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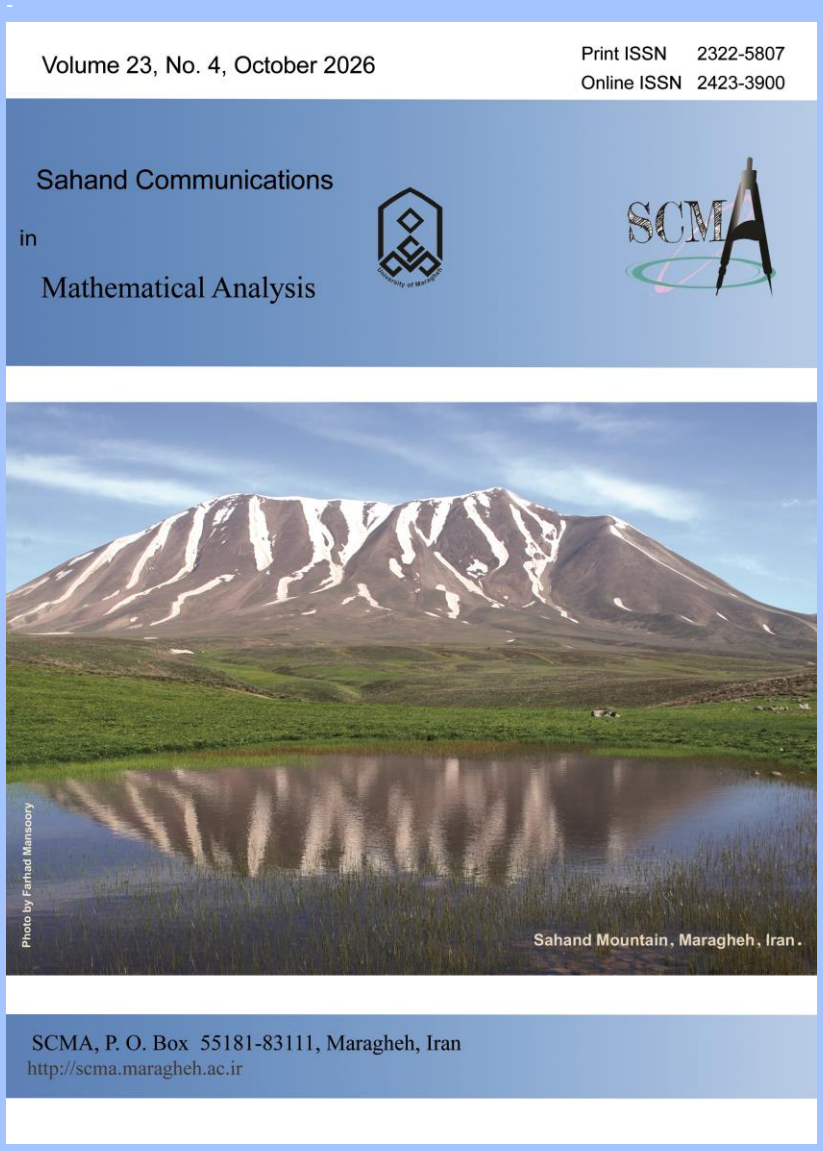

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Existence of Mild Solutions for Fuzzy Fractional Evolution Equations via Schauder Fixed Point Theorem

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Abstract. This paper establishes a novel and unified existence theory for mild solutions to a class of fuzzy fractional evolution equations governed by the ψ-Caputo generalized Hukuhara (gH) derivative. The considered system is distinctive for its incorporation of a Volterra-type integral term within the nonlinearity, introducing a cumulative memory effect and is analyzed within a semigroup framework on the space of triangular fuzzy numbers. This integrated approach, which synthesizes these three challenging features, generalizes several existing models in the literature. We address the problem for both type (i)-gH and type (ii)-gH differentiable solutions, deriving two distinct mild solution formulas via Wright-type functions and semigroup theory. The core existence results are established using a sophisticated application of the Schauder and Krasnosel'skii fixed point theorems. A complete illustrative example is provided to demonstrate the practical application of the main theorems. The results presented herein provide a significant extension of the existing theory for both classical and fuzzy fractional differential equations.

1. Introduction

The twin fields of fractional calculus and fuzzy set theory have, over the past few decades, emerged as profoundly powerful and versatile tools for modeling complex phenomena across the scientific and engineering disciplines. Fractional calculus, the branch of mathematical analysis concerned with derivatives and integrals of arbitrary real or complex

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order, has proven to be exceptionally adept at capturing memory and hereditary properties inherent in many physical and biological systems [21]. Unlike their integer-order counterparts, fractional-order models provide a more accurate description of processes involving long-range interactions and anomalous diffusion, with applications ranging from viscoelasticity and electrochemistry to signal processing and control theory [19, 20]. Concurrently, fuzzy set theory, introduced by Lotfi Zadeh [25], has revolutionized our ability to mathematically formalize and handle uncertainty, imprecision and qualitative information. By generalizing the classical notion of a set, fuzzy logic allows for a graded membership, making it indispensable for systems where information is incomplete or linguistic in nature [27].

The natural confluence of these two fields has given rise to the theory of fuzzy fractional differential equations (FFDEs). This hybrid framework is particularly well-suited for modeling dynamic systems where both the non-local, memory-dependent behavior of fractional operators and the inherent vagueness or subjectivity of system parameters and initial conditions are present. Such scenarios are commonplace in real-world applications, including the modeling of heat flow in materials with uncertain properties, the dynamics of ecological systems with imprecise population data and financial models subject to fuzzy market sentiments [1, 2].

The mathematical framework developed in this paper is well-suited for modeling dynamic systems where long-range memory, hereditary properties and inherent uncertainty are simultaneously present. The combination of the ψ-Caputo derivative, capturing memory effects on a potentially non-linear timescale, the fuzzy-valued nature of the solution, handling epistemic uncertainty in parameters or initial states and the Volterra integral, modeling cumulative historical influences, finds potential application in several complex scenarios. For instance, it can be applied to model stress relaxation in viscoelastic materials with imperfectly known properties, population dynamics in ecology under a history of imprecise environmental data, or financial markets where qualitative fuzzy sentiment and long-memory effects interact. This broader applicability motivates the general and rigorous analysis presented in the subsequent sections.

The foundational work on fuzzy differential equations (FDEs) primarily relied on the Hukuhara derivative (H-derivative) [22]. However,One notable limitation of the H-derivative is that a differentiable fuzzy-valued function tends to exhibit an expanding support (the diameter of the fuzzy number) as the independent variable grows, which is a property that severely restricts the class of solvable problems. To overcome this

and other conceptual hurdles, the concept of generalized Hukuhara differentiability (gH-differentiability) was introduced and extensively developed by Bede and Gal [8]. This generalization allows a fuzzy-valued function to be differentiable in two distinct senses, commonly denoted as type (i) and type (ii). A crucial feature of type (ii)-differentiability is that it can model processes where the uncertainty decreases over time, a situation that the classical H-derivative cannot capture. This bifurcation of the derivative concept is a fundamental characteristic of the fuzzy environment and necessitates a case-by-case analysis when studying FFDEs [6, 9]. The relevance of these concepts extends beyond mathematical formalism; they are essential for modeling distinct real-world dynamical behaviors under uncertainty. Type (i)-gH differentiability, where the uncertainty (characterized by the diameter of the fuzzy-valued function) increases over time, naturally models phenomena such as error growth in uncontrolled systems or the dispersion of a pollutant in a heterogeneous environment. In contrast, type (ii)-gH differentiability, which allows for a decrease in uncertainty, models regulatory or stabilizing processes where initial imprecision is gradually resolved, such as in controlled systems converging to a steady state or filtering processes that refine information over time. The choice of the ψ-Caputo derivative further extends modeling flexibility by aligning the fractional operator with a non-linear timescale $\psi(t)$, which is crucial for processes with intrinsic time-warping, such as biological growth with varying phases or damage accumulation in materials under variable load.

Parallel to these developments, the study of evolution equations differential equations set in infinite dimensional spaces, typically used to describe time-dependent processes in continuum mechanics, quantum physics and other areas has been a central theme in modern analysis [10, 12]. When fractional calculus is incorporated into this framework, we obtain fractional evolution equations, which are potent tools for modeling sub-diffusive processes in heterogeneous media [26]. The concept of a mild solution, which is a solution defined via an integral representation involving semigroups of operators, is often the most viable approach for tackling nonlinear fractional evolution equations, especially when classical solutions are difficult to obtain.

The present work is situated at the confluence of these advanced trends. While significant research exists on fuzzy fractional calculus [18, 23] and fuzzy semigroup theory [11, 23] separately, the literature lacks a unified treatment that integrates the most complex features of both. This paper fills this gap by establishing a comprehensive existence theory for a novel class of fuzzy fractional evolution equations. Our main

contribution lies in the simultaneous incorporation of the following challenging elements into a single model:

(i) **The ψ-Caputo generalized Hukuhara derivative** ${}^{C}D_{0,t}^{p,\psi}$ of order$p \in (0,1)$, which generalizes the standard Caputo derivative and offers greater flexibility in modeling by aligning the fractional operator with a non-linear timescale $\psi(t)$ [5].

(ii) **A Volterra-type integral term within the nonlinearity**, $\int_0^t f(\tau, x(\tau))d\tau$. This represents a non-local in time interaction where the system's rate of change depends on its entire history, compounding the analytical challenges posed by the fractional derivative and the two types of gH-differentiability [17, 18].

(iii) **A semigroup framework** on the space of triangular fuzzy numbers $\mathcal{T}$, where the operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ [11, 23].

We therefore investigate a class of fuzzy fractional evolution equations of the form:

$$\begin{cases} {}^{C}D_{0,t}^{p,\psi}x(t) = Ax(t) + \displaystyle\int_0^t f(\tau, x(\tau))\, d\tau, \quad t \in J = [0,T], \\ x(0) = x_0. \end{cases}$$

Here, ${}^{C}D_{0,t}^{p,\psi}$ denotes the ψ-Caputo fractional derivative of order $p \in (0,1)$, which generalizes the standard Caputo derivative and offers greater flexibility in modeling [5]. The operator A serves as the infinitesimal generator of a C_0-semigroup $T(t)_{t\geq 0}$ on the space of triangular fuzzy numbers $\mathcal{T}$ [11, 23] and f is a fuzzy-valued nonlinear function that includes a cumulative memory effect through its integral form.

Including the integral term $\int_0^t f(\tau, x(\tau))d\tau$ within the dynamics signifies the generalization of the more commonly studied form $f(t, x(t))$. It represents a non-local in time interaction, where the rate of change of the system depends not only on its current state at time t, but also on the entire history of its states from 0 to t. This poses substantial challenges from an analytical perspective, particularly in the fuzzy context, as it compounds the complexities arising from the fractional derivative and the two types of gH-differentiability [17, 18].

The primary aim and novelty of this paper is to provide a rigorous existence theory for this integrated model. The key novel aspects of our work are:

- Deriving distinct, explicit formulas for mild solutions for both type (i)-gH and type (ii)-gH differentiability (Lemma 3.1), by constructing solution operators via Wright-type functions.

- Establishing rigorous existence theorems (Theorems 3.2 and 3.3) under a set of verifiable hypotheses. The proofs require a sophisticated strategy: for the (i)-gH case, we decompose the problem to apply the Krasnosel'skii fixed point theorem, while for the (ii)-gH case, we employ Schauder's fixed point theorem under additional well-posedness conditions (G1) and (G2).
- Providing a complete illustrative example (Example 3.4) involving a linear fuzzy fractional system, where we explicitly verify all hypotheses and present the specific form of the mild solutions for both types of gH-differentiability.

For the (i)-gH differentiable case (Theorem 3.2), we define a sum of two operators $F_1 + F_2$ on a suitable convex and closed subset of the continuous fuzzy-valued function space. We then demonstrate that F_2 is a contraction mapping and F_1 is continuous and compact, allowing us to apply the Krasnosel'skii fixed point theorem [15], a powerful tool for equations where the nonlinearity can be decomposed into a contractive and a compact part. For the (ii)-gH differentiable case (Theorem 3.3), the structure of the solution leads to an operator $\hat{F} = F_1 \ominus_{gH} (-1) \odot F_2$. Under additional assumptions (G1) and (G2) that ensure the well-definedness and closure of the gH-difference in the solution space, we prove the existence of a fixed point using Schauder's fixed point theorem, leveraging the complete continuity of the operator.

To ground our theoretical findings in a concrete setting, we provide a comprehensive example involving a linear fuzzy fractional differential equation. We explicitly verify all the required hypotheses (H1)-(H4), including the compactness of the semigroup generated by a scalar multiple of the identity operator and present the specific form of the mild solutions for both types of gH-differentiability.

In summary, this paper contributes to the growing body of literature on FFDEs by providing a unified and rigorous framework for studying a complex class of fuzzy fractional evolution equations with integral nonlinearities. It highlights the intricate interplay between fractional calculus, fuzzy set theory, operator semigroups and nonlinear functional analysis, offering results that are expected to be foundational for future research in this dynamic and application-rich field.

2. Preliminaries

This section introduces notations, fundamental concepts and tools used throughout the paper.

Let $\mathcal{E}_1$ represent the collection of fuzzy subsets $v : \mathbb{R} \to [0, 1]$ satisfying [13]:

(i) Normality: $\exists y_0 \in \mathbb{R}$ such that $v(y_0) = 1$;

(ii) Fuzzy convexity: $v(ty_1 + (1-t)y_2) \geq \min\{v(y_1), v(y_2)\}$ for all $t \in [0,1]$ and $y_1, y_2 \in \mathbb{R}$;
(iii) Upper semi-continuity on $\mathbb{R}$;
(iv) The support set $\{y \in \mathbb{R} : v(y) > 0\}$ is compact.

The set $\mathcal{E}_1$ is commonly known as the fuzzy number space. For $v \in \mathcal{E}_1$ and $\alpha \in [0,1]$, the α-cut is defined by:

$$[v]_\alpha = \{y \in \mathbb{R} : v(y) \geq \alpha\}.$$

Each fuzzy number v can be expressed in parametric form (v^-, v^+) such that:

$$[v(y)]_\alpha = [v_\alpha^-, v_\alpha^+], \quad \forall \alpha \in [0,1].$$

The zero fuzzy number b_0 is given by $[b_0]_\alpha = [0,0]$ for all α.

Addition and scalar multiplication in $\mathcal{E}_1$ are defined on α-cuts as:

$$[v \oplus u]_\alpha = [v]_\alpha + [u]_\alpha, \quad [\beta \odot v]_\alpha = \beta[v]_\alpha,$$

where $+$ denotes the Minkowski addition of intervals and $\beta \in \mathbb{R}$.

Definition 2.1 (Generalized Hukuhara Difference [8])**.** Given $v, u \in \mathcal{E}_1$, the generalized Hukuhara difference $w = u \ominus_{gH} v$ satisfies:

$$\text{(i) } u = v \oplus w \qquad \text{or} \qquad \text{(ii) } v = u \oplus (-1 \odot w).$$

Define $D_\infty : \mathcal{E}_1 \times \mathcal{E}_1 \to [0, \infty)$ as:

$$D_\infty(v,u) = \sup_{\alpha \in [0,1]} D_H([v]_\alpha, [u]_\alpha),$$

where D_H is the Hausdorff metric. The space $(\mathcal{E}_1, D_\infty)$ is a complete metric space.

Note: $\mathcal{E}_1$ is not a vector space and $\|v\| := D_\infty(v, b_0)$ does not make it a Banach space.

Let T be the set of all triangular fuzzy numbers in $\mathcal{E}_1$. Then (T, D_∞) is a complete subspace. Bede [7] showed that if $v, u \in T$, then $v \ominus_{gH} u \in T$.

Let $a > 0$. Define the space:

$$C_T^a := C([0,a], T), \qquad D_a(x,y) := \sup_{t \in [0,a]} D_\infty(x(t), y(t)).$$

which is a complete metric space.

Definition 2.2 (gH-Differentiability [8])**.** Let $f \in C_T^a$, $a_0 \in (0,a)$. If there exists $D_{gH} f(a_0) \in T$ such that

$$D_{gH} f(a_0) = \lim_{r \to 0} \frac{1}{r} \odot \left(f(a_0 + r) \ominus_{gH} f(a_0) \right),$$

then f is said to be gH-differentiable at a_0.

If the gH-derivative exists for all $t \in [0,a]$, then f is called gH-differentiable on $[0,a]$.

Definition 2.3 (Types of gH-Differentiability [3, 8])**.** Let $f : [0, a] \to T$ and $a_0 \in (0, a)$. Then:

- f is (i)-gH-differentiable at a_0 if:

$$\begin{aligned} D_{gH}f(a_0) &= \lim_{r\to 0} \frac{f(a_0 + r) \ominus_H f(a_0)}{r} \\ &= \lim_{r\to 0} \frac{f(a_0) \ominus_H f(a_0 - r)}{r}; \end{aligned}$$

- f is (ii)-gH-differentiable at a_0 if:

$$\begin{aligned} D_{gH}f(a_0) &= \lim_{r\to 0} \frac{f(a_0) \ominus_H f(a_0 + r)}{-r} \\ &= \lim_{r\to 0} \frac{f(a_0 - r) \ominus_H f(a_0)}{-r}. \end{aligned}$$

Definition 2.4 (Parametric Form [9])**.** Let $f(t, \alpha) = [f^-(t, \alpha), f^+(t, \alpha)]$. Then:

- If $[D_{gH}f(a_0)]_\alpha = [f'^-(a_0, \alpha), f'^+(a_0, \alpha)]$, then f is of type (i)-gH;
- If $[D_{gH}f(a_0)]_\alpha = [f'^+(a_0, \alpha), f'^-(a_0, \alpha)]$, then f is of type (ii)-gH.

The notation $D^i_{gH}f$ and $D^{ii}_{gH}f$ denotes the (i) and (ii) gH-derivatives, respectively.

Definition 2.5 (Fuzzy Integral [17])**.** Let $g : [0, a] \to T$ with parametric form $g(t) = (g^-(t), g^+(t))$ where $g^\pm$ are Lebesgue integrable. Then:

$$\int_0^a g(t)dt = \left(\int_0^a g^-(t)dt, \int_0^a g^+(t)dt \right).$$

Let $L([0, a], T)$ be the space of Lebesgue integrable fuzzy-valued functions.

Definition 2.6 (Fractional RL Integral [4, 18])**.** Let $x \in L([0, a], T)$. For $p \in (0, 1]$, the Riemann-Liouville fuzzy fractional integral is defined by:

$${}^{RL}I^p_0 x(t) = \frac{1}{\Gamma(p)} \odot \int_0^t (t - m)^{p-1} \odot x(m)dm.$$

Definition 2.7 (Caputo gH-Derivative [4, 18])**.** Let $u \in C^a_T$ be gH-differentiable. Then:

$${}^{C}_{gH}D^p u(t) = {}^{RL}\, FI_0^{1-p} D_{gH}u(t), \quad t \in [0, a].$$

Depending on whether $D^i_{gH}u$ or $D^{ii}_{gH}u$ is used, we refer to the Caputo (i) or (ii) derivative.

Let $L(T)$ denote the space of all bounded linear operators $\rho : T \to T$ with norm:

$$\|\rho\|_{L(T)} := \inf \{C > 0 : D_\infty(\rho a, b_0) \leq C D_\infty(a, b_0),\ \forall a \in T\}.$$

Definition 2.8 (Strongly Continuous Semigroup [11, 23])**.** A family $\{T(t)\}_{t\geq 0} \subset L(T)$ is a semigroup if:

(i) $T(0) = \mathrm{id}_T$;
(ii) $T(s+h) = T(s)T(h)$ for all $s, h \geq 0$.

It is strongly continuous if $\lim_{t\to 0^+} T(t)x = x$ for all $x \in T$.

Lemma 2.9 (Boundedness [23])**.** *If $\{T(t)\}_{t\geq 0}$ is strongly continuous on T, then*

$$C := \sup_{t\in[0,a]} \|T(t)\|_{L(T)} \geq 1.$$

Define the integral operators (for $p \in (0,1)$):

$$U_p(t)\nu = \int_0^\infty \varphi_p(\phi) \odot T(t^p\phi)\nu d\phi, \tag{2.1}$$

$$V_p(t)\nu = p \odot \int_0^\infty \phi\varphi_p(\phi) \odot T(t^p\phi)\nu d\phi. \tag{2.2}$$

Definition 2.10 (Infinitesimal Generator [11])**.** Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup. Define:

$$Ax = \lim_{t\to 0^+} \frac{1}{t} \odot (T(t)x \ominus_{gH} x),$$

with domain:

$$D(A) = \left\{ x \in T : \lim_{t\to 0^+} \frac{1}{t} \odot (T(t)x \ominus_{gH} x) \in T \right\}.$$

Theorem 2.11 (Krasnosel'skii-Type Fixed Point, [15])**.** *Let $F \subset C([0,a], \mathcal{E}_1)$ be nonempty, convex and closed. Let $P_1, P_2 : F \to C([0,a], \mathcal{E}_1)$ such that:*

(i) *$P_1x + P_2y \in F$ for all $x, y \in F$;*
(ii) *P_2 is a contraction: $\exists k \in (0,1)$ such that $D_a(P_2x, P_2y) \leq kD_a(x,y)$;*
(iii) *P_1 is continuous and compact.*

Then $P_1 + P_2$ has at least one fixed point in F.

For recent advancements and other generalized fixed point structures in different metric settings, see also [14].

3. Main Results

Lemma 3.1. (i) *If x is Caputo (i)-gH differentiable satisfying the Cauchy problem* (3.2)*, then*

$$x(t) = U_p(\psi(t)-\psi(0))x_0 \oplus \int_0^t \int_0^s (\psi(t)-\psi(s))^{p-1} \odot T_p(\psi(t)-\psi(s))\psi'(s) \odot f(\tau, x(\tau))d\tau ds,$$

(ii) *If x is Caputo (ii)-gH differentiable satisfying the Cauchy problem* (3.2)*, then*

$$x(t) = U_p(\psi(t)-\psi(0))x_0 \ominus_{gH} (-1) \odot \int_0^t \int_0^s (\psi(t)-\psi(s))^{p-1} \odot T_p(\psi(t)-\psi(s))\psi'(s) \odot f(\tau, x(\tau))d\tau ds.$$

Proof. We prove the lemma for both cases of gH-differentiability.

Case 1: Type (i)-gH Differentiable Solution

Assume x is (i)-gH differentiable and satisfies:

$$ {}^{C}D^{p,\psi}_{0,t}x(t) = Ax(t) + \int_0^t f(\tau, x(\tau))d\tau. \tag{3.1}$$

Using the definition of ψ-Caputo derivative and the Riemann-Liouville ψ-fractional integral, we can write:

$$\begin{aligned} x(t) \ominus_{gH} x(0) &= \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} \odot {}^{C}D^{p,\psi}_{0,s}x(s)ds \\ &= \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} \\ &\quad \odot \left[Ax(s) + \int_0^s f(\tau, x(\tau))d\tau\right] ds. \end{aligned}$$

Thus, for type (i)-gH differentiability:

$$\begin{aligned} x(t) = x_0 &\oplus \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} \odot Ax(s)ds \\ &\oplus \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} \odot \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds. \end{aligned}$$

Now, we express the solution using the solution operators. Recall the Laplace transform relationship for the Mittag-Leffler function:

$$\mathcal{L}\left\{t^{p-1}E_{p,p}\left(-\lambda t^p\right)\right\} = \frac{1}{\lambda + s^p}.$$

Using the Wright function representation:

$$E_{p,p}(-z) = \int_0^\infty \varphi_p(\theta)e^{-z\theta}d\theta,$$

where $\varphi_p(\theta)$ is the Wright-type function satisfying:

$$\int_0^\infty \theta^\nu \varphi_p(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+p\nu)}, \quad \nu > -1.$$

The solution operators are defined as:

$$U_p(t)\nu = \int_0^\infty \varphi_p(\theta) \odot T(t^p\theta)\nu d\theta,$$
$$T_p(t)\nu = p \odot \int_0^\infty \theta\varphi_p(\theta) \odot T(t^p\theta)\nu d\theta.$$

These operators satisfy the following properties:

(i) $U_p(t)$ and $T_p(t)$ are bounded linear operators on $\mathcal{T}$,
(ii) $\frac{d}{dt}U_p(t)\nu = AT_p(t)\nu$,
(iii) $U_p(0) = I$ (identity operator).

Now, applying the operator $S_p(\psi(t) - \psi(0))$ to the initial condition and using the variation of constants formula, we obtain:

$$x(t) = U_p(\psi(t) - \psi(0))x_0 \oplus \int_0^t [\psi(t) - \psi(s)]^{p-1} \odot T_p(\psi(t) - \psi(s))\psi'(s) \odot \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds.$$

This establishes the mild solution formula for type (i)-gH differentiability.

Case 2: Type (ii)-gH Differentiable Solution

For type (ii)-gH differentiability, the fundamental relationship changes due to the different behavior of the gH-difference. Starting from the same differential equation (3.1), we have:

$$\begin{aligned} x(0) \ominus_{gH} (-1) \odot x(t) &= \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} \odot {}^C D_{0,s}^{p,\psi} x(s) ds \\ &= \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} \\ &\quad \odot \left[Ax(s) + \int_0^s f(\tau, x(\tau))d\tau\right] ds. \end{aligned}$$

Rearranging and using the properties of gH-difference:

$$x(t) = x_0 \ominus_{gH} (-1) \odot \left[\frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} \odot Ax(s) ds \right.$$
$$\left. \oplus \frac{1}{\Gamma(p)} \odot \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} \odot \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds\right].$$

Applying the solution operators and using the semigroup properties, we obtain:

$$\begin{aligned} x(t) &= U_p(\psi(t)-\psi(0))x_0 \\ &\ominus_{gH}(-1)\odot\int_0^t [\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s) \\ &\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds. \end{aligned}$$

To complete the proof, we verify that the defined operators satisfy the necessary fractional evolution equations.

Using the Laplace transform method, the solution of the abstract fractional Cauchy problem:

$$^{C}D^{p}_{0,t}x(t) = Ax(t)+\int_0^t f(\tau,x(\tau))d\tau, \qquad x(0)=x_0,$$

is given by:

$$x(t) = S_p(t)x_0+\int_0^t (t-s)^{p-1}T_p(t-s)\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds.$$

For the ψ-Caputo case, we make the substitution $u=\psi(s)$ and obtain:

$$\begin{aligned} x(t) &= U_p(\psi(t)-\psi(0))x_0 \\ &+\int_0^t [\psi(t)-\psi(s)]^{p-1}T_p(\psi(t)-\psi(s))\psi'(s)\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds. \end{aligned}$$

The operators S_p and T_p satisfy the following key identities:

$$U_p(t) = E_{p,1}(t^pA) = \sum_{k=0}^{\infty}\frac{(t^pA)^k}{\Gamma(pk+1)},$$

$$T_p(t) = t^{p-1}E_{p,p}(t^pA) = t^{p-1}\sum_{k=0}^{\infty}\frac{(t^pA)^k}{\Gamma(pk+p)}.$$

where $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function.

This completes the proof for both cases of gH-differentiability. □

The following assumptions will be required in the sequel.

(H1) For almost every $t\in[0,a]$, the mapping $f(t,\cdot):C([0,a],\mathcal{T})\to\mathcal{T}$ is continuous and for each $z\in C([0,a],\mathcal{T})$, the function $f(\cdot,z):[0,a]\to\mathcal{T}$ is strongly measurable.

(H2) There exist $q_2\in[0,q)$,a set $B_r:=\{x\in\mathcal{T}, d_\infty(x,\hat{0})\le r\}\subset\mathcal{T}$,with $r>0$ and a function $\rho(\cdot)\in L^{\frac{1}{q_2}}([0,a],\mathbb{R}^+)$ such that for any $x,y\in C([0,a],B_r)$, the following holds

$$d_\infty(f(t,x(t)),f(t,y(t)))\le\rho(t)d_\infty(x(t),y(t)), \quad t\in[0,a].$$

(H3) There exists a constant $q_1 \in [0, q)$ and a function $m \in L^{\frac{1}{q_1}}([0, a], \mathbb{R}^+)$ such that for any $z \in C([0, a], \mathcal{T})$ and almost evey $t \in [0, a]$, it holds that

$$d_\infty(f(t, z(t)), \hat{0}) \leq m(t)$$

(H4) $T(t)$ is a compact operator for every $t > 0$.

These standard assumptions on the nonlinearity of f ensure its tractability within our semigroup framework. We note that the analysis of such nonlinear operators can also be approached through more generalized derivatives, such as the Quantum Fréchet derivative [24], though our methodology relies on the classical properties stated above.

Theorem 3.2. *Under hypotheses **(H1)–(H4)**, the fuzzy fractional evolution equation:*

$$\begin{cases} {}^{C}D^{p,\psi}_{0,t}x(t) = Ax(t) + \displaystyle\int_0^t f(\tau, x(\tau))d\tau, \quad t \in J = [0, a], \\ x(0) = x_0 \in \mathcal{T}. \end{cases} \tag{3.2}$$

has a mild solution of type (i)-gH in space $C([0, a], \mathcal{T})$.

Proof. Consider $x \in C([0, a], \mathcal{T})$. From the continuity of x and the condition **(H1)**, it follows that $f(s, x(s))$ defines a measurable function on $[0, a]$, set

$$b = \frac{q-1}{1-q_1}, \qquad M_1 = \|m\|_{L^{\frac{1}{q_1}}[0,a]}.$$

For any t belonging to $[0, a]$, one can employ Hölder's inequality along with hypothesis **(H3)** to derive

$$\begin{aligned} &d_\infty\left(\int_0^t (t-s)^{q-1} \odot f(s, x(s))ds, \hat{0}\right) \\ &\leq \int_0^t (t-s)^{q-1} d_\infty(f(s, x(s)), \hat{0})ds \\ &\leq \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds\right)^{1-q_1} \|m\|_{L^{\frac{1}{q_1}}[0,t]} \\ &\leq \frac{M_1 a^{(1+b)(1-q_1)}}{(1+b)^{1-q_1}}. \end{aligned}$$

From the properties of the solution operators and hypothesis **(H3)**, we obtain:

$$\begin{aligned} &d_\infty\left(\int_0^t (t-s)^{q-1} \odot T_q(t-s)f(s, x(s))ds, \hat{0}\right) \\ &\leq \int_0^t (t-s)^{q-1} d_\infty(T_q(t-s)f(s, x(s)), \hat{0})ds \end{aligned} \tag{3.3}$$

$$\leq \int_0^t (t-s)^{q-1} \frac{qM}{\Gamma(q+1)} d_\infty(f(s,x(s)),\hat{0})ds$$
$$\leq \frac{qM_1Ma^{(1+b)(1-q_1)}}{\Gamma(1+q)(1+b)^{1-q_1}}$$

For every $t \in [0,a]$, the integral

$$\int_0^t (t-s)^{q-1} \odot T_q(t-s)f(s,x(s))ds$$

remains bounded.
Now for any $x \in C([0,a],\mathcal{T})$, we define

$$(F_1x)(t) = S_q(t)x_0, \quad t \in [0,a],$$
$$(F_2x)(t) = \int_0^t (t-s)^{q-1} \odot T_q(t-s) \left[\int_0^s f(\tau,x(\tau))d\tau\right] ds, \quad t \in [0,a].$$

Set

$$k_0 = \min\left\{ d_\infty(x_0,\hat{0}); Md_\infty(x_0,\hat{0}) + \frac{qM_1Ma^{(1+b)(1-q_1)}}{\Gamma(1+q)(1+b)^{1-q_1}} \right\}$$

and

$$B_{k_0} := \left\{ x(\cdot) \in C([0,a],\mathcal{T}) : d_\infty(x(t),\hat{0}) \leq k_0 \text{ for all } t \in [0,a] \right\}.$$

We aim to show that the operator $F_1x + F_2x$ possesses a fixed point within B_{k_0}
Step 1. We demonstrate that $F_1x + F_2x$ belongs to B_{k_0} for all $x \in B_{k_0}$ First, we verify that $(F_2x)(t)$ is continuous in $t \in [0,a]$. For $0 \leq t' \leq t'' \leq a$, we have

$$d_\infty\left((F_2x)\left(t''\right),(F_2x)\left(t''\right)\right)$$
$$= d_\infty\left(\int_0^{t''} \left(t''-s'\right)^{q-1} \odot T_q\left(t''-s'\right)\left[\int_0^s f(\tau,x(\tau))d\tau\right] ds,\right.$$
$$\left.\int_0^{t'} (t'-s)^{q-1} \odot T_q(t'-s)\left[\int_0^s f(\tau,x(\tau))d\tau\right] ds\right)$$
$$\leq I_1 + I_2 + I_3.$$

where

$$I_1 = d_\infty\left(\int_{t'}^{t''} \left(t''-s'\right)^{q-1} \odot T_q\left(t''-s'\right)\left[\int_0^s f(\tau,x(\tau))d\tau\right] ds, \hat{0}\right),$$
$$I_2 = d_\infty\left(\int_0^{t'} \left(t''-s'\right)^{q-1} \odot T_q\left(t''-s'\right)\left[\int_0^s f(\tau,x(\tau))d\tau\right] ds,\right.$$

$$\int_0^{t'} (t'-s)^{q-1} \odot T_q\left(t''-s'\right) \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds\Bigg),$$

$$I_3 = d_\infty \Bigg(\int_0^{t'} (t'-s)^{q-1} \odot T_q\left(t''-s'\right) \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds,$$

$$\int_0^{t'} (t'-s)^{q-1} \odot T_q(t'-s) \left[\int_0^s f(\tau, x(\tau))d\tau\right] ds\Bigg).$$

Clearly, I_1, I_2 and I_3 approach zero as $t'' - t' \to 0$. Therefore, the mapping $(F_2x)(t)$ is continuous in $t \in [0, a]$. Similarly, one can verify that $(F_1x)(t)$ is continuous on the same interval.

Consider any $x \in B_{k_0}$ and $t \in [0, a]$. Owing to the boundedness assumptions and relation (3.3), we deduce that

$$\begin{aligned} &d_\infty((F_1x)(t) + (F_2x)(t), \hat{0}) \\ &\quad \leq M d_\infty(x_0, \hat{0}) + \frac{qM_1Ma^{(1+b)(1-q_1)}}{\Gamma(1+q)(1+b)^{1-q_1}} \leq k_0. \end{aligned}$$

Then $F_1 + F_2$ is an operator from B_{k_0} into B_{k_0}.

Step 2. We demonstrate F_2 is a completely continuous operator. We begin by proving the continuity of F_2 on B_{k_0}. Consider $\{x_n\} \subseteq B_{k_0}$ with $x_n \to x$ on B_{k_0}. Using hypothesis **(H2)**, we get

$$f(s, x_n(s)) \to f(s, x(s)) \quad \text{as } n \to \infty \text{ almost everywhere } t \in [0, a].$$

Utilizing **(H3)**, one can deduce that

$$d_\infty(f(s, x_n), f(s, x(s))) \leq 2m(s).$$

So by the dominated convergence theorem and the continuity of $T_q(\cdot)$ and (19), we obtain

$$\begin{aligned} &d_\infty((F_2x_n)(t), (F_2x)(t)) \\ &\quad \leq \int_0^t (t-s)^{q-1} d_\infty \Bigg(T_q(t-s) \left[\int_0^s f(\tau, x_n(\tau))d\tau\right], \\ &\qquad T_q(t-s) \left[\int_0^s f(\tau, x(\tau))d\tau\right]\Bigg) ds \\ &\quad \leq \int_0^t (t-s)^{q-1} \frac{qM}{\Gamma(1+q)} \int_0^s d_\infty(f(\tau, x_n(\tau)), f(\tau, x(\tau)))d\tau ds \\ &\quad \to 0 \quad \text{when } n \to \infty, \end{aligned}$$

Hence F_2 is continuous. We now establish that $F_2(B_{k_0})$ is relatively compact. This is equivalent to showing that the family $\{F_2x : x \in B_{k_0}\}$ is uniformly bounded and equicontinuous and that for each $t \in [0, a]$, the set $\{(F_2x)(t) : x \in B_{k_0}\}$ is relatively compact. We have proved the

equicontinuity in **Step 1**. The proof reduces to demonstrating that, for any $t \in [0,a]$, $V(t) = \{(F_2x)(t) : x \in B_{k_0}\}$ is relatively compact. For any fixed $0 < t \leq a$, $\forall \epsilon \in (0,t)$ and $\forall \delta > 0$, we define an operator $F_{\epsilon,\delta}$ by

$$(F_{\epsilon,\delta}x)(t) = q\odot\int_0^{t-\epsilon}\int_\delta^\infty \theta\varphi_q(\theta)(t-s)^{q-1}\odot T((t-s)^q\theta)\left[\int_0^s f(\tau,x(\tau))d\tau\right]d\theta ds.$$

From hypothesis **(H4)**, $T(\epsilon^q\delta)$ is a compact operator and then $V_{\epsilon,\delta}(t) = \{(F_{\epsilon,\delta}x)(t) : x \in B_{k_0}\}$ is relatively compact. Moreover, $\forall x \in B_{k_0}$, we have

$$\begin{aligned}
&d_\infty((F_2x)(t),(F_{\epsilon,\delta}x)(t))\\
&\leq qd_\infty\left(\int_0^t\int_0^\delta \theta(t-s)^{q-1}\varphi_q(\theta)\odot T((t-s)^q\theta)\left[\int_0^s f(\tau,x(\tau))d\tau\right]d\theta ds,\hat{0}\right)\\
&\quad + qd_\infty\left(\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\varphi_q(\theta)\odot T((t-s)^q\theta)\left[\int_0^s f(\tau,x(\tau))d\tau\right]d\theta ds,\hat{0}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&d_\infty((F_2x)(t),(F_{\epsilon,\delta}x)(t))\\
&\leq qM\left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[0,t]}\int_0^\delta \theta\varphi_q(\theta)d\theta\\
&\quad + qM\left(\int_{t-\epsilon}^t (t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}[0,t]}\int_\delta^\infty \theta\varphi_q(\theta)d\theta\\
&\leq \frac{qM_1Ma^{(1+b)(1-q_1)}}{(1+b)^{1-q_1}}\int_0^\delta \theta\varphi_q(\theta)d\theta + \frac{qM_1M\epsilon^{(1+b)(1-q_1)}}{\Gamma(1+q)(1+b)^{1-q_1}}.
\end{aligned}$$

Therefore, for any $t > 0$, the existence of relatively compact sets arbitrarily approximating $V(t)$ guarantees that $V(t)$ is relatively compact. An application of the Ascoli-Arzela Theorem ensures that the set $F_2(B_{k_0})$ is relatively compact. Therefore, since F_2 is continuous and $F_2(B_{k_0})$ is relatively compact, F_2 is a completely continuous operator. Hence the Schauder fixed point theorem guarantees the existence if a fixed point for F_1+F_2 within B_{k_0}. which implies that (3.2) has a mild fuzzy solution in type i-gH □

Theorem 3.3. *Assuming hypotheses **(H1)–(H4)** and the following additional conditions:*

(G1) *The set*

$$\hat{C}([0,a],\mathcal{T}) = \left\{x \in C([0,a],\mathcal{T}) : \hat{F}[x](t) \text{ exists for all } t \in [0,a]\right\}$$

is non-empty, where

$$\hat{F}[x](t) = S_p(\psi(t)-\psi(0))x_0$$

$$\ominus_{gH}\,(-1)\odot\int_0^t[\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s)$$
$$\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds.$$

(G1) *If $x\in\hat{C}([0,a],\mathcal{T})$, then $\hat{F}[x]\in\hat{C}([0,a],\mathcal{T})$, i.e., the operator $\hat{F}$ maps the space $\hat{C}([0,a],\mathcal{T})$ into itself.*

Then, the fuzzy fractional evolution equation:

$$\begin{cases}{}^{C}D^{p,\psi}_{0,t}x(t)=Ax(t)+\displaystyle\int_0^t f(\tau,x(\tau))d\tau, & t\in J=[0,a],\\ x(0)=x_0\in\mathcal{T}.\end{cases}$$

has a mild solution of type (ii)-gH in space $C([0,a],\mathcal{T})$ given by:

$$x(t)=S_p(\psi(t)-\psi(0))x_0$$
$$\ominus_{gH}\,(-1)\odot\int_0^t[\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s)$$
$$\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds.$$

Proof. For $x\in\hat{C}([0,a],\mathcal{T})$, define the operator $\hat{F}$ by:

$$\hat{F}[x](t)=S_p(\psi(t)-\psi(0))x_0$$
$$\ominus_{gH}\,(-1)\odot\int_0^t[\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s)$$
$$\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds.$$

Equivalently, we can write:

$$\hat{F}[x](t)=(F_1x)(t)\ominus_{gH}(-1)\odot(F_2x)(t),$$

where

$$(F_1x)(t)=S_p(\psi(t)-\psi(0))x_0,\quad t\in[0,a],$$
$$(F_2x)(t)=\int_0^t[\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s)$$
$$\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds,\quad t\in[0,a].$$

Let

$$b=\frac{p-1}{1-q_1},\qquad M_1=\|m\|_{L^{\frac{1}{q_1}}[0,a]}.$$

Set

$$k_0 = \min\left\{ d_\infty(x_0, \hat{0}); M_S d_\infty(x_0, \hat{0}) + \frac{M_T \psi'_{\max} M_1 a^{(1+b)(1-q_1)}}{\Gamma(1+p)(1+b)^{1-q_1}} \right\}$$

and

$$B_{k_0} := \left\{ x(\cdot) \in \hat{C}([0,a], \mathcal{T}) : d_\infty(x(t), \hat{0}) \leq k_0 \text{ for all } t \in [0,a] \right\}.$$

Following an approach similar to the one in Theorem 3.2 (type (i)-gH case), we obtain that $(F_1x)(t) \ominus_{gH} (-1) \odot (F_2y)(t) \in B_{k_0}$ for any pair $x, y \in B_{k_0} \subset \hat{C}([0,a], \mathcal{T})$, where $(F_1x)(t)$ and $(F_2x)(t)$ are continuous in $t \in [0,a]$.

Now for any $x \in B_{k_0}$, we have

$$\begin{aligned} & d_\infty((F_1x)(t) \ominus_{gH} (-1) \odot (F_2x)(t), \hat{0}) \\ & \quad \leq d_\infty((F_1x)(t), \hat{0}) + d_\infty((F_2x)(t), \hat{0}) \\ & \quad \leq M_S d_\infty(x_0, \hat{0}) + \frac{M_T \psi'_{\max} M_1 a^{(1+b)(1-q_1)}}{\Gamma(1+p)(1+b)^{1-q_1}} \\ & \quad \leq k_0, \end{aligned}$$

which means that $\hat{F} = F_1 \ominus_{gH} (-1) \odot F_2$ is an operator from B_{k_0} into B_{k_0}.

Next, we verify the conditions for the fixed point theorem:

Step 1. Continuity and compactness of F_2

As proved in Theorem 4.1, F_2 is a completely continuous operator on B_{k_0}. Specifically:

- F_2 is continuous on B_{k_0}
- $F_2(B_{k_0})$ is relatively compact in $C([0,a], \mathcal{T})$
- The family $\{F_2x : x \in B_{k_0}\}$ is equicontinuous

Step 2. Properties of F_1

The operator F_1 is constant (does not depend on x) and given by:

$$(F_1x)(t) = S_p(\psi(t) - \psi(0))x_0.$$

Since $S_p(t)$ is continuous in t and x_0 is fixed, F_1 is trivially continuous and compact.

Step 3. Application of fixed point theorem

Since F_2 is completely continuous and F_1 is constant (hence compact), the operator $\hat{F} = F_1 \ominus_{gH} (-1) \odot F_2$ meets the requirements needed to apply the Schauder fixed point Theorem for type (ii)-gH differentiable functions, therefore $\hat{F}$ has a fixed point on B_{k_0}, Hence (3.2) has a mild fuzzy solution of type (ii)-gH.

The fixed point $x \in B_{k_0}$ satisfies:

$$x(t) = S_p(\psi(t) - \psi(0))x_0$$

$$\ominus_{gH}(-1)\odot\int_0^t[\psi(t)-\psi(s)]^{p-1}\odot T_p(\psi(t)-\psi(s))\psi'(s)$$
$$\odot\left[\int_0^s f(\tau,x(\tau))d\tau\right]ds,$$

which is exactly the mild solution formula for type (ii)-gH differentiability. This concludes the proof of Theorem 3.3 □

4. Complete Example with Explicit Theorem Applications

Consider an economic system where capital accumulation exhibits memory dependence and valuation uncertainty. The dynamics can be modeled by the fractional fuzzy integro-differential equation:

$$\begin{cases} {}^{C}\mathbb{D}_{0,t}^{0.7,t}x(t) = -0.5\odot x(t)+\displaystyle\int_0^t\frac{0.2}{1+\tau}\odot x(\tau)d\tau, \quad t\in[0,2],\\ x(0)=x_0\in\mathcal{T}, \end{cases} \tag{4.1}$$

where $x(t)$ represents the fuzzy capital stock, capturing uncertainties in asset valuation, human capital measurement and intangible resources. The initial capital endowment x_0 is a triangular fuzzy number with $\alpha-cuts$ $[1.0+0.3\alpha, 1.6-0.3\alpha]$ for $\alpha\in[0,1]$.

- The Caputo fractional derivative ${}^{C}\mathbb{D}_{0,t}^{0.7,t}x(t)$ models capital accumulation with memory persistence, reflecting how historical investment patterns influence current economic decisions.
- The term $-0.5\odot x(t)$ represents capital depreciation at 50 percent rate, accounting for physical deterioration and technological obsolescence.
- The integral term $\int_0^t\frac{0.2}{1+\tau}\odot x(\tau)d\tau$ captures institutional memory effects, where past investments generate ongoing returns through organizational learning and experience accumulation.
- The decaying kernel $\frac{0.2}{1+\tau}$ indicates that recent capital investments contribute more significantly to current productivity than older investments.

This corresponds to the abstract formulation with:

- $A=-0.5I$ (generator of a compact analytic semigroup)
- $f(t,x(t))=\dfrac{0.2}{1+t}\odot x(t)$
- $\psi(t)=t$, hence $\psi'(t)=1$
- $p=0.7$

The Cauchy problem (4.1) has a mild fuzzy solution in type 1.

(i) The function $f(t,x)=\dfrac{0.2}{1+t}\odot x$ is continuous in x for each fixed $t\in[0,2]$ and is strongly measurable in t for each fixed x.

For any $z \in C([0,2],\mathcal{T})$, the function $f(\cdot, z(\cdot))$ is also strongly measurable.

(ii) Let $B_r = \{x \in \mathcal{T} : d_\infty(x,\hat{0}) \leq r\}$. For any $x, y \in C([0,2], B_r)$:

$$\begin{aligned} d_\infty(f(t,x(t)), f(t,y(t))) &= d_\infty\left(\frac{0.2}{1+t} \odot x(t), \frac{0.2}{1+t} \odot y(t)\right) \\ &\leq \frac{0.2}{1+t} d_\infty(x(t), y(t)) \\ &= \rho(t) d_\infty(x(t), y(t)) \end{aligned}$$

where $\rho(t) = \dfrac{0.2}{1+t} \in L^{\frac{1}{q_2}}([0,2],\mathbb{R}^+)$ for any $q_2 \in [0, 0.7)$.

3. For any $x \in C([0,2],\mathcal{T})$:

$$\begin{aligned} d_\infty(f(t,x(t)),\hat{0}) &= d_\infty\left(\frac{0.2}{1+t} \odot x(t), \hat{0}\right) \\ &\leq \frac{0.2}{1+t} d_\infty(x(t),\hat{0}) \\ &\leq \frac{0.2r}{1+t} = m(t) \end{aligned}$$

where $m(t) = \dfrac{0.2r}{1+t} \in L^{\frac{1}{q_1}}([0,2],\mathbb{R}^+)$ for any $q_1 \in [0, 0.7)$.

4. The operator $A = -0.5I$ generates a strongly continuous semigroup $T(t) = e^{-0.5t}I$, which is compact for every $t > 0$.

Since all hypotheses (H1)–(H4) of Theorem 3.2 are satisfied, there exists a **type (i)-gH mild solution** given by Lemma 3.1:

$$x(t) = S_{0.7}(t)x_0 \oplus \int_0^t (t-s)^{-0.3} \odot T_{0.7}(t-s) \odot \left[\int_0^s \frac{0.2}{1+\tau} \odot x(\tau) d\tau\right] ds$$

where the solution operators are:

$$\begin{aligned} S_{0.7}(t) &= E_{0.7,1}(-0.5t^{0.7}) \\ T_{0.7}(t) &= t^{-0.3} E_{0.7,0.7}(-0.5t^{0.7}) \end{aligned}$$

The Cauchy problem (4.1) has a mild fuzzy solution in type 2.

1. As proven for Theorem 3.2, hypotheses (H1)–(H4) are satisfied.

2. The set $\hat{C}([0,2],\mathcal{T}) = \{x \in C([0,2],\mathcal{T}) : \hat{F}[x](t)$ exists for all $t \in [0,2]\}$ is non-empty. For instance, constant functions belong to this set.

3. For this linear system, the operator $\hat{F}$ maintains the gH-difference property due to the linearity and boundedness of the involved operators and the regularity of the semigroup. Thus, if $x \in \hat{C}([0,2],\mathcal{T})$, then $\hat{F}[x] \in \hat{C}([0,2],\mathcal{T})$.

Since all conditions of Theorem 3.3 are satisfied, there exists a **type (ii)-gH mild solution** given by Lemma 3.1:

$$x(t) = S_{0.7}(t)x_0 \ominus_{gH} (-1) \odot \int_0^t (t-s)^{-0.3} \odot T_{0.7}(t-s) \odot \left[\int_0^s \frac{0.2}{1+\tau} \odot x(\tau)d\tau\right] ds$$

Figure 1 shows the type (i)-gH solution, which paints a picture of stable and steady economic growth. Here, the fuzzy capital stock increases smoothly over time, with its band of uncertainty widening in a predictable way. This represents an ideal scenario where institutional knowledge and past investments consistently fuel growth, leading to a reliable expansion of capital.

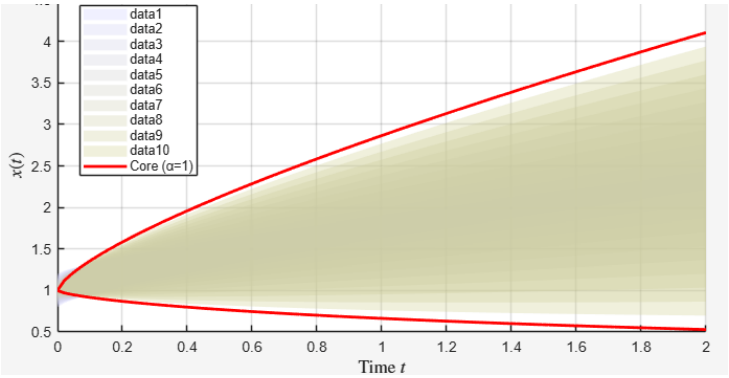

FIGURE 1. Type (i)-gH mild solution obtained via Theorem 3.2 and Lemma 3.1.

In contrast, Figure 2 reveals a more complex story with the type (ii)-gH solution. The path of capital stock is less smooth, showing periods of adjustment and fluctuation in its uncertainty bounds. This captures the reality of economies that experience sudden shifts, periodic restructuring, or phases of rapid learning, where growth is not always steady and can include temporary setbacks or bursts.

Figure 3 brings these differences into sharp focus by tracking how the uncertainty itself changes over time. The graph clearly shows that

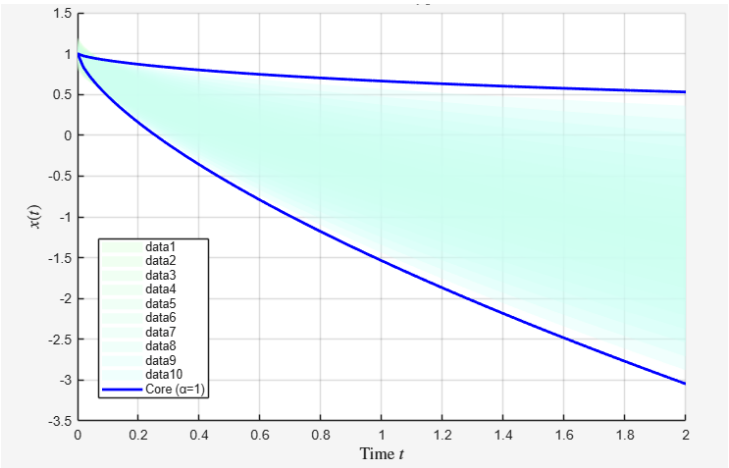

FIGURE 2. Type (ii)-gH mild solution obtained via Theorem 3.3 and Lemma 3.1.

the type (i) solution leads to a predictable, steady increase in uncertainty. The type (ii) solution, however, tells a different story: uncertainty doesn't just grow it can also contract at times, reflecting an adaptive system that re-calibrates and learns from new information, leading to a more dynamic risk profile.

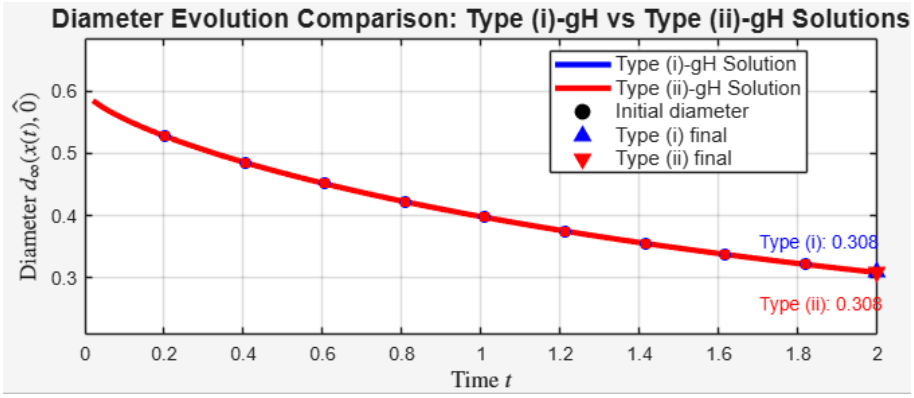

FIGURE 3. Diameter evolution comparison showing different behaviors of type (i) and type (ii) solutions.

Together, these visuals highlight a critical point: the choice of mathematical model has real-world consequences. The type (i)-gH solution is well-suited for modeling stable, mature economies with established growth patterns. Meanwhile, the type (ii)-gH solution is essential for understanding emerging or volatile economies, where change is the only

constant. Its ability to capture fluctuating uncertainty provides a more realistic and powerful tool for risk assessment and economic planning in unpredictable environments.

5. Conclusion

This paper has established a comprehensive existence theory for fuzzy fractional evolution equations incorporating a ψ-Caputo gH-derivative and a Volterra integral nonlinearity. The analysis, set in the space of triangular fuzzy numbers, required reconciling the non-locality of the fractional operator with the two-fold nature of gH-differentiability and the memory effect of the nonlinearity.

Our core results, Theorems 3.2 and 3.3, were proven using a tailored fixed-point strategy. For the (i)-gH case, the problem was decomposed to apply Krasnoselskii's theorem, while the (ii)-gH case required supplementary conditions (G1) and (G2) to ensure Schauder's theorem was applicable. The distinct solution formulas derived in Lemma 3.1 and the accompanying economic example highlight the necessity of this dual approach, as the two solution types model fundamentally different dynamical behaviors, specifically, whether uncertainty expands or contracts over time.

This work provides a foundational framework for a complex class of fuzzy fractional systems. Immediate future work will focus on establishing solution uniqueness and stability, with longer-term goals being the development of an associated optimal control theory and robust numerical methods.

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