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Dynamics of a C_0-Semigroup of Translations on Locally Compact Groups

Ibrahim Akbarbaglu[1], Mohammad Reza Azimi[2*] and Zeinab Naghdi[3]

Abstract. Let G be a locally compact group and $1 \leq p < \infty$. In this paper, we show that the mapping $\pi : G \times L^p(G) \to L^p(G)$ with the assignment $(t, f) \mapsto T_t f = \frac{\phi}{\delta_t * \phi} \delta_t * f$, is separately continuous which ensures that $\{T_t\}_{t \in G}$ forms a C_0-semigroup. Also, it is proved that the C_0-semigroup $\{T_t\}_{t \in G}$ admits a universal vector only when G is noncompact and second countable. The hypercyclic phenomenon is studied for the single weighted translation as well. Finally, we give examples of universal C_0-semigroups.

1. Introduction

Recall that a bounded linear operator T on a Fréchet space X is hypercyclic if there is some vector $x \in X$ such that its orbit, i.e., $orb(T, x) := \{x, Tx, \ldots, T^n x, \ldots\}$ is dense in X, where $T^n = T \circ \cdots \circ T$ (n-times).

Besides, T is topologically transitive, when for non-empty open sets U and V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$.

Assume that G is a locally compact group with the left Haar measure λ and the unit element e. Separability and infinite dimensionality are necessary conditions for the hypercyclic operators to occur in the complex Lebesgue space $L^p(G)$ with respect to $\lambda (1 \leq p < \infty)$. By taking into account this fact, throughout this paper we will focus on the second countable locally compact groups.

Dynamics of weighted translations on locally compact groups and weighted shifts have been studied in various works [5, 6, 9, 10, 27].

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In [9, 10] Chen and Chu characterized hypercyclic weighted translations with aperiodic elements on discrete and non-discrete coset spaces. Moreover, the dynamical properties of composition operators, including Kitai's criterion, have been extensively studied in [7, 8, 19].

Suppose that $B(L^p(G))$ denotes the algebra of all bounded and linear operators on $L^p(G)$, where $1 \leq p < \infty$ and G is a locally compact group. Let F be a semigroup of G containing e. A family $\{S_g : g \in F\}$ in $B(L^p(G))$ is called a C_0-semigroup (or strongly continuous semigroup) when

- $S_e = I$, where I is an identity operator;
- $S_g S_h = S_{gh}$ for all $g, h \in F$;
- $\lim_{g \to h} S_g f = S_h f$ for all $f \in L^p(G)$.

For more details concerning C_0-semigroups we refer the reader to the references [11, 17, 20, 22–24, 26].

Let $g \in G$ and let δ_g be the unit point mass at g. A Borel measurable function $\phi : G \to (0, \infty)$ is said to be a weight on G. For $F \subseteq G$, the weight ϕ is said to be F-admissible whenever, for each $t \in F$, there exists a constant $c > 0$ (depending on t) such that

$$\phi(tx) \leq c\, \phi(x) \quad \text{for almost every } x \in G.$$

Let ϕ be F-admissible and ϕ with $\frac{1}{\phi}$ be locally bounded. Define for any $g \in F$, a weighted translation semigroup T_g as follows:

$$T_g : L^p(G) \to L^p(G)$$
$$T_g(f) = \frac{\phi}{\delta_g * \phi} \delta_g * f$$

for all f in $L^p(G)$, where the convolution operator $\delta_g * f$ is the left translation of f by g^{-1}, precisely:

$$(\delta_g * f)(x) := \int_G f\left(y^{-1}x\right) d\delta_g(y) = f\left(g^{-1}x\right), \quad (x \in G).$$

For $g, h \in F$, note that $T_g T_h(f)(x) = T_{gh}(f)(x)$ for each $x \in G$.

Take $F \subseteq G$ and set up the family of translation operators $\{T_g\}_{g \in F}$. This family is called F-universal if there exists some function $f \in L^p(G)$, referred to as an F-universal vector, for which the set $\{T_g f : g \in F\}$ is dense in $L^p(G)$. In the case $F = G$, then we prefer to use only the universal term.

If $g \in G$, then by a routine calculation, the orbit of any $f \in L^p(G)$ under the single weighted translation T_g is obtained as follows:

$$orb(T_g, f) = \left\{ f, \frac{\phi}{\delta_g * \phi} \delta_g * f, \frac{\phi}{\delta_g^2 * \phi} \delta_g^2 * f, \ldots, \frac{\phi}{\delta_g^n * \phi} \delta_g^n * f, \ldots \right\}.$$

The translation operator cannot be hypercyclic by itself, therefore we will survey the hypercyclicity of the weighted translations. It's clear that if $\left\|\frac{\phi}{\delta_g * \phi}\right\|_\infty \leq 1$, then $\|T_g\| \leq 1$ and T_g won't be hypercyclic as well.

Throughout this paper, the support of any function $f \in L^p(G)$ is defined as $\{x \in G : f(x) \neq 0\}$ and denoted by $supp(f)$. Also, the characteristic function of a subset $A \subseteq G$ is denoted by χ_A.

The study of hypercyclic semigroups was initiated by Desch, Schappacher and Webb in [13]. Then some mathematicians have attracted their attention to this subject. For this topic, interested readers are referred to [11, 12, 15, 16, 21, 23]. Regarding dynamics of translations the references [1–6, 9, 10] are cited.

In any locally compact group, the torsion elements are those that have finite order. When the closed subgroup generated by an element is compact, then it is called periodic. Further, an element in G is aperiodic if it is not periodic. When dealing with the torsion elements, the hypercyclic phenomena for the weighted translations is rarely discussed. Indeed, if $g \in G$ is a torsion element, then the weighted translation T_g is never hypercyclic on $L^p(G)$. Recently, Chen and Chu in their paper [9], have stated an applied hypercyclic criterion for the weighted translations, induced by aperiodic elements on discrete groups. Afterwards, they generalized it to the non-discrete groups, in their later paper [10].

To satisfy the third condition of semigroups, as stated earlier in the introduction, we have to establish the continuity of the weighted translations with respect to their indices. So in this paper, first we prove that the assignment $(t, f) \mapsto T_t(f)$ is separately continuous. Then we show that second countability of the locally compact groups is a necessary condition for the universal semigroups. The hypercyclic behavior of weighted translations in C_0-semigroup is studied as well. At the end, we give examples of universal C_0-semigroups.

2. Harmonic Approaching to the C_0-Semigroup of Translations

In the following proposition, we shall see how the admissibility of the weight function guarantees the boundedness of the weighted translation operators.

Proposition 2.1. *Let G be a σ-compact locally compact group and for $g \in G$, $T_g : L^p(G) \to L^p(G)$ be a weighted translation operator. If ϕ is g-admissible, then*

$$\|T_g\| = \left\|\frac{\delta_{g^{-1}} * \phi}{\phi}\right\|_\infty.$$

Proof. Note that for an arbitrary $f \in L^p(G)$, by the change of variable formula we have

$$\begin{aligned}\|T_g f\|_p^p &= \int_G \left| \frac{\phi}{\delta_g * \phi} \delta_g * f \right|^p d\lambda \\ &= \int_G \left| \frac{\delta_{g^{-1}} * \phi}{\phi} f \right|^p d\lambda \\ &\leq \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty^p \int_G |f|^p d\lambda \\ &= \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty^p \|f\|_p^p.\end{aligned}$$

Hence, $\|T_g\| \leq \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty$. But for the reverse inequality, for every $n \in \mathbb{N}$ define

$$E_n := \left\{ t \in G : \left| \frac{\phi(gt)}{\phi(t)} \right| \geq \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty - \frac{1}{n} \right\}.$$

Note that each E_n is a non-empty set with positive measure. That λ is a regular measure, one may approximate $\lambda(E_n) = \sup\{\lambda(F) : \ F \subseteq E_n, \ F \text{ is compact}\}$. So we may find a compact subset $F_n \subseteq E_n$ with non-zero measure and very close to E_n in terms of measure. Then readily $\chi_{F_n} \in L^p(G)$ and for an arbitrary $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\|T_g \chi_{F_n}\|_p^p &= \int_G \left| \frac{\phi}{\delta_g * \phi} \delta_g * \chi_{F_n} \right|^p d\lambda \\ &= \int_G \left| \frac{\delta_{g^{-1}} * \phi(t)}{\phi(t)} \chi_{F_n}(t) \right|^p d\lambda(t) \\ &\geq \int_{F_n} \left| \frac{\delta_{g^{-1}} * \phi(t)}{\phi(t)} \right|^p d\lambda(t) \\ &\geq \left(\left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty - \frac{1}{n} \right)^p \|\chi_{F_n}\|_p^p.\end{aligned}$$

More explicitly, $\|T_g\| \geq \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty - \frac{1}{n}$ and by tending $n \to \infty$, we get that $\|T_g\| \geq \left\| \frac{\delta_{g^{-1}} * \phi}{\phi} \right\|_\infty$ and the proof is completed. □

Lemma 2.2 ([14, Proposition 1.16])**.** *Assume that a measurable function $L : G \to (0, \infty)$ is sub-multiplicative, i.e, for all $s, t \in G$, $L(st) \leq L(s)L(t)$. Then L is bounded and bounded away from zero on any compact set.*

The following theorem shows that a family $\{T_t\}_{t\in G}$ forms a C_0-semigroup or strongly continuous semigroup, indeed.

Theorem 2.3. *Let $1 \leq p < \infty$ and ϕ be an admissible weight on G. Then the mapping $\pi : G \times L^p(G) \to L^p(G)$ with the assignment $(t, f) \mapsto T_t(f)$ is separately continuous.*

Proof. Since for any $s \in G$, $\|T_s\| < \infty$, it is sufficient to show that for any $f \in L^p(G)$ and $\epsilon > 0$ there exists a neighborhood of the identity V such that for all $t \in V$, $\|f - T_t f\|_p < \epsilon$. For this reason, suppose first that $g \in C_c(G)$ and $F = supp(g)$. Pick a relatively compact neighborhood of the identity V. Then $supp(\delta_t * g) \subseteq VF$, whenever $t \in V$. Since ϕ is locally bounded, then it is bounded on every compact set, so that $\sup\{\phi(x) : x \in VF\} < \infty$. Now

$$\|g - T_t g\|_p \leq \left(\sup_{VF} \phi\right) \left\| \frac{g}{\phi} - \frac{\delta_t * g}{\delta_t * \phi} \right\|_p,$$

where the latter norm by [18, Proposition 2.41] is less than ϵ for t in a sufficiently small neighborhood of the identity $U \subset V$.

Let now $f \in L^p(G)$ be arbitrary. For any $\epsilon > 0$, there is some $g \in C_c(G)$ such that $\|f - g\|_p < \epsilon$. Now assume that V is a symmetric relatively compact neighborhood of the identity. By Lemma 2.2, the function $L : G \to (0, \infty)$ defined by $L(s) = \operatorname{ess\,sup}_{x\in G} \dfrac{\phi(sx)}{\phi(x)} < \infty$ is bounded on every compact set, so that $\sup_V L < \infty$. Now, for $t \in V$, $\|T_t f - T_t g\|_p \leq (\sup_V L) \|f - g\|_p$ and

$$\begin{aligned} \|f - T_t f\|_p &\leq \|f - g\|_p + \|g - T_t g\|_p + \|T_t f - T_t g\|_p \\ &< \epsilon + \left(\sup_{VF} \phi\right) \left\| \frac{g}{\phi} - \frac{\delta_t * g}{\delta_t * \phi} \right\|_p + \left(\sup_V L\right) \epsilon \end{aligned}$$

which is less than $\epsilon(1 + \sup_{VF} \phi + \sup_V L)$ for t in a sufficiently small neighborhood of the identity $U \subset V$. □

Proposition 2.4. *Let G be a locally compact group and let ϕ be an admissible weight on G. If the family $\{T_s\}_{s\in G}$ admits a universal vector in $L^p(G)$, then G must be second countable.*

Proof. Assume that $f \in L^p(G)$ is a universal vector for the family $\{T_s\}_{s\in G}$. By [18, Propositions 2.4 and 2.22], the support of f is contained in an open σ-compact subgroup H of G.

Suppose that $H \neq G$. Choose an element $t \in G \setminus H$ and let $U \subseteq H$ be a compact neighborhood of the identity. Then $U \cap tU = \emptyset$. Since f is universal, there exists $s \in G$ such that

$$\|\chi_U + \chi_{tU} - T_s f\|_p < \|\chi_U\|_p.$$

If the support of $T_s f$ meets only one of the sets U or tU (for instance U), then we obtain

$$\|\chi_{tU}\|_p^p + \|\chi_U - T_s f\|_p^p = \|\chi_{tU} + \chi_U - T_s f\|_p^p < \|\chi_{tU}\|_p^p,$$

which is impossible. Hence,

$$\operatorname{supp}(T_s f) \cap U \cap tU \neq \emptyset,$$

contradicting the choice of H. Therefore, G is σ-compact.

It is well known that for locally compact groups, σ-compactness together with metrizability is equivalent to second countability (see [25, Theorem 2.11]). Consider the action

$$\pi : G \times L^p(G) \to L^p(G),$$
$$\pi(g, f) = T_g f.$$

Since this action is separately continuous, it follows from [29, Theorem 3] that G is metrizable whenever it acts effectively on a separable first countable space. Consequently, it suffices to prove that $L^p(G)$ is separable.

Assume to the contrary that $L^p(G)$ is not separable. Then there exist an uncountable index set I, a constant $r > 0$, and a family $\{h_\alpha\}_{\alpha \in I} \subseteq L^p(G)$ such that

$$\|h_\alpha - h_\beta\|_p > r, \quad (\alpha \neq \beta).$$

Without loss of generality, take $r = 3$. For each $\alpha \in I$, the universality of f yields an element $s_\alpha \in G$ such that

$$\|h_\alpha - T_{s_\alpha} f\|_p < \frac{1}{2}.$$

Hence,

$$\|T_{s_\alpha} f - T_{s_\beta} f\|_p > 2, \quad (\alpha \neq \beta),$$

and thus $\{T_{s_\alpha} f\}_{\alpha \in I}$ is an uncountable family with pairwise distance greater than 2.

Define $A_k = \left\{x \in G : \frac{1}{k} < \phi(x) < k\right\}$ for each $k \in \mathbb{N} - \{1\}$. Then $\{A_k\}_{k \in \mathbb{N}}$ is an increasing sequence of measurable sets whose union equals G. By Levi's theorem, for each $\alpha \in I$,

$$\int_G |T_{s_\alpha} f|^p \, d\lambda = \lim_{k \to \infty} \int_{A_k} |T_{s_\alpha} f|^p \, d\lambda.$$

Thus, there exists $k_\alpha \in \mathbb{N}$ such that

$$\int_{G \setminus A_{k_\alpha}} |T_{s_\alpha} f|^p \, d\lambda < \frac{1}{2^p},$$

which implies

$$\left\| T_{s_\alpha} f \, \chi_{G \setminus A_{k_\alpha}} \right\|_p < \frac{1}{2}.$$

Since the mapping $\alpha \mapsto k_\alpha$ sends the uncountable set I into $\mathbb{N}$, there exists $k \in \mathbb{N}$ such that the subset

$$J = \{\alpha \in I : k_\alpha = k\}$$

is uncountable. For $\alpha, \beta \in J$ with $\alpha \neq \beta$, we obtain

$$\left\| \left(T_{s_\alpha} f - T_{s_\beta} f\right) \chi_{A_k} \right\|_p > 1,$$

which contradicts [29, Theorem 3]. This contradiction shows that $L^p(G)$ must be separable, and therefore G is second countable. □

The theorem below confirms that the universal phenomenon cannot occur on compact groups.

Theorem 2.5. *Let $1 \leq p < \infty$ and $S \subseteq G$. Let ϕ be an S-admissible weight on G. Let $f \in L^p(G)$ be an S-universal vector and $F \subseteq G$ be a compact set of positive measure. Then for every $\epsilon > 0$, $\alpha \in \mathbb{R}$ and every compact set $K \subseteq G$, there exists $s \in S \setminus K$ such that $\|\alpha\phi\chi_F - T_s f\|_p < \epsilon$.*

Proof. Let $\epsilon > 0$, $\alpha \in \mathbb{R}$, and let K be an arbitrary compact subset of G. We may assume without loss of generality that $\epsilon < \lambda(F)$ and that K is symmetric. Indeed, if this is not the case, one can replace K by the compact symmetric set $K \cup K^{-1}$.

Since

$$G = \bigcup_{n=2}^{\infty} \phi^{-1}\left(\tfrac{1}{n}, n\right),$$

there exists a sufficiently large integer $n_0 \in \mathbb{N}$ such that the set

$$F_{n_0} = \left\{t \in F : \phi(t) > \tfrac{1}{n_0}\right\}$$

satisfies

$$\lambda(F_{n_0}) > \tfrac{3}{4}\lambda(F).$$

Moreover, the set KF is compact and therefore has finite Haar measure. Choose an integer $M \in \mathbb{N}$ such that

$$M > \frac{2\lambda(KF)}{\lambda(F)}.$$

For $j = 0, 1, \ldots, M-1$, define

$$\alpha_j = \alpha + \frac{j\epsilon^2}{2M(1 + \|\phi\chi_F\|_p)\lambda(F)}.$$

Next, select $\delta > 0$ satisfying

$$\delta < \frac{\epsilon}{8M\left(1 + \|\phi\chi_F\|_p\right)\lambda(F)},$$

and choose $\beta > 1$ such that

$$\frac{1}{\beta^p} < \frac{\lambda(F)}{4}.$$

Since the family $\{T_s f : s \in S\}$ is dense in $L^p(G)$, for each $j = 0, 1, \ldots, M-1$ there exists an element $s_j \in S$ with

$$\left\|\alpha_j \phi \chi_F - T_{s_j} f\right\|_p < \frac{\delta}{\beta n_0}.$$

This yields that

$$\begin{aligned}\left\|\alpha\phi\chi_F - T_{s_j} f\right\|_p &\leq \left\|\alpha_j\phi\chi_F - T_{s_j} f\right\|_p + \|(\alpha_j - \alpha)\phi\chi_F\|_p \\ &< \frac{\delta}{\beta n_0} + |\alpha_j - \alpha|\|\phi\chi_F\|_p \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

We now claim that there exists an index $j \in \{0, 1, \ldots, M-1\}$ such that $s_j \notin K$. Suppose, on the contrary, that

$$s_j \in K, \quad \text{for all } j = 0, 1, \ldots, M-1.$$

Therefore,

$$\left(\frac{\delta}{\beta n_0}\right)^p > \int_G \left|\alpha_j \phi x_F - T_{s_j} f\right|^p d\lambda > \int_{F_{n_0}} \frac{1}{n_0^p} \left|\alpha_j \chi_F - \frac{\delta_{s_j} * f}{\delta_{s_j} * \phi}\right|^p d\lambda.$$

Then $\left\|(\alpha_j \chi_F - \dfrac{\delta_{s_j} * f}{\delta_{s_j} * \phi})\chi_{F_{n_0}}\right\|_p < \dfrac{\delta}{\beta}$. Define

$$F_j = \left\{t \in G : \left|\alpha_j \chi_F - \frac{\delta_{s_j} * f}{\delta_{s_j} * \phi}\right| \chi_{F_{n_0}} \geq \delta\right\}.$$

Hence, we have

$$\left(\frac{\delta}{\beta}\right)^p > \int_{F_{n_0}} \left|\alpha_j \chi_F - \frac{\delta_{s_j} * f}{\delta_{s_j} * \phi}\right|^p d\lambda > \int_{F_j} \delta^p d\lambda,$$

which implies $\lambda(F_j) < \dfrac{1}{\beta^p} < \dfrac{\lambda(F)}{4}$. Now, set $F_j' = s_j^{-1}(F_{n_0} \setminus F_j)$. Thus,

$$\left|\alpha_j \chi_F - \frac{\delta_{s_j} * f}{\delta_{s_j} * \phi}\right| \chi_{sjF_j'} < \delta < \frac{\epsilon}{8M(1 + \|\phi\chi_F\|_p)\lambda(F)}.$$

For each $0 \leq i, j \leq M-1$ with $i \neq j$, we have

$$|\alpha_j - \alpha_i| \geq \frac{\epsilon}{2M(1 + \|\phi\chi_F\|_p)\lambda(F)},$$

which implies in turn that

$$\begin{aligned}|\alpha_j - \alpha_i|\chi_{F'_j \cap F'_i} &\leq \left|\alpha_j - \frac{f}{\phi}\right| \chi_{F'_j} + \left|\alpha_i - \frac{f}{\phi}\right| \chi_{F'_i} \\ &< \delta + \delta \\ &< \frac{\epsilon}{4M(1 + \|\phi\chi_F\|_\infty)\lambda(F)}.\end{aligned}$$

Therefore, $F'_j \cap F'_i = \emptyset$. Moreover,

$$\lambda\left(F'_j\right) = \lambda(F_{n_0}) - \lambda(F_j) > \frac{3}{4}\lambda(F) - \frac{1}{4}\lambda(F) = \frac{1}{2}\lambda(F).$$

Eventually,

$$\lambda\left(\bigcup_{j=0}^{M-1} F'_j\right) > \frac{M}{2}\lambda(F) > \lambda(KF).$$

But this contradicts the fact that $F'_j \subset s_j^{-1}F \subset KF$ for each $j = 0, 1, \ldots, M-1$. □

By [10, Lemma 2.1], on a locally compact group G an element g is aperiodic if and only if for every compact subset $K \subset G$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $g^n K \cap K = \emptyset$ (or, $g^{-n}K \cap K = \emptyset$). In the following theorem, we shall examine the hypercyclicity of a single weighted translation by using this fact.

Theorem 2.6. *Let G be a locally compact group and let g be an aperiodic element in G. If T_g is a weighted translation semigroup on $L^p(G)$ where $1 \leq p < \infty$, then the following conditions are equivalent:*

(i) *T_g is hypercyclic,*

(ii) *For every compact $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (B_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(B_k)$. Also, the sequences*

$$\phi_n := \frac{\delta_{g^{-1}}^n * \phi}{\phi}, \qquad \widetilde{\phi}_n := \frac{\delta_g^n * \phi}{\phi}$$

have subsequences (ϕ_{n_k}) and $\left(\widetilde{\phi}_{n_k}\right)$, respectively such that

$$\lim_{k\to\infty} \|\phi_{n_k}|_{B_k}\|_\infty = \lim_{k\to\infty} \left\|\widetilde{\phi}_{n_k}|_{B_k}\right\|_\infty = 0.$$

Proof. $(i) \Rightarrow (ii)$. Let T_g be hypercyclic and $K \subset G$ be a compact set such that $\lambda(K) > 0$. Since g is aperiodic, it can be concluded that there is some $N \in \mathbb{N}$ such that for $n > N$, $g^{-n}K \cap K = \emptyset$ holds. Let $\epsilon > 0$.

We take δ such that $0 < \delta < \frac{\epsilon}{1+\epsilon}$. Since T_g is hypercyclic, there exists a hypercyclic vector $f \in L^p(G)$ for T_g such that for some $m > N$,

$$\|f - \chi_K\|_p < \delta^2, \qquad \left\|T_g^m f - \chi_K\right\|_p < \delta^2.$$

Let $D_\delta = \{x \in K : \ |f(x) - 1| \geq \delta\}$. Therefore,

$$|f(x)| > 1 - \delta, \quad (x \in K \setminus D_\delta).$$

Also, we have $\lambda(D_\delta) = 0$ or

$$\begin{aligned}
\delta^{2p} &\geq \|f - \chi_K\|_p^p \\
&= \int_G |f(x) - \chi_K(x)|^p d\lambda(x) \\
&\geq \int_K |f(x) - 1|^p d\lambda(x) \\
&\geq \int_{D_\delta} |f(x) - 1|^p d\lambda(x) \\
&\geq \delta^p \lambda(D_\delta),
\end{aligned}$$

and so $\lambda(D_\delta) < \delta^p$. Now assume that $F_\delta = \{x \in G \setminus K : \ |f(x)| \geq \delta\}$. Then $|f(x)| < \delta$ for $x \in (G \setminus K) \setminus F_\delta$. Again it can be written:

$$\delta^{2p} > \int_{G\setminus K} |f(x) - \chi_K(x)|^p d\lambda(x) \geq \int_{F_\delta} |f(x)|^p d\lambda(x) \geq \delta^p \lambda(F_\delta),$$

hence, $0 \leq \lambda(F_\delta) < \delta^p$.

Suppose that $M_{m,\delta} = \left\{x \in K : \ \left|\widetilde{\phi}_m(x)^{-1} f(g^{-m}x) - 1\right| \geq \delta\right\}$. Thus

$$\widetilde{\phi}_m(x)^{-1} \left|f\left(g^{-m}x\right)\right| > 1 - \delta, \quad (x \in K \setminus M_{m,\delta}).$$

Also, we have

$$\begin{aligned}
\delta^{2p} &> \left\|T_g^m(f) - \chi_K\right\|_p^p \\
&= \int_G \left|T_g^m(f(x)) - \chi_K(x)\right|^p d\lambda(x) \\
&\geq \int_{M_{m,\delta}} \left|\frac{\phi(x)}{\phi(g^{-m}x)} f(g^{-m}x) - 1\right|^p d\lambda(x) \\
&\geq \delta^p \lambda(M_{m,\delta}),
\end{aligned}$$

therefore, $\lambda(M_{m,\delta}) < \delta^p$. As we know $g^{-m}K \cap K = \emptyset$, so we can write

$$\widetilde{\phi}_m(x) < \frac{f(g^{-m}x)}{1-\delta} < \frac{\delta}{1-\delta} < \epsilon, \quad x \in K \setminus \left(M_{m,\delta} \cup g^m F_\delta\right).$$

Let $N_{m,\delta} = \{x \in K : \ |\phi_m(x) f(x)| \geq \delta\}$. Consequently,

$$\phi_m(x)|f(x)| < \delta, \quad (x \in K \setminus N_{m,\delta}).$$

Furthermore, since $g^m K \cap K = \emptyset$, it follows that

$$\begin{aligned}
\delta^{2p} &> \left\|T_g^m f - \chi_K\right\|_p^p \\
&= \int_G \left|\frac{\phi(x)}{\phi(g^{-m}x)} f(g^{-m}x) - \chi_K(x)\right|^p d\lambda(x) \\
&= \int_G \left|\frac{\phi(g^m x)}{\phi(x)} f(x) - \chi_K(g^m x)\right|^p d\lambda(x) \\
&\geq \int_{N_{m,\delta}} \left|\frac{(\delta_{g^{-1}}^m * \phi)(x)}{\phi(x)} f(x)\right|^p d\lambda(x) \\
&\geq \delta^p \lambda(N_{m,\delta}),
\end{aligned}$$

and this means that $\lambda(N_{m,\delta}) < \delta^p$. Thus we have

$$\phi_m(x) < \frac{\delta}{|f(x)|} < \frac{\delta}{1-\delta} < \epsilon, \quad x \in K \setminus (N_{m,\delta} \cup D_\delta).$$

Let $B_{m,\delta} = K \setminus (D_\delta \cup g^m F_\delta \cup M_{m,\delta} \cup N_{m,\delta})$. Then according to what we argued $\lambda(K \setminus B_{m,\delta}) < 4\delta^p$, $\|\phi|_{B_{m,\delta}}\|_\infty < \epsilon$ and $\left\|\widetilde{\phi}|_{B_{m,\delta}}\right\|_\infty < \epsilon$.

Now, for $k = 1, 2, \ldots$ we have a sequence of Borel sets $B_k \subset K$. We choose $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\lambda(K \setminus B_k) < 4\left(\frac{1}{k}\right)^p$, $\|\phi_{n_k}|_{B_k}\|_\infty < \frac{1}{k}$ and $\left\|\widetilde{\phi}_{n_k}|_{B_K}\right\|_\infty < \frac{1}{k}$. This means that the conditions of (ii) hold.

$(ii) \Rightarrow (i)$. To prove this, it is enough to show that T_g is topologically transitive. So, assume that U and V are non-empty open subsets of $L^p(G)$. Let $C_c(G)$ be the set of continuous functions on G with compact supports. By the density of $C_c(G)$ in $L^p(G)$, we are allowed to choose $h_1, h_2 \in C_c(G)$ such that $h_1 \in U$ and $h_2 \in V$. Let $K = supp(h_1) \cup supp(h_2)$. Now, suppose that $B_k \subset K$ and the subsequences (ϕ_{n_k}) and $\left(\widetilde{\phi}_{n_k}\right)$ satisfy condition (ii). Since g is an aperiodic element, there is an integer $N_1 \in \mathbb{N}$ such that $g^n K \cap K = \emptyset$ $(g^{-n} K \cap K = \emptyset)$ for all $n > N_1$.

Let $\epsilon > 0$. There is $N_2 \in \mathbb{N}$ such that $n_k > N_1$ and $\phi_{n_k}^p < \frac{\epsilon}{\|h_1\|_p^p}$ on B_k for $k > N_2$. Then we have

$$\begin{aligned}
\left\|T_g^{n_k}(h_1 \chi_{B_k})\right\|_p^p &= \int_{g^{n_k} B_k} \left|\frac{\phi(x)}{\phi\left(g^{-n_k}x\right)}\right|^p \left|h_1\left(g^{-n_k}x\right)\right|^p d\lambda(x) \\
&= \int_{B_k} \left|\frac{\phi(g^{n_k}x)}{\phi(x)}\right|^p |h_1(x)|^p d\lambda(x) \\
&= \int_{B_k} \phi_{n_k}^p(x) |h_1(x)|^p d\lambda(x) < \epsilon, \quad k > N_2.
\end{aligned}$$

So, $\left\|T_g^{n_k}\left(h_1\chi_{B_k}\right)\right\|_p \to 0$ as $k \to \infty$. Now let L_g be the weighted translation on the subspace $L_c^p(G)$ of the all functions in $L^p(G)$ with compact supports defined by

$$L_g(h) = \frac{\phi}{\delta_{g^{-1}} * \phi}\delta_{g^{-1}} * h, \quad (h \in L_c^p(G)).$$

Thus

$$T_g L_g(h) = h, \quad (h \in L_c^p(G)).$$

Again, with the same argument for $L_g^{n_k}$ and the sequence $\left(\widetilde{\phi}_{n_k}\right)$, we have that

$$\begin{aligned}
\lim_{k\to\infty}\left\|L_g^{n_k}(h_2\chi_{B_k})\right\|_p^p &= \lim_{k\to\infty}\int_{g^{-n_k}B_k}\left|\frac{\phi(x)}{\phi(g^{n_k}x)}\right|^p |h_2(g^{n_k}x)|^p\, d\lambda(x)\\
&= \lim_{k\to\infty}\int_{B_k}\left|\frac{\phi\left(g^{-n_k}x\right)}{\phi(x)}\right|^p |h_2(x)|^p\, d\lambda(x)\\
&= \lim_{k\to\infty}\int_{B_k}\left(\widetilde{\phi}_{n_k}(x)\right)^p |h_2(x)|^p d\lambda(x)\\
&= 0.
\end{aligned}$$

Let $z_k = h_1\chi_{B_k} + L_g^{n_k}(h_2\chi_{B_k})$ in $L^p(G)$ for all $k \in \mathbb{N}$. Therefore, we have

$$\|z_k - h_1\|_p^p \le \|h_1\|_\infty^p \lambda(K \setminus B_k) + \|L_g^{n_k}(h_2\chi_{B_k})\|_p^p$$

and

$$\|T_g^{n_k}z_k - h_2\|_p^p \le \left\|T_g^{n_k}\left(h_1\chi_{B_k}\right)\right\|_p^p + \|h_2\|_\infty^p\lambda(K \setminus B_k).$$

Thus, $\lim_{k\to\infty} z_k = h_1$ and $\lim_{k\to\infty} T_g^{n_k}z_k = h_2$. It means that $T_g^{n_k}(U) \cap V \neq \emptyset$ for some k. □

Corollary 2.7. *Let $\{T_g\}_{g\in F}$ be a C_0-semigroup and let an operator T_h be hypercyclic for some aperiodic element $h \in F$. Then the semigroup $\{T_g\}_{g\in F}$ is universal for the semigroup F generated by h and e.*

Proof. Let $f \in L^p(G)$ be a hypercyclic vector. Note that $\{T_h^n(f)\}_{n=0}^\infty = \{T_{h^n}(f)\}_{n=0}^\infty$ and now the rest of the proof is routine. □

Example 2.8. Let $G = (\mathbb{R}, +)$ be the group of real numbers with the Lebesgue measure on the Borel σ-algebra. Define the measurable weight

function ϕ on $\mathbb{R}$ by

$$\phi(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n} \chi_{(n,n+1]}(x) & 1 < x \\ 1 & -1 \leq x \leq 1 \\ \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[-n-1,-n)}(x) & x < -1. \end{cases}$$

Then $\frac{\phi(x)}{\phi(x-1)} \leq 2$ for all $x \in \mathbb{R}$, and hence by Theorem 2.6, the corresponding weighted translation $T_1 f(x) = \frac{\phi(x)}{\phi(x-1)} f(x-1)$ is hypercyclic. While, the weighted translation $T_1^\sharp f(x) = \phi(x) f(x-1)$ is not hypercyclic alone according to [10, Theorem 2.3].

Example 2.9. Let $(\mathbb{R}, \Sigma, \lambda)$ be a measure space with the Lebesgue measure. If we consider $\phi(x) = \frac{1}{1+x^2}$, then $\frac{\phi(x)}{\phi(x-t)} \leq 2(1+t^2)$ for all $x \in \mathbb{R}$ and any nonzero real t. It follows by Theorem 2.6 that for all $t \in \mathbb{R} \setminus \{0\}$, the weighted translation $T_t f(x) = \frac{\phi(x)}{\phi(x-t)} f(x-t)$ is hypercyclic. However, for any $t \in \mathbb{R} \setminus \{0\}$, the weighted translation $T_t^\sharp f(x) = \phi(x) f(x-t)$ is not hypercyclic by [10, Theorem 2.3].

Remark 2.10. It is concluded that with all admissible positive functions in $C_0(G)$ i.e., consisting of all continuous functions on G vanishing at infinity, one may construct a universal C_0-semigroup.

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