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Godunova-Levin-Chebaevskaya Inequality for Sugeno Integrals

Bayaz Daraby[1*] and Fariba Karimzadeh[2]

Abstract. This article concerns the well known Steffenson ineqaulity for fuzzy integrals. The Steffenson inequality has many refinenments such as the Fourier transform inequality, Abel inequality, Gauss inequality and Moment ineqialities. Besides the mentioned refinements, there is an another refinement so called Godunova-Levin-Chebaevskaya (GLC) inequality. The main aim to choose the GLC inequality to be inevstigated is the absence of research works on the GLC inequality neither in its classic form nor its fuzzy version. In order to obtain the fuzzy GLC inequality, we will attempt in the first step to obtain the fuzzy Steffenson inequality and relying on it fuzzy version of the fuzzy GLC inequality will be concluded. Several numerical example are given to prove validity of the obtained theoretical GLC inequality in pratice.

1. Introduction

For many conturies the mathematical sciences were known as tools to chracterize and description of the objective phnomena. L. A. Zadeh in 1965 broke down this believe by introducing the fuzzy set theory governing subjective phenomena.

Thanks to this innovation, all kind of non-deterministic phenomena can be investigated and modeled by the use of fuzzy theory. So it is reasonable that why nowadays fuzzy sets and fuzzy systems are applied in information sciences, uncertain control, pattern recognition, uncertain decision making and so on. For more consultation on the applications

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of fuzzy theory, the interested follower can see [11] and bibliography whithin.

Unerring, the fuzzy measure theory can be considered as the most applicable branch of fuzzy set theory. This is well known that one of the most common tools in mathematical modeling is integrals. Now, having the fuzzy measures in hand, one may define the fuzzy integrals and apply them in the models of non-deterministic phenomena and their engineering approaches. On the other hand, the fuzzy integrals are by themselves capable to be investigated as an independent research environment. In this way, the fuzzy oriented integration has some special behavior that distinguishes it from the classic integration. Actually comparing the classic integration with its fuzzy corresponding, it is figured out that fuzzy proper integrals do not obey the rules for classic integrals. So, this is worthy opportunity that we review the general history of fuzzy measures and fuzzy integrals. Indeed, fuzzy measures are special generalized measures for which the weaker condition of monotonocity is replaced by the additive property. The fuzzy measures that will be defined in the next section constitute the fuzzy integrals. The concept of fuzzy integral was initiated by Michio Sugeno in 1974 in grame of his PhD thesis and continued by other mathematicians. (See [3, 6, 14–17, 19, 22]).

Having a glance at the literature, one may find out that there are several definitions for the fuzzy integrals. Between this variety, the Sugeno integrals are the most considered approach for the fuzzy integration. These type of integrals are relied directly on the fuzzy measures and consequently characterized by them. In order to deeper dive in the ocean of fuzzy measures we suggest the following research works see: [19, 20].

In this position we are going to take a distence for a while from fuzzy environment and have a quick overview on the mathematical inequalities. Mathematical inequalities are one of the widest and at the same time one of the most important resarch fields in both of the applied and pure mathematical investigation. Actually, disregarding the use of inequalities, stops all kind of research ability in the mathematical as well as engineering societies. This is why we need to organize investigations on mathematical inequalities from both of the applied and abstract perspectives. During the past century specialists in the field of mathematical inequalities have been collected unbelievable number of inequalites together with their applications. See for instance [1, 2, 10, 12].

One of the most important and applicable mathematical inequalities is known as the Steffenson inequality. The Steffenson inequality has generated some of the important mathematical inequalities such as Fourier transform inequality, Abel inequality, Gauss inequality and moments

inequalities. But there is another refinement of the Steffenson inequality so-called Godunova-Levin-Chebaevskaya. The GLC inequality in the integral form represents an evaluation for the product of integrals with integral of products of functions. The literature review shows that investigation on the GLC inequality is rare even in the classic manner. In order to find more details on the Steffenson and GLC inequalities, we suggest the following bibliography, [5, 7, 8, 13, 18, 21].

So, based on the lack of research works on the fuzzy approach of the GLC inequality this reseanable that why we are interested in the initiating of investigation on the GLC inequality in the fuzzy space.

Here we state the organization of rest of paper as follows. In Section 2, all of requirements on the fuzzy integrals such as fuzzy measures and Sugeno integrals are presented. Also, we collect all necessary information related to the GLC inequalities that will be needed in the next section. Section 3 is considered as the main body of this paper, where we obtain a fuzzy version of GLC inequality. Each step of this section will be supported by numerical examples to show the efficiency of the obtained theoretical result. At last, we have Section 4 to summarize the undertaken article.

2. Technical Background

This section can be considered as the technical tool box to evaluate integral inequalities in the fuzzy measure spaces. In this way, the first tool is fuzzy measure that is given as follows.

Definition 2.1 ([9, 20])**.** Suppose that $\sum$ stands for the σ-algebra consisting of all subsets of $\mathbb{R}$. In this case, the non-negative real-valued set function $\nu : \sum \longrightarrow [0, 1]$ is said to be a fuzzy measure provided that the following properties are fulfilled:

($\mathfrak{M}_1$) $\nu(\emptyset) = 0$.

($\mathfrak{M}_2$) The monotonicity property:

$$\Theta_1 \subseteq \Theta_2 \quad \Rightarrow \quad \nu(\Theta_1) \leq \nu(\Theta_2), \quad \Theta_1, \Theta_2 \in \sum.$$

($\mathfrak{M}_3$) The continuity property from above:

$$\Theta_i \subseteq \Theta_{i+1} \quad \Rightarrow \quad \lim_{n\to\infty} \nu(\Theta_n) = \nu\left(\bigcup_{n=1}^{\infty} \Theta_n\right),$$
$$\Theta_n \in \sum, \quad i, n = 1, 2, \ldots.$$

($\mathfrak{M}_4$) The continuity property from below:

$$\Theta_i \supseteq \Theta_{i+1}, \quad \nu(\Theta_1) < \infty \quad \Rightarrow \quad \lim_{n\to\infty} \nu(\Theta_n) = \nu\left(\bigcap_{n=1}^{\infty} \Theta_n\right),$$

$$\Theta_n \in \sum, \quad i, n = 1, 2, \ldots.$$

In this case the triplet $(\sum, \nu, \mathbb{R})$ is called, the fuzzy measure space. Next, we have definition of ρ-level of a real-valued functions that will play an essential role in the computational approach of the fuzzy integrals.

Definition 2.2. ([4]) Suppose the non-negative and real-valued function Θ belngs to the given set V. By ρ-level of Θ that is denoted by $L_\rho\Theta$ for non-negative constant ρ, we mean

$$\begin{aligned} L_\rho\Theta &= \{z \in V | \Theta(z) \geq \rho\} \\ &= \{\Theta \geq \rho\} \\ &= \Theta_\rho. \end{aligned}$$

In the following a comparison result for the ρ-level functions is presented.

Corollary 2.3 ([4, 7])**.** *Assume ρ and ϱ are two non-negative constants with $\rho \leq \varrho$. In this case, for a given real-valued function Θ we have*

$$\Theta_\varrho \subseteq \Theta_\rho,$$

that is

$$L_\varrho\Theta \subseteq L_\rho\Theta.$$

We are now ready to define fuzzy integration. But, priorly we unify the fuzzy-measurable functions as below.

Definition 2.4 ([4])**.** Suppose ν is a fuzzy measure on $X \subseteq \mathbb{R}$. The set of ν-measurable functions $\mathfrak{F}^\nu$ is defined as follows:

$$\mathfrak{F}^\nu(X) = \{\Theta : X \longrightarrow \mathbb{R}_+ | \Theta : \nu - \text{measurable function}\}.$$

Next, we define the Sugeno integral as one of the most applied approaches of the fuzzy integration.

Definition 2.5 ([4],[19])**.** Suppose ν is a fuzzy measure belonging to the fuzzy measure space $(\sum, \nu, \mathbb{R})$. In the case that $\Theta \in \mathfrak{F}^\nu(\mathbb{R})$ with $I \in \sum$, the Sugeno integral of Θ with respect to the fuzzy measure ν on I is defined by:

$$\oint \Theta d\nu = \sup_{\rho \in [0, +\infty)} [\rho \wedge \nu(I \cap L_\rho\Theta)].$$

As special case f the Sugeno integral, we consider the case when $I = \mathbb{R}$. In this case, the Sugeno integral takes the form

$$\oint_{\mathbb{R}} \Theta d\nu = \sup_{\rho \in [0, +\infty)} [\rho \wedge \nu(L_\rho\Theta)].$$

It is time to present properties of the Sugeno interals. To this aim, we have the following technical lemma.

Lemma 2.6 ([4]). *Suppose that ν be a fuzzy measure on $\mathbb{R}$ and $\Theta, \Lambda \in \mathfrak{F}^{\nu}(Z)$. Then, the Sugeno interals satisfy the following properties:*

(S_1) $\oint \Theta d\nu \leq \nu(I)$,

(S_2) $\oint k d\nu = k \wedge \nu(I), \quad k \in \mathbb{R}_+$,

(S_3) *If* $\Theta \leq \Lambda$ *on* I, *then,* $\oint \Theta d\nu \leq \oint \Lambda d\nu$,

(S_4) $\nu(I \cap L_\rho \Theta) \geq \rho \quad \Rightarrow \quad \oint \Theta d\nu \geq \rho$,

(S_5) $\nu(I \cap L_\rho \Theta) \leq \rho \quad \Rightarrow \quad \oint \Theta d\nu \leq \rho$,

At the end of this section, we give a fully technical computational approach for the Sugeno integrals. Making use of the properties (S_4) and (S_5) in Lemma 2.6 and taking the notation

$$\Theta(\rho) = \nu(I \cap L_\rho \Theta),$$

it is clear that

$$\Theta(\rho) = \rho \quad \Leftrightarrow \quad \oint \Theta d\nu = \rho.$$

Equivalently, we can solve the fixed point problem

$$\Theta(\rho) = \rho,$$

instead of the Sugeno integral

$$\oint \Theta d\nu = \rho.$$

3. Steffenson Inequality

This section studies the theoretical backbone of this investigation. In order to reach the aim of this paper, we need to state and prove both of the classical and fuzzy Steffenson inequalities .

3.1. **Classical Case.** The first part of this section is devoted to the classic version of the steffenson inequality. This inequality initiated by the work of J.F. Steffenson (1918). In the following the classic version of this inequality is presented.

Theorem 3.1 ([19]). *Suppose ϕ and ψ both are integrable functions on the interval $[a, b]$, with ϕ is a decreasing function and $\psi \in [0, 1]$. Then,*

$$\int_{b-\lambda}^{b} \phi(z)dz \leq \int_{a}^{b} \phi(z)\psi(z)dz \leq \int_{a}^{a+\lambda} \phi(z)dz, \tag{3.1}$$

where

$$\lambda = \int_a^b \psi(z)dz.$$

Example 3.2. Let us consider the following setting:

$$\phi : [0,1] \longrightarrow [0,+\infty), \quad \phi(z) = 2 - z,$$
$$\psi : [0,1] \longrightarrow [0,+\infty), \quad \psi(z) = 0.5.$$

Clearly, ϕ and ψ, both meet the conditions of Theorem 3.1. A sequence of direct integrations gives us

$$\frac{5}{8} = \int_{0.5}^1 \phi(z)dz \leq \int_0^1 \phi(z)\psi(z)dz = \frac{6}{8} \leq \int_0^{0.5} \phi(z)dz = \frac{7}{8},$$

where

$$\lambda = \int_0^1 \psi(z)dz = 0.5 \quad .$$

This shows that the Steffenson inequality (3.1) is satisfied for the setting chosen in this application.

Since we are going to investigate on the GLC inequalities, so, we need to obtain special refinement of the Steffenson inequality (3.1) that keeps us in such way to reach the GLC inequality. To this aim, we state the following useful citeria for which the Steffenson inequality (3.1) is satisfied.

Remark 3.3. If we consider the statement of Theorem 3.1, this is easy to check that the Steffenson inequality (3.1) holds provided that $0 \leq \psi(z) \leq 1$. Based on this key point, we can now make the following criteria:
(3.2)

$$0 \leq \psi(z) \leq 1 \quad \Rightarrow \quad 0 \leq \int_a^x \psi(z)dz \leq x-a, \text{(second inequality in (3.1))}$$

also
(3.3)

$$0 \leq \psi(z) \leq 1 \quad \Rightarrow \quad 0 \leq \int_x^b \psi(z)dz \leq b-x, \text{(first inequality in (3.1)).}$$

In the following we choose an interesting setting to have a useful refinement of the Steffenson inequality.

$$\psi(z) = \frac{\lambda H(z)}{\int_a^b H(z)dz}, \quad \lambda > 0, \quad \int_a^b H(z)dz > 0.$$

Relying on this setting, the Steffenson inequality (3.1) takes the new form

$$\int_{b-\lambda}^{b}\phi(z)dz\leq\int_{a}^{b}\frac{\lambda\phi(z)H(z)}{\displaystyle\int_{a}^{b}H(z)dz}dz\leq\int_{a}^{a+\lambda}\phi(z)dz.$$

Equivalently, we have

$$\frac{1}{\lambda}\int_{b-\lambda}^{b}\phi(z)dz\leq\int_{a}^{b}\frac{\phi(z)H(z)}{\displaystyle\int_{a}^{b}\phi(z)dz}dz\leq\frac{1}{\lambda}\int_{a}^{a+\lambda}\phi(z)dz. \tag{3.4}$$

Concentrating on the inequalities (3.2) and (3.3), we come to the conclusion that the inequality (3.4) is satisfied provided that

$$0\leq\int_{a}^{x}\frac{\lambda H(z)dz}{\displaystyle\int_{a}^{b}H(z)dz}\leq x-a$$

$$\Rightarrow\quad 0\leq\lambda\int_{a}^{x}H(z)dz\leq(x-a)\int_{a}^{b}H(z)dz,\qquad\int_{x}^{b}H(z)dz\geq 0,$$

as well as

$$0\leq\int_{x}^{b}\frac{\lambda H(z)dz}{\displaystyle\int_{a}^{b}H(z)dz}\leq b-x$$

$$\Rightarrow\quad 0\leq\lambda\int_{x}^{b}H(z)dz\leq(b-x)\int_{a}^{b}H(z)dz,\qquad\int_{a}^{x}H(z)dz\geq 0.$$

3.2. **Fuzzy Case.** As the first step, we shall show that the second part of the Steffenson inequality is not satisfied in the fuzzy environment. In this way, we are going to use the setting of Example 3.2. So, let us have the following counter example.

Example 3.4. Consider both of the real-valued functions ϕ and ψ as definied in Example 3.2. Furthermore, from now on we take the fuzzy measure ν as the Lebesgue measure. In this case, based on the previous instructions on solvability of the Sugeno integrals, we get the following results:

$$\begin{aligned}\int_{0}^{1}\phi(z)\psi(z)d\nu&=\int_{0}^{1}\left(1-\frac{z}{2}\right)d\nu\\&=\sup_{\rho\geq 0}\left(\rho\wedge\nu\left([0,1]\cap\left\{1-\frac{z}{2}\geq\rho\right\}\right)\right)\\&=\sup_{\rho\geq 0}\left(\rho\wedge\nu\left([0,1]\cap\{z\leq 2(1-\rho)\}\right)\right)\end{aligned}$$

$$= \sup_{\rho \geq 0} (\rho \wedge \nu([0, 2(1-\rho)])) \\ = \sup_{\rho \geq 0} (\rho \wedge 2 - 2\rho).$$

Here, it just is needed to solve the fixed point problem

$$\rho = 2 - 2\rho,$$

that gives us the solution of the Sugeno integral

$$\oint_0^1 \phi(z)\psi(z)d\nu = \frac{2}{3}.$$

On the other hand, making use of the property (S_2) in Lemma 2.6, it follows that

$$\lambda = \oint_0^1 \psi(z)d\nu = \oint_0^1 \frac{1}{2}d\nu = \frac{1}{2} \wedge \nu[0,1] = \min\left(\frac{1}{2}, 1\right) = \frac{1}{2}.$$

A similar analysis shows that

$$\oint_0^\lambda \phi(z)d\nu = \frac{1}{2}.$$

This is obvious that

$$\oint_0^1 \phi(z)\psi(z)d\nu > \oint_0^\lambda \phi(z)d\nu = \frac{1}{2},$$

that contradicts the second part of the Steffenson inequality. This shows that the Steffenson inequality is not satisfied in the fuzzy platform.

In order to overcome this inappropriate situation, we need some additional hypothesis that enables us to reach the second part of the Stefenson inequality in the sense of Sugeno. This hypothesis is given as follows:

$$(\mathfrak{L}_{\mathfrak{S}}) \qquad \oint_a^b \phi(z)\psi(z)d\nu \leq \lambda = \min\left\{b-a, \oint_a^b \psi(z)d\nu\right\}.$$

Now, we can state and prove a fuzzy Steffenson inequality as follows.

Theorem 3.5. *Suppose the real-valued function $\phi : [a,b] \longrightarrow \mathbb{R}_+$ be nonincreasing and left continuous. Also, assume that $\psi : [a,b] \longrightarrow \mathbb{R}_+$ be a real-valued function with $\psi(z) \in [0,1]$. Then, the fuzzy Steffenson inequality*

$$\oint_a^b \phi(z)\psi(z)d\nu \leq \oint_a^{a+\lambda} \phi(z)d\nu, \tag{3.5}$$

holds, if and only if the general assumption $(\mathfrak{L}_{\mathfrak{S}})$ holds.

Proof. At the first step, we need to use this assumtion that ϕ is nonincreasing and left continuous function. In this case,

$$\{\phi \geq \rho\} = \left[a, \phi^{-1}(\rho)\right], \qquad \phi^{-1}(\rho) = \sup\{z : \phi(z) \geq \rho\}.$$

Relying on the definition of the Sugeno integral, we have

$$\begin{aligned} \oint_a^{a+\lambda} \phi(z) d\nu &= \sup_\rho \left[\rho \wedge \nu\left([a, a+\lambda] \cap \{\phi(z) \geq \rho\}\right)\right] \\ &= \sup_\rho \left[\rho \wedge \nu[a, a+\lambda] \wedge \left[a, \phi^{-1}(\rho)\right]\right] \\ &= \nu[a, a+\lambda] \sup_\rho \left[\rho \wedge \left[a, \phi^{-1}(\rho)\right]\right] \\ &= \lambda \wedge \sup_\rho \left[\rho \wedge \nu\left[a, \phi^{-1}(\rho)\right]\right]. \end{aligned} \tag{3.6}$$

The equalities (3.6) guarantee this fact that

$$\oint_a^{a+\lambda} \phi(z) d\nu = \lambda \wedge \oint_a^b \phi(z) d\nu. \tag{3.7}$$

It is time to use the monotonicity properties of ϕ and Sugeno integrals. Since $0 \leq \psi(z) \leq 1$, so, $0 \leq \phi(z)\psi(z) \leq \phi(z)$. Therefore, based on the monotonicity property (S_3) in Lemma 2.6, we arrive at the following integral inequality:

$$\oint_a^b \phi(z)\psi(z) d\nu \leq \oint_a^b \phi(z) d\nu. \tag{3.8}$$

On the other hand, in the case that the general assumption $(\mathfrak{L}_{\mathfrak{S}})$ holds, then, combination of this assumption and (3.8), yields

$$\oint_a^b \phi(z)\psi(z) d\nu \leq \lambda \wedge \oint_a^b \phi(z) d\nu. \tag{3.9}$$

Using (3.7), the Sugeno integral inequality (3.9) turns to

$$\oint_a^b \phi(z)\psi(z) d\nu \leq \oint_a^{a+\lambda} \phi(z) d\nu. \tag{3.10}$$

This completes the necessity part of the proof. In reverse, we prove the sufficiency part. To this aim, assume that the fuzzy Steffenson inequality (3.5) is satisfied. According to (3.10), we have

$$\begin{aligned} \oint_a^b \phi(z)\psi(z) d\nu &\leq \oint_a^{a+\lambda} \phi(z) d\nu \\ &= \lambda \wedge \oint_a^b \phi(z) d\nu \\ &= \min\left\{\lambda, \oint_a^b \phi(z) d\nu\right\}. \end{aligned}$$

Hence, we directly find out that

$$\int_a^b \phi(z)\psi(z)d\nu \leq \lambda.$$

Now the sufficienccy part of the proof is completed. □

Example 3.6. In this position, we give a numerical example that demonstrates validity of the obtained theoretical criterion in Theorem 3.5. To this aim, we shoose the following setting:

$$\phi : [0,1] \longrightarrow [0,+\infty], \quad \phi(z) = \exp(-z), \psi \quad : [0,1] \longrightarrow [0,+\infty], \quad \psi(z) = 0.9.$$

Now, we begin to solve the Sugeno integrals as follows:

$$\begin{aligned}
\int_0^1 \phi(z)\psi(z)d\nu &= \int_0^1 0.9\exp(-z)d\nu \\
&= \sup_{\rho\geq 0}[\rho \wedge \nu[[0,1] \cap \{0.9\exp -z \geq \rho\}]] \\
&= \sup_{\rho\geq 0}\left[\rho \wedge \nu\left[[0,1] \cap \left\{z \leq -\ln\left(\frac{10}{9}\rho\right)\right\}\right]\right] \\
&= \sup_{\rho\geq 0}\left[\rho \wedge \nu\left[0, -\ln\left(\frac{10}{9}\rho\right)\right]\right] \\
&= \sup_{\rho\geq 0}\left[\rho \wedge -\ln\left(\frac{10}{9}\rho\right)\right].
\end{aligned}$$

Next, we have to solve the fixed point problem

$$-\ln\left(\frac{10}{9}\rho\right) = \rho.$$

After some simplification, we arrive at the following

$$\rho\exp(\rho) = 0.9.$$

Approximates solution of this algebraic equation is 0.529833. So, we have

$$\int_0^1 \phi(z)\psi(z)d\nu \approx 0.529833.$$

With a similar computational approach, it is easy to check that

$$\int_0^{0.9} \phi(z)d\nu = 0.9.$$

Also, based on the property (S_2) in Lemma 2.6, we get that

$$\lambda = \int_0^1 \psi(z)d\nu = \int_0^1 0.9d\nu = \min\{1, 0.9\} = 0.9 \quad .$$

We now collect the obtained numerical data to reach this fact the fuzzy steffenson inequality

$$0.529833 \approx \fint_0^1 \phi(z)\psi(z)d\nu \leq \fint_0^\lambda \phi(z)d\nu = 0.9, \tag{3.11}$$

holds. On the other hand, considering the general assumption $(\mathfrak{L}_{\mathfrak{S}})$, we have

$$\begin{aligned}\lambda &= \min\left\{b-a, \fint_a^b \psi(z)d\nu\right\}\\ &= \min\left\{1, \fint_0^1 0.9d\nu\right\}\\ &= 0.9.\end{aligned}$$

This proves that the fuzzy steffenson inequality (3.11) is satisfied if and only if the $(\mathfrak{L}_{\mathfrak{S}})$ holds, that is results of Theorem 3.5 is valid in practice.

4. Godunova-Levin-Chebaevskaya Inequality

All of the results obtained for the fuzzy Steffenson inequalities in the previous section enable us to study Sugeno approach of the GLC inequality. But, priorly we have to focus on the classical version of the GLC inequality.

4.1. **Classical Case.** In the following, we state the original version of the GLC inequality and give a detailed proof for it that will make the foundation of its fuzzy refinement.

Theorem 4.1 ([2])**.** *Suppose $\phi(z)$ be a nonnegative and nonincreasing function on $[a,b]$, and $\varphi(u)$ be an increasing convex operator for $u \geq 0$ with $\varphi(0) = 0$. If $\psi(z)$ is a nonnegative nondecreasing function on $[a,b]$ such that there exist a nonnegative function $\psi_1(z)$ defined by the equation*

$$\psi_1(z)\varphi\left(\frac{\psi(z)}{\psi_1(z)}\right) = 1, \qquad \int_a^b \psi_1(z)dz = 1,$$

then, the following inequality is valid:

$$\varphi\left(\frac{\int_a^b \phi(z)\psi(z)dz}{\int_a^b \psi(z)dz}\right) \leq \frac{1}{\lambda}\int_a^{a+\lambda} \varphi(\phi(z))dz, \qquad \lambda = \varphi\left(\int_a^b \psi(z)dz\right). \tag{4.1}$$

Remark 4.2. Let us consider the following setting for the convex operator φ,

$$\varphi(u) = u^p, \quad p > 1.$$

Then, the original GLC inequality (4.1) takes the form

$$\left(\frac{\int_a^b \phi(z)\psi(z)dz}{\int_a^b \psi(z)dz}\right)^p \leq \frac{1}{\left(\int_a^b \psi(t)dz\right)^p}\int_a^{a+\lambda}\phi^p(z)dz. \tag{4.2}$$

Equivalently, we have the most common version of the GLC inequality as follows:

$$\int_a^b \phi(z)\psi(z)dz \leq \left(\int_a^{a+\lambda}\phi^p(z)dz\right)^{\frac{1}{p}}, \qquad \lambda = \left(\int_a^b \psi(z)dz\right)^p. \tag{4.3}$$

Example 4.3. We finalize investigation on the classical version of GLC inequality by validating the theoretical inequality (4.3) in the numerical manner. To this aim, we choose the following setting:

$$\begin{aligned}
&\phi : [0,1] \longrightarrow [0,+\infty], \qquad \phi(z) = 3 - \frac{z}{2},\\
&\psi : [0,1] \longrightarrow [0,+\infty], \qquad \psi(z) = \sqrt{3}z,\\
&\psi_1 : [0,1] \longrightarrow [0,+\infty], \qquad \psi_1(z) = 3z^2.
\end{aligned}$$

It is easy to check that ϕ and ψ, both satisfy the hypotheses of Theorem 4.1. In addition, Choosing $p = 2$, a direct computation shows that

$$\begin{aligned}
\phi_1\left(\frac{\psi}{\psi_1}\right)^2 &= \frac{\psi^2(z)}{\psi_1(z)} = \frac{\left(\sqrt{3}z\right)^2}{3z^2} = 1.\\
\int_0^1 \psi_1(z)dz &= \int_0^1 3z^2dz = 1.
\end{aligned}$$

Aforementioned setting and related computations reveal that all conditions of Theorem 4.1 hold. So, it has to be shown that the GLC inequality (4.2) is satisfied. To this aim, we have

$$\begin{aligned}
&\int_0^1 \phi(z)\psi(z)dz = \sqrt{3}\int_0^1 z\left(3 - \frac{z}{2}\right)dz = \frac{4}{\sqrt{3}} \approx 2.309401.\\
&\lambda = \left(\int_0^1 \psi(z)\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.\\
&\left(\int_0^\lambda \phi^2(z)dz\right)^{\frac{1}{2}} = \left(9z + \frac{z^3}{12} - \frac{3}{2}z^2\Big|_0^{\frac{3}{4}}\right)^{\frac{1}{2}} \approx 2.4375.
\end{aligned}$$

Considering the above calculations, we get that

$$2.309401 \approx \int_0^1 \phi(z)\psi(z)dz \leq \left(\int_0^\lambda \phi^2(z)dz\right)^{\frac{1}{2}} \approx 2.4375,$$

that is, the GLC inequality for the above setting is satisfied.

4.2. **Fuzzy Case.** This is the last part of our investigation, where we shall state and prove the fuzzy approach of the GLC inequality. But, priorly, it will be shown that under the classical setting the fuzzy version of the GLC inequality may not be satisfied. To this aim, we have the following counter example.

Example 4.4. Let us consider the same setting as have chsoen for the classical GLC inequality in Example 4.3. In this case, one has:

$$\begin{aligned}(4.4)\quad \oint_0^1 \phi(z)\psi(z)d\nu &= \sup_{\rho\geq 0}\left[\rho \wedge \nu\left[[0,1]\cap\left\{\sqrt{3}\left(3z-\frac{z^2}{2}\right)\geq\rho\right\}\right]\right]\\ &= \sup_{\rho\geq 0}\left[\rho \wedge \nu\left[[0,1]\cap\left\{z^2-6z+\frac{2}{\sqrt{3}\rho}\leq 0\right\}\right]\right]\\ &= \sup_{\rho\geq 0}\left[\rho\wedge 3-\sqrt{9-\frac{2}{\sqrt{3}}\rho}\right]\approx 4.845299,\end{aligned}$$

and

$$\lambda = \left(\oint_0^1 \psi(z)d\nu\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4},$$

also

(4.5)

$$\begin{aligned}\left(\oint_0^\lambda \phi^2(z)d\nu\right)^{\frac{1}{2}} &= \left(\sup_{\rho\geq 0}\left[\rho\wedge\nu\left[\left[0,\frac{3}{4}\right]\cap\left\{\frac{z^2-12z+36}{4}\geq\rho\right\}\right]\right]\right)^{\frac{1}{2}}\\ &= \left(\sup_{\rho\geq 0}\left[\rho\wedge\nu\left[\left[0,\frac{3}{4}\right]\cap\left\{z^2-12z+36\geq 4\rho\right\}\right]\right]\right)^{\frac{1}{2}}\\ &= \left(\sup_{\rho\geq 0}\left[\rho\wedge\nu\left[0,6-2\sqrt{\rho}\right]\right]\right)^{\frac{1}{2}}\\ &= \left(\sup_{\rho\geq 0}\left[\rho\wedge 6-2\sqrt{\rho}\right]\right)^{\frac{1}{2}}\end{aligned}$$

$$\approx 1.645751.$$

Sugeno integrals (4.4)-(4.5), indicate that the fuzzy approach of the GLC inequality (4.3) is no longer satisfied. So, we need such a strategy that guarantees the fulfilment of GLC inequality in the fuzzy measure spaces. To this aim, we have the following theorem.

Remark 4.5. Prior to present fuzzy approach for the GLC inequality, this is worth mentioning that if we consider the statement of Theorem 4.1, it is figured out that actually, both of the functions ϕ and ψ belong to the L_p functional spaces, that is $\phi \in \mathcal{L}^p$ and $\psi \in \mathcal{L}^q$. So, in the following, the statement of theorem is updated based on this remark.

Here we state and prove a technical lemma that will be useful in the proof of the main theorem of this investigations.

Lemma 4.6. *Let us define the function $T(z)$ as follows:*

$$T(z) = \frac{1}{\min\{1, \mu[a,z]\}} \oint_a^z \phi(t) d\nu = \begin{cases} \oint_a^z \phi(t) d\nu, & \mu[a,z] \geq 1, \\ \dfrac{\oint_a^z \phi(t) d\nu}{\nu[a,z]}, & 0 < \nu[a,z] < 1. \end{cases} \tag{4.6}$$

Then, $T(z)$ is an increasing function.

Proof. Based on the fundamental theorem of fuzzy calculus, we have that

$$T'(z) = \phi(z) \geq 0, \qquad \nu[a,z] > 1. \tag{4.7}$$

Now for the case $0 < \nu[a,z] < 1$, we choose $t_1, t_2 \in [a,z]$ such that $a \leq t_1 < t_2 \leq z$. Then,

(4.8)

$$\begin{aligned} T(t_2) - T(t_1) &= \frac{1}{\nu[a,t_2]} \oint_a^{t_2} \phi(z) d\nu - \frac{1}{\nu[a,t_1]} \oint_a^{t_1} \phi(z) d\nu \\ &= \frac{1}{\nu[a,t_1]\nu[a,t_2]} \left\{ \nu[a,t_1] \oint_a^{t_2} \phi(z) d\nu - \nu[a,t_2] \oint_a^{t_1} \phi(z) d\nu \right\} \\ &= \frac{1}{\nu[a,t_1]\nu[a,t_2]} \left\{ \nu[a,t_1] \oint_{t_1}^{t_2} \phi(z) d\nu + \nu[t_1,t_2] \oint_a^{t_1} \phi(z) d\nu \right\} \\ &\geq 0. \end{aligned}$$

This analysis shows that the function $T(z)$ defined by (4.6) is increasing that is

$$z \leq b \quad \Rightarrow \quad T(z) \leq T(b). \qquad \square$$

Theorem 4.7. *Suppose $\phi \in \mathcal{L}^p, \psi \in \mathcal{L}^q$, with*

$$p > 1, \qquad q = \frac{p}{p-1}, \qquad \oint_a^b \psi^q(z)d\nu \leq 1.$$

Let ϕ is nonnegative, nonincreasing and left continuous. Also asuume ψ is nonnegative and nondcreasing, for which there exists a nonnegative function ψ_1 such that

$$\psi_1(z)\varphi\left(\frac{\psi(z)}{\psi_1(z)}\right) = 1, \qquad \oint_a^b \psi_1(z)dz = 1.$$

Note that by $\varphi(u), u \in \mathbb{R}_+$, we mean an increasing convex operator with property $\varphi(0) = 0$. Then the fuzzy GLC inequality

$$\oint_a^b \phi(z)\psi(z)d\nu \leq \left(\oint_a^{a+\lambda} \phi^p(z)d\nu\right)^{\frac{1}{p}}, \qquad \lambda = \left(\oint_a^b \psi(z)d\nu\right)^p,$$

holds, provided that the assumption $(\mathfrak{L}_{\mathfrak{S}})$ is satisfied.

Proof. It has previously proven that the general assumption $(\mathfrak{L}_{\mathfrak{S}})$ holds if and only if the fuzzy Steffenson-type inequality (3.5) is satisfied. On the other hand, the fuzzy Steffenson-type inequality (3.5) holds provided that $0 \leq \psi \leq 1$, that is

$$0 \leq \oint_a^b \psi(z)d\nu \leq \oint_a^b d\nu = \min\{1, \nu[a,b]\}. \tag{4.9}$$

So, we will focus on proving the conditions (4.9). In this way, let us define the function ψ as follows:

$$\psi(z) = \frac{\lambda H(z)}{\oint_a^b H(z)d\nu}, \qquad \lambda > 0 \qquad , \oint_a^b H(z)d\nu > 0.$$

In this case, the fuzzy Steffenson-type inequality (3.5) takes the form

$$\oint_a^b \frac{\lambda\phi(z)H(z)}{\oint_a^b H(z)d\nu}d\nu \leq \oint_a^{a+\lambda} \phi(z)d\nu. \tag{4.10}$$

Consequently, the fuzzy Steffenson-type inequality (4.10) holds if and only if the inequality

$$\lambda \oint_a^z \psi(t)d\nu \leq \min\{1, \nu[a,b]\} \oint_a^b \psi(z)d\nu, \qquad \oint_z^b \psi(t)d\nu \geq 0, \tag{4.11}$$

is satisfied. By now, in the light of general assumtion $\mathfrak{L}_{\mathfrak{S}}$ it has shown that the fuzzy Steffenson inequality (3.5) is satisfied, and equivalently the conditions (4.9) hold.

In continuation, we follow the arguments presented in Remark 3.3, that yields the followings:

$$\lambda \oint_a^z \psi(t)d\nu \leq \min\{1, \nu[a,b]\} \oint_a^z \psi(t)d\nu, \qquad \lambda = \left(\oint_a^b \psi(z)d\nu \right)^p.$$

According to the Lemma 4.6, we have

$$\frac{1}{\min\{1, \nu[a,z]\}} \oint_a^z \phi(t)d\nu \leq \frac{1}{\min\{1, \nu[a,b]\}} \oint_a^b \phi(z)d\nu.$$

So, we have

$$\min\{1, \nu[a,b]\} \oint_a^z \phi(t)d\nu \leq \min\{1, \nu[a,z]\} \oint_a^b \phi(Z)d\nu. \tag{4.12}$$

Now, we just need to combine the inequalities (4.11) and (4.12) to reach the

$$\lambda \oint_a^z \psi(t)d\nu \leq \min\{1, \nu[a,z]\} \oint_a^b \phi(z)d\nu, \qquad \oint_z^b \psi(t)d\nu \geq 0. \tag{4.13}$$

The inequality (4.13) equivalently implies the fuzzy Steffenson-type inequality (4.10). In other words, we have reached the following inequality:

$$\oint_a^b \phi(z)\psi(z)d\nu \leq \int_a^{a+\lambda} \phi(z)d\nu, \qquad \lambda = \left(\oint_a^b \psi(z)d\nu \right)^p.$$

At the final step, considering the convex operator $\varphi(u) = u^p, p > 1$, and apply it directly on both sides of the fuzzy Steffenson inequality (4.10), we come to the conclusion that

$$\oint_a^b \phi(z)\psi(z)d\nu \leq \left(\int_a^{a+\lambda} \phi^p(z)d\nu \right)^{\frac{1}{p}}, \qquad \lambda = \left(\oint_a^b \phi(z)d\nu \right)^p.$$

Now, the proof is completed. □

Here we give an example to indicate that Theorem 4.7 is valid in practice.

Example 4.8. We consider the following setting to examine the validity of Theorem 4.7,

$$\begin{aligned} &\phi : [0,1] \longrightarrow [0, +\infty], \quad \phi(z) = 1 - z, \\ &\psi : [0,1] \longrightarrow [0, +\infty], \quad \psi(z) = \sqrt{2}, \\ &\psi_1 : [0,1] \longrightarrow [0, +\infty], \quad \psi_z = 2. \end{aligned}$$

Clearly, the setting chosen above meet all conditions of Theorem 4.7. Now, chossing $p = 2$, a direct calculation indicates that

$$\begin{aligned}
\int_0^1 \phi(z)\psi(z)d\nu &= \int_0^1 \sqrt{2}(1-z)d\nu \\
&= \sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[[0,1]\cap\{\sqrt{2}(1-z)\geq\rho\}\right]\right\} \\
&= \sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[[0,1]\cap\left\{z\leq\left(1-\frac{1}{\sqrt{2}}\right)\rho\right\}\right]\right\} \\
&= \sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[0,\left(1-\frac{1}{\sqrt{2}}\right)\rho\right]\right\} \\
&= \sup_{\rho\geq 0}\left\{\rho \wedge \left(1-\frac{1}{\sqrt{2}}\right)\rho\right\} \\
&\approx 0.585786,
\end{aligned}$$

and

$$\lambda = \left(\int_0^1 \psi(z)d\nu\right)^2 = \left(\min\{1,\sqrt{2}\}\right)^2 = 1,$$

also

$$\begin{aligned}
\left(\int_0^1 \phi^2(z)d\nu\right)^{\frac{1}{2}} &= \left(\int_0^1 (1-z)^2 d\nu\right)^{\frac{1}{2}} \\
&= \left(\sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[[0,1]\cap\left\{(1-z)^2\geq\rho\right\}\right]\right\}\right)^{\frac{1}{2}} \\
&= \left(\sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[[0,1]\cap\left\{z\leq 1-\sqrt{\rho}\right\}\right]\right\}\right)^{\frac{1}{2}} \\
&= \left(\sup_{\rho\geq 0}\left\{\rho \wedge \nu\left[0,1-\sqrt{\rho}\right]\right\}\right)^{\frac{1}{2}} \\
&\approx 1.236068.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\int_0^1 \phi(z)\psi(z)d\nu &\approx 0.585786 \\
&\leq \min\left\{1-0,\int_0^1 \psi(z)d\nu\right\}
\end{aligned}$$

$$= \min\left\{1, \fint_0^1 \sqrt{2} d\nu\right\}$$
$$= \min\{1, \sqrt{2}\} = 1,$$

Since, the general assumption $\mathfrak{L}_{\mathfrak{S}}$ is satisfied, so, this is obvious that the fuzzy GLC inequality

$$\fint_0^1 \phi(z)\psi(z)d\nu \leq \left(\fint_0^1 \phi^2(z)d\nu\right)^{\frac{1}{2}},$$

holds.

5. Concluding Remarks

The Godunova-Levin-Chebaevskaya inequality has chosen to be refined in the fuzzy measure space. The Steffenson inequalties are the corner-stone of the GLC inequalities. So, we had to obtain the fuzzy Steffenson inequalities. To this aim, provding some expressive examples it has shown that we should not expect that all of the classical functional inequalities have to be satisfied in the fuzzy spaces. Therefore, we need to some additional assumptions that help us to obtain fuzzy refinement of the Steffenson inequality. This process has done in Section 3. Next, we studied the classical GLC inequality and it has shown that the existing results for the GLC inequality are not satisfied in the fuzzy case. To overcome this difficulty, we have used the new fuzzy Steffenson inequality to prove the fuzzy GLC inequality. Some numerical examples showed that the obtained results are valid in practice. Since, investigation on the fuzzy inequalities is a dynamic research field, so in the future works wide range of classical inequalities will be considered to be refined in the fuzzy measure spaces. As the main research area, we will be concentrated on the inequalities in control theory.

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