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Retarded Half-Linear Integral Inequalities on Time Scales: Advances and Applications

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Abstract. This paper investigates retarded half-linear integral inequalities on time scales. Our results extend several known integral inequalities as well as recent contributions in the literature. Moreover, they provide useful tools for analyzing the qualitative behavior of solutions to dynamic equations on time scales.

1. Introduction

The study of dynamic inequalities on time scales plays a fundamental role in analyzing the qualitative properties of dynamic equations, such as stability, boundedness and oscillatory behavior [2, 4]. Integral inequalities, particularly those of the Gronwall–Bellman type, are essential tools in the study of differential and difference equations, providing explicit estimates for solutions and enhancing the understanding of dynamic systems.

Throughout this paper, we denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+ = [0, +\infty)$ represents the set of nonnegative real numbers. Let $\mathbb{T}$ be an arbitrary time scale, i.e., a nonempty closed subset of $\mathbb{R}$. The set $\mathbb{T}^k$ is defined as follows: If $\mathbb{T}$ has a maximum m and m is left-scattered, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}$, $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$, $\mathbb{T}_0 \subset \mathbb{T}^k$. The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is defined by

$$\mu(t) = \sigma(t) - t,$$

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where the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}.$$

Furthermore, the circle plus addition $\oplus$ is defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all } t \in \mathbb{T}^k.$$

In recent years, many researchers have focused on developing new integral inequalities to establish explicit solution bounds and deepen the understanding of dynamic systems. For instance, Sun and Hassan [9] investigated a class of nonlinear integral inequalities on time scales of the form:

(1.1)

$$y(t) \leq f(t) + g(t) \int_{t_0}^{t} \left[a(s)y(s) + b(s)y^{\alpha}(\sigma(s)) - c(s)y^{\beta}(\sigma(s))\right] \Delta s,$$

where $y, f, g, a, b, c : \mathbb{T}^k \to \mathbb{R}_+$ are rd-continuous functions and α, β are positive constants such that $0 < \alpha < 1 < \beta$.

Similarly, Tian et al. [10] extended these results by considering the more complex inequality:

(1.2)

$$y^p(t) \leq f(t) + g(t) \int_{t_0}^{t} \Bigg[a(s)y^p(s) + b(s)y^q(s) + c(s) + \int_{t_0}^{s} m(\tau)y^r(\tau)\,\Delta\tau + h_1(s)y^{\alpha}(\sigma(s)) - h_2(s)y^{\beta}(\sigma(s))\Bigg] \Delta s,$$

where $p \geq q > 0$, $p \geq r > 0$, $0 < \alpha < p < \beta$, p, q, r, α and β are real constants and $y, f, g, a, b, c, m, h_1, h_2 : \mathbb{T}^k \to \mathbb{R}_+$ are rd-continuous functions.

It is noted that the inequalities (1.1) and (1.2) have been proven in the cases when $0 < \alpha < 1 < \beta$ and $0 < \alpha < p < \beta$, respectively. Therefore, it would be of interest to find the explicit bound for u in inequality (1.2) in the following two cases: $0 < p < \alpha < \beta$ or $0 < \beta < \alpha < p$.

In addition, Liu [7] established a new class of half-linear integral inequalities on time scales, incorporating a functional delay term:

$$\begin{aligned} x^p(t) \leq f(t) + g(t) \int_{t_0}^{t} \Bigg[& a(s)x^p(s) + b(s)x^q(s) + c(s) \\ & + m(s) \int_{\theta(t_0)}^{\theta(s)} n(\xi)\, x^r(\xi)\, \Delta\xi \Bigg] \Delta s \\ & + l(t) \int_{t_0}^{t} \left[h(s)x^{\alpha}(\sigma(s)) - k(s)x^{\beta}(\sigma(s))\right] \Delta s. \end{aligned}$$

In the present study, we extend and generalize the findings of [3] by exploring a broader class of integral inequalities on time scales. While previous research has primarily focused on $\mathbb{R}_+$, our approach introduces the study of half-linear integral inequalities within the context of time scales, thereby offering novel insights into the qualitative behavior of solutions to dynamic equations. The structure of the paper is as follows: Section 2 is devoted to preliminaries and essential background on time scales and integral inequalities. Section 3 presents the formulation and proof of the main results, while Section 4 provides an application of our results. to study the boundedness and qualitative behavior of solutions of a retarded half-linear dynamic equation.

2. Preliminaries

Lemma 2.1 ([7]). *Let $m > 0, n > 0, p > 0, \alpha > 0$, and $\beta > 0$ be given. Then, for each $x \geq 0$,*

$$mx^{\alpha} - nx^{\beta} \leq \frac{m(\beta-\alpha)}{\beta-p}\left(\frac{(\beta-p)n}{(\alpha-p)m}\right)^{\frac{\alpha-p}{\alpha-\beta}} x^p,$$

holds for the cases when $0 < p < \alpha < \beta$ or $0 < \beta < \alpha < p$.

Lemma 2.2 ([5]). *Let $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \quad \text{for any } K > 0.$$

We can get the following exceptional cases.

- For $K = 1$, we get

$$a^{\frac{q}{p}} \leq \frac{q}{p} a + \frac{p-q}{p}, \quad a \geq 0,\ p \geq q > 0.$$

- For $K = 1$ and $p = 1$, we have

$$a^q \leq qa + (1-q), \quad a \geq 0,\ 0 < q \leq 1.$$

Lemma 2.3 ([11]). *Assume that $u, v \geq 0$ and $p > 0$. Then*

$$(u+v)^p \leq K_p(u^p + v^p),$$

where $K_p = 1$, $0 \leq p \leq 1$, and $K_p = 2^{p-1}$, $p \geq 1$.

Lemma 2.4 ([1]). *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Lemma 2.5 ([6]). *Let $\alpha : \mathbb{T}_0 \to \mathbb{T}_0$ be a continuous and strictly increasing function such that $\alpha(t) \leq t$ and α^{Δ} is rd-continuous. Assume that $f : \mathbb{T}_0 \to \mathbb{R}$ is an rd-continuous function. Then*

$$g(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(s)\,\Delta s, \quad t \in \mathbb{T}_0,$$

implies

$$g^{\Delta}(t) = f(\alpha(t))\,\alpha^{\Delta}(t), \quad t \in \mathbb{T}_0.$$

Lemma 2.6 ([1]). *Suppose that for each $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left|w(\sigma(t),\tau) - w(s,\tau) - w_t^{\Delta}(t,\tau)\,(\sigma(t) - s)\right| \leq \varepsilon\,|\sigma(t) - s|, \quad s \in \mathcal{U},$$

where $w : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), $t \in \mathbb{T}^k$ with $t > t_0$ and $w_t^{\Delta}(t,.)$ are rd continuous on $[t_0, \sigma(t)]$. Then

$$g(t) := \int_{t_0}^{t} w(t,\tau)\Delta\tau,$$

implies

$$g^{\Delta}(t) = \int_{t_0}^{t} w_t^{\Delta}(t,\tau)\Delta\tau + w(\sigma(t),t), \quad t \in \mathbb{T}^k.$$

Lemma 2.7 ([1]). *Suppose that y and f are rd-continuous functions and $p \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t), \quad t \in \mathbb{T},$$

implies

$$y(t) \leq y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))\,f(\tau)\,\Delta\tau, \quad t \in \mathbb{T}.$$

Lemma 2.8 ([8]). *Let $u(t), f(t), g(t) \in C_{rd}$ be nonnegative functions and assume that $f(t)$ is nondecreasing. Then*

$$u(t) \leq f(t) + \int_{t_0}^{t} g(\tau)\,u(\tau)\,\Delta\tau, \quad t \in \mathbb{T}_0,$$

implies

$$u(t) \leq f(t)\,e_g(t,t_0), \quad t \in \mathbb{T}_0.$$

We state the following auxiliary lemma for later use.

Lemma 2.9. *Let $u(t), f(t), g(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. Let $\alpha(t)$ be a continuous strictly increasing function on $\mathbb{T}_0$ such that $\alpha(t) \leq t$ and α^{Δ} is rd-continuous. If*

$$u(t) \leq f(t) + \int_{\alpha(t_0)}^{\alpha(t)} g(\tau)\,u(\tau)\,\Delta\tau, \quad t \in \mathbb{T}_0,$$

then

$$u(t) \leq e_G(t, t_0)\, f(t),$$

where

$$G(t) = \alpha^{\Delta}(t)\, g(t).$$

Proof. The proof follows directly by applying the ε-perturbation argument as in Lemma 2.8 from [8], with the upper limit of integration t replaced by the retarded function $\alpha(t)$. □

3. Main Results

Lemma 3.1. *Let $m > 0, n > 0, p > 0, \alpha > 0$ and $\beta > 0$ be constants. Suppose that $N : [0, \infty) \to [0, \infty)$ be a continuously differentiable function on $[0, b]$ $(b > 0)$ with $N(0) = 0$, then for all $x \geq 0$ with $x^{\alpha}, x^{\beta} \in [0, b]$ we have*

$$mN\left(x^{\alpha}\right) - nN\left(x^{\beta}\right) \leq \xi\left(m, n, k, \alpha, \beta, p\right) x^{p},$$

where

$$\xi(m, n, k, \alpha, \beta, p) = \frac{mk(\beta - \alpha)}{\beta - p} \left(\frac{(\beta - p)n}{(\alpha - p)m}\right)^{\frac{\alpha - p}{\alpha - \beta}} \tag{3.1}$$

if $0 < p < \alpha < \beta$ or $0 < \beta < \alpha < p$.

Proof. Case $x = 0$. Then

$$mN(0) - nN(0) = 0,$$

so the inequality clearly holds.

Assume that $x > 0$. Then

$$\begin{aligned} mN(x^{\alpha}) - nN\left(x^{\beta}\right) = {} & \frac{m - n}{2}\left(N(x^{\alpha}) + N\left(x^{\beta}\right)\right) \\ & + \frac{m + n}{2}\left(N(x^{\alpha}) - N\left(x^{\beta}\right)\right). \end{aligned} \tag{3.2}$$

Since N is continuously differentiable, then it is k-Lipschitzian. Using $N(0) = 0$ and the non-negativity of N, we have

$$N\left(x^{\alpha}\right) = N\left(x^{\alpha}\right) - N\left(0\right) \leq kx^{\alpha}, \tag{3.3}$$

and

$$N\left(x^{\beta}\right) = N\left(x^{\beta}\right) - N\left(0\right) \leq kx^{\beta}. \tag{3.4}$$

On the other hand from the Mean Value Theorem, there exists some c between x^{α} and x^{β} such that

$$N\left(x^{\alpha}\right) - N\left(x^{\beta}\right) = N'\left(c\right)\left(x^{\alpha} - x^{\beta}\right). \tag{3.5}$$

Substituting (3.3)–(3.5) into (3.2), we obtain

$$mN(x^{\alpha}) - nN\left(x^{\beta}\right) \leq \tfrac{m-n}{2} k\left(x^{\alpha} + x^{\beta}\right) + \tfrac{m+n}{2} N'(c)\left(x^{\alpha} - x^{\beta}\right) \tag{3.6}$$

$$\leq \tfrac{m-n}{2}k\left(x^{\alpha}+x^{\beta}\right)+\tfrac{m+n}{2}k\left(x^{\alpha}-x^{\beta}\right)$$
$$= mkx^{\alpha}-nkx^{\beta},$$

where $k=\sup_{u\in[a,b]} N'(u)$. Finally, applying Lemma 2.1 to (3.6), we deduce $mN(x^{\alpha})-nN\left(x^{\beta}\right)\leq \xi(m,n,k,\alpha,\beta,p)\,x^{p}$, where $\xi(m,n,k,\alpha,\beta,p)$ is defined as in (3.1). □

Theorem 3.2. *Assume that $u,a,c,f,g,b,d\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. Let a be a non-decreasing function and let $\alpha:\mathbb{T}_0\to\mathbb{T}_0$ be continuous and strictly increasing satisfying $\alpha(t)\leq t$, with $\alpha^{\Delta}\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$ and, $\alpha(t_0)=t_0$.*

Let λ,β be positive constants satisfying $1<\lambda<\beta$ or $0<\beta<\lambda<1$. Suppose that u satisfies

$$u(t)\leq a(t)+\int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)u^{\lambda}(s)-d(s)u^{\beta}(s)+c(s)\right]^{p}\Delta s, \tag{3.7}$$

then

$$u(t)\leq \begin{cases} a(t)+k(t)+\int_{\alpha(t_0)}^{\alpha(t)} h(s)k(s)e_h(\alpha(t),s)\Delta s, & 0<p\leq 1 \\ a(t)+\left[\eta^{-1}(t)\left(j^{1-p}(T)+(1-p)2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^{p}(s)\Delta s\right)\right]^{\frac{1}{1-p}}, & p>1, \end{cases} \tag{3.8}$$

where

$$k(t)=\int_{\alpha(t_0)}^{\alpha(t)} f(s)a(s)\Delta s+\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[p\left(M(s)a(s)+c(s)\right)+(1-p)\right]\Delta s, \tag{3.9}$$

$$h(t)=f(t)+pg(t)M(t), \tag{3.10}$$

$$M(t)=\frac{b(t)(\beta-\lambda)}{\beta-1}\left(\frac{(\beta-1)\,d(t)}{(\lambda-1)b(t)}\right)^{\frac{\lambda-1}{\lambda-\beta}}, \tag{3.11}$$

$$j(t)=\int_{\alpha(t_0)}^{\alpha(t)} f(s)a(s)\Delta s+2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[M(s)a(s)+c(s)\right]^{p}\Delta s,$$

$$\eta(t) = \exp\left(-(1-p)\int_{t_0}^{t} \alpha^{\Delta}(s) f(\alpha(s))\Delta s\right).$$

Proof. Using Lemma 2.1 and (3.7), we obtain

$$\begin{aligned} u(t) &\le a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)\,\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)u^{\lambda}(s) - d(s)u^{\beta}(s) + c(s)\right]^{p}\,\Delta s \\ &\le a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)\,\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[M(s)u(s) + c(s)\right]^{p}\,\Delta s, \end{aligned}$$

where $M(t)$ is defined as in (3.11).

Denote

$$y(t) = \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)\,\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[M(s)u(s) + c(s)\right]^{p}\,\Delta s. \tag{3.12}$$

Clearly, $y(t_0) = 0$ and $y(t)$ is a nondecreasing function and

$$u(t) \le a(t) + y(t). \tag{3.13}$$

Case 1: $0 < p \le 1$. Applying Lemma 2.2 to (3.12), we get

$$\begin{aligned} y(t) &\le \int_{\alpha(t_0)}^{\alpha(t)} f(s)\left(a(s) + y(s)\right)\,\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[M(s)(a(s) + y(s)) + c(s)\right]^{p}\,\Delta s \\ &\le \int_{\alpha(t_0)}^{\alpha(t)} f(s)a(s)\,\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} f(s)y(s)\,\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[p\left(M(s)a(s) + M(s)y(s) + c(s)\right) + (1-p)\right]\,\Delta s \\ &\le k(t) + \int_{\alpha(t_0)}^{\alpha(t)} h(s)y(s)\,\Delta s, \end{aligned}$$

where $k(t), h(t)$ are defined by (3.9), (3.10), respectively. Define

$$v(t) = \int_{\alpha(t_0)}^{\alpha(t)} h(s)y(s)\Delta s, \tag{3.14}$$

then $v(t_0) = 0$, $v(t)$ is nondecreasing and

$$y(t) \le k(t) + v(t). \tag{3.15}$$

Applying Lemma 2.5 and taking the delta derivative of (3.14), we have

$$v^{\Delta}(t) = \alpha^{\Delta}(t)\, h(\alpha(t))\, y(\alpha(t))$$

$$\leq \alpha^{\Delta}(t)\, h(\alpha(t))\, [k(\alpha(t)) + v(\alpha(t))]$$
$$\leq \alpha^{\Delta}(t)\, h(\alpha(t))\, [k(\alpha(t)) + v(t)]\,.$$

Thus:

$$v^{\Delta}(t) - \alpha^{\Delta}(t)h(\alpha(t))v(t) \leq \alpha^{\Delta}(t)h(\alpha(t))k(\alpha(t)).$$

Multiplying both sides of the above inequality by the generalized exponential $e_{-h\circ\alpha}(t, t_0)$, we obtain

$$[v(t)e_{-h\circ\alpha}(t, t_0)]^{\Delta} \leq \alpha^{\Delta}(t)h(\alpha(t))k(\alpha(t))e_{-h\circ\alpha}(t, t_0).$$

Now integrate both sides from t_0 to t :

$$v(t) \leq \int_{t_0}^{t} \alpha^{\Delta}(s)h(\alpha(s))k(\alpha(s))e_{h\circ\alpha}(t, s)\Delta s.$$

Let $r = \alpha(s)$, we get

$$v(t) \leq \int_{\alpha(t_0)}^{\alpha(t)} h(r)k(r)e_{h\circ\alpha}\left(t, \alpha^{-1}(r)\right)\Delta r. \tag{3.16}$$

From (3.13), (3.15) and (3.16) we can obtain the desired inequality (3.8).

Case 2: $p > 1$. Applying Lemma 2.3 to (3.12), we have

$$\begin{aligned} y(t) &\leq \int_{\alpha(t_0)}^{\alpha(t)} f(s)a(s)\,\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} f(s)y(s)\,\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} 2^{p-1}g(s)\left[(M(s)y(s))^p + (M(s)a(s) + c(s))^p\right]\Delta s \\ &\leq j(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)y(s)\,\Delta s + 2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)y^p(s)\,\Delta s. \end{aligned}$$

Let us fix $T > 0$. Then, for all $t \in [t_0, T]$, we have

$$y(t) \leq j(T) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)y(s)\Delta s + 2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)y^p(s)\Delta s.$$

Now, define an auxiliary function:

$$\varphi(t) = j(T) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)y(s)\Delta s + 2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)y^p(s)\Delta s.$$

Then

$$y(t) \leq \varphi(t),\ y(\alpha(t)) \leq y(t) \leq \varphi(t), \varphi(t_0) = j(T). \tag{3.17}$$

Taking the delta derivative of φ, we get

$$\varphi^{\Delta}(t) = \alpha^{\Delta}(t)f(\alpha(t))y(\alpha(t)) + 2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^p(\alpha(t))y^p(\alpha(t)).$$

Since $y(\alpha(t)) \leq \varphi(t)$, it follows that:

$$\varphi^{\Delta}(t) = \alpha^{\Delta}(t)f(\alpha(t))\varphi(t) + 2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^p(\alpha(t))\varphi^p(t).$$

Let

$$\theta(t) = \varphi^{1-p}(t).$$

Hence, we obtain

$$\theta^{\Delta}(t) = (1-p)\varphi^{-p}(t)\varphi^{\Delta}(t).$$

It follows that

$$\begin{aligned}
\theta^{\Delta}(t) &\leq (1-p)\phi^{-p}(t)\Big[\alpha^{\Delta}(t)f(\alpha(t))\phi(t) \\
&\qquad + 2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^{p}(\alpha(t))\phi^{p}(t)\Big] \\
&\leq (1-p)\phi^{1-p}(t)\alpha^{\Delta}(t)f(\alpha(t)) \\
&\qquad + (1-p)2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^{p}(\alpha(t)) \\
&\leq (1-p)\alpha^{\Delta}(t)f(\alpha(t))\theta(t) \\
&\qquad + (1-p)2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^{p}(\alpha(t)).
\end{aligned}$$

Define a function

$$\eta(t) = \exp\left(-(1-p)\int_{t_0}^{t}\alpha^{\Delta}(s)f(\alpha^{\Delta}(s))\Delta s\right).$$

We had the delta derivative of the product

$$[\eta(t)\theta(t)]^{\Delta} = \eta(t)\theta^{\Delta}(t) + \theta(\sigma(t))\eta^{\Delta}(t).$$

Since $\theta(\sigma(t)) \leq \theta(t)$, it follows that:

$$[\eta(t)\theta(t)]^{\Delta} \leq \eta(t)\theta^{\Delta}(t) + \theta(t)\eta^{\Delta}(t),$$

and

$$\eta^{\Delta}(t) = -(1-p)\alpha^{\Delta}(t)f(\alpha^{\Delta}(t))\eta(t).$$

Then we get

$$\begin{aligned}
\theta^{\Delta}(t)\eta(t) \leq \Big[&(1-p)\theta(t)\alpha^{\Delta}(t)f(\alpha(t)) \\
&+ (1-p)2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^{p}(\alpha(t))\Big]\eta(t),
\end{aligned}$$

and

$$\begin{aligned}
&\theta^{\Delta}(t)\eta(t) - (1-p)\theta(t)\alpha^{\Delta}(t)f(\alpha(t))\eta(t) \\
&\quad \leq (1-p)2^{p-1}\alpha^{\Delta}(t)g(\alpha(t))M^{p}(\alpha(t))\eta(t),
\end{aligned}$$

then, we get

$$[\eta(t)\theta(t)]^{\Delta} \leq (1-p)\,2^{p-1}\,\alpha^{\Delta}(t)\,g(\alpha(t))\,M^{p}(\alpha(t))\,\eta(t).$$

Integrating the above inequality from t_0 to t, we have
(3.18)

$$\eta(t)\theta(t) \leq \eta(t_0)\theta(t_0) + (1-p)2^{p-1}\int_{t_0}^{t} \alpha^{\Delta}(s)g(\alpha(s))M^p(\alpha(s))\eta(s)\Delta s.$$

Define $r = \alpha(s)$, then we have

$$\int_{t_0}^{t} \alpha^{\Delta}(s)g(\alpha(s))M^p(\alpha(s))\eta(s)\Delta s = \int_{\alpha(t_0)}^{\alpha(t)} g(r)M^p(r)\eta(\alpha^{-1}(r))\Delta r.$$

Thus inequality (3.18) becomes:

$$\theta(t) \leq \eta^{-1}(t)\left[j^{1-p}(T) + (1-p)2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)\eta(\alpha^{-1}(s))\Delta s\right].$$

Since $\theta(t) = \varphi^{1-p}(t)$, we obtain

$$\varphi(t) \leq \left[\eta^{-1}(t)\left(j^{1-p}(T) + (1-p)2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)\eta(\alpha^{-1}(s))\Delta s\right)\right]^{\frac{1}{1-p}}.$$

Since T is chosen arbitrarily, then we have
(3.19)

$$\varphi(t) \leq \left[\eta^{-1}(t)\left(j^{1-p}(t) + (1-p)2^{p-1}\int_{\alpha(t_0)}^{\alpha(t)} g(s)M^p(s)\eta(\alpha^{-1}(s))\Delta s\right)\right]^{\frac{1}{1-p}}.$$

Then, from (3.13), (3.17) and (3.19) yields the estimate of $u(t)$ in (3.8). This completes the proof. □

Remark 3.3. If we take $\mathbb{T} = \mathbb{R}$, $\alpha(t) = t$, $\lambda = 1$ and $d(t) = 0$, then for $0 < p \leq 1$, our result reduces to the inequality established in Theorem 2.3 of [3]. However, for $p > 1$, the inequality is essentially new, as it involves nonlinear terms with powers λ and β through the function $M(t)$ on time scales, which are not considered in the cited work. This illustrates the novelty and generality of our theorem.

Corollary 3.4. *Let $\mathbb{T} = \mathbb{R}$ and let $u, a, c, f, g, b, d \in C(\mathbb{R}_+, \mathbb{R}_+)$ be continuous nonnegative functions defined on $\mathbb{R}_+$. Assume that a is a non-decreasing function and let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function satisfying $\alpha(t) \leq t$, with $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha(0) = 0$. Let λ and β be positive constants satisfying $1 < \lambda < \beta$ or $0 < \beta < \lambda < 1$. Suppose that u satisfies*

$$u(t) \leq a(t) + \int_0^{\alpha(t)} f(s)\,u(s)\,ds + \int_0^{\alpha(t)} g(s)\left[b(s)u^{\lambda}(s) - d(s)u^{\beta}(s) + c(s)\right]^p ds,$$

then

$$u(t) \leq \begin{cases} a(t) + k(t) + \displaystyle\int_0^{\alpha(t)} h(s)k(s) \exp\left(\int_s^{\alpha(t)} h(\tau)\, d\tau \right) ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 < p \leq 1, \\ a(t) + \left[\eta^{-1}(t) \left(j^{1-p}(T) \right.\right. \\ \qquad \left.\left. + (1-p)2^{p-1} \displaystyle\int_0^{\alpha(t)} g(s) M^p(s) \eta\big(\alpha^{-1}(s)\big)\ ds \right) \right]^{\frac{1}{1-p}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p > 1, \end{cases}$$

where

$$k(t) = \int_0^{\alpha(t)} f(s)a(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} g(s) \left[p\left(M(s)a(s) + c(s) \right) + (1-p) \right] ds,$$

$$h(t) = f(t) + pg(t)M(t),$$

$$M(t) = \frac{b(t)(\beta - \lambda)}{\beta - 1} \left(\frac{(\beta - 1)\, d(t)}{(\lambda - 1)b(t)} \right)^{\frac{\lambda - 1}{\lambda - \beta}},$$

$$j(t) = \int_0^{\alpha(t)} f(s)a(s)ds + 2^{p-1} \int_0^{\alpha(t)} g(s) \left[M(s)a(s) + c(s) \right]^p ds,$$

$$\eta(t) = \exp\left(-(1-p \int_0^t \alpha^{'}(s) f(\alpha(s))ds \right).$$

Theorem 3.5. *Let $u, b, c, f, g, h, l \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $k_1, k_2 \in C_{rd}(\mathbb{T}_0, (0, \infty))$. Assume that a is non-decreasing function and $h^{\Delta}(t) \geq 0$, $\alpha : \mathbb{T}_0 \to \mathbb{T}_0$ is a continuous, strictly increasing function such that $\alpha(t) \leq t$, and $\alpha^{\Delta} \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $\mu(t)A(t) < 1$, $m, n, r, p, \lambda, \beta$ are constants satisfying*

(i) $0 \leq n \leq m$, $0 \leq r \leq m, m < \lambda < \beta$ *or*
(ii) $0 \leq n \leq m$, $0 \leq r \leq m, 0 < \beta < \lambda < m$ *and* $p \in (0, 1]$.

Suppose that u satisfies

$$u^m(t) \leq a(t) + h(t) \int_{\alpha(t_0)}^{\alpha(t)} \left[f(s)u^n(s) + l(s) \int_{\alpha(t_0)}^{s} \left(k_1(\tau)u^{\lambda}(\tau) - k_2(\tau)u^{\beta}(\tau) \right) \Delta\tau \right] \Delta s + \int_{\alpha(t_0)}^{\alpha(t)} g(s) \left[b(s)u^r(s) + c(s) \right]^p \Delta s, \tag{3.20}$$

then

$$u(t) \leq \left(a(t) + \int_{t_0}^{t} e_{B\oplus A_1}(t,\sigma(s))C_1(s)\Delta s\right)^{\frac{1}{m}}, \quad 0 < p \leq 1, \tag{3.21}$$

where

$$A(t) = h^{\Delta}(t)\left[\int_{\alpha(t_0)}^{\alpha(\sigma(t))}\left(\frac{n}{m}f(s) + l(s)\int_{\alpha(t_0)}^{s} M_1(\tau)\Delta\tau\right)\Delta s\right], \tag{3.22}$$

$$\begin{aligned} B(t) = h(t)\Bigg(& f(\alpha(t))\frac{n}{m} + l(\alpha(t))\int_{\alpha(t_0)}^{\alpha(t)} M_1(\tau)\,\Delta\tau \\ &+ g(\alpha(t))pb(\alpha(t))\frac{r}{m}\Bigg)\alpha^{\Delta}(t), \end{aligned} \tag{3.23}$$

$$\begin{aligned} C(t) = {} & h^{\Delta}(t)\int_{\alpha(t_0)}^{\alpha(\sigma(t))}\left[f(s)\left(\frac{n}{m}a(s) + \frac{m-n}{m}\right)\right. \\ & \left. + l(s)\int_{\alpha(t_0)}^{s} M_1(\tau)a(\sigma(\tau))\,\Delta\tau\right]\Delta s \\ & + h(t)\left[f(\alpha(t))\left(\frac{n}{m}a(\alpha(t)) + \frac{m-n}{m}\right)\right. \\ & \left. + l(\alpha(t))\int_{\alpha(t_0)}^{\alpha(t)} M_1(\tau)a(\sigma(\tau))\,\Delta\tau\right]\alpha^{\Delta}(t) \\ & + g(\alpha(t))\left[p\left(b(\alpha(t))\left(\frac{r}{m}a(\alpha(t)) + \frac{m-r}{m}\right)\right.\right. \\ & \left.\left. + c(\alpha(t))\right) + (1-p)\right]\alpha^{\Delta}(t), \end{aligned} \tag{3.24}$$

$$A_1(t) = \frac{A(t)}{1-\mu(t)A(t)}, \tag{3.25}$$

$$M_1(t) = \frac{k_1(t)(\beta-\lambda)}{\beta-m}\left(\frac{(\beta-m)\,k_2(t)}{(\lambda-m)\,k_1(t)}\right)^{\frac{\lambda-m}{\lambda-\beta}}, \tag{3.26}$$

$$C_1(t) = [1+\mu(t)A_1(t)]\,C(t). \tag{3.27}$$

Proof. Using Lemma 2.1 and (3.20), we obtain

(3.28)

$$u^m(t) \leq a(t) + h(t)\int_{\alpha(t_0)}^{\alpha(t)}\left[f(s)u^n(s)\right.$$

$$
\begin{aligned}
&+ l(s)\int_{\alpha(t_0)}^{s}\Big(k_1(\tau)u^{\lambda}(\tau)-k_2(\tau)u^{\beta}(\tau)\Big)\,\Delta\tau\Big]\Delta s\\
&+\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)u^{r}(s)+c(s)\right]^{p}\Delta s\\
&\le a(t)+h(t)\int_{\alpha(t_0)}^{\alpha(t)}\left[f(s)u^{n}(s)+l(s)\int_{\alpha(t_0)}^{s}M_1(\tau)\,u^{m}(\tau)\Delta\tau\right]\Delta s\\
&+\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)u^{r}(s)+c(s)\right]^{p}\Delta s,
\end{aligned}
$$

where $M_1(t)$ is defined as in (3.26).

Define a function y by

$$
\begin{aligned}
y(t) = h(t)\int_{\alpha(t_0)}^{\alpha(t)}&\left[f(s)u^{n}(s)+l(s)\int_{\alpha(t_0)}^{s}M_1(\tau)u^{m}(\tau)\Delta\tau\right]\Delta s\\
&+\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)u^{r}(s)+c(s)\right]^{p}\Delta s,
\end{aligned}
\tag{3.29}
$$

for all $t\in\mathbb{T}_0$. It is clear that $y(t)>0$ is a nondecreasing function with $y(t_0)=0$. Then (3.28) can be rewritten as follows:

$$
u^{m}(t)\le a(t)+y(t) \quad \text{or} \quad u(t)\le (a(t)+y(t))^{\frac{1}{m}}. \tag{3.30}
$$

By Lemma 2.2 ($K=1$) and (3.30), we have

$$
\begin{aligned}
u^{n}(t)&\le [a(t)+y(t)]^{\frac{n}{m}}\le \frac{n}{m}\left[a(t)+y(t)\right]+\frac{m-n}{m},\\
u^{r}(t)&\le [a(t)+y(t)]^{\frac{r}{m}}\le \frac{r}{m}\left[a(t)+y(t)\right]+\frac{m-r}{m}.
\end{aligned}
$$

Substituting the last relations into (3.29), we get

(3.31)

$$
\begin{aligned}
y(t)\le& h(t)\int_{\alpha(t_0)}^{\alpha(t)}\left[f(s)\left(\frac{n}{m}\left(a(s)+y(s)\right)+\frac{m-n}{m}\right)\right.\\
&\left.+l(s)\int_{\alpha(t_0)}^{s}M_1(\tau)\left[a(\tau)+y(\tau)\right]\Delta\tau\right]\Delta s\\
&+\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[b(s)\left(\frac{r}{m}\left(a(s)+y(s)\right)+\frac{m-r}{m}\right)+c(s)\right]^{p}\Delta s\\
\le& h(t)\int_{\alpha(t_0)}^{\alpha(t)}\left[f(s)\left(\frac{n}{m}(a(s)+y(s))+\frac{m-n}{m}\right)\right.
\end{aligned}
$$

$$+l(s)\int_{\alpha(t_0)}^{s} M_1(\tau)\left[a(\tau)+y(\tau)\right]\Delta\tau\Bigg]\Delta s$$
$$+\int_{\alpha(t_0)}^{\alpha(t)}\Bigg\{g(s)p\left(b(s)\left(\frac{r}{m}(a(s)+y(s))+\frac{m-r}{m}\right)+c(s)\right)$$
$$+(1-p)g(s)\Bigg\}\Delta s.$$

Applying Lemma 2.5 and taking the delta derivative of (3.31), we obtain

$$y^{\Delta}(t)=h^{\Delta}(t)\int_{\alpha(t_0)}^{\alpha(\sigma(t))}\Bigg[f(s)\left(\frac{n}{m}(a(s)+y(s))+\frac{m-n}{m}\right)$$
$$+l(s)\int_{\alpha(t_0)}^{s} M_1(\tau)\left(a(\tau)+y(\tau)\right)\Delta\tau\Bigg]\Delta s$$
$$+h(t)\Bigg[f(\alpha(t))\left(\frac{n}{m}(a(\alpha(t))+y(\alpha(t)))+\frac{m-n}{m}\right)$$
$$+l(\alpha(t))\int_{\alpha(t_0)}^{\alpha(t)} M_1(\tau)\left(a(\tau)+y(\tau)\right)\Delta\tau\Bigg]\alpha^{\Delta}(t)$$
$$+g(\alpha(t))\Bigg[p\Bigg(b(\alpha(t))\left(\frac{r}{m}(a(\alpha(t))+y(\alpha(t)))+\frac{m-r}{m}\right)$$
$$+c(\alpha(t))\Bigg)+(1-p)\Bigg]\alpha^{\Delta}(t).$$

Thus, the above inequality can be reformulated as

$$y^{\Delta}(t)\leq y(\sigma(t))A(t)+B(t)y(t)+C(t), \tag{3.32}$$

where $A(t), B(t)$ and $C(t)$ are defined as in (3.22), (3.23) and (3.24), respectively. From (3.25), we get

$$A(t)=\frac{A_1(t)}{1+\mu(t)A_1(t)},$$

and by (3.32), we obtain

$$y^{\Delta}(t)=\left(y(t)+\mu(t)y^{\Delta}(t)\right)A(t)+B(t)y(t)+C(t)$$
$$\leq\frac{A_1(t)}{1+\mu(t)A_1(t)}\left(y(t)+\mu(t)y^{\Delta}(t)\right)+B(t)y(t)+C(t),$$

which yields

$$\frac{1}{1+\mu(t)A_1(t)}y^{\Delta}(t)\leq\left(B(t)+\frac{A_1(t)}{1+\mu(t)A_1(t)}\right)y(t)+C(t),$$

i.e.,

$$y^{\Delta}(t) \leq (B \oplus A_1)(t)\, y(t) + C_1(t),$$

where $C_1(t)$ is defined as in (3.27). Note that $y, C_1 \in C_{rd}$ and $B \oplus A_1 \in \mathcal{R}^+$. By Lemma 2.7, we get

$$y(t) \leq \int_{t_0}^{t} e_{B\oplus A_1}(t, \sigma(s)) C_1(s) \Delta s.$$

Combining this with (3.30), we obtain the desired inequality given in (3.21). The proof is therefore complete. □

Remark 3.6. This work builds on Liu (2019) [7], extending his half-linear integral inequalities on time scales to a more general setting with additional nonlinear terms, parameters and conditions, thereby yielding novel and original results.

Corollary 3.7. *Let $\mathbb{T} = \mathbb{R}$ and assume that $u, a, b, c, f, g, h, l \in C(\mathbb{R}_+, \mathbb{R}_+)$, $k_1, k_2 \in C(\mathbb{R}_+, (0, \infty))$. Assume also a is non-decreasing function for $t \in \mathbb{R}_+$ and $h^{'}(t) \geq 0$, $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, strictly increasing function such that $\alpha(t) \leq t$, and $\alpha' \geq 0$, $m, n, r, p, \lambda, \beta$ are constants satisfying*

(i) $0 \leq n \leq m$, $0 \leq r \leq m, m < \lambda < \beta$ or

(ii) $0 \leq n \leq m$, $0 \leq r \leq m, 0 < \beta < \lambda < m$ and $p \in (0, 1]$.

$$\begin{aligned} u^m(t) \leq a(t) + h(t) \int_0^{\alpha(t)} \Bigg[& f(s) u^n(s) \\ & + l(s) \int_0^s \left(k_1(\tau) u^{\lambda}(\tau) - k_2(\tau) u^{\beta}(\tau) \right) d\tau \Bigg] ds \\ & + \int_0^{\alpha(t)} g(s) \left[b(s) u^r(s) + c(s) \right]^p ds, \end{aligned}$$

then

$$u(t) \leq \left(a(t) + \int_0^t \exp\left(\int_s^t \left(B(\tau) + A(\tau) \right) d\tau \right) C(s) ds \right)^{\frac{1}{m}}, \quad 0 < p \leq 1,$$

where

$$A(t) = h^{'}(t) \left[\int_0^{\alpha(t)} \left(\frac{n}{m} f(s) + l(s) \int_0^s M_1(\tau) d\tau \right) ds \right],$$

$$\begin{aligned} B(t) = h(t) \Bigg(& f(\alpha(t)) \frac{n}{m} + l(\alpha(t)) \int_0^{\alpha(t)} M_1(\tau)\, d\tau \\ & + g(\alpha(t))\, p\, b(\alpha(t)) \frac{r}{m} \Bigg) \alpha'(t), \end{aligned}$$

$$
\begin{aligned}
C(t) = h'(t) \int_0^{\alpha(t)} & \left[f(s) \left(\frac{n}{m} \left(a(s) + \frac{m-n}{m} \right) \right) \right. \\
& \left. + l(s) \int_0^s M_1(\tau) a(\tau)\, d\tau \right] ds \\
& + h(t) \left[f(\alpha(t)) \left(\frac{n}{m} \left(a(\alpha(t)) + \frac{m-n}{m} \right) \right) \right. \\
& \left. + l(\alpha(t)) \int_0^{\alpha(t)} M_1(\tau) a(\tau)\, d\tau \right] \alpha'(t) \\
& + g(\alpha(t)) \left[p \left(b(\alpha(t)) \left(\frac{r}{m} a(\alpha(t)) + \frac{m-r}{m} \right) + c(\alpha(t)) \right) \right. \\
& \left. + (1-p) \right] \alpha'(t), \\
M_1(t) = & \frac{k_1(t)(\beta-\lambda)}{\beta-m} \left(\frac{(\beta-m)\, k_2(t)}{(\lambda-m)\, k_1(t)} \right)^{\frac{\lambda-m}{\lambda-\beta}}, \\
C_1(t) = & \left[1 + \mu(t) A(t) \right] C(t).
\end{aligned}
$$

Corollary 3.8. *Let $\mathbb{T} = \mathbb{Z}$ and assume that u, a, b, c, f, g, h, l are non-negative functions defined for $t \in \mathbb{N}_0$, $k_1, k_2 \in C_{rd}(\mathbb{N}_0, \mathbb{R}_+)$. Assume also a is non-decreasing function in $\mathbb{N}_0$ and $h^{\Delta}(t) \geq 0$, $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ is a continuous, strictly increasing function such that $\alpha(t) \leq t$, and $\alpha^{\Delta} \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $A(t) < 1$, $m, n, r, p, \lambda, \beta$ are constants satisfying*

(i) $0 \leq n \leq m$, $0 \leq r \leq m, m < \lambda < \beta$ or

(ii) $0 \leq n \leq m$, $0 \leq r \leq m, 0 < \beta < \lambda < m$ and $p \in (0, 1]$.

Suppose that u satisfies

$$
\begin{aligned}
u^m(t) \leq a(t) + h(t) \sum_{s=0}^{\alpha(t)-1} & \left[f(s) u^n(s) \right. \\
& \left. + l(s) \sum_{\tau=0}^{s-1} \left(k_1(\tau) u^{\lambda}(\tau) - k_2(\tau) u^{\beta}(\tau) \right) \right] \\
& + \sum_{s=0}^{\alpha(t)-1} g(s) \left[b(s) u^r(s) + c(s) \right]^p,
\end{aligned}
$$

then

$$
u(t) \leq \left(a(t) + \sum_{s=t_0}^{t-1} \left(\prod_{k=s+1}^{t-1} \exp\left(1 + (B(k) \oplus A_1(k)) \right) \right) C_1(s) \right)^{\frac{1}{m}}.
$$

where

$$A(t) = h^{\Delta}(t) \sum_{s=\alpha(t_0)}^{\alpha(t+1)-1} \left(\frac{n}{m} f(s) + l(s) \sum_{\tau=\alpha(t_0)}^{s-1} M_1(\tau) \right),$$

$$B(t) = h(t) \Bigg(f(\alpha(t)) \frac{n}{m} + l(\alpha(t)) \sum_{\tau=0}^{\alpha(t+1)-1} M_1(\tau) + g(\alpha(t))\, p\, b(\alpha(t)) \frac{r}{m} \Bigg) \alpha^{\Delta}(t),$$

$$\begin{aligned} C(t) = {} & h^{\Delta}(t) \sum_{s=0}^{\alpha(t+1)-1} \Bigg[f(s) \left(\frac{n}{m} \left(a(s) + \frac{m-n}{m} \right) \right) + l(s) \sum_{\tau=0}^{s-1} M_1(\tau) a(\tau+1) \Bigg] \\ & + h(t) \Bigg[f(\alpha(t)) \left(\frac{n}{m} \left(a(\alpha(t)) + \frac{m-n}{m} \right) \right) + l(\alpha(t)) \sum_{\tau=0}^{\alpha(t+1)-1} M_1(\tau) a(\tau+1) \Bigg] \alpha^{\Delta}(t) \\ & + g(\alpha(t)) \Bigg[p \Big(b(\alpha(t)) \left(\frac{r}{m} a(\alpha(t)) + \frac{m-r}{m} \right) + c(\alpha(t)) \Big) + (1-p) \Bigg] \alpha^{\Delta}(t), \end{aligned}$$

$$A_1(t) = \frac{A(t)}{1 - A(t)},$$

$$M_1(t) = \frac{k_1(t)(\beta - \lambda)}{\beta - m} \left(\frac{(\beta - m)\, k_2(t)}{(\lambda - m)\, k_1(t)} \right)^{\frac{\lambda - m}{\lambda - \beta}},$$

$$C_1(t) = [1 + A_1(t)]\, C(t).$$

Theorem 3.9. *Assume that $u, a, f, g, l, \alpha, k_1, k_2, \lambda, \beta, p, m, n, M_1$ are defined the same as in Theorem 3.5, $h^{\Delta}(t) \geq 0$ and $\mu(t)\widetilde{A}(t) < 1$. Assume also that $a(t)$ is nondecreasing for $t \in \mathbb{T}_0$. Let $v(t,s)$ be defined as in Lemma 2.6 such that $v_t^{\Delta}(t,s) \geq 0$ for $t \geq s$.*

If $u(t)$ satisfies

$$u^m(t) \leq a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) u^m(s)\, \Delta s \tag{3.33}$$

$$+\Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left(u^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\,\Delta\tau\right)^p\Delta s$$
$$+l(t)\int_{t_0}^{t} v(t,s)\Big[k_1(s)u^\lambda(\sigma(s))-k_2(s)u^\beta(\sigma(s))\Big]\Delta s,$$

then

$$u(t)\le\left[\left(a(t)+\int_{t_0}^{t} e_{\widetilde{B}\oplus\widetilde{A}_1}(t,\sigma(s))\widetilde{C}_1(s)\Delta s\right)e_F(t,t_0)\right]^{\frac{1}{q}},$$

where

$$\widetilde{A}(t)=\Omega^\Delta(t)\int_{\alpha(t_0)}^{\alpha(\sigma(t))} pg(s)\left[\frac{n}{m}e_F^{(n/m)}(s,t_0)+\int_{\alpha(t_0)}^{\alpha(s)}\frac{r}{m}h(\tau)e_F^{(r/m)}(\tau,t_0)\Delta\tau\right]\Delta s$$
$$+l^\Delta(t)\int_{t_0}^{\sigma(t)} v(\sigma(t),s)M_1(s)e_F(\sigma(s),t_0)\Delta s+l(t)\int_{t_0}^{t} v_t^\Delta(t,s)M_1(s)e_F(\sigma(s),t_0)\Delta s$$
$$+l(t)v(\sigma(t),t)M_1(t)e_F(\sigma(t),t_0),\tag{3.34}$$

$$\widetilde{B}(t)=\alpha^\Delta(t)pg\left(\alpha(t)\right)\left[\frac{n}{m}e_F^{\frac{n}{m}}\left(\alpha(t),t_0\right)+\int_{\alpha(t_0)}^{\alpha(t)}\frac{r}{m}e_F^{\frac{r}{m}}\left(\tau,t_0\right)h\left(\tau\right)\Delta\tau\right]\Omega(t),\tag{3.35}$$

$$\widetilde{C}(t)=\Omega^\Delta(t)\int_{\alpha(t_0)}^{\alpha(\sigma(t))}\left\{pg(s)\left[\left(\frac{n}{m}a(s)+\frac{m-n}{m}\right)e_F^{\frac{n}{m}}(s,t_0)+\int_{\alpha(t_0)}^{\alpha(s)}h(\tau)\left(\frac{r}{m}a(\tau)+\frac{m-r}{m}\right)e_F^{\frac{r}{m}}(\tau,t_0)\,\Delta\tau\right]+(1-p)g(s)\right\}\Delta s$$
$$+\alpha^\Delta(t)\left\{pg(\alpha(t))\left[\left(\frac{n}{m}a(\alpha(t))+\frac{n-m}{m}\right)e_F^{\frac{n}{m}}(\alpha(t),t_0)+\int_{\alpha(t_0)}^{\alpha(t)}h(\tau)\left(\frac{r}{m}a(\tau)+\frac{m-r}{m}\right)e_F^{\frac{r}{m}}(\tau,t_0)\,\Delta\tau\right]\right.$$
$$\left.+(1-p)g(\alpha(t))\right\}\Delta\alpha(t),\tag{3.36}$$

$$
\begin{aligned}
&+ (1-p)g(t)\Big\}\Omega(t)\\
&+ l^{\Delta}(t)\int_{t_0}^{\sigma(t)} v(\sigma(t),s)\,M_1(s)\left[a(\sigma(s))e_F(\sigma(s),t_0)\right]\Delta s\\
&+ l(t)\int_{t_0}^{t} v_t^{\Delta}(t,s)\,M_1(s)\left[a(\sigma(s))e_F(\sigma(s),t_0)\right]\Delta s\\
&+ l(t)\,v(\sigma(t),t)\,M_1(t)\left[a(\sigma(t))e_F(\sigma(t),t_0)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{A}_1(t) = \frac{\widetilde{A}(t)}{1-\mu(t)\widetilde{A}(t)},\\
&\widetilde{C}_1(t) = \left[1+\mu(t)\widetilde{A}_1(t)\right]\widetilde{C}(t).
\end{aligned}
\tag{3.37}
$$

Proof. From Lemma 2.1 and (3.33), we get

$$
\begin{aligned}
u^m(t) \le a(t) &+ \int_{\alpha(t_0)}^{\alpha(t)} f(s)u^m(s)\,\Delta s\\
&+ \Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left(u^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\,\Delta\tau\right)^p\Delta s\\
&+ l(t)\int_{t_0}^{t} v(t,s)\,M_1(s)\,u^m(\sigma(s))\,\Delta s,
\end{aligned}
$$

where M_1 is defined as in (3.26). Denote

$$
\begin{aligned}
z(t) = \Bigg\{ & a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u^m(s)\,\Delta s\\
&+ \Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[u^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\,\Delta\tau\right]^p\Delta s\\
&+ l(t)\int_{t_0}^{t} v(t,s)\,M_1(s)\,u^m(\sigma(s))\,\Delta s\Bigg\}^{\frac{1}{m}},
\end{aligned}
\tag{3.38}
$$

then

$$
u(t) \le z(t). \tag{3.39}
$$

It can be deduced from (3.38), (3.39) that

$$
z^m(t) \le a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)z^m(s)\,\Delta s \tag{3.40}
$$

$$+\Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left(z^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)z^r(\tau)\,\Delta\tau\right)^p\Delta s$$
$$+\,l(t)\int_{t_0}^{t} v(t,s)\,M_1(s)\,z^m(\sigma(s))\,\Delta s,$$

Define a function $v(t)$ by

$$v(t)=a(t)+w(t), \tag{3.41}$$

where

$$w(t)=\Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[z^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)z^r(\tau)\,\Delta\tau\right]^p\Delta s \tag{3.42}$$
$$+\,l(t)\int_{t_0}^{t} v(t,s)\,M_1(s)\,z^m(\sigma(s))\,\Delta s.$$

Then (3.40) can be restated as

$$z^m(t)\le\int_{\alpha(t_0)}^{\alpha(t)} f(s)z^m(s)\Delta s+v(t). \tag{3.43}$$

Noting that $w(t)$ is nondecreasing, from (3.43) and Lemma 2.9, we obtain

$$z^m(t)\le v(t)e_F(t,t_0), \tag{3.44}$$

where

$$F(t)=\alpha(t)f(t).$$

From (3.41) and (3.44), we have

$$z(t)\le\left[(a(t)+w(t))\,e_F(t,t_0)\right]^{\frac{1}{m}}. \tag{3.45}$$

It can be derived from (3.42), (3.45) that

(3.46)

$$w(t)=\Omega(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left[\Big((a(s)+w(s))e_F(s,t_0)\Big)^{\frac{n}{m}}\right.$$
$$\left.+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\Big((a(\tau)+w(\tau))e_F(\tau,t_0)\Big)^{\frac{r}{m}}\Delta\tau\right]^p\Delta s$$
$$+\,l(t)\int_{t_0}^{t} v(t,s)\,M_1(s)\Big((a(\sigma(s))+w(\sigma(s)))e_F(\sigma(s),t_0)\Big)\Delta s.$$

By Lemma 2.2 ($K=1$) and (3.46), we obtain

(3.47)

$$w(t)=\Omega(t)\int_{\alpha(t_0)}^{\alpha(t)}\left\{pg(s)\left[\Big(\frac{n}{m}(a(s)+w(s))+\frac{m-n}{m}\Big)e_F^{\frac{n}{m}}(s,t_0)\right.\right.$$

$$+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\Big(\frac{r}{m}(a(\tau)+w(\tau))+\frac{m-r}{m}\Big)e_F^{\frac{r}{m}}(\tau,t_0)\Delta\tau\Big]$$
$$+(1-p)g(s)\Big\}\Delta s$$
$$+l(t)\int_{t_0}^{t} v(t,s)M_1(s)\Big(a(\sigma(s))+w(\sigma(s))\Big)e_F(\sigma(s),t_0)\Delta s.$$

Using Lemma 2.5, taking delta derivative of (3.47), we have

(3.48)

$$\begin{aligned}
w^{\Delta}(t) &= \Omega^{\Delta}(t)\int_{\alpha(t_0)}^{\alpha(\sigma(t))}\Bigg\{pg(s)\Bigg[\Big(\frac{n}{m}(a(s)+w(s))+\frac{m-n}{m}\Big)e_F^{n/m}(s,t_0)\\
&\quad+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\Big(\frac{r}{m}(a(\tau)+w(\tau))+\frac{m-r}{m}\Big)e_F^{r/m}(\tau,t_0)\Delta\tau\Bigg]\\
&\quad+(1-p)g(s)\Bigg\}\Delta s+\alpha^{\Delta}(t)\\
&\quad\times\Bigg[pg(\alpha(t))\Bigg[\Big(\frac{n}{m}(a(\alpha(t))+w(\alpha(t)))+\frac{m-n}{m}\Big)e_F^{n/m}(\alpha(t),t_0)\\
&\quad+\int_{\alpha(t_0)}^{\alpha(t)} h(\tau)\Big(\frac{r}{m}(a(\tau)+w(\tau))+\frac{m-r}{m}\Big)e_F^{r/m}(\tau,t_0)\Delta\tau\Bigg]\\
&\quad+(1-p)g(t)\Bigg]\Omega(t)\\
&\quad+l^{\Delta}(t)\int_{t_0}^{\sigma(t)} v(\sigma(t),s)M_1(s)\Big[a(\sigma(s))+w(\sigma(s))\Big]e_F(\sigma(s),t_0)\Delta s\\
&\quad+l(t)\int_{t_0}^{t} v_t^{\Delta}(t,s)M_1(s)\Big[a(\sigma(s))+w(\sigma(s))\Big]e_F(\sigma(s),t_0)\Delta s\\
&\quad+l(t)\,v(\sigma(t),t)\,M_1(t)\Big[a(\sigma(t))+w(\sigma(t))\Big]e_F(\sigma(t),t_0)\\
&= w(\sigma(t))\widetilde{A}(t)+\widetilde{B}(t)w(t)+\widetilde{C}(t),
\end{aligned}$$

where $\widetilde{A}(t),\widetilde{B}(t)$, and $\widetilde{C}(t)$ are defined as in (3.34), (3.35) and (3.36).

The rest of the proof is analogous to that of Theorem 3.5 and therefore, we omit it here. □

Corollary 3.10. *Let $\mathbb{T} = q^{\mathbb{N}_0}$, where $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0,\ q > 1\}$. Assume that all assumptions of Theorem 3.9, $h^{\Delta}(t) \geq 0$ and $\widetilde{A}(t) < \frac{1}{(q-1)t}$. Assume also that $a(t)$ is nondecreasing for $t \in \mathbb{T}_0$. Let $v(t,s)$ be defined as in Lemma 2.7 such that $v_t^{\Delta}(t,s) \geq 0$ for $t \geq s$.*

If $u(t)$ satisfies

$$\begin{aligned} u^m(t) \leq a(t) &+ \int_0^{\alpha(t)} f(s)u^m(s)\,\Delta s \\ &+ \Omega(t)\int_0^{\alpha(t)} g(s)\left(u^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\,u^r(\tau)\,\Delta\tau\right)^p \Delta s \\ &+ l(t)\int_0^t v(t,s)\Big[k_1(s)u^{\lambda}(qs) - k_2(s)u^{\beta}(qs)\Big]\Delta s, \end{aligned}$$

then

$$u(t) \leq \left[\left(a(t) + \int_0^t E_q(\widetilde{B} \oplus \widetilde{A}_1)(t,qs)\widetilde{C}_1(s)\Delta s\right)\right]^{\frac{1}{m}},$$

where

$$E_q(F)(t,0) = \prod_{s\in[0,t)} \Big(1 + (q-1)tF(s)\Big),$$

$$\begin{aligned} A(t) = \Omega^{\Delta}(t)\int_0^{\alpha(qt)} pg(s)\Bigg[&\frac{n}{m}\big(E_q(F)(s,0)\big)^{\frac{n}{m}} \\ &+ \int_0^{\alpha(s)} \frac{r}{m}h(\tau)\big(E_q(F)(\tau,0)\big)^{\frac{r}{m}}\Delta\tau\Bigg]\Delta s \\ &+ l^{\Delta}(t)\int_0^{qt} v(\sigma(t),s)K(s)E_q(F)(qt,0)\,\Delta s \\ &+ l(t)\int_{t_0}^t v_t^{\Delta}(t,s)K(s)E_q(F)(qs,0)\,\Delta s \\ &+ l(t)\,v(\sigma(t),t)\,K(t)\,E_q(F)(qt,t_0), \end{aligned}$$

$$\begin{aligned} \widetilde{B}(t) = \alpha^{\Delta}(t)\,pg(\alpha(t))\Bigg[&\frac{n}{m}\,(E_q(F)(\alpha(t),0))^{\frac{n}{m}} \\ &+ \int_0^{\alpha(t)} \frac{r}{m}h(\tau)\,(E_q(F)(\tau,0))^{\frac{r}{m}}\ \Delta_q\tau\Bigg]\Omega(t), \end{aligned}$$

$$\widetilde{C}(t) = \Omega^{\Delta}(t)\int_0^{\alpha(qt)}\Bigg\{pg(s)\Bigg[\Big(\frac{n}{m}a(s) + \frac{m-n}{m}\Big)\,(E_q(F)(s,0))^{\frac{n}{m}}$$

$$+\int_0^{\alpha(s)} h(\tau)\left(\left(\frac{r}{m}a(\tau)+\frac{m-r}{m}\right)(E_q(F)(\tau,0))^{\frac{r}{m}}\right)\Delta\tau$$
$$+(1-p)g(s)\Bigg]\Bigg\}\Delta s+\alpha^{\Delta}(t)$$
$$\times\left\{pg(\alpha(t))\left[\left(\frac{n}{m}a(\alpha(t))+\frac{n-m}{m}\right)(E_q(F)(\alpha(t),t_0))^{\frac{n}{m}}\right]\right\}\Omega(t)$$

$$\widetilde{A}_1(t)=\frac{\widetilde{A}(t)}{1-(q-1)t\widetilde{A}(t)}.$$

Theorem 3.11. *Assume that $u,a,f,g,h,l\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$, $b,k_1,k_2:\mathbb{T}_0\to(0,\infty)$ are rd-continuous functions, $\left(\frac{l}{b}\right)^{\Delta}(t)\geq 0$, $\mu(t)\widetilde{F}(t)<1$, and α,λ,β,p are defined the same as in Theorem 3.5. Assume also that $a(t)$ is nondecreasing for $t\in\mathbb{T}_0$. Let $v(t,s)$ be defined as in Lemma 2.6 such that $v_t^{\Delta}(t,s)\geq 0$ for $t\geq s$. Suppose that $N:\mathbb{R}_+\to\mathbb{R}_+$be defined as in Lemma 3.1.*

If $u(t)$ satisfies

(3.49)

$$u^m(t)\leq a(t)+\int_{\alpha(t_0)}^{\alpha(t)} f(s)u^m(s)\Delta s$$
$$+b(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\Big(u^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\Delta\tau\Big)^p\Delta s$$
$$+l(t)\int_{t_0}^{t} v(t,s)\Big[k_1(s)\,N(u^{\lambda}(\sigma(s)))-k_2(s)\,N(u^{\beta}(\sigma(s)))\Big]\Delta s,$$

then

$$u(t)\leq\left(a(t)+\int_{t_0}^{t} e_{\widetilde{L}\oplus\widetilde{F_1}}(t,\sigma(s))\widetilde{M_1}(s)\Delta s\right)^{\frac{1}{m}},\quad 0<p\leq 1,$$

where

(3.50)

$$\xi_1\big(k_1(t),k_2(t),k,\lambda,\beta,m\big)=\xi_1(t)$$
$$=\frac{k_1(t)\,k(\beta-\lambda)}{\beta-m}\left(\frac{(\beta-m)\,k_2(t)}{(\lambda-m)\,k_1(t)}\right)^{\frac{\alpha-m}{\alpha-\beta}}$$

if $m>\lambda>\beta$ or $m<\lambda<\beta$, and

$$\widetilde{L}(t)=\left\{\frac{n}{m}e_F^{\frac{n}{m}}(\alpha(t),t_0)+\int_{\alpha(t_0)}^{\alpha(t)} h(\tau)e_F^{\frac{r}{m}}(\tau,t_0)\,\Delta\tau\right\}\tag{3.51}$$

$$\times p\, g(\alpha(t))\, \alpha^{\Delta}(t),$$

$$\widetilde{F}(t) = \left(\frac{l}{b}\right)^{\Delta}(t) \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, e_F(s,t_0)\, \Delta s + \frac{l(\sigma(t))}{b(\sigma(t))} \left[\int_{t_0}^{t} v_t^{\Delta}(t,s)\, e_F(s,t_0)\, \xi_1(s)\, \Delta s + v(\sigma(t),t)\, \xi_1(t)\, e_F(t,t_0) \right], \tag{3.52}$$

$$\begin{aligned} \widetilde{M}(t) = {} & \left\{ e_F^{\frac{n}{m}}(\alpha(t),t_0) \left(\frac{n}{m} a(\alpha(t)) + \frac{m-n}{m}\right) \right. \\ & \left. + \int_{\alpha(t_0)}^{\alpha(t)} h(\tau) e_F^{\frac{r}{m}}(\tau,t_0) \left(\frac{r}{m} a(\tau) + \frac{m-r}{m}\right) \Delta\tau \right\} \\ & \times p\, g(\alpha(t))\, \alpha^{\Delta}(t) + \alpha^{\Delta}(t)(1-p)\, g(\alpha(t)) \\ & + \left(\frac{l}{b}\right)^{\Delta}(t) \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, e_F(s,t_0)\, a(\sigma(s))\, \Delta s \\ & + \frac{l(\sigma(t))}{b(\sigma(t))} \left[\int_{t_0}^{t} v_t^{\Delta}(t,s)\, e_F(s,t_0)\, a(\sigma(s))\, \xi_1(s)\, \Delta s \right. \\ & \left. + v(\sigma(t),t)\, \xi_1(t)\, e_F(t,t_0)\, a(\sigma(t)) \right], \end{aligned} \tag{3.53}$$

$$\widetilde{F}_1(t) = \frac{\widetilde{F}(t)}{1-\mu(t)\widetilde{F}(t)},$$

$$\widetilde{M}_1(t) = \left[1 + \mu(t)\widetilde{F}_1(t)\right] \widetilde{M}(t).$$

Proof. From Lemma 2.9 and (3.49), we get

$$\begin{aligned} u^m(t) \le {} & a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) u^m(s) \Delta s \\ & + b(t) \int_{\alpha(t_0)}^{\alpha(t)} g(s) \Big(u^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau) u^r(\tau) \Delta\tau \Big)^p \Delta s \\ & + l(t) \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, u^m(\sigma(s)) \Delta s, \end{aligned}$$

where ξ_1 is defined as in (3.50). Denote

$$\begin{aligned} z(t) = {} & \left\{ a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) u^m(s) \Delta s \right. \\ & \left. + b(t) \int_{\alpha(t_0)}^{\alpha(t)} g(s) \Big(u^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau) u^r(\tau) \Delta\tau \Big)^p \Delta s \right. \end{aligned} \tag{3.54}$$

$$+ l(t) \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, u^m(\sigma(s))\Delta s \Bigg\}^{\frac{1}{m}}.$$

Clearly, $z(t)$ is nonnegative and nondecreasing function and

$$u(t) \le z(t). \tag{3.55}$$

It follows from (3.54) and (3.55) that

$$\begin{aligned} z^m(t) \le a(t) &+ \int_{\alpha(t_0)}^{\alpha(t)} f(s) z^m(s)\Delta s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} g(s)\Big(z^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau) z^r(\tau)\Delta\tau\Big)^p \Delta s \\ &+ \frac{l(t)}{b(t)} \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, z^m(\sigma(s))\Delta s. \end{aligned} \tag{3.56}$$

Define a function $v(t)$ by

$$v(t) = a(t) + w(t), \tag{3.57}$$

where

$$\begin{aligned} w(t) = &\int_{\alpha(t_0)}^{\alpha(t)} g(s)\Big(z^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau) z^r(\tau)\Delta\tau\Big)^p \Delta s \\ &+ \frac{l(t)}{b(t)} \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, z^m(\sigma(s))\Delta s. \end{aligned} \tag{3.58}$$

Then (3.56) can be restated as

$$z^m(t) \le \int_{\alpha(t_0)}^{\alpha(t)} f(s) z^m(s)\Delta s + v(t). \tag{3.59}$$

Noting that $v(t)$ is nondecreasing, from (3.59) and Lemma 2.9, we obtain

$$z^m(t) \le v(t) e_F(t, t_0), \tag{3.60}$$

where

$$F(t) = \alpha^{\Delta}(t) f(t).$$

From (3.57) and (3.60), we have

$$z(t) \le e_F^{\frac{1}{m}}(t, t_0)\, [a(t) + w(t)]^{\frac{1}{m}}. \tag{3.61}$$

It can be derived from (3.58), (3.61) that

$$\begin{aligned} w(t) \le \int_{\alpha(t_0)}^{\alpha(t)} g(s) \Bigg(& e_F^{n/m}(s, t_0)\, [a(s) + w(s)]^{n/m} \\ &+ \int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\, e_F^{r/m}(\tau, t_0)\, [a(\tau) + w(\tau)]^{r/m} \Delta\tau \Bigg)^p \Delta s \end{aligned}$$

$$
\begin{aligned}
&+\frac{l(t)}{b(t)}\int_{t_0}^{t} v(t,s)\,\xi_1(s)\,e_F(s,t_0)\,[a(\sigma(s))+w(\sigma(s))]\Delta s\\
&\leq \int_{\alpha(t_0)}^{\alpha(t)}\Bigg\{p\,g(s)\Big(e_F^{n/m}(s,t_0)\,[a(s)+w(s)]^{n/m}\\
&\quad+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)\,e_F^{r/m}(\tau,t_0)\,[a(\tau)+w(\tau)]^{r/m}\Delta\tau\Big)\Bigg\}\Delta s\\
&\quad+(1-p)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\Delta s\\
&\quad+\frac{l(t)}{b(t)}\int_{t_0}^{t} v(t,s)\,\xi_1(s)\,e_F(s,t_0)\,[a(\sigma(s))+w(\sigma(s))]\Delta s.
\end{aligned}
$$

Taking the derivative of $w(t)$ and by Lemma 2.2, we get

(3.62)

$$
\begin{aligned}
w^{\Delta}(t) &\leq \Bigg\{e_F^{\frac{n}{m}}(\alpha(t),t_0)\Big(\frac{n}{m}(a(\alpha(t))+w(\alpha(t)))+\frac{m-n}{m}\Big)\\
&\quad+\int_{\alpha(t_0)}^{\alpha(t)} h(\tau)e_F^{\frac{r}{m}}(\tau,t_0)\Big(\frac{r}{m}(a(\tau)+w(\tau))+\frac{m-r}{m}\Big)\Delta\tau\Bigg\}\\
&\quad\times p\,g(\alpha(t))\alpha^{\Delta}(t)+\alpha^{\Delta}(t)(1-p)g(\alpha(t))\\
&\quad+\left(\frac{l}{b}\right)^{\Delta}(t)\int_{t_0}^{t} v(t,s)\xi_1(s)e_F(s,t_0)(a(\sigma(s))+w(\sigma(s)))\Delta s\\
&\quad+\frac{l(\sigma(t))}{b(\sigma(t))}\Bigg[\int_{t_0}^{t} v_t^{\Delta}(t,s)e_F(s,t_0)\xi_1(s)(a(\sigma(s))+w(\sigma(s)))\Delta s\\
&\quad+v(\sigma(t),t)\xi_1(t)e_F(t,t_0)(a(\sigma(t))+w(\sigma(t)))\Bigg]\\
&=\widetilde{L}(t)w(t)+\widetilde{F}(t)w(\sigma(t))+\widetilde{M}(t),
\end{aligned}
$$

where $\widetilde{L}(t)$, $\widetilde{F}(t)$ and $\widetilde{M}(t)$ are defined as in (3.51), (3.52) and (3.53). By a similar argument with Theorem 3.5 in the remaining part of the proof, the result follows. This completes the proof. □

Corollary 3.12. *Assume that all assumptions of Theorem 3.11 hold. Let* $N(t)=\arctan t$ *If* $u(t)$ *satisfies*

$$
u^m(t)\leq a(t)+\int_{\alpha(t_0)}^{\alpha(t)} f(s)u^m(s)\Delta s
$$

$$
\begin{aligned}
&+ b(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\Big(u^n(s)+\int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\Delta\tau\Big)^p\Delta s\\
&+ l(t)\int_{t_0}^{t} v(t,s)\Big[k_1(s)\arctan(u^\lambda(\sigma(s)))\\
&- k_2(s)\arctan(u^\beta(\sigma(s)))\Big]\Delta s,
\end{aligned}
$$

then

$$
u(t)\le\left(a(t)+\int_{t_0}^{t} e_{\widetilde{L}\oplus\widetilde{F_1}}(t,\sigma(s))\widetilde{M}_1(s)\Delta s\right)^{\frac{1}{m}},\quad 0<p\le 1,
$$

where

$$
\begin{aligned}
\xi_1\big(k_1(t),k_2(t),1,\lambda,\beta,m\big)&=\xi_1(t)\\
&=\frac{k_1(t)(\beta-\lambda)}{\beta-m}\left(\frac{(\beta-m)\,k_2(t)}{(\lambda-m)\,k_1(t)}\right)^{\frac{\alpha-m}{\alpha-\beta}}
\end{aligned}
$$

if $0<m<\lambda<\beta$ *or* $0<\lambda<\beta<m$, *and*

$$
\begin{aligned}
\widetilde{L}(t)&=\left\{\frac{n}{m}e_F^{\frac{n}{m}}(\alpha(t),t_0)+\int_{\alpha(t_0)}^{\alpha(t)} h(\tau)e_F^{\frac{r}{m}}(\tau,t_0)\Delta\tau\right\}pg(\alpha(t))\alpha^\Delta(t),\\
\widetilde{F}(t)&=\left(\frac{l}{b}\right)^\Delta(t)\int_{t_0}^{t} v(t,s)\,\xi_1(s)\,e_F(s,t_0)\Delta s\\
&\quad+\frac{l(\sigma(t))}{b(\sigma(t))}\Bigg[\int_{t_0}^{t} v_t^\Delta(t,s)\,e_F(s,t_0)\,\xi_1(s)\Delta s\\
&\quad+v(\sigma(t),t)\,\xi_1(t)\,e_F(t,t_0)\Bigg],\\
\widetilde{M}(t)&=\Bigg\{e_F^{\frac{n}{m}}(\alpha(t),t_0)\Big(\frac{n}{m}a(\alpha(t))+\frac{m-n}{m}\Big)\\
&\quad+\int_{\alpha(t_0)}^{\alpha(t)} h(\tau)e_F^{\frac{r}{m}}(\tau,t_0)\Big(\frac{r}{m}a(\tau)+\frac{m-r}{m}\Big)\Delta\tau\Bigg\}p\,g(\alpha(t))\alpha^\Delta(t)\\
&\quad+\alpha^\Delta(t)(1-p)g(\alpha(t))\\
&\quad+\left(\frac{l}{b}\right)^\Delta(t)\int_{t_0}^{t} v(t,s)\,\xi_1(s)\,e_F(s,t_0)\,a(\sigma(s))\Delta s\\
&\quad+\frac{l(\sigma(t))}{b(\sigma(t))}\Bigg[\int_{t_0}^{t} v_t^\Delta(t,s)\,e_F(s,t_0)\,a(\sigma(s))\,\xi_1(s)\Delta s
\end{aligned}
$$

$$+ v(\sigma(t),t)\,\xi_1(t)\,e_F(t,t_0)\,a(\sigma(t))\Bigg],$$

$$\widetilde{F}_1(t) = \frac{\widetilde{F}(t)}{1-\mu(t)\widetilde{F}(t)},$$

$$\widetilde{M}_1(t) = \Big[1+\mu(t)\widetilde{F}_1(t)\Big]\,\widetilde{M}(t).$$

Corollary 3.13. *Assume that all assumptions of Theorem 3.11 hold. Let* $N(t) = \ln(1+2t)$.

If $u(t)$ *satisfies*

$$\begin{aligned} u^m(t) \le a(t) &+ \int_{\alpha(t_0)}^{\alpha(t)} f(s)u^m(s)\Delta s \\ &+ b(t)\int_{\alpha(t_0)}^{\alpha(t)} g(s)\left(u^n(s) + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau)u^r(\tau)\Delta\tau\right)^p \Delta s + l(t) \\ &+ \int_{t_0}^{t} v(t,s)\left[k_1(s)\ln\big(1+2u^\lambda(\sigma(s))\big) - k_2(s)\ln\big(1+2u^\beta(\sigma(s))\big)\right]\Delta s, \end{aligned}$$

then

$$u(t) \le \left(a(t) + \int_{t_0}^{t} e_{\widetilde{L}\oplus\widetilde{F_1}}(t,\sigma(s))\widetilde{M}_1(s)\Delta s\right)^{\frac{1}{m}},\ 0<p\le 1,$$

where

$$\begin{aligned} \xi_1\big(k_1(t),k_2(t),2,\lambda,\beta,m\big) &= \xi_1(t) \\ &= \frac{2k_1(t)(\beta-\lambda)}{\beta-m}\left(\frac{(\beta-m)\,k_2(t)}{(\lambda-m)\,k_1(t)}\right)^{\frac{\alpha-m}{\alpha-\beta}} \end{aligned}$$

if $0<m<\lambda<\beta$ *or* $0<\lambda<\beta<m$ *, and*

$$\widetilde{L}(t) = \left\{\frac{n}{m}e_F^{\frac{n}{m}}(\alpha(t),t_0) + \int_{\alpha(t_0)}^{\alpha(t)} h(\tau)e_F^{\frac{r}{m}}(\tau,t_0)\Delta\tau\right\} pg(\alpha(t))\alpha^\Delta(t),$$

$$\begin{aligned} \widetilde{F}(t) = &\left(\frac{l}{b}\right)^\Delta(t)\int_{t_0}^{t} v(t,s)\,\xi_1(s)\,e_F(s,t_0)\Delta s \\ &+ \frac{l(\sigma(t))}{b(\sigma(t))}\Bigg[\int_{t_0}^{t} v_t^\Delta(t,s)\,e_F(s,t_0)\,\xi_1(s)\Delta s \\ &+ v(\sigma(t),t)\,\xi_1(t)\,e_F(t,t_0)\Bigg], \end{aligned}$$

$$
\begin{aligned}
\widetilde{M}(t) = &\Bigg\{ e_F^{\frac{n}{m}}(\alpha(t), t_0)\Big(\frac{n}{m}a(\alpha(t)) + \frac{m-n}{m}\Big) \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} h(\tau) e_F^{\frac{r}{m}}(\tau, t_0)\Big(\frac{r}{m}a(\tau) + \frac{m-r}{m}\Big)\Delta\tau \Bigg\} p\, g(\alpha(t))\alpha^{\Delta}(t) \\
&+ \alpha^{\Delta}(t)(1-p)g(\alpha(t)) \\
&+ \left(\frac{l}{b}\right)^{\Delta}(t) \int_{t_0}^{t} v(t,s)\, \xi_1(s)\, e_F(s, t_0)\, a(\sigma(s))\Delta s \\
&+ \frac{l(\sigma(t))}{b(\sigma(t))}\Bigg[\int_{t_0}^{t} v_t^{\Delta}(t,s)\, e_F(s, t_0)\, a(\sigma(s))\, \xi_1(s)\Delta s \\
&+ v(\sigma(t), t)\, \xi_1(t)\, e_F(t, t_0)\, a(\sigma(t))\Bigg],
\end{aligned}
$$

$$
\widetilde{F}_1(t) = \frac{\widetilde{F}(t)}{1 - \mu(t)\widetilde{F}(t)},
$$

$$
\widetilde{M}_1(t) = \Big[1 + \mu(t)\widetilde{F}_1(t)\Big]\, \widetilde{M}(t).
$$

4. Applications

Consider the following dynamic equation on time scales:

$$
(u^m(t))^{\Delta} = P\left(t, u(t), u(\sigma(t))), \int_{\alpha(t_0)}^{\alpha(t)} Q(s, u(s))\Delta s\right), \tag{4.1}
$$

with initial condition:

$$
u(t_0) = u_0, t_0, \quad t \in \mathbb{T}^k,
$$

where $m > 0$ and u_0 are constants, $\alpha : \mathbb{T}^k \to \mathbb{T}^k$ be a continuous function satisfying $\alpha(t) \leq t$, $\alpha(t_0) = t_0$ and $P : \mathbb{T}^k \times \mathbb{R}^3 \to \mathbb{R}$, $Q : \mathbb{T}^k \times \mathbb{R}$ are continuous functions.

Theorem 4.1. *Suppose that the functions P and Q in (4.1) satisfy the conditions*

(4.2)

$$
\begin{aligned}
|P(t, U, V, W)| \leq\ & f(t)|U|^m + b(t)\left(|V|^n + |W|\right)^p \\
&+ \Big[k_1 N\left(|U|^{\lambda}\right) - k_2(t) N\left(|U|^{\beta}\right)\Big], \quad t \in \mathbb{T}^k,\ U, V, W \in \mathbb{R},
\end{aligned}
$$

and

$$
|Q(t, U| \leq h(t)\,|U|^r, \quad t \in \mathbb{T}^k, U \in \mathbb{R} \tag{4.3}
$$

where $f, g, h \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $b, k_1, k_2 : \mathbb{T}_0 \to (0, \infty)$ *are rd-continuous functions,* $\left(\frac{1}{b}\right)^{\Delta}(t) \geq 0$, $N : \mathbb{R}^+ \to \mathbb{R}$ *is continuous, differentiable, non-negative and* m, n, r, p *are constants satisfying*

(i) $0 \leq n \leq m$, $0 \leq r \leq m, m < \lambda < \beta$ *or*

(ii) $0 \leq n \leq m$, $0 \leq r \leq m, 0 < \beta < \lambda < m$.

If $u(t)$ *is a solution of (4.1) and the function* $\mu(t)F(t) < 1$, *then*

$$u(t) \leq \left(|u_0|^m + \int_{t_0}^{t} e_{\widetilde{L} \oplus \widetilde{F_1}}(t, \sigma(s)) \widetilde{M_1}(s) \right)^{\frac{1}{m}}, \quad 0 < p \leq 1, \tag{4.4}$$

where

$$\xi_1\big(k_1(t), k_2(t), k, \lambda, \beta, m\big) = \xi_1(t) = \frac{k_1(t) k (\beta - \lambda)}{\beta - m} \cdot \left(\frac{(\beta - m) k_2(t)}{(\lambda - m) k_1(t)} \right)^{\frac{\lambda - m}{\lambda - \beta}}$$

if $m < \lambda < \beta$ *or* $0 < x < 1$ *and* $0 < \lambda < \beta < m$, *and*

$$\widetilde{L}(t) = \left\{ \frac{n}{m} e_F^{\frac{n}{m}}(\alpha(t), t_0) + \int_{\alpha(t_0)}^{\alpha(t)} h(\tau) e_F^{\frac{r}{m}}(\tau, t_0) \Delta\tau \right\} p \alpha^{\Delta}(t),$$

$$\widetilde{F}(t) = \frac{-b^{\Delta}(t)}{b^2(t)} \left[\int_{t_0}^{t} \xi_1(s) e_F(s, t_0) \Delta s + \xi_1(t) e_F(t, t_0) \right],$$

$$\begin{aligned} \widetilde{M}(t) = & \left\{ e_F^{\frac{n}{m}}(\alpha(t), t_0) \left(\frac{n}{m} |u_0|^m + \frac{m - n}{m} \right) \right. \\ & \left. + \int_{\alpha(t_0)}^{\alpha(t)} h(\tau) e_F^{\frac{r}{m}}(\tau, t_0) \left(\frac{r}{m} |u_0|^m + \frac{m - r}{m} \right) \Delta\tau \right\} p\, \alpha^{\Delta}(t) \\ & + \alpha^{\Delta}(t)(1 - p) \\ & + \left(\frac{1}{b} \right)^{\Delta}(t) \int_{t_0}^{t} \xi_1(s)\, e_F(s, t_0)\, a(\sigma(s)) \Delta s \\ & + \frac{1}{b(\sigma(t))} |u_0|^m\, \xi_1(t)\, e_F(t, t_0), \end{aligned}$$

$$\widetilde{F_1}(t) = \frac{\widetilde{F}(t)}{1 - \mu(t)\widetilde{F}(t)},$$

$$\widetilde{M_1}(t) = \left[1 + \mu(t) \widetilde{F_1}(t) \right] \widetilde{M}(t).$$

Proof. The equivalent integral equation of (4.1) is denoted by

$$u^m(t) = u_0^m + \int_{\alpha(t_0)}^{\alpha(t)} P\left(t, u(t), v(\sigma(t)), \int_{\alpha(t_0)}^{\alpha(t)} Q(s, u(s)) \Delta s \right).$$

Using assumptions (4.2) and (4.3), we get the following inequality:

$$
\begin{aligned}
u^m(t) \le |u_0|^m &+ \int_{\alpha(t_0)}^{\alpha(t)} f(s)\,|u(s)|^m \Delta s \\
&+ b(t) \int_{\alpha(t_0)}^{\alpha(t)} g(s)\Big(|u(s)|^n + \int_{\alpha(t_0)}^{\alpha(s)} h(\tau)|u(\tau)|^r \Delta\tau\Big)^p \Delta s \\
&+ \int_{t_0}^{t} \Big[k_1(s)N(|u(\sigma(s))|^\lambda) - k_2(s)N(|u(\sigma(s))|^\beta)\Big]\,\Delta s.
\end{aligned} \tag{4.5}
$$

Then, a suitable application of Theorem 3.11 to (4.5) yields (4.4) . □

Example 4.2. Consider the following dynamic equation on time scales $\mathbb{T} = \mathbb{N}_0$, $N(x) = \ln(1+x)$. If

$$
\begin{aligned}
u(t) = K_0 &+ \sum_{s=1}^{\alpha(t)-1} \frac{1}{(s+1)^2}\, u(s) \\
&+ \frac{3}{4} \sum_{s=1}^{\alpha(t)-1} \left(u(s) + \sum_{\tau=0}^{\alpha(t)-1} u(\tau) \right)^{\frac{1}{2}} \\
&+ \sum_{s=1}^{t} \left[\log\big(1+u^2(s+1)\big) - \log\big(1+u^3(s+1)\big) \right],
\end{aligned} \tag{4.6}
$$

where $K_0 = \frac{1}{2}, \alpha(t) = t-1$.

Theorem 4.3. *Suppose that u is a solution of Eq.(4.6) on $\mathbb{N}_0$. Then*

$$
u(t) \le \left(\frac{1}{2} + \sum_{s=1}^{t-1} \prod_{j=s+1}^{t-1} (1 + L(j) + F_1(j)) \widetilde{M}_1(s) \right), \tag{4.7}
$$

where

$$
\begin{aligned}
&\xi^*(1,1,1,2,3,1) = \frac{1}{4}, \\
&\widetilde{L}(t) = \frac{1}{2} \left\{ \prod_{s=1}^{t-2} (1+F(s)) + \sum_{s=0}^{t-2} \left(\prod_{s=1}^{\tau-1} (1+F(s)) \right) \right\}, \\
&\widetilde{F}(t) = \frac{1}{4} \prod_{s=1}^{t-1} (1+F(s)), F(t) = f(t) = \frac{1}{(t+1)^2}, \\
&\widetilde{M}(t) = \frac{1}{4} \left[\left(\prod_{s=1}^{t-2} (1+F(s)) \right) + \sum_{\tau=0}^{t-2} \left(\prod_{s=1}^{\tau-1} (1+F(s)) \right) \right]
\end{aligned}
$$

$$+\frac{1}{2}+\frac{1}{8}\left(\prod_{s=1}^{t-1}(1+F(s))\right),$$

$$\widetilde{F}_1(t)=\frac{\widetilde{F}(t)}{1-\widetilde{F}(t)},$$

$$\widetilde{M_1}(t)=\left[1+\widetilde{F}(t)\right]\widetilde{M}(t).$$

Proof. From (4.6), we have

$$\begin{aligned}|u(t)| = K_0 &+ \sum_{s=0}^{\alpha(t)-1}\frac{1}{(s+1)^3}|u(s)| \\ &+\frac{3}{4}\sum_{s=0}^{\alpha(t)-1}\left(|u(s)|+\sum_{\tau=0}^{\alpha(t)-1}|u(\tau)|\right)^{\frac{1}{2}} \\ &+\sum_{s=0}^{t}\Big[(s+1)\log\left(1+\left|u^2(s+1)\right|\right) \\ &-\frac{1}{(s+1)^2}\log\left(1+\left|u^3(s+1)\right|\right)\Big]\end{aligned}$$

Using Theorem 3.11, we obtain the desired inequality (4.7). □

Conclusion

In this paper, we have studied some new retarded half-linear integral inequalities within the framework of time scales by incorporating additional nonlinear terms, parameters and more general conditions. The obtained results generalize those of Boudeliou and Liu and they also provide novel bounds and inequalities not found in the literature. Overall, this study expands the scope of understanding and paves the way for future research in the development of integral inequalities in the context of time scales calculus.

Results and Discussion

This study generalizes Boudeliou's half-linear integral inequalities to the time scales framework. By including additional nonlinear terms, parameters, and broader conditions, the derived bounds are more general and sharper than previous results. These extensions cover a wider class of functions and dynamic equations, bridging continuous and discrete analyses and provide practical advantages in applications involving nonlinearities.

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