I_\nu(y - x_0, t/2),\ I_\nu(x_0, t/2)\right\}.$$

Because $I_\nu(\cdot,\cdot)$ is non-decreasing in the second variable, we have

$$I_\nu(y - x_0, t/2) \geq I_\nu(y - x_0, t) > 1 - r.$$

From (IN8) ($I_\mu + I_\nu \geq 1$) we obtain

$$\begin{aligned} I_\nu(x_0, t/2) &\geq 1 - I_\mu(x_0, t/2) \\ &\geq 1 - I_\mu(x_0, t) \\ &> 1 - (\alpha + r). \end{aligned}$$

Thus, $I_\nu(y,t) \geq \min\{1-r,\ 1-\alpha-r\} = 1-\alpha-r$, since $1-\alpha-r \leq 1-r$ (because $\alpha \geq 0$). Using (IN8) again,

$$\begin{aligned} I_\mu(y,t) &\geq 1 - I_\nu(y,t) \\ &\geq 1-(1-\alpha-r) \\ &= \alpha + r \\ &> \alpha. \end{aligned}$$

Hence, $y \in B$, so $U \subseteq B$. Therefore, B is open and $I_\mu(\cdot,t)$ is lower semi-continuous.

The proofs for F_μ, T_ν, I_ν and F_ν are completely similar; they use the corresponding inequalities in (IN3) together with the consistency relations (IN8). □

Theorem 3.27. *Let (X,N) be an intuitionistic – neutrosophic fuzzy normed linear space.*

a) *If (X,N) reduces to an $IFNLS$, then τ_N coincides with the topology induced by the corresponding intuitionistic fuzzy norm.*
b) *If (X,N) reduces to an $NFNLS$, then τ_N coincides with the topology induced by the corresponding neutrosophic fuzzy norm.*
c) *Assume, in addition to the axioms of an $INFNLS$, the followings hold*
   (i) $T_\mu(x,t) = 1$ *if and only if* $x = 0$ *(condition (IN6)),*
   (ii) $T_\mu(x,\cdot)$ *is strictly increasing for each* $x \neq 0$*,*
   (iii) $\lim\limits_{t\to 0^+} T_\mu(x,t) = 0$ *for every* $x \neq 0$*.*

Then the mapping $\|\cdot\|_{1/2} : X \to [0,\infty)$ defined by

$$\|x\|_{1/2} = \inf\{t > 0 : T_\mu(x,t) \geq 1/2\}$$

is a norm on X and the topology τ_N coincides with the norm topology induced by $\|\cdot\|_{1/2}$.

Proof. a) Assume (X,N) reduces to an intuitionistic fuzzy normed linear space ($IFNLS$). This means, according to Remark 3.3, that $I_\mu = I_\nu = F_\mu = F_\nu \equiv 0$, and $T_\mu(x,t) + T_\nu(x,t) \leq 1$.

In this case, a neighborhood $U(x_0,r,t)$ in the $INFNLS$ simplifies to

$$U_{\mathrm{IF}}(x_0,r,t) = \left\{ x \in X : \begin{matrix} T_\mu(x-x_0,t) > 1-r, \\ T_\nu(x-x_0,t) < r, \end{matrix} \right\},$$

By $T_\mu + T_\nu \leq 1$, the condition $T_\nu(x-x_0,t) < r$ is automatically implied by $T_\mu(x-x_0,t) > 1-r$ (since $T_\nu \leq 1 - T_\mu < 1-(1-r) = r$). Thus, $U_{\mathrm{IF}}(x_0,r,t) = \{x \in X : T_\mu(x-x_0,t) > 1-r\}$, which is exactly the standard neighborhood in an intuitionistic

fuzzy normed space (where the intuitionistic fuzzy norm is given by $(\mu, \nu) = (T_\mu, T_\nu)$). Consequently, the family $\{U_{\mathrm{IF}}(x_0, r, t)\}$ generates the usual intuitionistic fuzzy topology and it coincides with τ_N. Hence, τ_N equals the topology induced by the $IFNLS$.

b) Now suppose (X, N) reduces to a neutrosophic normed linear space, i.e., $(T_\nu, I_\nu, F_\nu) \equiv (0, 1, 1)$ (see Remark 3.3). Then a neighbourhood $U(x_0, r, t)$ becomes

because the conditions involving ν-components are automatically satisfied: $T_\nu(x - x_0, t) = 0 < r$, $I_\nu(x - x_0, t) = 1 > 1 - r$, $F_\nu(x - x_0, t) = 1 > 1 - r$. This is precisely the neighbourhood used in the theory of neutrosophic normed spaces with the neutrosophic norm (T_μ, I_μ, F_μ). Therefore, τ_N coincides with the topology induced by the $NFNLS$.

c) We verify the norm axioms. By definition, it is clear $\|x\|_{1/2} \geq 0$. If $x = 0$, then $T_\mu(0, t) = 1$ for all $t > 0$, so $\|0\|_{1/2} = 0$. Conversely, suppose $\|x\|_{1/2} = 0$, then for every $\varepsilon > 0$ there exists $t_\varepsilon < \varepsilon$ such that $T_\mu(x, t_\varepsilon) \geq 1/2$. By monotonicity, $T_\mu(x, \varepsilon) \geq T_\mu(x, t_\varepsilon) \geq 1/2$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ and using hypothesis (iii), we obtain $0 = \lim_{\varepsilon \to 0^+} T_\mu(x, \varepsilon) \geq 1/2$, a contradiction unless $x = 0$. Hence, $\|x\|_{1/2} = 0$ implies $x = 0$. For $\alpha \neq 0$, by using (IN2), we have:

$$\begin{aligned} \|\alpha x\|_{1/2} &= \inf\{t > 0 : T_\mu(\alpha x, t) \geq 1/2\} \\ &= \inf\{t > 0 : T_\mu(x, t/|\alpha|) \geq 1/2\} \\ &= \inf\{|\alpha| s : s > 0,\ T_\mu(x, s) \geq 1/2\} \\ &= |\alpha|\, \|x\|_{1/2}. \end{aligned}$$

The case $\alpha = 0$ is trivial.

For the triangle inequality, let $x, y \in X$ and $\varepsilon > 0$. By definition of $\|x\|_{1/2}$, there exists t_x such that

$$\|x\|_{1/2} \leq t_x < \|x\|_{1/2} + \varepsilon/2 \quad \text{and} \quad T_\mu(x, t_x) \geq 1/2.$$

Similarly, choose t_y with

$$\|y\|_{1/2} \leq t_y < \|y\|_{1/2} + \varepsilon/2 \quad \text{and} \quad T_\mu(y, t_y) \geq 1/2.$$

Set $a = \|x\|_{1/2} + \varepsilon/2$ and $b = \|y\|_{1/2} + \varepsilon/2$. Since $t_x < a$ and $T_\mu(x, \cdot)$ is strictly increasing, we have

$$T_\mu(x, a) > T_\mu(x, t_x) \geq 1/2.$$

Analogously, $T_\mu(y, b) > T_\mu(y, t_y) \geq 1/2$.

Now apply (IN3) with $t = a$, $s = b$:

$$T_\mu(x + y, a + b) \geq \min\{T_\mu(x, a),\ T_\mu(y, b)\} > 1/2.$$

By definition of infimum, $\|x+y\|_{1/2} \leq a+b = \|x\|_{1/2}+\|y\|_{1/2}+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|x+y\|_{1/2} \leq \|x\|_{1/2} + \|y\|_{1/2}$.

To prove topological equivalence, we want to show that a sequence $\{x_n\}$ converges to 0 in τ_N if and only if it converges to 0 in the norm $\|\cdot\|_{1/2}$.

($\Rightarrow$) Assume $x_n \to 0$ in τ_N. Then for $t = 1$ we have $T_\mu(x_n, 1)$-$\to 1$. Hence, there exists N such that $T_\mu(x_n, 1) > 1/2$ for all $n \geq N$, which means $\|x_n\|_{1/2} < 1$ for $n \geq N$. In fact, for any $\varepsilon > 0$, choose $t_\varepsilon > 0$ with $T_\mu(x_n, t_\varepsilon) > 1/2$ for large n, then $\|x_n\|_{1/2} < t_\varepsilon$. Since t_ε can be made arbitrarily small by taking n large enough, we obtain $\|x_n\|_{1/2} \to 0$.

($\Leftarrow$) Assume $\|x_n\|_{1/2} \to 0$. Fix $t > 0$ and $\delta \in (0, 1)$. Since $T_\mu(0, \cdot) \equiv 1$ and T_μ is continuous in t, there exists $\eta > 0$ such that $T_\mu(y, \eta) > 1 - \delta$ whenever $\|y\|_{1/2} < \eta$. (Indeed, by the strict monotonicity, $\|y\|_{1/2} < \eta$ implies $T_\mu(y, \eta) > 1/2$; by choosing η sufficiently small, the continuity of T_μ guarantees $T_\mu(y, \eta) > 1 - \delta$.) Since $\|x_n\|_{1/2} \to 0$, there exists N such that $\|x_n\|_{1/2} < \eta$ for all $n \geq N$. Then for $n \geq N$, $T_\mu(x_n, t) \geq T_\mu(x_n, \eta) > 1 - \delta$.

As $\delta > 0$ was arbitrary, $T_\mu(x_n, t) \to 1$. Analogous arguments using the corresponding inequalities in (IN3) give the convergence of the remaining five components.Hence, $x_n \to 0$ in τ_N, thus, convergence in $|\cdot|_{1/2}$ implies convergence in τ_N. □

4. Conclusion

To address the challenge of simultaneously modeling hesitation-based and multi-component uncertainty within a single framework, this paper has introduced the notion of an intuitionistic–neutrosophic fuzzy normed linear space ($INFNLS$), which unifies intuitionistic and neutrosophic fuzzy normed linear spaces. Its six-component definition captures dual uncertainty aspects through two neutrosophic triplets, allowing each traditional intuitionistic component to undergo neutrosophic analysis.

It has been shown that every $INFNLS$ induces a first-countable Hausdorff topology under which the algebraic operations are continuous, thereby providing the abstract structure with a concrete topological foundation. Completeness in finite dimensions has been established under the mild additional conditions (IN6) and (IN7). Moreover, the induced topology has been proved to reduce to those of $IFNLS$, $NFNLS$ and classical normed spaces in the appropriate special cases, confirming that $INFNLS$ offers a genuine generalization of these established frameworks.

The significance of this work is that it fills a recognized gap in the literature by providing a single analytical setting where layered uncertainty can be treated rigorously. By embedding both intuitionistic and neutrosophic structures within the well-developed theory of normed linear spaces, the $INFNLS$ framework opens several directions for future research, including infinite-dimensional analysis, fixed point theory and potential applications in decision making and engineering under uncertainty.

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