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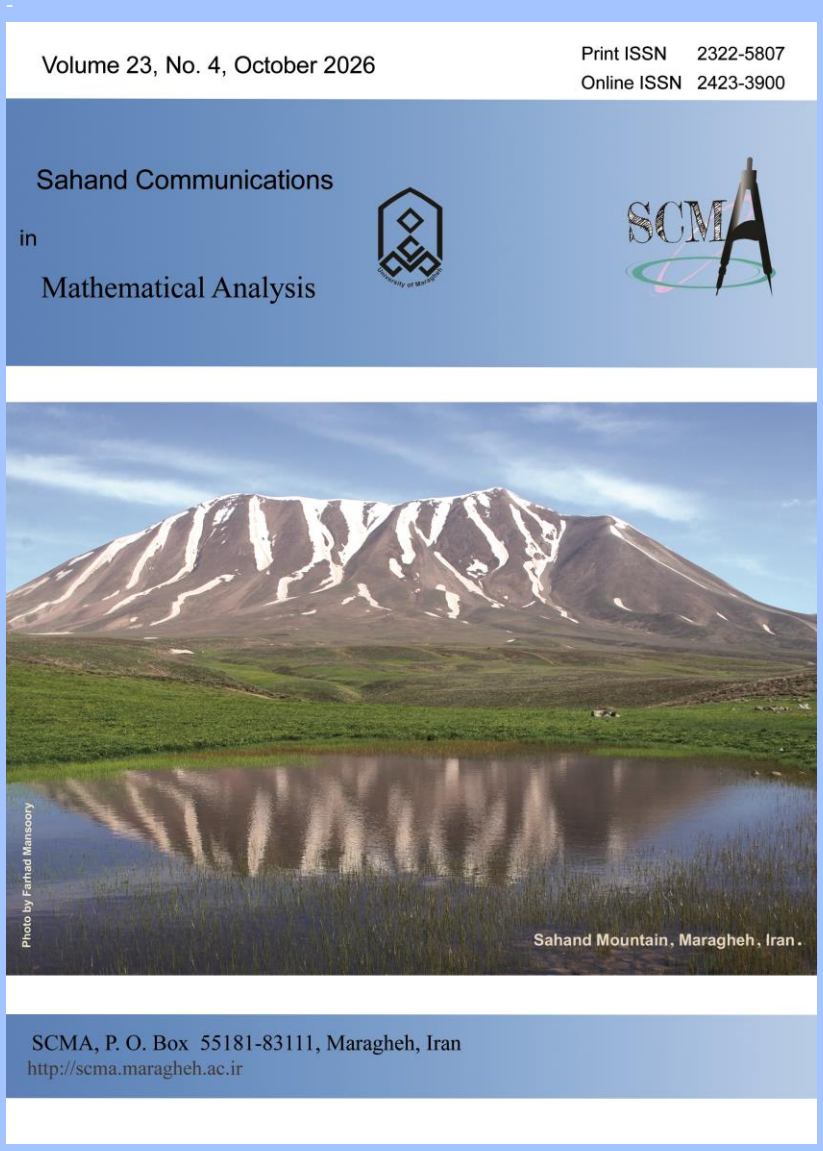
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 4
Pages: 37-55

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2026.2081140.2456



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 4 (2026), 37-55
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2081140.2456

Starlikeness Criteria for a Generalized Class of Univalent Functions Defined by Second-Order Differential Inequalities

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Abstract. Let $\mathcal{A}$ denote the class of functions analytic in the unit disk $\mathbb{D}$, normalized by $f(0) = 0$ and $f'(0) = 1$. We study the starlikeness of a generalized class of univalent functions defined by a second-order differential inequality involving the operator $(z/f(z))^{\mu}$, where $0 < \mu < 1$. This class extends the well-known class $\mathcal{M}(\lambda)$ introduced by Obradović and Ponnusamy and contains several previously studied subclasses as special cases. By applying methods of differential subordination and refined analytic estimates, we obtain explicit sufficient conditions on the parameter λ ensuring that functions in this class are starlike in $\mathbb{D}$. In particular, sharp bounds for λ are derived to guarantee inclusion in the classes $\mathcal{S}^*$ and $\mathcal{S}^*(\beta)$ and corresponding starlikeness results are established for an associated integral operator. These findings generalize and refine earlier starlikeness criteria for univalent functions defined by second-order differential inequalities.

1. Introduction

The theory of univalent functions is a central topic in geometric function theory due to its deep connections with conformal mappings, differential equations and various applications in complex analysis. Among the most extensively studied subclasses of univalent functions are the classes of starlike and convex functions, which are characterized by strong geometric properties and admit numerous analytic descriptions. In particular, the class of starlike functions plays a fundamental role

2020 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Univalent functions, Starlike functions, Differential inequalities, Differential subordination, Geometric function theory.
Received: 2025-12-20, Accepted: 2026-01-31.
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in the study of geometric behavior of analytic functions in the unit disk. Therefore, many mathematicians have investigated the starlikeness of certain classes of univalent functions (see recent studies, e.g., [1, 3, 5, 12–19]).

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} \subset \mathcal{H}$ be the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

For $0 \leq \beta < 1$, a function $f \in \mathcal{A}$ is said to be *starlike of order* β if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{S}^*(\beta)$ and the case $\beta = 0$ corresponds to the classical class of starlike functions $\mathcal{S}^*$. It is well known that every starlike function is univalent in $\mathbb{D}$ (see, for example, [2]).

A fruitful approach in the study of univalent and starlike functions is to characterize such classes by means of differential inequalities. In this direction, Obradović and Ponnusamy introduced a class of analytic functions defined by a second-order differential inequality involving the operator $z/f(z)$, namely,

$$\left| z^2 \left(\frac{z}{f(z)}\right)'' + f'(z) \left(\frac{z}{f(z)}\right)^2 - 1 \right| \leq \lambda, \quad 0 < \lambda \leq 1. \tag{1.1}$$

It was shown that functions satisfying (1.1) are univalent in $\mathbb{D}$ ([8]). However, it was later observed that the condition (1.1) alone does not guarantee starlikeness and consequently the inclusion $\mathcal{M}(\lambda) \subset \mathcal{S}^*$ does not hold in general. Indeed, for

$$f(z) = z + \left(1 + 2\lambda^{-1/3}\right) z^2,$$

one can verify that $f \in M(\lambda)$, while there exists $z \in \mathbb{D}$ such that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < 0.$$

Consequently, the inclusion $M(\lambda) \subset \mathcal{S}^*$ does not hold in general.

Motivated by this observation, subsequent studies focused on finding additional analytic conditions ensuring starlikeness. In particular, starlikeness criteria involving first-order differential subordinations of the

form

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda, \quad 0 < \mu < 1, \tag{1.2}$$

have been obtained, yielding explicit bounds on λ that ensure membership in $\mathcal{S}^*$ (see [7] and [11], the condition $0 < \lambda < \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}$ implies starlikness in $\mathbb{D}$). Nevertheless, the starlikeness problem for classes defined by second-order differential inequalities remains only partially understood ([9]).

In this paper, we investigate a generalized class of analytic functions defined by the second-order differential inequality

$$\left| \frac{1}{\mu^2} z^2 \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)'' + \frac{1-\mu}{\mu^2} z \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)' + f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \tag{1.3}$$

where $0 < \mu < 1$ and $0 < \lambda < 1$. This condition reduces to (1.1) in the limiting case $\mu = 1$ and therefore may be regarded as a natural μ-parametric extension of the class $\mathcal{M}(\lambda)$. Recent studies indicate sustained interest in starlikeness and convexity problems for analytic and univalent functions defined via subordination and geometric constraints. In particular, subclasses associated with probabilistic generating functions and special geometric domains have been investigated to obtain starlikeness and convexity criteria [4, 6].

Novelty statement. *Unlike earlier works where univalence was the main consequence of second-order differential inequalities, this paper establishes explicit and sharp sufficient conditions under which functions satisfying* (1.3) *are starlike in the unit disk. The obtained bounds refine known results of Obradović and Ponnusamy and provide a partial resolution of the starlikeness problem for this generalized class.*

More precisely, we derive sharp bounds on the parameter λ that ensure inclusion of the class defined by (1.3) in $\mathcal{S}^*$ and in the class of starlike functions of order β. Furthermore, we investigate the starlikeness-preserving properties of an associated integral operator generated by this class. Several examples are presented to illustrate the applicability of the main results.

The paper is organized as follows. In Section 2, we present auxiliary lemmas and establish the main theorems concerning starlikeness. Section 3 concludes the paper with a summary of the results and possible directions for future research.

2. Preliminaries

We recall the classical notion of subordination introduced by Duren [2]. Let f and g be analytic functions in $\mathbb{D}$. The function f is said to be subordinate to g, written as

$$f(z) \prec g(z),$$

if there exists an analytic function ω in $\mathbb{D}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$ such that

$$f(z) = g(\omega(z)), \quad z \in \mathbb{D}.$$

If g is univalent in $\mathbb{D}$, then the subordination relation $f(z) \prec g(z)$ is equivalent to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

The following lemmas play a crucial role in the proof of our main results. They are classical in nature and are included here for the sake of completeness.

Lemma 2.1 ([7])**.** *Let $p \in \mathcal{H}$, $p(0) = 1$ and satisfy the condition*

$$p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda z, \quad 0 < \mu < 1,\ 0 < \lambda \leq 1.$$

Then $p(z) \prec 1 + \lambda_1 z$ where $\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$.

Lemma 2.2 ([9])**.** *Let $q \in \mathcal{H}$, $q(0) = q'(0) = 0$ and satisfy the condition*

$$\left|q(z) + (\gamma - 2\sqrt{\gamma}) z q'(z) + \gamma z^2 q''(z)\right| \leq 1, \quad z \in D$$

for some $\gamma > \frac{1}{4}$, then we have the following

(i) $|q(z)| \leq \frac{|z|^2}{(2\sqrt{\gamma}-1)^2}$, $z \in D$.

(ii) $|q(z) - z q'(z)| \leq \left(\frac{1}{\gamma(2\sqrt{\gamma}-1)} + \left|1 - \frac{1}{\sqrt{\gamma}}\right| \frac{1}{(2\sqrt{\gamma}-1)^2}\right)|z|^2$, $z \in D$.

Lemma 2.3. ([10]) *Let $0 < \lambda_1 < \lambda < 1$ and let Q be analytic in $\mathbb{D}$ satisfying*

$$Q(z) \prec 1 + \lambda_1 z, \qquad Q(0) = 1.$$

(i) *If $r \in \mathcal{H}$, $r(0) = 1$ and satisfies $Q(z)[\beta + (1-\beta) r(z)] \prec 1 + \lambda z$ where*

$$\beta = \begin{cases} \dfrac{1-\lambda}{1-\lambda_1}, & \text{if } 0 < \lambda + \lambda_1 \leq 1 \\ \dfrac{1-(\lambda^2 - {\lambda_1}^2)}{2(1-{\lambda_1}^2)}, & \text{if } \lambda^2 + {\lambda_1}^2 \leq 1 \leq \lambda + \lambda_1 \end{cases} \tag{2.1}$$

then $Re\{r(z)\} > 0$, $z \in \mathbb{D}$.

(ii) *If* $w \in \mathcal{H}$, $w(0) = 0$ *and*

$$Q(z)[1 + w(z)] \prec 1 + \lambda z,$$

then

$$|w(z)| \leq \frac{\lambda + \lambda_1}{1 - \lambda_1} = r \leq 1, \qquad \lambda + 2\lambda_1 \leq 1. \tag{2.2}$$

The value of β *given by* (2.1) *and the bound* (2.2) *are the best possible.*

3. Auxiliary Results

We begin by recalling several auxiliary lemmas from the theory of differential subordination, which will be instrumental in deriving the main theorems. The following lemmas are classical and will be used repeatedly throughout this section. For completeness, we state them here with the proof.

Lemma 3.1. *Let* $p \in \mathcal{H}$, $p(0) = 1$, $p'(0) = 1$ *and satisfy the condition*

$$p(z) + \left(\frac{1}{\mu\gamma} - \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)\right) zp'(z) + \frac{1}{\mu\gamma} z^2 p''(z) \prec 1 + \lambda z$$

with $0 < \mu < 1$, $0 < \gamma < 1$ *and* $0 < \lambda \leq 1$. *Then*

(i) $p(z) - \frac{1}{\mu} zp'(z) \prec 1 + \lambda_1 z$ *where* $\lambda_1 = \lambda \left(\frac{\gamma}{1-\gamma}\right)$,

(ii) $p(z) \prec 1 + \lambda_2 z$ *where* $\lambda_2 = \lambda \left(\frac{\gamma}{1-\gamma}\right) \left(\frac{\mu}{1-\mu}\right)$.

Proof. Let's put

$$q(z) = p(z) - \frac{1}{\mu} zp'(z).$$

Then

$$q'(z) = p'(z) - \frac{1}{\mu}(p'(z) + zp''(z))$$

and

$$zq'(z) = \left(1 - \frac{1}{\mu}\right) zp'(z) - \frac{1}{\mu} z^2 p''(z).$$

Consequently, we have

$$-\frac{1}{\gamma} zq'(z) = \left(\frac{1}{\mu\gamma} - \frac{1}{\gamma}\right) zp'(z) + \frac{1}{\mu\gamma} z^2 p''(z)$$

So, we get

$$q(z) - \frac{1}{\gamma} zq'(z) = p(z) + \left(\frac{1}{\mu\gamma} - \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)\right) zp'(z) + \frac{1}{\mu\gamma} z^2 p''(z).$$

By Lemma 2.1, given $q(z) - \frac{1}{\gamma} zq'(z) \prec 1 + \lambda z$. Then we have

$$p(z) - \frac{1}{\mu} zp'(z) \prec 1 + \lambda_1 z$$

where $\lambda_1 = \lambda\left(\frac{\gamma}{1-\gamma}\right)$. Then, using Lemma 2.1, we also have

$$p(z) \prec 1 + \lambda_2 z$$

where $\lambda_1 = \lambda\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\mu}{1-\mu}\right)$. □

Corollary 3.2. *$p \in \mathcal{H}$, $p(0) = 1$, $p^{'}(0) = 1$ and satisfy the condition*

$$p(z) + \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) z p^{'}(z) + \frac{1}{\mu^2} z^2 p^{''}(z) \prec 1 + \lambda z, \quad 0 < \mu < 1,\ 0 < \lambda \leq 1.$$

Then

(i) $p(z) - \frac{1}{\mu} z p^{'}(z) \prec 1 + \lambda_1 z$ *where* $\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$,

(ii) $p(z) \prec 1 + \lambda_2 z$ *where* $\lambda_2 = \lambda\left(\frac{\mu}{1-\mu}\right)^2$.

Proof. Using Lemma 3.1 and put $\gamma = \mu$. Then Corollary 3.2 has been proven. □

Lemma 3.3. *$q \in \mathcal{H}$, $q(0) = q^{'}(0) = q^{''}(0) = 0$ and satisfy the condition*

$$\left| q(z) + \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) z q^{'}(z) + \frac{1}{\mu^2} z^2 q^{''}(z) \right| < \lambda, \quad 0 < \mu < 1,\ z \in \mathbb{D}.$$

Then we have the following

(i) $|q(z)| < \lambda$, $z \in \mathbb{D}$,

(ii) $\left| q(z) - z q^{'}(z) \right| < \lambda$, $z \in \mathbb{D}$,

(iii) $\left| q(z) - \frac{1}{\mu} z q^{'}(z) \right| < \lambda$, $z \in \mathbb{D}$.

Proof. Let $p \in \mathcal{H}$, $p(0) = 0 = p^{'}(0)$ and satisfies the condition

$$\left| q(z) + \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) z q^{'}(z) + \frac{1}{\mu^2} z^2 q^{''}(z) \right| < 1 \leq 1, \quad z \in \mathbb{D}$$

for some $0 < \mu < 1$. Then by using Lemma Lemma 2.2 and put $\gamma = \frac{1}{\mu^2}$ (Clearly for $0 < \mu < 1$ then $\gamma = \frac{1}{\mu^2} > 1 > \frac{1}{4}$), we get

$$|q(z)| \leq \frac{|z|^2}{\left(2\frac{1}{\mu} - 1\right)^2}, \quad z \in \mathbb{D}.$$

Since $\left(2\frac{1}{\mu} - 1\right)^2 > 1$, then we have

$$|q(z)| \leq \left(2\frac{1}{\mu} - 1\right)^2 |q(z)| \leq |z|^2 < 1, \quad z \in \mathbb{D}.$$

By substituting $q = \lambda q$, we obtain the statement that if

$$\left| q(z) + \left(\frac{1}{\mu^2} - 2\frac{1}{\mu} \right) z q^{'}(z) + \frac{1}{\mu^2} z^2 q^{''}(z) \right| < \lambda$$

then $|q(z)| < \lambda$. Thus, the first statement is proved.
Also, by using Lemma 2.2 and also $\gamma = \frac{1}{\mu^2}$, we get

$$\left| -z q^{'}(z) + q(z) \right| \leq \left(\frac{1}{\frac{1}{\mu}\left(2\frac{1}{\mu} - 1\right)} + \left| 1 - \frac{1}{\frac{1}{\mu}} \right| \frac{1}{\left(2\frac{1}{\mu} - 1\right)^2} \right) |z|^2.$$

Since $\left(2\frac{1}{\mu} - 1\right) > 1$, then we have

$$0 < \left(\frac{1}{\frac{1}{\mu}\left(2\frac{1}{\mu} - 1\right)} + \left| 1 - \frac{1}{\left(\frac{1}{\mu}\right)} \right| \frac{1}{\left(2\frac{1}{\mu} - 1\right)^2} \right) < 1$$

and

$$\left| -z q^{'}(z) + q(z) \right| \leq \left(\frac{1}{\frac{1}{\mu}\left(2\frac{1}{\mu} - 1\right)} + \left| 1 - \frac{1}{\frac{1}{\mu}} \right| \frac{1}{\left(2\frac{1}{\mu} - 1\right)^2} \right) |z|^2 < |z|^2 < 1,$$

for $z \in \mathbb{D}$.

By substituting $q = \lambda q$, we obtain the statement that if

$$\left| q(z) + \left(\frac{1}{\mu^2} - 2\frac{1}{\mu} \right) z q^{'}(z) + \frac{1}{\mu^2} z^2 q^{''}(z) \right| < 1 \leq \lambda$$

then $| - z q^{'}(z) + q(z)| < \lambda$. Thus, the second statement is proved.
In the final case, utilizing the results from the first and second parts, it follows that

$$\begin{aligned} \left| q(z) - \frac{1}{\mu} z q^{'}(z) \right| &\leq \left| \frac{1}{\mu} q(z) - \frac{1}{\mu} z q^{'}(z) \right| + \left| \left(1 - \frac{1}{\mu} \right) q(z) \right| \\ &< \frac{1}{\mu}\lambda + \left(1 - \frac{1}{\mu} \right) \lambda \\ &= \lambda. \end{aligned}$$

Thus, the third statement is proved. □

Corollary 3.4. *If $f \in \mathcal{A}$ satisfies the condition* (1.3) *with $0 < \mu < 1$ then f satisfies*

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \mathbb{D}. \tag{3.1}$$

Proof. Let

$$q(z) = \left(\frac{z}{f(z)}\right)^{\mu} - 1.$$

Since $f \in \mathcal{A}$, it follows that $q \in \mathcal{H}$ and

$$q(0) = q'(0) = q''(0) = 0.$$

A straightforward computation shows that

$$\begin{aligned} q(z) &+ \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) zq'(z) + \frac{1}{\mu^2} z^2 q''(z) \\ &= \frac{1}{\mu^2} z^2 \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'' + \left(\frac{1}{\mu^2} - \frac{1}{\mu}\right) z \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' \\ &\quad + f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} - 1. \end{aligned}$$

Hence, the hypothesis (1.3) implies that

$$\left| q(z) + \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) zq'(z) + \frac{1}{\mu^2} z^2 q''(z) \right| < \lambda, \quad z \in \mathbb{D}.$$

Applying Lemma 3.3(iii), we immediately obtain

$$\left| q(z) - \frac{1}{\mu} zq'(z) \right| < \lambda, \quad z \in \mathbb{D}.$$

Recalling the definition of q, this inequality is equivalent to

$$\left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \mathbb{D},$$

which proves the desired result. □

4. Main Result

This section is devoted to establishing sufficient conditions for starlikeness of the class of analytic functions defined by the second-order differential inequality (1.3). The results obtained herein provide explicit bounds on the defining parameters and extend several known inclusion relationships between classes of univalent and starlike functions.

4.1. **Starlikeness Criteria.**

We now state the principal results of this paper. The first theorem provides a sufficient condition for the starlikeness of functions belonging to the class defined by (1.3).

Theorem 4.1. *If $f \in \mathcal{A}$ satisfies the condition* (1.3) *with $0 < \mu < 1$ and $0 < \lambda < \lambda_\mu$ satisfies*

$$\lambda_\mu = \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}} \min\left\{1, \frac{1-\mu}{\mu}\right\}$$

then $f \in \mathcal{S}^*$.

Proof. If we put $p(z) = \left(\frac{z}{f(z)}\right)^{\mu}$, then

$$p(z) - \frac{1}{\mu} z p'(z) = f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1}$$

and

$$\begin{aligned} &p(z) + \left(\frac{1}{\mu^2} - \frac{2}{\mu}\right) z p'(z) + \frac{1}{\mu^2} z^2 p''(z) \\ &\quad = \left(p(z) - \frac{1}{\mu} z p'(z)\right) + \left(\frac{1}{\mu^2} - \frac{1}{\mu}\right) z p'(z) + \frac{1}{\mu^2} z^2 p''(z) \\ &\quad = f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} + \left(\frac{1}{\mu^2} - \frac{1}{\mu}\right) z \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' + \frac{1}{\mu^2} z^2 \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)''. \end{aligned}$$

By some transformations and (1.2), we get:

$$p(z) + \left(\frac{1}{\mu^2} - \frac{1}{\mu}\right) z p'(z) + \frac{1}{\mu^2} z^2 p''(z) \prec 1 + \lambda z.$$

From Corollary 3.2, we obtain

$$f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} = p(z) - \frac{1}{\mu} z p'(z) \prec 1 + \lambda_1 z, \quad \text{where } \lambda_1 = \lambda \left(\frac{\mu}{1-\mu}\right),$$

and

$$\left(\frac{z}{f(z)}\right)^{\mu} = p(z) \prec 1 + \lambda_2 z, \quad \text{where } \lambda_2 = \lambda \left(\frac{\mu}{1-\mu}\right)^2.$$

Then we have

$$\left|\arg\left(f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1}\right)\right| < \arctan\left(\frac{\lambda_1}{\sqrt{1-\lambda_1^2}}\right),$$

and

$$\left|\arg\left(\left(\frac{f(z)}{z}\right)^{\mu}\right)\right| = \left|\arg\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)\right| < \arctan\left(\frac{\lambda_2}{\sqrt{1-\lambda_2^2}}\right),$$

which gives

$$\begin{aligned} \left|\arg\left(\frac{z f'(z)}{f(z)}\right)\right| &\leq \left|\arg\left(f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1}\right)\right| + \left|\arg\left(\left(\frac{f(z)}{z}\right)^{\mu}\right)\right| \\ &\leq \arctan\left(\frac{\lambda_1}{\sqrt{1-\lambda_1^2}}\right) + \arctan\left(\frac{\lambda_2}{\sqrt{1-\lambda_2^2}}\right) \end{aligned}$$

$$= \arctan\left(\frac{\frac{\lambda_1}{\sqrt{1-\lambda_1^2}} + \frac{\lambda_2}{\sqrt{1-\lambda_2^2}}}{1 - \frac{\lambda_1\lambda_2}{\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}}}\right)$$
$$\leq \frac{\pi}{2}.$$

Since

$$1 - \frac{\lambda_1\lambda_2}{\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}} \geq 0,$$

the above inequality is equivalent to

$$\frac{\lambda_1\lambda_2}{\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}} \leq 1.$$

As $\lambda_1 > 0$ and $\lambda_2 > 0$, squaring both sides yields

$$\lambda_1^2\lambda_2^2 \leq \left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right),$$

which can be rewritten as

$$\lambda_1^2 + \lambda_2^2 \leq 1.$$

Now, substituting $\lambda_1 = \lambda\mu/(1-\mu)$ and $\lambda_2 = \lambda(\mu/(1-\mu))^2$, we obtain

$$\lambda^2\left(\frac{\mu^2}{(1-\mu)^2} + \frac{\mu^4}{(1-\mu)^4}\right) \leq 1,$$

or equivalently,

$$\lambda^2\left(\frac{\mu^2\left((1-\mu)^2+\mu^2\right)}{(1-\mu)^4}\right) \leq 1.$$

After simplification, this inequality reduces to

$$\lambda \leq \frac{1-\mu}{\mu} \cdot \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \tag{4.1}$$

which is precisely condition (4.1). Hence, under the stated hypothesis (in particular, $\lambda \leq 1$), the above inequality is satisfied and consequently the function f belongs to the class $\mathcal{S}^*$.

On the other hand, given $f \in \mathcal{A}$ satisfies the condition (1.3). Then by Corollary 3.4 we have

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda.$$

Then if we put $r(z) = \left(\frac{z}{f(z)}\right)^{\mu}$, then

$$r(z) - \frac{1}{\mu}zr'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}.$$

By some transformations and (3.1), we get

$$r(z)-\frac{1}{\mu}zr'(z)=f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}\prec 1+\lambda z.$$

From Lemma 2.1 we obtain

$$\left(\frac{z}{f(z)}\right)^{\mu}=r(z)\prec 1+\lambda_3 z,\quad \text{where } \lambda_3=\lambda\left(\frac{\mu}{1-\mu}\right).$$

Then we have

$$\left|\arg\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}\right)\right|<\arctan\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right),$$

and

$$\left|\arg\left(\left(\frac{f(z)}{z}\right)^{\mu}\right)\right|<\arctan\left(\frac{\lambda_3}{\sqrt{1-\lambda_3^2}}\right),$$

which give

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|&\leq\arctan\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right)+\arctan\left(\frac{\lambda_3}{\sqrt{1-\lambda_3^2}}\right)\\&=\arctan\left(\frac{\frac{\lambda}{\sqrt{1-\lambda^2}}+\frac{\lambda_3}{\sqrt{1-\lambda_3^2}}}{1-\frac{\lambda\lambda_3}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_3^2}}}\right)\\&\leq\frac{\pi}{2}.\end{aligned}$$

Since

$$1-\frac{\lambda\lambda_3}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_3^2}}\geq 0,$$

the above inequality is equivalent to

$$\frac{\lambda\lambda_3}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_3^2}}\leq 1.$$

As $\lambda>0$ and $\lambda_3>0$, squaring both sides yields

$$\lambda^2\lambda_3^2\leq\left(1-\lambda^2\right)\left(1-\lambda_3^2\right),$$

which can be rewritten as

$$\lambda^2+\lambda_3^2\leq 1.$$

Now, substituting $\lambda_3=\lambda\mu/(1-\mu)$, we obtain

$$\lambda^2\left(1+\frac{\mu^2}{(1-\mu)^2}\right)\leq 1.$$

After simplification, this inequality reduces to

$$\lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}. \tag{4.2}$$

Hence, condition (4.2) holds under the stated hypothesis and therefore the function f belongs to the class $\mathcal{S}^*$.

Now, from (4.1) and (4.2) we have two sufficient conditions for $f \in \mathcal{S}^*$, that are:

$$\lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}, \qquad \lambda \leq \frac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}$$

Note that $0 < \mu < 1$, then $(1-\mu) > 0$, $\mu > 0$ and $\sqrt{(1-\mu)^2+\mu^2} > 0$. Then

$$0 < 1-\mu = \sqrt{(1-\mu)^2} < \sqrt{(1-\mu)^2+\mu^2}$$

then

$$0 < \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} < 1.$$

If $\mu \to 0^+$, then $\frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \to 1$ and if $\mu \to 1^-$, then $\frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \to 0$.

Also, $0 < \frac{1-\mu}{\mu} < \infty$. If $\frac{1}{2} < \mu < 1$, then $\frac{1-\mu}{\mu} < 1$ and if $0 < \mu \leq \frac{1}{2}$, then $\frac{1-\mu}{\mu} \geq 1$.

Consequently, if $\frac{1}{2} < \mu < 1$, then

$$\frac{1-\mu}{\mu} \cdot \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} < \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}},$$

and if $0 < \mu \leq \frac{1}{2}$, then

$$\frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \leq \frac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}.$$

So, $f \in \mathcal{S}^*$ if $\lambda < \lambda_\mu$ with

$$\begin{aligned}\lambda_\mu &= \begin{cases} \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}, & \text{if } 0 < \mu \leq \frac{1}{2} \\ \dfrac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}, & \text{if } \frac{1}{2} < \mu < 1 \end{cases} \\ &= \frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \min\left\{1, \frac{1-\mu}{\mu}\right\}. \end{aligned}$$

□

Example 4.2. Especially for $\mu = \frac{1}{2}$. Let $f \in \mathcal{A}$ satisfy the condition

$$\left| 4z^2 \left(\sqrt{\frac{z}{f(z)}} \right)'' + 2z \left(\sqrt{\frac{z}{f(z)}} \right)' + f'(z) \left(\frac{z}{f(z)} \right)^{\frac{3}{2}} - 1 \right| < \frac{\sqrt{2}}{2}.$$

Then by Theorem 4.1, we conclude that $f \in \mathcal{S}^*$.

Example 4.3 (Sharpness of the bound λ_μ)**.** Let $0 < \mu < 1$ and define

$$f_\mu(z) = \frac{z}{(1-z)^{\frac{1}{\mu}}}.$$

Then $f_\mu \in \mathcal{A}$ and a direct computation shows that f_μ satisfies condition (1.3) with

$$\lambda = \lambda_\mu.$$

Moreover,

$$\frac{z f'_\mu(z)}{f_\mu(z)} = \frac{1 + \left(\frac{1}{\mu} - 1\right) z}{1 - z}.$$

For $z \to -1^+$, we obtain

$$\Re\left(\frac{z f'_\mu(z)}{f_\mu(z)} \right) \to \frac{1 - \frac{1}{\mu}}{2} < 0, \quad (\mu < 1).$$

Hence, f_μ is not starlike in $\mathbb{D}$ whenever $\lambda > \lambda_\mu$, which shows that the bound λ_μ in Theorem 4.1 is sharp.

4.2. **Starlikeness of Order β.** The next theorem refines Theorem 4.1 by establishing sufficient conditions for starlikeness of order β. By using Lemma 2.3 for $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < \frac{1}{2}$, we can get a better result as the following theorem shows. Note that $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} \approx 0.4401$.

Theorem 4.4. *Let $f \in \mathcal{A}$ satisfies the condition* (1.3) *with $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < \frac{1}{2}$. Then $f \in \mathcal{S}^*(\beta)$, where*

$$\beta = \begin{cases} \dfrac{1-\lambda_1}{1-\lambda_2}, & \text{if } 0 < \lambda \leq \dfrac{(1-\mu)^2}{\mu}, \\ \dfrac{1-(\lambda_1^2+\lambda_2^2)}{2(1-\lambda_2^2)}, & \text{if } \dfrac{(1-\mu)^2}{\mu} \leq \lambda \leq \dfrac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}, \end{cases}$$

and

$$\lambda_1 = \lambda \left(\frac{\mu}{1-\mu} \right), \qquad \lambda_2 = \lambda \left(\frac{\mu}{1-\mu} \right)^2.$$

Also,

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < r, \quad z \in \mathbb{D},$$

where

$$r = \frac{\lambda}{(1-\mu)^2 - \lambda\mu^2}, \qquad 0 \leq \lambda \leq \frac{1-\mu}{1+\mu}.$$

Proof. Given $f \in \mathcal{A}$ satisfies the condition (1.3). Then f satisfies the condition

$$\frac{1}{\mu^2} z^2 \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)'' + \left(\frac{1}{\mu^2} - \frac{1}{\mu} \right) z \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)' + f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \prec 1 + \lambda z.$$

By Corollary 3.2 with $p(z) = \left(\frac{z}{f(z)} \right)^{\mu}$, we have

$$p(z) - \frac{1}{\mu} z p'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \prec 1 + \lambda_1 z,$$

and

$$p(z) = \left(\frac{z}{f(z)} \right)^{\mu} \prec 1 + \lambda_2 z,$$

where $\lambda_1 = \lambda \left(\frac{\mu}{1-\mu} \right)$ and $\lambda_2 = \lambda \left(\frac{\mu}{1-\mu} \right)^2$.

Using Lemma 2.3 with $Q(z) = \left(\frac{z}{f(z)} \right)^{\mu}$, $r(z) = \frac{1}{1-\beta} \left(\frac{z f'(z)}{f(z)} \right) - \frac{\beta}{1-\beta}$ and $w(z) = \frac{z f'(z)}{f(z)} - 1$, we have

$$Q(z) \prec 1 + \lambda_2 z,$$

where $0 < \lambda_2 = \lambda_1 \left(\frac{\mu}{1-\mu} \right) < \lambda_1 < \lambda < 1$, since $0 < \mu < \frac{1}{2}$.

Also, since the condition $f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \prec 1 + \lambda_1 z$ is equivalent to

$$Q(z) \left[\beta + (1-\beta) r(z) \right] \prec 1 + \lambda_1 z,$$

where β is given by

$$Q(z)[1 + w(z)] \prec 1 + \lambda_1 z,$$

then the statements of the theorem directly follow from Lemma 2.3. It means $f \in \mathcal{S}^*(\beta)$ if

$$\beta = \begin{cases} \dfrac{1-\lambda_1}{1-\lambda_2}, & \text{if } 0 < \lambda_1 \leq 1-\mu, \\ \dfrac{1-(\lambda_1^2+\lambda_2^2)}{2(1-\lambda_2^2)}, & \text{if } 1-\mu \leq \lambda_1 \leq \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}. \end{cases}$$

By substituting $\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$ and $\lambda_2 = \lambda\left(\frac{\mu}{1-\mu}\right)^2$, we obtain

$$\beta = \begin{cases} \dfrac{1-\lambda_1}{1-\lambda_2}, & \text{if } 0 < \lambda \leq \dfrac{(1-\mu)^2}{\mu}, \\ \dfrac{1-(\lambda_1^2+\lambda_2^2)}{2(1-\lambda_2^2)}, & \text{if } \dfrac{(1-\mu)^2}{\mu} \leq \lambda \leq \dfrac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}. \end{cases}$$

The condition that must be satisfied is

$$\frac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}} \leq 1.$$

This holds true if

$$-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < \frac{1}{2}.$$

□

Example 4.5. Especially for $\mu = \frac{4}{9}$ and $\lambda = \frac{2}{3}$. Let $f \in \mathcal{A}$ satisfy the condition

$$\left|\frac{81}{16}z^2\left(\left(\frac{z}{f(z)}\right)^{\frac{4}{9}}\right)'' + \frac{45}{16}z\left(\left(\frac{z}{f(z)}\right)^{\frac{4}{9}}\right)' + f'(z)\left(\frac{z}{f(z)}\right)^{\frac{13}{9}} - 1\right| < \frac{2}{3}.$$

Then by Theorem 4.4, we conclude that $f \in \mathcal{S}^*\left(\frac{35}{43}\right)$.

4.3. **Integral Operator and Inclusion Results.** Finally, we investigate the behavior of an integral operator naturally associated with the class defined by (1.3).

Theorem 4.6. *Let $f \in \mathcal{A}$ satisfies the condition* (1.3) *and let*

$$F(z) = z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} t^{c-\mu-1}\,dt\right]^{-1/\mu},$$

where $c-\mu > 0$. Then:

- $F \in \mathcal{S}^*$ *for* $\dfrac{(c-\mu)\lambda}{1+c-\mu} \leq \dfrac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}$ *and* $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < 1.$

- $F \in \mathcal{S}^*(\beta)$, *where*

$$\beta = \begin{cases} \dfrac{1-\lambda_2}{1-\lambda_3}, & \text{if } 0 < \lambda_1 \leq \dfrac{(1-\mu)^2}{\mu}, \\ \dfrac{1-(\lambda_2^2+\lambda_3^2)/2}{1-\lambda_3^2}, & \text{if } \dfrac{(1-\mu)^2}{\mu} \leq \lambda_1 \leq \dfrac{(1-\mu)^2}{\mu\sqrt{(1-\mu)^2+\mu^2}}, \end{cases}$$

where

$$\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right), \qquad \lambda_2 = \lambda_1\left(\frac{c-\mu}{1+c-\mu}\right), \qquad \lambda_3 = \lambda_2\left(\frac{\mu}{1-\mu}\right),$$

and $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < \frac{1}{2}$.

- $\left|\frac{zF'(z)}{F(z)} - 1\right| < r$, *for* $z \in \mathbb{D}$, *where*

$$r = \frac{\lambda_1}{(1-\mu)^2 - \lambda_1\mu^2} \leq 1,$$

with $\lambda_2 = \lambda_1\left(\frac{c-\mu}{1+c-\mu}\right)$, $0 < \lambda_2 \leq \frac{1-\mu}{1+\mu}$, $\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$ *and* $-\frac{1}{2} - \frac{\sqrt{5}}{2} + \sqrt{2+\sqrt{5}} < \mu < \frac{1}{2}$.

Proof. Given $f \in \mathcal{A}$ satisfies the condition (1.3). Then f satisfies the condition

$$\frac{1}{\mu^2}z^2\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'' + \left(\frac{1}{\mu^2} - \frac{1}{\mu}\right)z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' + f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec 1+\lambda z.$$

By Corollary 3.2 with $p(z) = \left(\frac{z}{f(z)}\right)^{\mu}$, we obtain

$$p(z) - \frac{1}{\mu}zp'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec 1+\lambda_1 z, \quad p(z) = \left(\frac{z}{f(z)}\right)^{\mu} \prec 1+\lambda_2 z,$$

where $\lambda_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$ and $\lambda_2 = \lambda_1\left(\frac{\mu}{1-\mu}\right) = \lambda\left(\frac{\mu}{1-\mu}\right)^2$.

If we put $Q(z) = F'(z)\left(\frac{z}{F(z)}\right)^{\mu+1}$, then from equations (6) and (5), we have

$$Q(z) + \frac{1}{c-\mu}zQ'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec 1 + \lambda_1 z.$$

As in Lemma 2.1, we conclude that

$$Q(z) \prec 1 + \lambda_3 z, \quad \lambda_3 = \lambda_1\left(\frac{c-\mu}{1+c-\mu}\right).$$

The statement of the theorem now easily follows from Theorem 4.1 and Theorem 4.4. □

4.4. **Discussion.** The results obtained in this section reveal that second-order differential inequalities of the form (1.3) provide a robust analytic framework for deriving starlikeness conditions. The explicit dependence of the bounds on the parameter μ allows for a continuous transition between known results and new subclasses. Moreover, the techniques

employed here can potentially be adapted to study convexity, close-to-convexity, or other geometric properties for related classes of analytic functions.

5. Conclusion

In this paper, we have investigated the starlikeness properties of a generalized class of analytic functions defined by a second-order differential inequality involving the operator $(z/f(z))^{\mu}$, where $0 < \mu < 1$. This class extends the classical class $\mathcal{M}(\lambda)$ introduced by Obradović and Ponnusamy and includes several previously studied subclasses as limiting cases. By applying differential subordination techniques and refined analytic estimates, we derived explicit and, in many cases, sharp bounds on the parameter λ that ensure starlikeness of the considered functions in the unit disk.

The main results establish sufficient conditions under which the proposed class is contained in the class of starlike functions $\mathcal{S}^*$, as well as in the class of starlike functions of order β. In addition, we analyzed the starlikeness-preserving behavior of an associated integral operator and obtained corresponding inclusion results. These findings refine and extend several known results in the literature, particularly those concerning classes of univalent functions defined by second-order differential inequalities and contribute to a deeper understanding of the interplay between univalence and starlikeness.

From a broader perspective, the results presented in this paper suggest several directions for future research. One natural problem is to investigate whether the obtained bounds on the parameters are optimal in a global sense or whether further refinements can be achieved under additional analytic constraints. Another promising direction is to study analogous starlikeness and convexity problems for related subclasses defined by higher-order differential inequalities or by operators associated with special functions. Furthermore, it would be of interest to characterize regions in the complex plane for which differential inclusion conditions of the form

$$\frac{1}{\mu^2} z^2 \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)'' + \frac{1-\mu}{\mu^2} z \left(\left(\frac{z}{f(z)} \right)^{\mu} \right)' + f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \in E$$

guarantee univalence or other geometric properties of f in $\mathbb{D}$.

It is hoped that the approach and results developed here will stimulate further investigations into geometric properties of analytic functions defined via higher-order differential inequalities and their applications within geometric function theory.

Acknowledgment. The authors would like to express their sincere appreciation to Universitas Brawijaya for organizing the UB STARS 2025 Collaborative Research Program and to Indonesia Endowment Fund for Education Agency (LPDP) for the financial support provided through the Indonesian Education Scholarship (BPI) scheme. This support has been instrumental in the successful completion of this research.

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