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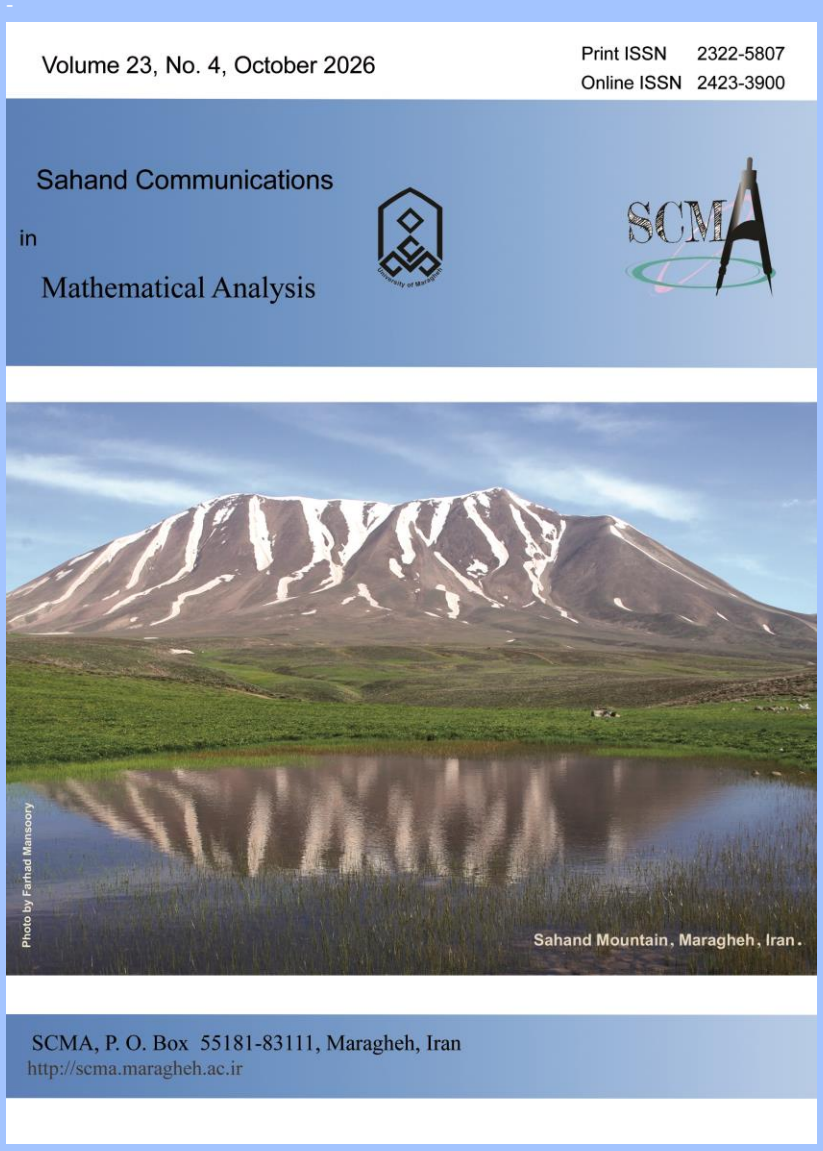

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# Applications of $q$-Calculus in Coefficient Problems for Normalized and Bi-Univalent Function Classes

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Abstract. In this work, we introduce a distinguished subclass of analytic functions whose structure is based on $q$-calculus, providing a generalization of several well-known classes previously studied in the literature. This subclass represents one of the notable applications of fractional calculus within geometric function theory and the coefficient problems of analytic functions. The main contribution of our study lies in establishing sharp or near-optimal upper bounds for several significant problems, despite the presence of the parameter $q$, achieved through novel and effective proof techniques.

## 1. Introduction

Let $\mathbf{A}$ denote the class of normalized analytic functions defined in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

which can be represented by the series expansion

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad z \in \mathbb{U}. \tag{1.1}$$

A notable subclass of $\mathbf{A}$, denoted by $\mathcal{S}$, consists of functions that are univalent (one-to-one) in $\mathbb{U}$. By the celebrated Koebe one-quarter theorem [18], each $f \in \mathcal{S}$ possesses an inverse function $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z \quad \text{for} \quad z \in \mathbb{U},$$

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$$f\left(f^{-1}(\xi)\right)=\xi \quad \text{for} \quad |\xi|<r_0(f), \quad r_0(f)\geq\frac{1}{4}.$$

The inverse function $g=f^{-1}$ admits the power series representation

(1.2)
$$g(\xi)=f^{-1}(\xi)=\xi-a_2\xi^2+\left(2a_2^2-a_3\right)\xi^3-\left(5a_2^3-5a_2a_3+a_4\right)\xi^4+\cdots.$$

A function $f\in\mathbf{A}$ is called bi-univalent in $\mathbb{U}$ if both $f$ and its inverse $f^{-1}$ are univalent in $\mathbb{U}$. The collection of all such functions is denoted by $\Sigma$.

The systematic study of analytic and bi-univalent functions was revitalized by Srivastava et al. [42] and has subsequently been extended by numerous researchers, including Çağlar et al. [17], Bulut [11] and Adegani et al. [1], among others (see also [3, 4, 15, 21, 43, 44, 47, 50, 56, 60]).

Prominent examples of bi-univalent functions presented in [42] include

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\big(\log(1+z)-\log(1-z)\big),$$

which illustrate that the class $\Sigma$ is nonempty, although the classical Koebe function does not belong to it.

Despite substantial progress over the past decades, deriving sharp estimates for the Taylor–Maclaurin coefficients $|a_s|$ $(s\geq 3)$ for functions in $\Sigma$ remains a challenging and largely open problem for many of its subclasses (see, for instance, [47–49]).

The renowned Fekete–Szegö functional, commonly given by $a_3-\delta a_2^2$ for a function $f\in\mathcal{S}$, holds a central place in the theory of geometric functions. Its historical importance stems from the classical work of Fekete and Szegö [22], who disproved the celebrated Littlewood–Paley conjecture, which asserted that the coefficients of odd univalent functions are bounded by one. Since this seminal contribution, the functional has continued to attract significant attention due to its pivotal role in estimating coefficients for various distinguished subclasses of univalent functions. Consequently, the Fekete–Szegö problem remains an active and rich area of research within Geometric Function Theory (see, for example, [16, 33, 38, 45, 46]).

To recall the classical notion of subordination, let $f$ and $F$ be analytic functions defined on $\mathbb{U}$. We say that $f$ is subordinate to $F$, denoted by $f\prec F$ (or equivalently $f(z)\prec F(z)$ for all $z\in\mathbb{U}$), if there exists a Schwarz function $\omega$, analytic in $\mathbb{U}$, satisfying

$$\omega(0)=0 \quad \text{and} \quad |\omega(z)|<1, \quad (z\in\mathbb{U}),$$

such that

$$f(z)=F(\omega(z))$$

holds for all $z\in\mathbb{U}$.

Geometrically, this indicates that the image of $f$ is contained within the image of $F$ under a rotation-preserving analytic mapping of $\mathbb{U}$ onto itself. A fundamental result [35] states that if $F$ is univalent in $\mathbb{U}$, then the subordination relation can be equivalently characterized as

$$f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq F(\mathbb{U}) \quad \Longleftrightarrow \quad f \prec F.$$

In recent years, significant research efforts have focused on developing and investigating diverse subclasses of analytic and bi-univalent functions through the use of analytic generators that map the open unit disk $\mathbb{U}$ onto geometrically meaningful domains represented by $\varphi(z)$. Through this approach, numerous authors have succeeded in deriving essential geometric characteristics of such function families and in establishing sharp coefficient estimates associated with their initial Taylor–Maclaurin coefficients. These developments have not only enriched the structural understanding of analytic mappings but have also contributed significantly to the ongoing progress of coefficient problems within Geometric Function Theory. For detailed discussions and further illustrative examples, one may refer to [14, 34, 36, 40, 57, 58].

The principal motivation for the present investigation arises from several analytically generated domains associated with specific mappings $\varphi(z)$, which serve as geometric templates for constructing new subclasses of univalent and bi-univalent functions. Some notable examples are outlined below:

(i) For the analytic function $\varphi_1(z) = \sqrt{1+z}$, one obtains the class $\mathcal{S}_L^*$, originally introduced by Sokol and Stankiewicz [41]. This class consists of all functions $f \in \mathbf{A}$ that

$$\frac{zf'(z)}{f(z)} \in \Omega_L,$$

where $\Omega_L$ denotes the region bounded by the right half of the Bernoulli lemniscate given by

$$|\xi^2 - 1| < 1.$$

(ii) For the analytic function $\varphi_2(z) = \dfrac{2}{1+e^{-z}}$, one obtains the class $\mathcal{S}_{\text{sig}}^*$, introduced and investigated by Geol et al. [23]. A function $f \in \mathcal{S}_{\text{sig}}^*$ if and only if

$$\frac{zf'(z)}{f(z)} \in \Omega_{\text{sig}},$$

where $\Omega_{\text{sig}}$ is the region characterized by

$$\Omega_{\text{sig}} = \left\{ \xi \in \mathbb{C} : \left|\log(\xi^2 - \xi)\right| < 1 \right\}.$$

Inspired by these classes, Ahmad et al. [2] introduced a new analytic function that combines the above two cases via the analytic product using the ratio

$$\varphi_1(z) = \sqrt{1+z}$$

and

$$\varphi_2(z) = \frac{2}{1+e^{-z}}.$$

Let $f \in \mathbf{A}$ be as in (1.1). We say $f \in \mathcal{R}_{sl}$ if and only if it satisfies the following condition:

$$f'(z) \prec \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathbb{U}.$$

From a geometric perspective, every function $f \in \mathcal{R}_{sl}$ maps the open unit disk $\mathbb{U}$ onto a balloon-shaped domain that is symmetric with respect to the real axis, as illustrated in Figure 1.

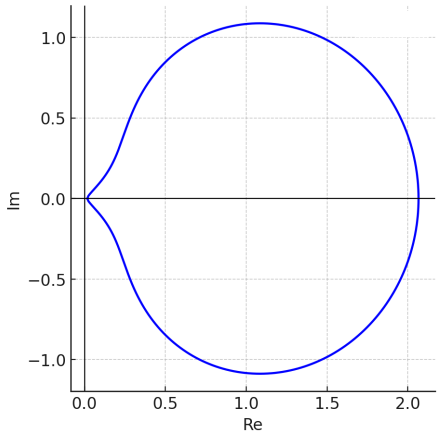


FIGURE 1. Image of the unit disk under the mapping $w = \dfrac{2\sqrt{1+z}}{1+e^{-z}}$, producing a balloon-shaped domain in the complex plane.

In the present investigation, we extend the function class proposed by Ahmad et al. [2] by employing the framework of $q$-calculus. Before proceeding to our main results, it is instructive to recall the historical background, foundational principles and major developments of this topic, particularly emphasizing its applications within Geometric Function Theory.

Quantum calculus–commonly known as $q$-calculus or $q$-analysis constitutes a generalization of classical differential and integral calculus wherein the notion of limits is replaced by $q$-analogues of fundamental

operations. Its early foundations were laid by Euler and Jacobi and the theory was later formalized by Jackson [26, 27] through the introduction of the $q$-derivative and the $q$-integral. The importance of $q$-calculus stems from its broad applicability across various mathematical and physical disciplines, including analysis, quantum physics and operator theory (see, for instance, Aral and Gupta [7, 8]).

Several special functions within this framework–such as the $q$-Gamma, $q$-Beta and $q$-Mittag Leffler functions have been studied extensively in the literature [9, 10, 12, 13, 30, 51–53]. In the context of Geometric Function Theory, the $q$-calculus has proved to be a powerful analytical tool in exploring the structural and coefficient properties of functions belonging to the class $\mathcal{S}$ and its numerous subclasses (see, e.g., Ismail et al. [25], Kanas and Răducanu [29], Srivastava et al. [54, 55]). In particular, the $q$-derivative operator has played a pivotal role in defining and characterizing families of $q$-starlike and related analytic functions.

Throughout this study, we assume $0 < q < 1$ and utilize $q$-calculus operators to introduce and examine a new subclass of normalized bi-univalent analytic functions. Several recent contributions have incorporated similar techniques in the development of analytic operator theory and its geometric applications (see, for example, [5, 6, 20, 32]).

The derivative operator $D_q$ is defined by

$$D_q f(z) := \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)\,z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

provided that $f'(0)$ exists. In the limit as $q \to 1^-$, the $q$-derivative reduces to the usual derivative:

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{d}{dz} f(z) = f'(z), \quad z \neq 0.$$

Clearly, for a function $f \in \mathcal{S}$ represented by (1.1), we have

$$D_q f(z) = D_q \left( z + \sum_{s=2}^{\infty} a_s z^s \right) = 1 + \sum_{s=2}^{\infty} [s]_q a_s z^{s-1}, \quad z \in \mathbb{U}.$$

where

$$[s]_q = \frac{1 - q^s}{1 - q}, \quad s \in \mathbb{N},$$

Jackson [27] defined the $q$-integral of a function $f(z)$ as

$$\int_0^z f(\vartheta)\, d_q\vartheta = (1-q) z \sum_{s=0}^{\infty} q^s f(vq^s).$$

It is evident that the $q$-derivative and the $q$-integral act as inverse operators, effectively canceling each other when applied consecutively, as

will be demonstrated below.

$$\begin{aligned} D_q \int_0^z f(\vartheta)\, d_q\vartheta &= \frac{1}{(1-q)z}\Bigg[(1-q)z\sum_{s=0}^{\infty} q^s f(zq^s) \\ &\quad - (1-q)qz\sum_{s=0}^{\infty} q^s f\left(zq^{s+1}\right)\Bigg] \\ &= \sum_{s=0}^{\infty} q^s f(zq^s) - \sum_{s=0}^{\infty} q^{s+1} f\left(zq^{s+1}\right) \\ &= f(z). \end{aligned}$$

In contrast,

$$\begin{aligned} \int_0^z D_q f(\vartheta)\, d_q\vartheta &= (1-q)z\sum_{s=0}^{\infty} q^s D_q f\left(zq^s\right) \\ &= (1-q)z\sum_{s=0}^{\infty} q^s \frac{f\left(zq^s\right) - f\left(zq^{s+1}\right)}{(1-q)q^s z} \\ &= \sum_{s=0}^{\infty}\left(f\left(zq^s\right) - f\left(zq^{s+1}\right)\right) \\ &= f(z). \end{aligned}$$

and

$$\begin{aligned} D_q \int_0^z f(\vartheta)\, d_q\vartheta &= \frac{1}{(1-q)z}\Bigg[(1-q)z\sum_{s=0}^{\infty} q^s f(zq^s) \\ &\quad - (1-q)qz\sum_{s=0}^{\infty} q^s f\left(zq^{s+1}\right)\Bigg] \\ &= \sum_{s=0}^{\infty} q^s f(zq^s) - \sum_{s=0}^{\infty} q^{s+1} f\left(zq^{s+1}\right) \\ &= f(z). \end{aligned}$$

We now introduce a generalized form of the Balloon function, denoted by $\mathbf{B}(z;q)$, which is defined as follows:

$$\mathbf{B}(z;q) = \frac{2\sqrt{1+\dfrac{(1+q)z}{2+(1-q)z}}}{1+e_q^{-z}}, \quad z \in \mathbb{U}.$$

Expanding $\mathbf{B}(z;q)$ into its power series, we obtain the following.

$$\mathbf{B}(z;q) = 1 + \frac{3+q}{4}z + \frac{-9+9q+5q^2+3q^3}{32(1+q)}z^2 \tag{1.3}$$

$$+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{128(1+q+q^2)}z^3+\cdots.$$

Here, $e_q^z$ represents the $q$-exponential function, which is given by

$$e_q^z=\sum_{s=0}^{\infty}\frac{z^s}{[s]_q!},$$

where

$$[s]_q!=\begin{cases}[s]_q[s-1]_q\cdots[2]_q[1]_q, & s\in\mathbb{N},\\ 1, & s=0.\end{cases}$$

The function $e_q^z$ serves as a natural generalization of the classical exponential function when $q\to 1^-$. Figure 2 illustrates the mapping behavior of the function $\mathbf{B}(z;q)$ on the unit disk $\mathbb{U}$ for three representative values of $q$, highlighting the variation in the image regions. The figure clearly demonstrates how variations in the parameter $q$ affect the geometric structure and distortion of the image domain.

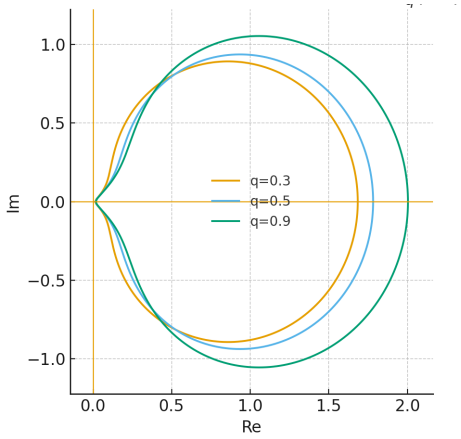


FIGURE 2. Mapping of $\mathbf{B}(z;q)$ over the unit disk $\mathbb{U}$ for three values of $q$, showing the variation of the balloon-shaped region.

**Definition 1.1.** A function $f\in\mathbf{A}$ is said to belong to the class $A_{sl}^q$, where $q\in(0,1)$, if

$$D_qf(z)\prec\mathbf{B}(z;q).$$

It is evident that $f\in A_{sl}^q$ if and only if it can be represented in the form

$$f(z)=\int_0^z\mathcal{X}(\vartheta,q)\,d_q\vartheta,\quad z\in\mathbb{U},\tag{1.4}$$

for some analytic function $\mathcal{X}(z,q)$ that is subordinate to $\mathbf{B}(z;q)$. For example, (1.4) produces the following functions:

$$\mathbb{K}_s(z) = \int_0^z \mathcal{X}(\vartheta^s, q)\, d_q\vartheta, \quad z \in \mathbb{U},\ s = 1, 2, 3, \ldots, \tag{1.5}$$

which all belong to the class $A_{sl}^q$. Their corresponding series representations are given by:

$$\mathbb{K}_1(z) = z + \frac{3+q}{4(1+q)}z^2 + \frac{-9+9q+5q^2+3q^3}{32(1+q)\,(1+q+q^2)}z^3 + \cdots, \tag{1.6}$$

$$\mathbb{K}_2(z) = z + \frac{3+q}{4\,(1+q+q^2)}z^3 + \frac{-9+9q+5q^2+3q^3}{32(1+q)\,(1+q+q^2+q^3)}z^4 + \cdots, \tag{1.7}$$

$$\mathbb{K}_3(z) = z + \frac{3+q}{4\,(1+q+q^2+q^3)}z^4 + \cdots. \tag{1.8}$$

1.1. **Aims and Scope.** The theory of analytic and bi-univalent functions has been an active area of research due to its profound connections and diverse applications within Geometric Function Theory. A major focus lies in obtaining sharp coefficient estimates, including classical problems such as the Fekete–Szegö inequality and Hankel determinant bounds.

The present work aims to introduce and analyze a new subclass of normalized and bi-univalent analytic functions generated via $q$-differential operators. This approach establishes an intrinsic connection between $q$-calculus and Geometric Function Theory, providing natural generalizations of conventional differential operators and function classes.

Specifically, the study derives sharp coefficient estimates for the proposed class, the corresponding Fekete–Szegö inequalities and bounds for the second-order Hankel determinant. The results encompass and generalize several previously known findings as special or limiting cases for particular parameter choices, thereby broadening the applicability and theoretical understanding of $q$-calculus within contemporary Geometric Function Theory.

1.2. **Basic Tools.** As a foundational concept, the Hankel determinant serves as a significant analytic instrument to examine the structure of the coefficients of the analytic functions. For a function $f \in \mathbf{A}$ and integers $s, k \geq 0$, Pommerenke [37] introduced the $k$th-order Hankel

determinant, defined by

$$H_{k,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+k-1} \\ a_{s+1} & a_{s+2} & \cdots & a_{s+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s+k-1} & a_{s+k} & \cdots & a_{s+2(k-1)} \end{vmatrix}.$$

In recent years, the determination of sharp upper bounds for $H_{k,s}(f)$ corresponding to various subclasses of normalized analytic functions has become a topic of considerable interest. Numerous authors have obtained precise estimates for these determinants, reflecting the geometric and analytic properties of the function classes under consideration.

In particular, the second-order Hankel determinant $H_{2,2}(f)$ has received substantial attention in the literature. Hayman [24] was the first to establish a sharp inequality for functions belonging to the class $\mathcal{S}$, showing that

$$|H_{2,s}(f)| \leq \sqrt{\gamma_s}, \quad \text{and in particular,} \quad |H_{2,2}(f)| \leq \gamma, \quad 1 \leq \gamma \leq \tfrac{1}{13}.$$

Subsequently, Janteng et al. [28] derived precise upper bounds for $H_{2,2}(f)$ corresponding to the convex and starlike subclasses of $\mathcal{S}$, thereby contributing significantly to the understanding of their geometric and analytic behavior.

We now recall that the class $\mathcal{P}$ denotes the family of analytic functions $p$ defined on the open unit disk $\mathbb{U}$ having a positive real part and being represented by the Taylor–Maclaurin series expansion

$$p(z) = 1 + \sum_{s=1}^{\infty} \varepsilon_s z^s, \quad z \in \mathbb{U}. \tag{1.9}$$

The following lemmas, which provide coefficient estimates and interrelations for functions belonging to the class $\mathcal{P}$, will be utilized in the subsequent theoretical development.

**Lemma 1.2** ([31, 59])**.** *Let $p \in \mathcal{P}$ be represented by the power series expansion given in* (1.9)*. Then the coefficients $\varepsilon_n$ of $p$ satisfy the following sharp inequalities:*

$$|\varepsilon_s| \leq 2, \quad s \geq 1, \tag{1.10}$$

$$|\varepsilon_{k+s} - \delta\, \varepsilon_k \varepsilon_s| \leq 2, \quad 0 < \delta \leq 1, \tag{1.11}$$

$$|\varepsilon_2 - \varrho\, \varepsilon_1^2| \leq 2\, \max\{1, |2\varrho - 1|\}, \quad \varrho \in \mathbb{C}. \tag{1.12}$$

**Lemma 1.3** ([19])**.** *Let $p \in \mathcal{P}$ be expressed by the power series expansion in* (1.9) *and let $L_1 \in [0, 1]$ satisfy $L_1(2L_1 - 1) \leq L_2 \leq L_1$. Then the following sharp inequality holds:*

$$\left|L_2 \varepsilon_1^3 - 2L_1 \varepsilon_1 \varepsilon_2 + \varepsilon_3\right| \leq 2. \tag{1.13}$$

**Lemma 1.4** ([39]). *Let $p \in \mathcal{P}$ be represented by the expansion in* (1.9). *If $x, z \in \mathbb{U}$, then the coefficients of $p$ satisfy the following functional relations:*

$$2\varepsilon_2 = \varepsilon_1^2 + x\left(4 - \varepsilon_1^2\right),$$
$$4\varepsilon_3 = 2x\left(4 - \varepsilon_1^2\right)\varepsilon_1 - x^2\left(4 - \varepsilon_1^2\right)\varepsilon_1 + 2z\left(1 - |x|^2\right)\left(4 - \varepsilon_1^2\right) + \varepsilon_1^3.$$

## 2. Coefficient Problems of the Class $A_{sl}^q$

In this section, we establish sharp estimates for the initial coefficients of the functions belonging to class $A_{sl}^q$. Furthermore, we derive the corresponding Fekete–Szegö inequality and obtain the sharp bound for the second-order Hankel determinant associated with this class.

**Theorem 2.1.** *Let $f \in A_{sl}^q$. Then the initial coefficients of $f$ satisfy the following sharp inequalities:*

$$|a_2| \leq \frac{3+q}{4(1+q)}, \tag{2.1}$$

$$|a_3| \leq \frac{3+q}{4(1+q+q^2)}, \tag{2.2}$$

$$|a_4| \leq \frac{3+q}{4(1+q+q^2+q^3)}. \tag{2.3}$$

*Proof.* Let $f \in A_{sl}^q$. Then

$$D_q f(z) = \mathbf{B}(u(z); q), \tag{2.4}$$

where $r \in \mathcal{P}$ and is defined by

$$r(z) = \frac{1+u(z)}{1-u(z)} = 1 + \varepsilon_1 z + \varepsilon_2 z^2 + \varepsilon_3 z^3 + \cdots. \tag{2.5}$$

It follows from (2.5) that

$$u(z) = \frac{r(z)-1}{r(z)+1} = \frac{1}{2}\left[\varepsilon_1 z + \left(\varepsilon_2 - \frac{\varepsilon_1^2}{2}\right) z^2 + \left(\varepsilon_3 - \varepsilon_1\varepsilon_2 + \frac{\varepsilon_1^3}{4}\right) z^3 + \cdots\right]. \tag{2.6}$$

Substituting (2.6) into (1.3), we obtain

$$\begin{aligned}\mathbf{B}(u(z); q) = 1 &+ \frac{3+q}{8}\varepsilon_1 z \\ &+ \left[\frac{3+4q+q^2}{8(1+q)}\left(\varepsilon_2 - \frac{\varepsilon_1^2}{2}\right) + \frac{-9+9q+5q^2+3q^3}{128(1+q)}\varepsilon_1^2\right] z^2 \\ &+ \left[\frac{3+q}{8}\left(\varepsilon_3 - \varepsilon_1\varepsilon_2 + \frac{\varepsilon_1^3}{4}\right)\right.\end{aligned} \tag{2.7}$$

$$+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\varepsilon_1\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024(1+q+q^2)}\varepsilon_1^3\Bigg]z^3+\cdots.$$

In addition, the left-hand side of (2.4) can be expressed as

$$D_qf(z)=1+(1+q)a_2z+\left(1+q+q^2\right)a_3z^2+\left(1+q+q^2+q^3\right)a_4z^3+\cdots.$$

It follows from (2.7) and (2) that

(2.8)
$$a_2=\frac{3+q}{8(1+q)}\,\varepsilon_1,$$

(2.9)
$$a_3=\frac{1}{1+q+q^2}\left[\frac{3+4q+q^2}{8(1+q)}\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\varepsilon_1^2\right],$$

(2.10)
$$a_4=\frac{1}{1+q+q^2+q^3}\left\{\frac{3+q}{8}\left(\varepsilon_3-\varepsilon_1\varepsilon_2+\tfrac{\varepsilon_1^3}{4}\right)+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\,\varepsilon_1\left(\varepsilon_2-\tfrac{\varepsilon_1^2}{2}\right)+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024(1+q+q^2)}\,\varepsilon_1^3\right\}.$$

By (1.10) and (2.8), we have

$$|a_2|\le\frac{3+q}{4(1+q)}.$$

From (2.9), we get

$$a_3=\frac{3+4q+q^2}{8(1+q)(1+q+q^2)}\left(\varepsilon_2-\left[\frac{1}{2}-\frac{-9+9q+5q^2+3q^3}{16(3+4q+q^2)}\right]\varepsilon_1^2\right).\tag{2.11}$$

Put

$$\varrho=\frac{1}{2}-\frac{-9+9q+5q^2+3q^3}{16(3+4q+q^2)},$$

and define

$$J(q)=|2\varrho-1|=\frac{|-9+9q+5q^2+3q^3|}{8(3+4q+q^2)}.$$

Set

$$N(q)=3q^3+5q^2+9q-9,\qquad M(q)=q^2+4q+3,$$

so that

$$J(q)=\frac{1}{8}|h(q)|,\quad\text{with}\quad h(q)=\frac{N(q)}{M(q)}.$$

Note that $M(q) = (1+q)(3+q)$ for every $0 < q < 1$, hence the sign and monotonicity of $h$ are governed by $N$ and its derivative. Since

$$h'(q) = \frac{N'(q)M(q) - N(q)M'(q)}{M^2(q)},$$

observe that the numerator equals

$$S(q) = N'(q)M(q) - N(q)M'(q) = 3q^4 + 24q^3 + 38q^2 + 48q + 63.$$

Clearly, $S(q) > 0$ for all $0 < q < 1$. Consequently, $h$ is strictly increasing in $(0, 1)$. Since $h(0) = -3$ and $h(1) = 1$, the range of $h$ is $(-3, 1)$. Hence,

$$\max_{0<q<1} J(q) = \frac{1}{8} \max_{0<q<1} |h(q)| = \frac{3}{8},$$

so, it follows from (1.12) and (2.11) that

$$|a_3| \leq \frac{3 + 4q + q^2}{4(1+q)\,(1+q+q^2)} = \frac{3+q}{4\,(1+q+q^2)}.$$

To determine the bound for $a_4$, it follows from (2.10) that

$$a_4 = \frac{3+q}{8(1+q+q^2+q^3)}\left[\frac{179 + 185q + 177q^2 + 40q^3 - 25q^4 - 17q^5 + 5q^6}{128(1+q)(3+q)\,(1+q+q^2)}\varepsilon_1^3 - \frac{33 + 23q + 3q^2 - 3q^3}{8(3+4q+q^2)}\varepsilon_1\varepsilon_2 + \varepsilon_3\right].$$

Set

$$L_2 := \frac{179 + 185q + 177q^2 + 40q^3 - 25q^4 - 17q^5 + 5q^6}{128(1+q)(3+q)\,(1+q+q^2)},$$

$$L_1 := \frac{33 + 23q + 3q^2 - 3q^3}{16(3+4q+q^2)} \in [0, 1],$$

then we have

$$a_4 = \frac{3+q}{8(1+q+q^2+q^3)}\left(L_2\varepsilon_1^3 - 2L_1\varepsilon_1\varepsilon_2 + \varepsilon_3\right). \tag{2.12}$$

Consider

$$L_2 - L_1 = \frac{N_1(q)}{M_1(q)},$$

where a direct simplification gives

$$N_1(q) = 5q^5 + 2q^4 - 27q^3 - 117q^2 - 178q - 85,$$

$$M_1(q) = 128q^3 + 512q^2 + 512q + 384 > 0, \quad \text{for all } 0 < q < 1.$$

We note that

$$\begin{aligned} N_1(q) &= 5q^5 + 2q^4 - \left(27q^3 + 117q^2 + 178q + 85\right) \\ &\leq 5 + 2 - \left(27q^3 + 117q^2 + 178q + 85\right) \end{aligned}$$

$$\begin{aligned}
&\leq 7-\left(27q^3+117q^2+178q+85\right)\\
&\leq 7-85\\
&=-78\\
&<0.
\end{aligned}$$

Therefore $L_2 < L_1$. On the other hand, set

$$L_2 - L_1(2L_1-1) = \frac{N_2(q)}{M_2(q)},$$

and after clearing the denominators and simplifying one obtains

$$\begin{aligned}
N_2(q) &= -q^8-3q^7-9q^6+52q^5+233q^4+429q^3+397q^2+266q+60,\\
M_2(q) &= 32q^6+288q^5+992q^4+1728q^3+1760q^2+1056q+288>0
\end{aligned}$$

for all $0<q<1$.

We note that

$$\begin{aligned}
N_2(q) &= -(q^8+3q^7+9q^6)+52q^5+233q^4+429q^3+397q^2+266q+60\\
&\geq -13+52q^5+233q^4+429q^3+397q^2+266q+60\geq 47.
\end{aligned}$$

Hence,

$$L_1(2L_1-1) < L_2.$$

Now, it is possible for us to apply (1.13) and consequently (2.12) leads to the following.

$$|a_4| \leq \frac{3+q}{4(1+q+q^2+q^3)}.$$

It is evident that the bounds obtained are sharp, as can be seen from (1.6), (1.7) and (1.8). The proof is complete. □

**Conjecture.** Let $f \in A_{sl}^q$. Then for every integer $s \geq 2$ the following inequality holds:

$$|a_s| \leq \frac{3+q}{4\sum_{k=0}^{s-1} q^k}.$$

To provide a clear numerical illustration of the derived coefficient bounds and their dependence on the parameter $q$, we present Table 1, which lists the computed values of $|a_2|$, $|a_3|$ and $|a_4|$ for several representative values of $q$ ranging from 0.1 to 0.9. This allows the reader to observe the variation of the bounds and verify that they reduce to known classical results as $q \to 1^-$.

TABLE 1. Numerical values of the coefficient bounds $|a_2|$, $|a_3|$ and $|a_4|$ for representative values of $q$.

rraystretch1.5

| $q$ | $\lvert a_2\rvert$ | $\lvert a_3\rvert$ | $\lvert a_4\rvert$ |
|---|---|---|---|
| 0.1 | 0.7045 | 0.6982 | 0.6976 |
| 0.2 | 0.6667 | 0.6452 | 0.6410 |
| 0.3 | 0.6346 | 0.5935 | 0.5822 |
| 0.4 | 0.6071 | 0.5449 | 0.5234 |
| 0.5 | 0.5833 | 0.5000 | 0.4667 |
| 0.6 | 0.5625 | 0.4592 | 0.4136 |
| 0.7 | 0.5441 | 0.4224 | 0.3652 |
| 0.8 | 0.5278 | 0.3893 | 0.3218 |
| 0.9 | 0.5132 | 0.3598 | 0.2835 |

**Theorem 2.2.** *Let $f \in A_{sl}^q$. Then, for every complex number $\lambda \in \mathbb{C}$, the following sharp Fekete–Szegö type inequality holds:*

$$\left|a_3 - \lambda\, a_2^2\right| \leq \max\left\{ \frac{3+q}{4(1+q+q^2)},\ \left|\frac{9-14q^2-8q^3-3q^4+2(3+q)^2\lambda}{32(1+q)^2}\right| \right\}.$$

*Proof.* Using (2.8) and (2.9), after straightforward simplifications, we obtain

(2.13)

$$a_3 - \lambda a_2^2 = \frac{3+q}{8\,(1+q+q^2)} \times \left( \varepsilon_2 - \left[ \frac{33+56q+26q^2-3q^4+2(3+q)^2\lambda\left(1+q+q^2\right)}{16(1+q)^2(3+q)} \right] \varepsilon_1^2 \right).$$

Now, applying the estimate (1.12) for functions in the class $\mathcal{P}$ to the expression in (2.13), we obtain

$$\left|a_3 - \lambda a_2^2\right| < \max\left\{ \frac{3+q}{4\,(1+q+q^2)},\ \left|\frac{9-14q^2-8q^3-3q^4+2(3+q)^2\lambda}{32(1+q)^2}\right| \right\},$$

which proves the Theorem. □

**Corollary 2.3.** *Let $f \in A_{sl}^q$. Then*

$$H_{2,1}(f) \leq \max\left\{ \frac{3+q}{4\,(1+q+q^2)},\ \left|\frac{27+12q-12q^2-8q^3-3q^4}{32(1+q)^2}\right| \right\}.$$

**Theorem 2.4.** *Let $f \in A_{sl}^q$. Then*

$$H_{2,2}(f) \leq \frac{(3+q)^2}{8\,(1+q+q^2)^2},$$

*and this bound is the best possible.*

*Proof.* Using (2.8)–(2.10), together with some simplifications, we obtain

$$\begin{aligned}
&|a_2a_3 - a_4| \\
&= \left| \frac{-15 - 65q + 640q^2 + 220q^3 - 100q^4 - 148q^5 - 28q^6 - 8q^7 + 15q^8 + q^9}{16384(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} \varepsilon_1^4 \right. \\
&\quad + \frac{q^2(-99 - 102q - 32q^2 + 6q^3 + 3q^4)}{512(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} \varepsilon_1^2\varepsilon_2 \\
&\quad \left. + \frac{(3+q)^2}{64(1+q)^2(1+q^2)} \varepsilon_1\varepsilon_3 - \frac{(3+q)^2}{64\left(1+q+q^2\right)^2} \varepsilon_2^2 \right|.
\end{aligned}$$

Using Lemma 1.4 with $\varepsilon_1 = \varepsilon$, $|z| = 1$ and $|x| = y$, we obtain

$$\begin{aligned}
&|a_2a_3 - a_4| \\
&= \left| \left( \frac{-15 - 65q + 640q^2 + 220q^3 - 100q^4 - 148q^5 - 28q^6 - 8q^7 + 15q^8 + q^9}{16384(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} \right. \right. \\
&\quad + \frac{q^2(-99 - 102q - 32q^2 + 6q^3 + 3q^4)}{1024(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} + \frac{(3+q)^2}{256(1+q)^2(1+q^2)} \\
&\quad \left. - \frac{(3+q)^2}{128\left(1+q+q^2\right)^2} \right) \varepsilon^4 + \frac{(3+q)^2}{256(1+q)^2(1+q^2)} y^2(4-\varepsilon^2)\varepsilon^2 \\
&\quad + \left( \frac{q^2(-99 - 102q - 32q^2 + 6q^3 + 3q^4)}{1024(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} \right. \\
&\quad \left. + \frac{(3+q)^2}{128(1+q)^2(1+q^2)} - \frac{(3+q)^2}{64\left(1+q+q^2\right)^2} \right) y(4-\varepsilon^2)\varepsilon^2 \\
&\quad \left. + \frac{(3+q)^2}{256(1+q)^2(1+q^2)} 2(1-y^2)(4-\varepsilon^2)\varepsilon + \frac{(3+q)^2}{128\left(1+q+q^2\right)^2} y^2(4-\varepsilon^2)^2 \right|.
\end{aligned}$$

After some simplifications and by applying the triangular inequality for the modulus, we obtain

$$\begin{aligned}
&|a_2a_3 - a_4| \\
&\leq \frac{591 + 1586q + 2302q^2 + 2182q^3 + 1502q^4 + 470q^5 + 86q^6 - 14q^7 - q^8}{16384(1+q)^2(1+q^2)\left(1+q+q^2\right)^2} \varepsilon^4 \\
&\quad + \frac{(3+q)^2}{256(1+q)^2(1+q^2)} \varepsilon^2(4-\varepsilon^2)\, y^2 \\
&\quad + \frac{72 + 264q + 467q^2 + 486q^3 + 416q^4 + 234q^5 + 69q^6 + 8q^7}{1024(1+q)^3(1+q^2)\left(1+q+q^2\right)^2} \varepsilon^2(4-\varepsilon^2)\, y \\
&\quad + \frac{(3+q)^2}{128(1+q)^2(1+q^2)} \varepsilon(1-y^2)(4-\varepsilon^2) + \frac{(3+q)^2}{128\left(1+q+q^2\right)^2} y^2(4-\varepsilon^2)^2 \\
&=: E(y, \varepsilon, q).
\end{aligned}$$

Step 1 we determine the maximum of $E(y,\varepsilon,q)$, we consider its partial derivative with respect to $y$ :

$$\begin{aligned}E_y &= \frac{(3+q)^2}{128(1+q)^2(1+q^2)}\,\varepsilon^2(4-\varepsilon^2)\,y \\ &\quad + \frac{72+264q+467q^2+486q^3+416q^4+234q^5+69q^6+8q^7}{1024(1+q)^3(1+q^2)\left(1+q+q^2\right)^2}\,\varepsilon^2(4-\varepsilon^2) \\ &\quad - \frac{(3+q)^2}{64(1+q)^2(1+q^2)}\,\varepsilon(4-\varepsilon^2)\,y + \frac{(3+q)^2}{64\left(1+q+q^2\right)^2}\,(4-\varepsilon^2)^2\,y.\end{aligned}$$

Clearly, it follows that $E_y \geq 0$ for $y \in [0,1]$. Thus, the maximum value of $E(y,\varepsilon,q)$ is attained when $y=1$, which

$$\begin{aligned}E(\varepsilon,q) &:= E(1,\varepsilon,q) \\ &= \frac{591+1586q+2302q^2+2182q^3+1502q^4+470q^5+86q^6-14q^7-q^8}{16384(1+q)^2(1+q^2)\left(1+q+q^2\right)^2}\varepsilon^4 \\ &\quad + \frac{108+396q+723q^2+798q^3+664q^4+362q^5+105q^6+12q^7}{1024(1+q)^3(1+q^2)\left(1+q+q^2\right)^2}\varepsilon^2(4-\varepsilon^2) \\ &\quad + \frac{(3+q)^2}{128\left(1+q+q^2\right)^2}(4-\varepsilon^2)^2.\end{aligned}$$

Step 2 we want to find the value of $c$ that maximizes $E(\varepsilon,q)$. Therefore, we compute the derivative with respect to $\varepsilon$ and set it equal to 0; we obtain the three roots:

$$\varepsilon_0 = 0,$$

$$\varepsilon_1 = \frac{4\sqrt{\frac{72+264q+314q^2+420q^3+432q^4+268q^5+78q^6+8q^7}{(1+q)^3(1+q^2)}}}{\sqrt{\frac{15+65q-640q^2-220q^3+100q^4+148q^5+28q^6+8q^7-15q^8-q^9}{(1+q)^3(1+q^2)}}} \notin [0,2],$$

$$\varepsilon_2 = \frac{-4\sqrt{\frac{72+264q+314q^2+420q^3+432q^4+268q^5+78q^6+8q^7}{(1+q)^3(1+q^2)}}}{\sqrt{\frac{15+65q-640q^2-220q^3+100q^4+148q^5+28q^6+8q^7-15q^8-q^9}{(1+q)^3(1+q^2)}}} \notin [0,2].$$

The point $q_0 \approx 0.2039341973$ is not critical in the classical sense; rather, it denotes the threshold value at which the expression under the square root in the denominator of $\varepsilon_1$ changes sign. As a result, the root $\varepsilon_1$ is real for $0 < q < q_0$, whereas for $q > q_0$ it becomes complex and is therefore excluded from the real-valued analysis. This behavior is further illustrated in Figure 3, which depicts the sign change of the corresponding expression and highlights the location of $q_0$. Consequently, the root does not belong to the interval $[0,2]$, as clearly illustrated in Figure 4. By the same reasoning, $\varepsilon_2$ will also be excluded.

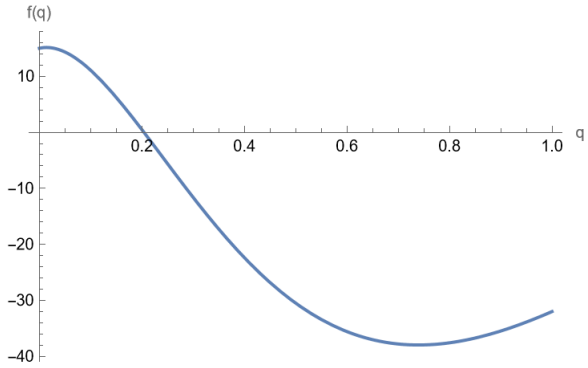


FIGURE 3. Graphical illustration of the expression under the square root in the denominator of $\varepsilon_1$ as a function of $q$, showing the threshold value $q_0 \approx 0.2039341973$ where the sign changes from positive to negative.

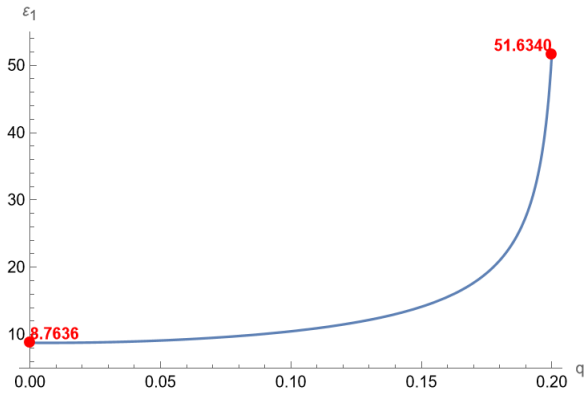


FIGURE 4. Plot of $\varepsilon_1$ for $0 \leq q \leq q_0$ with endpoints highlighted.

Thus, the only root that belongs to the interval $[0, 2]$ is $\varepsilon_0$.

Since

$$E_{\varepsilon\varepsilon}(0,q) = \frac{108 + 396q + 723q^2 + 798q^3 + 664q^4 + 362q^5 + 105q^6 + 12q^7}{128(1+q)^3(1+q^2)(1+q+q^2)^2} - 16 \cdot \frac{(3+q)^2}{128(1+q+q^2)^2}.$$

then $E_{\varepsilon\varepsilon}(0,q) < 0$ (see Figure 5), the maximum is attained at $\varepsilon = 0$ and this gives

$$|a_2a_4 - a_3^2| \leq E(1,0,q) = \frac{(3+q)^2}{8\,(1+q+q^2)^2}.$$

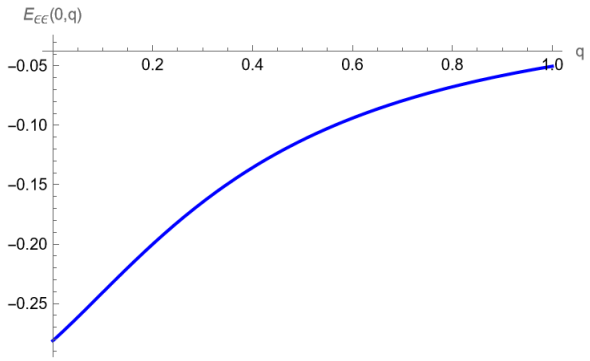


FIGURE 5. Plot of the second partial derivative $E_{\varepsilon\varepsilon}(0,q)$ for $0 < q < 1$, showing that it is negative throughout, which confirms that $\varepsilon = 0$ corresponds to a maximum.

Thus, the proof is established, affirming the validity of the result. □

## 3. Coefficient Problems of $A_{sl}^{q} \cap \Sigma$

In this section, we derive sharp upper bounds for the initial coefficients and establish the corresponding Fekete–Szegö inequality for a subclass of bi-univalent functions associated with the operator $D_q$.

**Definition 3.1.** Let $f \in \Sigma$. The function $f$ is said to belong to the class $\Sigma_{sl}^{q}$, with $q \in (0,1)$, if the following subordination conditions are satisfied:

$$D_q f(z) \prec \mathbf{B}(z;q), \quad z \in \mathbb{U}, \tag{3.1}$$

$$D_q g(\xi) \prec \mathbf{B}(\xi;q), \quad \xi \in \mathbb{U}, \tag{3.2}$$

where $g := f^{-1}$ denotes the inverse function of $f$.

**Example 3.2.** Since the parameter $q$ ranges over an infinite interval, it is not possible to provide explicit examples for all admissible values of $q$. Therefore, we illustrate the non-emptiness of the class by considering a representative case, namely $q = 0.5$. Other cases can be examined analogously by the reader, for instance, through graphical verification using computational software such as the Mathematica program.

Let us consider the family of functions

$$f_j(z) = \frac{z}{1 - jz}, \quad |j| \leq 1,$$

which are analytic in the unit disk $\mathbb{U}$. A straightforward computation shows that

$$D_q f_j(z) = 1 + \sum_{s=2}^{\infty} \frac{1 - q^s}{1 - q} j^{s-1} z^{s-1}.$$

For $q = 0.5$, it can be observed that

$$D_q f_j(\mathbb{U}) \subset \mathbf{B}(\mathbb{U}, q), \quad \text{whenever } |j| < 0.1,$$

as illustrated in Figure 6. This demonstrates the existence of non-trivial functions belonging to the class $\Sigma_{sl}^q$, thereby confirming that the class is non-empty.

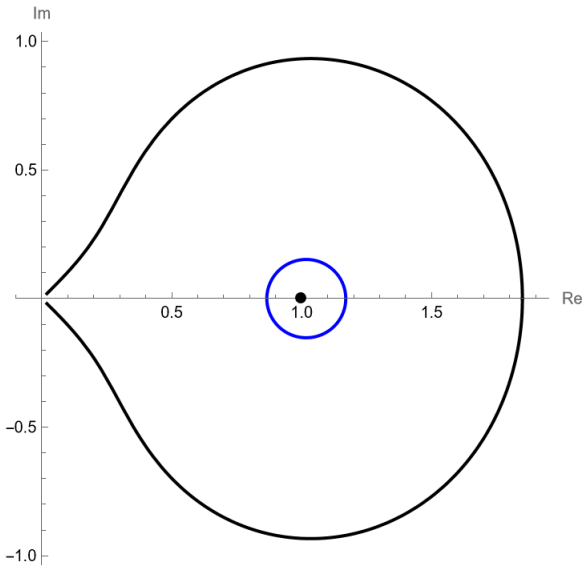


FIGURE 6. Image of the unit disk under $\mathbf{B}(z; 0.5)$ (black) and $D_{0.5} f_{0.1}(z)$ (blue).

**Example 3.3.** Consider the function

$$f(z) = \frac{z}{1-z},$$

which is well known to be bi-univalent in the unit disk. For this function, the second Taylor coefficient satisfies $a_2 = 1$. However, for all $q \in (0, 1)$, we have

$$|a_2| = 1 \geq \frac{3+q}{4(1+q)}.$$

Consequently, the function $f(z) = z/(1-z)$ does not satisfy the coefficient condition required for the class $\Sigma_{sl}^q$ in Theorem 3.4 below and therefore does not belong to this class.

**Theorem 3.4.** *Let $f \in \Sigma_{sl}^q$. Then*

$$|a_2| \leq \min\left\{ \frac{3+q}{4(1+q)}, \sqrt{\frac{(3+q)^3}{69+116q+90q^2+28q^3+q^4}} \right\},$$

*and*

$$|a_3| \leq \min\left\{\frac{(3+q)^2}{16(1+q)^2}+\frac{3+q}{4\,(1+q+q^2)},\right.$$
$$\left.\frac{(3+q)^3}{69+116q+90q^2+28q^3+q^4}+\frac{3+q}{4\,(1+q+q^2)}\right\}.$$

*Proof.* Let $f \in \Sigma_{sl}^{q}$. Then

$$D_q f(z) = \mathbf{B}(u(z); q), \tag{3.3}$$

and

$$D_q g(\xi) = \mathbf{B}(v(\xi); q), \tag{3.4}$$

where $r, s \in \mathcal{P}$ and are defined by

$$r(z) = \frac{1+u(z)}{1-u(z)} = 1+\varepsilon_1 z+\varepsilon_2 z^2+\varepsilon_3 z^3+\cdots, \tag{3.5}$$

and

$$s(\xi) = \frac{1+v(\xi)}{1-v(\xi)} = 1+\varsigma_1 \xi+\varsigma_2 \xi^2+\varsigma_3 \xi^3+\cdots. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$u(z) = \frac{r(z)-1}{r(z)+1} = \frac{1}{2}\left[\varepsilon_1 z+\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)z^2+\left(\varepsilon_3-\varepsilon_1\varepsilon_2+\frac{\varepsilon_1^3}{4}\right)z^3+\cdots\right], \tag{3.7}$$

and

$$v(\xi) = \frac{s(\xi)-1}{s(\xi)+1} = \frac{1}{2}\left[\varsigma_1 \xi+\left(\varsigma_2-\frac{\varsigma_1^2}{2}\right)\xi^2+\left(\varsigma_3-\varsigma_1\varsigma_2+\frac{\varsigma_1^3}{4}\right)\xi^3+\cdots\right]. \tag{3.8}$$

Substituting (3.7) and (3.8) into (1.3), we obtain

$$\begin{aligned}&\mathbf{B}(u(z); q)\\ &= 1+\frac{3+q}{8}\varepsilon_1 z+\left[\frac{3+4q+q^2}{8(1+q)}\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\varepsilon_1^2\right]z^2\\ &\quad+\left[\frac{3+q}{8}\left(\varepsilon_3-\varepsilon_1\varepsilon_2+\frac{\varepsilon_1^3}{4}\right)+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\varepsilon_1\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)\right.\\ &\quad\left.+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024\,(1+q+q^2)}\varepsilon_1^3\right]z^3+\cdots,\end{aligned} \tag{3.9}$$

and

(3.10)

$$\begin{aligned}&\mathbf{B}(v(\xi);q)\\&= 1+\frac{3+q}{8}\varsigma_1\xi+\left[\frac{3+4q+q^2}{8(1+q)}\left(\varsigma_2-\frac{\varsigma_1^2}{2}\right)+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\varsigma_1^2\right]\xi^2\\&\quad+\left[\frac{3+q}{8}\left(\varsigma_3-\varsigma_1\varsigma_2+\frac{\varsigma_1^3}{4}\right)+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\varsigma_1\left(\varsigma_2-\frac{\varsigma_1^2}{2}\right)\right.\\&\quad\left.+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024\,(1+q+q^2)}\varsigma_1^3\right]\xi^3+\cdots.\end{aligned}$$

Also, the left-hand sides of (3.3) and (3.4) are given by

(3.11)

$$D_qf(z)=1+(1+q)a_2z+\left(1+q+q^2\right)a_3z^2+\left(1+q+q^2+q^3\right)a_4z^3+\cdots,$$

and

$$\begin{aligned}D_qg(\xi)&=1-(1+q)a_2\xi+\left(1+q+q^2\right)\left(2a_2^2-a_3\right)\xi^2\\&\quad-\left(1+q+q^2+q^3\right)(5a_2^3-5a_2a_3+a_4)\xi^3+\cdots.\end{aligned}\tag{3.12}$$

It follows from (3.3), (3.9) and (3.11) that

(3.13)

$$(1+q)a_2=\frac{3+q}{8}\varepsilon_1,$$

(3.14)

$$\left(1+q+q^2\right)a_3=\frac{3+4q+q^2}{8(1+q)}\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\varepsilon_1^2,$$

(3.15)

$$\begin{aligned}\left(1+q+q^2+q^3\right)a_4&=\frac{3+q}{8}\left(\varepsilon_3-\varepsilon_1\varepsilon_2+\frac{\varepsilon_1^3}{4}\right)\\&\quad+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\,\varepsilon_1\left(\varepsilon_2-\frac{\varepsilon_1^2}{2}\right)\\&\quad+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024\,(1+q+q^2)}\,\varepsilon_1^3.\end{aligned}$$

Similarly, from (3.4), (3.10) and (3.12) it follows that

(3.16)

$$-(1+q)a_2=\frac{3+q}{8}\varsigma_1,$$

(3.17)

$$\left(1+q+q^2\right)\left(2a_2^2-a_3\right)=\frac{3+4q+q^2}{8(1+q)}\left(\varsigma_2-\frac{\varsigma_1^2}{2}\right)+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\varsigma_1^2,$$

(3.18)
$$\begin{aligned}
&-\left(1+q+q^2+q^3\right)\left(5a_2^3-5a_2a_3+a_4\right)\\
&\quad=\frac{3+q}{8}\left(\varsigma_3-\varsigma_1\varsigma_2+\frac{\varsigma_1^3}{4}\right)+\frac{-9+9q+5q^2+3q^3}{64(1+q)}\varsigma_1\left(\varsigma_2-\frac{\varsigma_1^2}{2}\right)\\
&\qquad+\frac{11-50q+11q^2+5q^3+2q^4+5q^5}{1024\left(1+q+q^2\right)}\varsigma_1^3.
\end{aligned}$$

From (3.13) and (3.16), we have

$$a_2=\frac{3+q}{8(1+q)}\varepsilon_1=-\frac{3+q}{8(1+q)}\varsigma_1, \tag{3.19}$$

which yields the following fundamental equation that underpins our proof

$$\varepsilon_1=-\varsigma_1. \tag{3.20}$$

Applying (1.10) to (3.19), we obtain

$$|a_2|\leq\frac{3+q}{4(1+q)}. \tag{3.21}$$

Squaring and adding (3.13) and (3.16) gives

$$a_2^2=\frac{(3+q)^2\left(\varepsilon_1^2+\varsigma_1^2\right)}{64(1+q)^2}. \tag{3.22}$$

Moreover, summing (3.14) and (3.17) leads to

$$\begin{aligned}
\left(1+q+q^2\right)2a_2^2&=\frac{3+4q+q^2}{8(1+q)}(\varepsilon_2+\varsigma_2)\\
&\quad+\left(-\frac{3+4q+q^2}{16(1+q)}+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\right)\left(\varepsilon_1^2+\varsigma_1^2\right).
\end{aligned}$$

From (3.22), we get

$$\begin{aligned}
\left(1+q+q^2\right)2a_2^2&=\frac{3+4q+q^2}{8(1+q)}(\varepsilon_2+\varsigma_2)\\
&\quad+\left(-\frac{3+4q+q^2}{16(1+q)}+\frac{-9+9q+5q^2+3q^3}{128(1+q)}\right)\frac{64(1+q)^2a_2^2}{(3+q)^2},
\end{aligned}$$

which implies

$$a_2^2=\frac{(\varepsilon_2+\varsigma_2)(3+q)^3}{4(69+116q+90q^2+28q^3+q^4)}. \tag{3.23}$$

Thus, (1.10) implies that

$$|a_2| \leq \sqrt{\frac{(3+q)^3}{69+116q+90q^2+28q^3+q^4}}. \tag{3.24}$$

Subsequently, by subtracting (3.17) from (3.14), we obtain

$$a_3 = a_2^2 + \frac{(\varepsilon_2 - \varsigma_2)(3+q)}{16\,(1+q+q^2)}, \tag{3.25}$$

and hence,

$$|a_3| \leq |a_2|^2 + \frac{3+q}{4\,(1+q+q^2)}. \tag{3.26}$$

Substituting (3.21) and (3.24) into (3.26), respectively, we have

$$|a_3| \leq \frac{(3+q)^2}{16(1+q)^2} + \frac{3+q}{4\,(1+q+q^2)},$$

and

$$|a_3| \leq \frac{(3+q)^3}{69+116q+90q^2+28q^3+q^4} + \frac{3+q}{4\,(1+q+q^2)}.$$

Hence, the desired result follows, completing the proof with elegance and clarity. □

The concluding Theorem of this section establishes the Fekete–Szegö functional for functions in the class $\Sigma_{\mathrm{sl}}^q$.

**Theorem 3.5.** *Let $f \in \Sigma_{\mathrm{sl}}^q$ and $\varrho \in \mathbb{R}$. Then*

$$|a_3 - \varrho a_2^2| \leq \begin{cases} \dfrac{3+q}{4\,(1+q+q^2)}, \\[2ex] 0 \leq |1-\varrho| \leq \dfrac{69+116q+90q^2+28q^3+q^4}{4(3+q)^2\,(1+q+q^2)}, \\[2ex] \dfrac{|1-\varrho|(3+q)^3}{69+116q+90q^2+28q^3+q^4}, \\[2ex] |1-\varrho| \geq \dfrac{69+116q+90q^2+28q^3+q^4}{4(3+q)^2\,(1+q+q^2)}. \end{cases} \tag{3.27}$$

*Proof.* Assuming that $f \in \Sigma_{\mathrm{sl}}^q$, it follows from (3.23) and (3.25) that

$$\begin{aligned} a_3 - \varrho a_2^2 &= \frac{(1-\varrho)(3+q)^3(\varepsilon_2+\varsigma_2)}{4(69+116q+90q^2+28q^3+q^4)} + \frac{(\varepsilon_2-\varsigma_2)(3+q)}{16\,(1+q+q^2)} \\ &= \left(\Xi(q,\varrho) + \frac{3+q}{16\,(1+q+q^2)}\right)\varepsilon_2 + \left(\Xi(q,\varrho) - \frac{3+q}{16\,(1+q+q^2)}\right)\varsigma_2, \end{aligned} \tag{3.28}$$

where

$$\Xi(q,\varrho)=\frac{(1-\varrho)(3+q)^3}{4(69+116q+90q^2+28q^3+q^4)}. \tag{3.29}$$

Taking the modulus of (3.28) , we obtain

$$|a_3-\varrho a_2^2|\leq\begin{cases}\dfrac{3+q}{4\,(1+q+q^2)}, & 0\leq\Xi(q,\varrho)\leq\frac{3+q}{16(1+q+q^2)},\\[2ex] 4\,\Xi(q,\varrho), & \Xi(q,\varrho)\geq\frac{3+q}{16(1+q+q^2)}.\end{cases} \tag{3.30}$$

After straightforward computations, we obtain(3.27). □

## 4. Conclusions

In this paper, we have investigated new subclasses of analytic and bi-univalent functions associated with the $q$-generalized balloon-type function. By employing tools from $q$-calculus and subordination theory, we derived sharp bounds for the initial coefficients, the Fekete–Szegö functional and the second-order Hankel determinant for the classes $A_{sl}^{q}$ and $\Sigma_{sl}^{q}$. Our results extend and unify several earlier findings as special or limiting cases, particularly when $q\to 1^{-}$, thereby establishing a clear connection with the corresponding classical function classes. Despite these advances, the present study has certain limitations. The analysis is primarily restricted to the initial coefficients and low-order Hankel determinants, while the behavior of higher-order coefficients remains open and technically challenging. Moreover, the sharpness of some bounds is established only for specific extremal functions and a complete characterization of extremal functions for all parameter ranges of $q$ is still unknown. From an applied perspective, the $q$-generalized balloon class provides a flexible framework that may be useful in studying geometric properties of analytic functions arising in quantum calculus and related deformation theories. The interplay between $q$-operators and geometric functionals suggests potential applications in areas where discrete or quantum analogues of classical operators naturally arise. Several open problems emerge from this work. These include the extension of the obtained results to higher-order Hankel determinants, the verification of the proposed conjecture for general coefficients and the investigation of analogous subclasses generated by alternative $q$-differential operators or other special function convolutions. A systematic comparison between different choices of $q$-operators may further clarify the role of operator structure in determining geometric properties. We hope that the methods and results presented here will stimulate further research in this direction and contribute to a deeper understanding of $q$-calculus techniques within contemporary Geometric Function Theory.

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