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Sk Nazmul and Barnali Dhang

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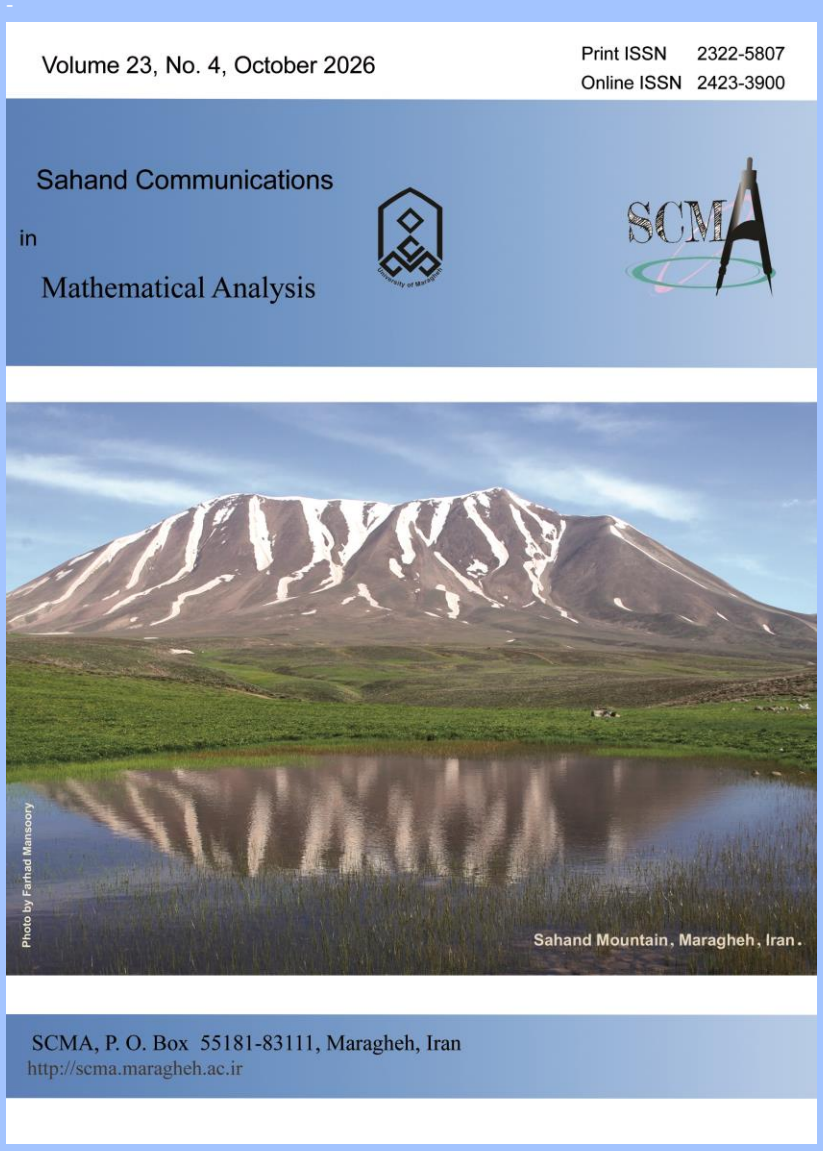

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Some Properties of Gradation of Openness of Soft Sets

Sk Nazmul[1*] and Barnali Dhang[2]

Abstract. The purpose of the current work is to analyze the fundamental behaviours of gradation of openness of soft sets, a new approach of fuzzy soft topological space, where the underlying topology is a gradation of openness of soft sets using soft elements and elementary soft operations. For this, firstly, for any real number $r \in [0, 1]$, the r-level neighbourhood system of a soft element, r-level interior, r-level open, r-level closed of a soft set etc. are defined, their basic properties and relationships including celebrated decomposition theorems are studied. Further to test continuity, openness, closedness of a soft mapping, the definitions of sgp-map, FS-open map, FS-homeomorphism are mentioned and their features are explored. Finally, subspace gradation of openness is commenced and some of their properties are discussed.

1. Introduction

In many real-world problems, uncertainty and imprecision are inherent challenges. To address these issues, various mathematical models have been developed, among which Zadeh in 1965 [22] presented the 'Fuzzy Set Theory', has gained significant attention. In 1968, Chang introduced fuzzy topology [7] by extending classical topology through fuzzy set theory. Later, in 1976, R Lowen [12] slightly modified Chang's definition, refining the structure of fuzzy topological spaces and enhancing their applicability in various mathematical and applied fields. Several generalizations of fuzzy topology have been presented [8] over time

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* Corresponding author.

to provide greater flexibility. Following the advancement in fuzzy topology, Hazra et al.[8] introduced 'gradation of openness' by considering the fuzzification of fuzzy sets.

During that period, Molodtsov [16] in 1999 gave a new mathematical strategy 'Soft Set', to handle uncertainty, vagueness and imprecise information. Next Maji et al. [13] defined several operations over soft sets and demonstrated an application in the problem of decision making. Aktas and Cagman [2] compared soft sets with fuzzy sets and rough sets, highlighting their differences in handling uncertainty. Building on this foundation, they unveiled the idea of soft group. Soft mapping (soft functions) between two soft sets is defined in [11, 14], providing a framework for studying functions in the soft set environment. As a continuation of this, researchers [3, 6, 19] have extensively worked on constructing various algebraic and analytic structures in both soft and fuzzy set settings to enhance their applicability in different mathematical and real problems. In 2018, M. R. Abed et al. [1], have introduced fuzzifying soft topological spaces and some key features are examined using soft points.

So, it is natural to extend this and to study their topological behaviours deeply. To do this, we have faced some issues and to overcome these we have considered soft elements and elementary soft operations instead of soft points and ordinary soft operations in the definition of gradation of openness of soft sets and studied the detailed topological behaviours. In this present work, firstly, for any real number $r \in [0,1]$, the r-level neighbourhood system of a soft element, r-level interior of a soft set, r-level open soft set, r-level closed soft set etc. are introduced, their basic properties and relationships including decomposition theorems are discussed. Further the definitions of elementary soft function, sgp-map, FS-open map, FS-homeomorphism are given and their features are explored. Finally, subspace gradation of openness of soft sets (SGOSS) is commenced and some of their properties including the behaviours of SGOSS of elementary soft functional image and pre image of soft sets are established.

2. Preliminaries

This part explores, following [6, 9, 11, 13, 16, 17], some core definitions and preliminary findings are outlined in our form. If not mentioned, consider X represent the initial universal set and A represent the parameters set.

Definition 2.1. F^A is termed as

(i) soft set over X, where F is a mapping from A to $2^X = P(X)$.
(ii) fuzzy soft set over X, where F is a mapping from A to I^X.

Definition 2.2. A soft set F^A over X is said to be a null soft set if for each $e \in A$, $F(e) = \phi$ and represented by $\overline{\phi}^A$ and an absolute soft set if for each $e \in A$, $F(e) = X$ and represented by $\overline{X}^A$.

Definition 2.3. Let F^A, G^A be two members of $S\left(\overline{X}^A\right)$, then their

(i) union is symbolized by $F^A \tilde{\cup} G^A$, where
$$\left[F^A \tilde{\cup} G^A\right](e) = F(e) \cup G(e), \quad \forall e \in A,$$
(ii) intersection is symbolized by $F^A \tilde{\cap} G^A$, where
$$\left[F^A \tilde{\cap} G^A\right](e) = F(e) \cap G(e), \quad \forall e \in A$$
(iii) and the complement of F^A is symbolized by $\left(F^A\right)^c$, where
$$\left(F^A\right)^c(e) = X \setminus F(e), \quad \forall e \in A.$$

Definition 2.4. A soft topology τ on X is a subcollection of $S\left(\overline{X}^A\right)$ under the succeeding criteria:

(i) $\overline{\phi}^A, \overline{X}^A \in \tau$;
(ii) $\tilde{\cap}_{i=1}^n F_i^A \in \tau$, for any finite subcollection $\left\{F_i^A\right\}_{i=1}^n$ of τ;
(iii) $\tilde{\cup}_{i\in\Delta} F_i^A \in \tau$ for any arbitrary subcollection $\left\{F_i^A\right\}_{i\in\Delta}$ of τ.

Also $\left(\overline{X}^A, \tau\right)$ is reffered as an STS (soft topological space).

Definition 2.5. A function $\tilde{\epsilon} : A \to X$ is known as a soft element of $\overline{X}^A$. $\tilde{\epsilon}$ is said to belong to $F^A \tilde{\subseteq} \overline{X}^A$, indicated by $\tilde{\epsilon} \tilde{\in} F^A$ if
$$\tilde{\epsilon}(\alpha) \in F(\alpha), \quad \forall \alpha \in A.$$
The collection of all soft elements of F^A is represented by $SE\left(F^A\right)$. Also, $\tilde{\epsilon}^A$ is the soft set containing only a single element $\tilde{\epsilon}$.

Rest of the paper, we use
$$\mathbb{S}\left(\overline{X}^A\right) = \overline{\phi}^A \cup \left\{F^A \in S\left(\overline{X}^A\right) : F(e) \neq \phi, \quad \forall e \in A\right\}$$
and $SS(\mathcal{F}) = \tilde{\cup}\{\tilde{\epsilon} \in \mathcal{F}\}$.

Definition 2.6. Let F^A and G^A are two members of $\mathbb{S}\left(\overline{X}^A\right)$. Then

(i) their elementary union is expressed using $F^A \Cup G^A$, which formulated as $F^A \Cup G^A = SS\left(SE\left(F^A\right) \cup SE\left(G^A\right)\right)$,
(ii) their elementary intersection is expressed using $F^A \Cap G^A$ which formulated as $F^A \Cap G^A = SS\left(SE\left(F^A\right) \cap SE\left(G^A\right)\right)$, and
(iii) their elementary complement of F^A is expressed using $\left(F^A\right)^{\mathbf{C}}$, which formulated as $\left(F^A\right)^{\mathbf{C}} = SS\left(\left\{\tilde{\epsilon} \tilde{\in} \overline{X}^A : \tilde{\epsilon} \tilde{\in} \left(F^A\right)^c\right\}\right)$.

Proposition 2.7. *Consider two arbitrary soft set* $F^A, G^A \in \mathbb{S}\left(\overline{X}^A\right)$,

(i) $F^A \Cup G^A = F^A \tilde{\cup} G^A$,
(ii) $F^A \Cap G^A = F^A \tilde{\cap} G^A$, *if* $F^A \Cap G^A \neq \overline{\phi}^A$ *and*
(ii) $\left(F^A\right)^{\mathbf{C}} \subseteq \left(F^A\right)^c$ *and* $\left(F^A\right)^{\mathbf{C}} = \left(F^A\right)^c$ *if* $\left(F^A\right)^{\mathbf{C}} \neq \overline{\phi}^A$.

Definition 2.8. A soft mapping $f_\varphi : \left(\overline{X}^A, \tau\right) \to \left(\overline{Y}^B, \nu\right)$ is soft continuous if $f_\varphi^{-1}\left[G^B\right] \in \tau, \forall G^B \in \nu$.

3. Gradation of Openness of Soft Sets and Some of Their Properties

Here, we've precisely describe a new notion of fuzzy soft topological space, in which the underlying topology is a gradation of openness using soft elements and elementary soft set operations and some of its momentous topological behaviers in fuzzy soft setting are discussed.

Definition 3.1. A mapping $\mathcal{T} : \mathbb{S}\left(\overline{X}^A\right) \to I$ fulfilling the criteria

(i) $\mathcal{T}\left(\overline{\phi}^A\right) = 1 = \mathcal{T}\left(\overline{X}^A\right)$;
(ii) $\mathcal{T}\left(\Cap_{i=1}^n F_i^A\right) \geq \wedge_{i=1}^n \mathcal{T}\left(F_i^A\right)$, for any finite sub collection $\left\{F_1^A, F_2^A, \ldots, F_n^A\right\}$ of $\mathbb{S}(X^A)$ and
(iii) $\mathcal{T}(\Cup_{i\in\Delta} F_i^A) \geq \wedge_{i\in\Delta}\mathcal{T}\left(F_i^A\right)$, for any sub collection $\left\{F_i^A : i \in \Delta\right\}$ of $\mathbb{S}\left(\overline{X}^A\right)$

is known as gradation of openness of soft set (briefy GOSS) on $\overline{X}^A$ and we refer to $\left(\overline{X}^A, \mathcal{T}\right)$ as a fuzzy soft topological space (briefly FSTS).

Theorem 3.2 (First Decomposition Theorem)**.** *Consider* $\mathcal{T}$ *be a GOSS on* $\overline{X}^A$. *Following that* $\mathcal{T}_r = \left\{F^A \in \mathbb{S}\left(\overline{X}^A\right) : \mathcal{T}\left(F^A\right) \geq r\right\}$ *is a soft topology on* $\overline{X}^A, \forall r \in I$.

Proof. Let r be any member of I. Since $\mathcal{T}\left(\overline{X}^A\right) = \mathcal{T}\left(\overline{\phi}^A\right) = 1 \geq r$ and hence $\overline{X}^A, \overline{\phi}^A \in \mathcal{T}_r$. Next let $\left\{F_1^A, F_2^A, \ldots, F_n^A\right\}$ be any finite subcollection of $\mathcal{T}_r$. Then $\mathcal{T}\left(F_i^A\right) \geq r, \forall i = 1, 2, \ldots, n$ and hence $\mathcal{T}\left(\Cap_{i=1}^n F_i^A\right) \geq \wedge_{i=1}^n \mathcal{T}\left(F_i^A\right) \geq r$. Thus $\Cap_{i=1}^n F_i^A \in \mathcal{T}_r$. Finally let $\left\{F_i^A : i \in \Delta\right\}$ be any subcollection of $\mathcal{T}_r$. Then $\mathcal{T}\left(F_i^A\right) \geq r, \forall i \in \Delta$ and hence $\mathcal{T}(\Cup_{i\in\Delta} F_i^A) \geq \wedge_{i\in\Delta}^n \mathcal{T}\left(F_i^A\right) \geq r$. So, $\Cup_{i\in\Delta} F_i^A \in \mathcal{T}_r$. Therefore, $\mathcal{T}_r$ is a soft topology on $\overline{X}^A$. □

Definition 3.3. For every $r \in I$, $\mathcal{T}_r$ mentioned in Proposition 3.2 is called the r-level soft topology on $\overline{X}^A$.

Remark 3.4. It is obvious that $\mathcal{T}_r \subseteq \mathcal{T}_s, \forall s \leq r$.

Theorem 3.5 (Second Decomposition Theorem)**.** *Let*

$$\left\{\left(\overline{X}^A, \mathcal{T}_r\right) : r \in [0,1]\right\}$$

be a family of soft topological spaces. Then $\mathcal{T} : \mathbb{S}\left(\overline{X}^A\right) \to I$ *be defined by*

$$\mathcal{T}\left(G^A\right) = \begin{cases} \vee\left\{r \in [0,1] : G^A \in \mathcal{T}^r\right\} \\ 0, \qquad \text{otherwise.} \end{cases}, \text{ is a GOSS on } \overline{X}^A.$$

Proof. Since, $\overline{\phi}^A, \overline{X}^A \in \mathcal{T}_r, \forall r \in [0,1]$ and hence

$$\mathcal{T}\left(\overline{X}^A\right) = \mathcal{T}\left(\overline{\phi}^A\right) = 1 \tag{3.1}$$

Next let F_1^A, F_2^A be any two member of $\mathbb{S}\left(\overline{X}^A\right)$ with $\mathcal{T}\left(F_1^A\right) = r_1$ and $\mathcal{T}\left(F_2^A\right) = r_2$. Then for any $\epsilon > 0, F_1^A \in \mathcal{T}_s, s \leq (r_1 - \epsilon)$ and $F_2^A \in \mathcal{T}_s, s \leq (r_2 - \epsilon)$. Thus, $F_1^A \Cap F_2^A \in \mathcal{T}_s, s \leq (r_1 \wedge r_2) - \epsilon$ and hence $\mathcal{T}\left[F_1^A \Cap F_2^A\right] \geq (r_1 \wedge r_2) - \epsilon$. As $\epsilon > 0$ is not restricted, it implies

$$\mathcal{T}\left[F_1^A \Cap F_2^A\right] \geq (r_1 \wedge r_2) \tag{3.2}$$

Finally let $\left\{F_i^A, i \in \Delta\right\}$ be any subcollection of $\mathbb{S}\left(\overline{X}^A\right)$ with $\mathcal{T}(F_i^A) = r_i, \forall i \in \Delta$. Then for any $\epsilon > 0, F_i^A \in \mathcal{T}_s, s \leq (r_i - \epsilon), \forall i \in \Delta$. Thus, $\Cup_{i\in\Delta} F_i^A \in \mathcal{T}_s, s \leq (\wedge_{i\in\Delta} r_i) - \epsilon$ and hence $\mathcal{T}\left[\Cup_{i\in\Delta} F_i^A\right] \geq (\wedge_{i\in\Delta} r_i) - \epsilon$. As $\epsilon > 0$ is not restricted, it implies

$$\mathcal{T}\left[\Cup F_i^A\right] \geq \wedge_{i\in\Delta} r_i \tag{3.3}$$

Therefore, $\mathcal{T}$, by (3.1), (3.2) and (3.3), is a GOSS on $\overline{X}^A$. □

Proposition 3.6. *Suppose* $\mathcal{T}$ *and* $\mathcal{T}^*$ *are two GOSSs on* $\overline{X}^A$*. Then* $(\mathcal{T} \sqcap \mathcal{T}^*)$*, where* $[\mathcal{T} \sqcap \mathcal{T}^*]\left(F^A\right) = \mathcal{T}\left(F^A\right) \wedge \mathcal{T}^*\left(F^A\right), \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$ *is also a GOSS on* $\overline{X}^A$*.*

Proof. (a) Since $\mathcal{T}\left(\overline{X}^A\right) = \mathcal{T}^*\left(\overline{X}^A\right) = \mathcal{T}\left(\overline{\phi}^A\right) = \mathcal{T}^*\left(\overline{\phi}^A\right) = 1$.

So, $(\mathcal{T} \sqcap \mathcal{T}^*)\left(\overline{X}^A\right) = 1$ and $(\mathcal{T} \sqcap \mathcal{T}^*)\left(\overline{\phi}^A\right) = 1$.

(b) Let F_1^A, F_2^A be two soft sets in $\mathbb{S}\left(\overline{X}^A\right)$. Now

$$\begin{aligned}(\mathcal{T} \sqcap \mathcal{T}^*)\left(F_1^A \Cap F_2^A\right) &= \mathcal{T}\left(F_1^A \Cap F_2^A\right) \wedge \mathcal{T}^*\left(F_1^A \Cap F_2^A\right) \\ &\geq \left[\mathcal{T}\left(F_1^A\right) \wedge \mathcal{T}\left(F_2^A\right)\right] \bigwedge \left[\mathcal{T}^*\left(F_1^A\right) \wedge \mathcal{T}^*\left(F_2^A\right)\right] \\ &= \mathcal{T}\left(F_1^A\right) \wedge \mathcal{T}^*\left(F_1^A\right) \wedge \mathcal{T}\left(F_2^A\right) \wedge \mathcal{T}^*\left(F_2^A\right)\end{aligned}$$

$$= (\mathcal{T} \sqcap \mathcal{T}^*)\left(F_1^A\right) \wedge (\mathcal{T} \sqcap \mathcal{T}^*)\left(F_2^A\right).$$

Thus, $(\mathcal{T} \sqcap \mathcal{T}^*)\left(F_1^A \Cap F_2^A\right) \geq (\mathcal{T} \sqcap \mathcal{T}^*)\left(F_1^A\right) \wedge (\mathcal{T} \sqcap \mathcal{T}^*)\left(F_2^A\right)$.

(c) Let, $F_i^A, i \in \Delta$ are soft sets in $\mathbb{S}\left(\overline{X}^A\right)$. So, $(\mathcal{T} \sqcap \mathcal{T}^*)\left(F_i^A\right) = \mathcal{T}\left(F_i^A\right) \wedge \mathcal{T}^*\left(F_i^A\right)$, for $i \in \Delta$. Now,

$$\begin{aligned}(\mathcal{T} \sqcap \mathcal{T}^*)\left(\Cup_{i\in\Delta} F_i^A\right) &= \mathcal{T}\left(\Cup_{i\in\Delta} F_i^A\right) \bigwedge \mathcal{T}^*\left(\Cup_{i\in\Delta} F_i^A\right)\\ &\geq \left[\wedge_{i\in\Delta}\mathcal{T}\left(F_i^A\right)\right] \bigwedge \left[\wedge_{i\in\Delta}\mathcal{T}^*\left(F_i^A\right)\right]\\ &= \bigwedge_{i\in\Delta}\left[\mathcal{T}\left(F_i^A\right) \wedge \mathcal{T}^*\left(F_i^A\right)\right]\\ &= \bigwedge_{i\in\Delta}(\mathcal{T} \sqcap \mathcal{T}^*)\left(F_i^A\right)\end{aligned}$$

Thus , $(\mathcal{T} \sqcap \mathcal{T}^*)\left(\Cup_{i\in\Delta} F_i^A\right) \geq \bigwedge_{i\in\Delta}(\mathcal{T} \sqcap \mathcal{T}^*)\left(F_i^A\right)$.

Therefore, from (a), (b) and (c) we get, $(\mathcal{T} \sqcap \mathcal{T}^*)$ is a GOSS on $\overline{X}^A$. □

Remark 3.7. For any two GOSSs $\mathcal{T}, \mathcal{T}^*$ on $\overline{X}^A$, $\mathcal{T} \sqcup \mathcal{T}^*$ defined by $[\mathcal{T} \sqcup \mathcal{T}^*]\left(F^A\right) = \mathcal{T}\left(F^A\right) \vee \mathcal{T}^*\left(F^A\right), \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$ may not be a GOSS in general, which is shown by the following example.

Example 3.8. Let $X = \{x, y\}$ and $A = \{\alpha_1, \alpha_2\}$.
Then $\mathbb{S}\left(\overline{X}^A\right) = \left\{\overline{X}^A, \overline{\phi}^A\right\} \cup \left\{U_i^A\right\}_{i=1}^8$. Where,

$$\begin{array}{ll} U_1(\alpha_1) = X & U_1(\alpha_2) = \{x\} \\ U_2(\alpha_1) = X & U_2(\alpha_2) = \{y\} \\ U_3(\alpha_1) = \{x\} & U_3(\alpha_2) = X \\ U_4(\alpha_1) = \{y\} & U_4(\alpha_2) = X \\ U_5(\alpha_1) = \{x\} & U_5(\alpha_2) = \{y\} \\ U_6(\alpha_1) = \{y\} & U_6(\alpha_2) = \{x\} \\ U_7(\alpha_1) = \{x\} & U_7(\alpha_2) = \{x\} \\ U_8(\alpha_1) = \{y\} & U_8(\alpha_2) = \{y\}. \end{array}$$

Now define $\mathcal{T} : \mathbb{S}\left(\overline{X}^A\right) \to I$ and $\mathcal{T}^* : \mathbb{S}\left(\overline{X}^A\right) \to I$ by

$$\begin{aligned}&\mathcal{T}\left(\overline{X}^A\right) = \mathcal{T}\left(\overline{\phi}^A\right) = \mathcal{T}\left(U_1^A\right) = \mathcal{T}\left(U_2^A\right) = 1;\\ &\mathcal{T}\left(U_4^A\right) = \mathcal{T}\left(U_6^A\right) = \mathcal{T}\left(U_8^A\right) = 1;\\ &\mathcal{T}\left(U_3^A\right) = \mathcal{T}\left(U_5^A\right) = 0.2; \mathcal{T}\left(U_7^A\right) = 0.5;\\ &\mathcal{T}^*\left(\overline{X}^A\right) = \mathcal{T}^*\left(\overline{\phi}^A\right) = \mathcal{T}^*\left(U_1^A\right) = \mathcal{T}^*\left(U_2^A\right) = \mathcal{T}^*\left(U_4^A\right) = \mathcal{T}^*\left(U_6^A\right)\\ &\qquad = \mathcal{T}^*\left(U_7^A\right) = \mathcal{T}^*\left(U_8^A\right) = 1;\end{aligned}$$

$\mathcal{T}^*\left(U_3^A\right) = \mathcal{T}^*\left(U_5^A\right) = 0.2.$

Hence $\mathcal{T}$ and $\mathcal{T}^*$ are both GOSSs on $\overline{X}^A$. But, $(\mathcal{T} \sqcup \mathcal{T}^*)\left(U_5^A \Cup U_7^A\right) = 0.2 \leq 0.5 = (\mathcal{T} \sqcup \mathcal{T}^*)\left(U_5^A\right) \wedge (\mathcal{T} \sqcup \mathcal{T}^*)\left(U_7^A\right)$. Thus, $(\mathcal{T} \sqcup \mathcal{T}^*)$ is not a GOSS on $\overline{X}^A$.

Definition 3.9. An FSTS $\left(\overline{X}^A, \mathcal{T}\right)$ is reffered to be smaller than an FSTS $\left(\overline{X}^A, \mathcal{T}^*\right)$ if $\mathcal{T}\left(F^A\right) \leq \mathcal{T}^*\left(F^A\right), \quad \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$, is expressed by $\mathcal{T} \leq \mathcal{T}^*$.

Theorem 3.10. *Let $\mathcal{T}$ and $\mathcal{T}^*$ are GOSSs on $\overline{X}^A$. When $\mathcal{T}^* \leq \mathcal{T}$ or $\mathcal{T} \leq \mathcal{T}^*$ consequently $(\mathcal{T} \sqcup \mathcal{T}^*)$ is a GOSS on $\overline{X}^A$.*

Remark 3.11. The opposite implication of the aforementioned theorem could be false. An illustration of this is below:

Example 3.12. Consider X, A and $\mathcal{T}$ are identical to those in the Example 3.8.
Now define $\mathcal{T}^* : \mathbb{S}\left(\overline{X}^A\right) \to I$ by

$$\begin{aligned}\mathcal{T}^*\left(\overline{X}^A\right) &= \mathcal{T}^*\left(\overline{\phi}^A\right) = \mathcal{T}^*\left(U_1^A\right) = \mathcal{T}^*\left(U_2^A\right) = \mathcal{T}^*\left(U_3^A\right) = \mathcal{T}^*\left(U_5^A\right)\\ &= \mathcal{T}^*\left(U_7^A\right) = 1;\\ \mathcal{T}^*\left(U_4^A\right) &= \mathcal{T}^*\left(U_6^A\right) = 0.2\\ \mathcal{T}^*\left(U_8^A\right) &= 0.5.\end{aligned}$$

So, $\mathcal{T}$ and $\mathcal{T}^*$ both are GOSS on X. Also neither $\mathcal{T} \leq \mathcal{T}^*$ nor $\mathcal{T}^* \leq \mathcal{T}$. But, $[\mathcal{T} \sqcup \mathcal{T}^*]$ is a GOSS on X since $[\mathcal{T} \sqcup \mathcal{T}^*]\left(U^A\right) = 1, \forall U^A \in \mathbb{S}\left(\overline{X}^A\right)$.

Definition 3.13. Consider $\left(\overline{X}^A, \mathcal{T}\right)$ is an FSTS, $\tilde{\epsilon}\tilde{\in}\overline{X}^A$ and $r \in I$. A soft set $F^A \in \mathbb{S}\left(\overline{X}^A\right)$ is called an r-level neighbourhood of $\tilde{\epsilon}$, if $\exists G^A \in \mathbb{S}\left(\overline{X}^A\right)$ with $\mathcal{T}\left(G^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$.

Proposition 3.14. *A soft set $G^A \in \mathbb{S}\left(\overline{X}^A\right)$ is an r-level neighbourhood of all its soft elements iff $\mathcal{T}\left(G^A\right) \geq r$.*

Proof. Let, G^A be an r-level neighbourhood of all its soft element. Then for each $\tilde{\epsilon}_i\tilde{\in}G^A, \exists F_i^A \in \mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(F_i^A\right) \geq r$ and $\tilde{\epsilon}_i\tilde{\in}F_i^A\tilde{\subseteq}G^A$. Now , $G^A = \Cup_{i\in\Delta}\tilde{\epsilon}\tilde{\subseteq} \Cup_i F_i^A\tilde{\subseteq}G^A$. So, $G^A = \Cup_{i\in\Delta}F_i^A$. Also $\mathcal{T}\left(G^A\right) = \mathcal{T}\left(\Cup_{i\in\Delta}F_i^A\right) \geq \wedge_{i\in\Delta}\mathcal{T}\left(F_i^A\right) \geq r$. Therefore, the given condition is satisfied.

Conversely let G^A be a soft set in $\mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(G^A\right) \geq r$. And $\tilde{\epsilon}\tilde{\in}G^A$ be choosen randomly. Then $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}G^A$ and $\mathcal{T}\left(G^A\right) \geq r$ and hence, G^A is an r-level-neighbourhood of $\tilde{\epsilon}$. As $\tilde{\epsilon}$ is random, we deduce that G^A is an r-level-neighbourhood of each of its soft element. □

Remark 3.15. Let $\left(\overline{X}^A, \mathcal{T}\right)$ be an FSTS, $\tilde{\epsilon}\tilde{\in}\overline{X}^A$ and $r \in I$. Then $\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$ indicates the family of all r-level neighbourhood of the soft element $\tilde{\epsilon}$ with respect to $\mathcal{T}$.

Proposition 3.16. *Let* $\left\{\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) : \tilde{\epsilon}\tilde{\in}\overline{X}^A, r \in I\right\}$ *be the r-level-neighbourhood system. Then*

(i) $\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) \neq \phi, \forall \tilde{\epsilon}\tilde{\in}\overline{X}^A, \forall r \in I$.
(ii) $\tilde{\epsilon}\tilde{\in}F^A$, $\forall F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$.
(iii) $F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$ *and* $F^A\tilde{\subseteq}G^A$ *implies that* $G^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$.
(iv) $F^A, G^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$ *implies that* $\left(F^A \Cap G^A\right) \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$.
(v) $F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$ *implies that* $\exists G^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$, *such that* $G^A\tilde{\subseteq}F^A$ *and* $G^A \in \mathcal{N}_{\mathcal{T}}^r\left(\tilde{\delta}\right), \forall \tilde{\delta}\tilde{\in}G^A$.

Proof. The varification of first three statements are clear.

(iv) Let $F^A, G^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$. Then, $\exists H_1^A, H_2^A$ in $\mathbb{S}\left(\overline{X}^A\right)$, where $\mathcal{T}\left(H_1^A\right) \geq r, \mathcal{T}\left(H_2^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}H_1^A\tilde{\subseteq}F^A, \tilde{\epsilon}\tilde{\in}H_2^A\tilde{\subseteq}G^A$. Now, $\mathcal{T}\left(H_1^A \Cap H_2^A\right) \geq \left[\mathcal{T}\left(H_1^A\right) \wedge \mathcal{T}\left(H_1^A\right)\right] \geq r$ and hence, $\tilde{\epsilon}\tilde{\in}\left(H_1^A \Cap H_2^A\right)\tilde{\subseteq}\left(F^A \Cap G^A\right)$. Therefore, $\left(F^A \Cap G^A\right) \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$, where $r \in (0, 1]$.

(v) Since $F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$. Then $\exists G^A \in \mathcal{S}\left(X^A\right)$, such that $\mathcal{T}\left(G^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$.

Also, $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}G^A$ implies that $G^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$. Thus by Proposition 3.14, G^A is an r-level neighbourhood of all its soft element. Therefore, $G^A \subseteq F^A$ and $G^A \in \mathcal{N}_{\mathcal{T}}^r\left(\tilde{\delta}\right), \forall \tilde{\delta}\tilde{\in}G^A$. □

Proposition 3.17. *Let* $\left\{\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) : \tilde{\epsilon}\tilde{\in}\overline{X}^A, r \in I\right\}$ *be the r-level neighbourhood system. Then* $\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) = \cap_{s<r}\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$.

Proof. Let $F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$. Then $\exists G^A \in \mathbb{S}\left(\overline{X}^A\right)$, such that $\mathcal{T}\left(G^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$. Hence $\mathcal{T}\left(G^A\right) \geq r > s, \forall s < r$. Thus, $F^A \in \mathcal{N}_{\mathcal{T}}^s(\tilde{\epsilon})$, and hence $\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) \subseteq \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}), \forall s < r$. Therefore, $\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}) \subseteq \cap_{s\leq r}\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$.

Again let $F^A \notin \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$. Then there does not exists any G^A in $\mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(G^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$. If possible let $F^A \in \cap_{s<r}\mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon})$ implies that $F^A \in \mathcal{N}_{\mathcal{T}}^r(\tilde{\epsilon}), \forall s < r$.

Let $r_1 = \left\{\mathcal{T}\left(H^A\right) : \tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A\right\} < r$. Then by density of real numbers, we can say $\exists r_2$ such that $r_1 < r_2 < r$. So there does not exists any $H^A \in \mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(G^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$. implies that $F^A \notin \mathcal{N}_{\mathcal{T}}^{r_2}$, which contradicts of our assumption $F^A \in \cap_{s<r}\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})$. So, $F^A \notin \cap_{s<r}\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})$. implies that $\cap_{s<r}\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon}) \subseteq \mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})$. Therefore, $\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon}) = \cap_{s<r}\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})$. □

Remark 3.18. It is visible that if $r > s$, then $\mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon}) \subseteq \mathcal{N}_{\mathcal{T}}^{s}(\tilde{\epsilon})$.

Proposition 3.19. *Suppose $\mathcal{T}$ be a GOSS on X and $\tilde{\epsilon}\tilde{\in}\overline{X}^A$. Then, $\mathcal{T}^* : \mathbb{S}\left(\overline{X}^A\right) \to I$ is define by*

$$\mathcal{T}^*\left(G^A\right) = \begin{cases} \vee\left\{r : G^A \in \mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})\right\}, & if \tilde{\epsilon} \in G^A \\ 1, & if \tilde{\epsilon}\tilde{\notin}G^A \end{cases}$$

is a GOSS on $\overline{X}^A$. Also $\mathcal{T} \leq \mathcal{T}^*$.

Proof. (a) Since $\mathcal{T}\left(\overline{X}^A\right) = 1$, so $\overline{X}^A \in \mathcal{N}_{\mathcal{T}}^{1}(\tilde{\epsilon}), \forall \tilde{\epsilon}\tilde{\in}\overline{X}^A$.

Hence, $\mathcal{T}^*\left(\overline{X}^A\right) = 1$. Also given $\mathcal{T}^*\left(\tilde{\phi}^A\right) = 1$.

(b) Let F_1^A, F_2^A be two soft set in $\mathbb{S}\left(\overline{X}^A\right)$. So,

$$\mathcal{T}\left(F_1^A \Cap F_2^A\right) \geq \wedge\left[\mathcal{T}\left(F_1^A\right), \mathcal{T}\left(F_2^A\right)\right].$$

Let $\mathcal{T}^*\left(F_1^A\right) = r_1$ and $\mathcal{T}^*\left(F_2^A\right) = r_2$. Then $F_1^A \in \mathcal{N}_{\mathcal{T}}^{s}(\tilde{\epsilon}), \forall s \leq r_1$ and $F_2^A \in \mathcal{N}_{\mathcal{T}}^{s}(\tilde{\epsilon}), \forall s \leq r_2$ respectively. Now, $F_1^A \in \mathcal{N}_{\mathcal{T}}^{r_1}(\tilde{\epsilon})$ implies that $\exists G_1^A$ in $\mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(G_1^A\right) \geq r_1$ and $\tilde{\epsilon}\tilde{\in}G_1^A\tilde{\subseteq}F_1^A$, also $F_2^A \in \mathcal{N}_{\mathcal{T}}^{r_2}(\tilde{\epsilon})$ implies that $\exists G_2^A$ in $\mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(G_2^A\right) \geq r_2$ and $\tilde{\epsilon}\tilde{\in}G_2^A\tilde{\subseteq}F_2^A$.

So, $\tilde{\epsilon}\tilde{\in}\left(G_1^A \Cap G_2^A\right)\tilde{\subseteq}\left(F_1^A \Cap F_2^A\right)$ implies

$$\mathcal{T}\left(G_1^A \Cap G_2^A\right) \geq \wedge[r_1, r_2].$$

Hence, $\left(F_1^A \Cap F_2^A\right) \in \mathcal{N}_{\mathcal{T}}^{q}(\tilde{\epsilon})$, where $q = \min(r_1, r_2)$. Hence,

$$\begin{aligned} \mathcal{T}^*\left(F_1^A \Cap F_2^A\right) &= \vee\left\{r : \left(F_1^A \Cap F_2^A\right) \in \mathcal{N}_{\mathcal{T}}^{r}(\tilde{\epsilon})\right\} \\ &\geq q \\ &= \min(r_1, r_2) \\ &= \wedge\left[\mathcal{T}^*\left(F_1^A\right), \mathcal{T}^*\left(F_2^A\right)\right]. \end{aligned}$$

(c) Let $\left\{F_i^A, i \in \Delta\right\}$ be any subfamily of $\mathbb{S}\left(\overline{X}^A\right)$ and $\mathcal{T}^*\left(F_i^A\right) = r_i$ (say), $\forall i \in \Delta$. Then $F_i^A \in \mathcal{N}_{\mathcal{T}}^{r_i}(\tilde{\epsilon}), \forall i \in \Delta$ and hence for each $i \in \Delta, \exists G_i^A$ such that $\mathcal{T}\left(G_i^A\right) \geq r_i$ and $\tilde{\epsilon}\tilde{\in}G_i^A\tilde{\subseteq}F_i^A$. Thus,

$\tilde{\epsilon}\tilde{\in}\left(\Cup_{i\in\Delta}G_i^A\right)\tilde{\subseteq}\left(\Cup_{i\in\Delta}F_i^A\right)$. Also, $\mathcal{T}\left(\Cup_{i\in\Delta}G_i^A\right)\geq\wedge_{i\in\Delta}\mathcal{T}\left(G_i^A\right)\geq\wedge_{i\in\Delta}r_i=q$(say) and hence, $\left(\Cup_{i\in\Delta}F_i^A\right)\in\mathcal{N}_{\mathcal{T}}^q\left(\tilde{\epsilon}\right)$. Thus,

$$\begin{aligned}\mathcal{T}^*\left(\Cup_{i\in\Delta}F_i^A\right)&=\vee\left\{r:\left(\Cup_{i\in\Delta}F_i^A\right)\in\mathcal{N}_{\mathcal{T}}^r\left(\tilde{\epsilon}\right)\right\}\\&\geq q\\&=\wedge_{i\in\Delta}r_i\\&=\wedge_{i\in\Delta}\mathcal{T}^*\left(F_i^A\right)\end{aligned}$$

Therefore, $\mathcal{T}^*\left(G^A\right)$ is a GOSS.

To show $\mathcal{T}\leq\mathcal{T}^*$. Let, $\mathcal{T}\left(F^A\right)=p$. Now two cases may arise.

- **Case 1:** Whenever $\tilde{\epsilon}\tilde{\notin}F^A$ then $\mathcal{T}^*\left(F^A\right)=1\geq\mathcal{T}^*\left(F^A\right)$.
- **Case 2:** Whenever $\tilde{\epsilon}\tilde{\in}F^A$. $F^A\in\mathcal{N}_{\mathcal{T}}^p\left(\tilde{\epsilon}\right)$ and hence $p\in\left\{r:F^A\in\mathcal{N}_{\mathcal{T}}^r\left(\tilde{\epsilon}\right)\right\}$. Thus

$$\mathcal{T}^*\left(F^A\right)=\vee\left\{r:F^A\in\mathcal{N}_{\mathcal{T}}^r\left(\tilde{\epsilon}\right)\right\}\geq p=\mathcal{T}\left(F^A\right).$$

Therefore, $\mathcal{T}\left(F^A\right)\leq\mathcal{T}^*\left(F^A\right),\forall F^A\in\mathbb{S}\left(\overline{X}^A\right)$ i.e. $\mathcal{T}\leq\mathcal{T}^*$. □

Remark 3.20. The above theorem shows that for an arbitrary GOSS $\mathcal{T}$ on $\overline{X}^A$, it is possible to define another GOSS $\mathcal{T}^*$ on $\overline{X}^A$ such that $\mathcal{T}\leq\mathcal{T}^*$.

Definition 3.21. Suppose $\left(\overline{X}^A,\mathcal{T}\right)$ is an FSTS and $F^A\in\mathbb{S}\left(\overline{X}^A\right)$

(i) F^A is termed as an r-level open soft set of $\mathcal{T}$ if $\mathcal{T}\left(F^A\right)\geq r$.
(ii) A soft element $\tilde{\epsilon}$ is known as an r-level interior element of F^A if $\exists G^A\in\mathbb{S}\left(\overline{X}^A\right)$, such that $\mathcal{T}\left(G^A\right)\geq r$ and $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$.
(iii) The r-level interior of F^A is represent by $\text{int}_r\left(F^A\right)$ and it is a soft set generated by the aggregation of all r-level soft interior elements of F^A.

Remark 3.22. Undoubtly, the r-level soft interior of $F^A\in\mathbb{S}\left(\overline{X}^A\right)$, is given by

$$\text{int}_r\left(F^A\right)=SS\left\{\tilde{\epsilon}\tilde{\in}F^A:\exists G^A\in\mathbb{S}\left(\overline{X}^A\right):\mathcal{T}\left(G^A\right)\geq r\text{ and }\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A\right\}.$$

Proposition 3.23. *Let* $\left(\overline{X}^A,\mathcal{T}\right)$ *be an FSTS and* $F^A,G^A\in\mathbb{S}\left(\overline{X}^A\right)$, *then,*

(i) $\text{int}_r\left(F^A\right)\tilde{\subseteq}F^A$;
(ii) $F^A\tilde{\subseteq}G^A$ *implies that* $\text{int}_r\left(F^A\right)\tilde{\subseteq}\text{int}_r\left(G^A\right)$;
(iii) $\text{int}_r\left(F^A\right)\Cup\text{int}_r\left(G^A\right)\tilde{\subseteq}\text{int}_r\left(F^A\Cup G^A\right)$;
(iv) $\text{int}_r\left(F^A\right)\Cap\text{int}_r\left(G^A\right)=\text{int}_r\left(F^A\Cap G^A\right)$.

Proof. Proving (i) and (ii) is elementary.

(iii) Let $\tilde{\epsilon}\tilde{\in}\left[\text{int}_r\left(F^A\right)\Cup\text{int}_r\left(G^A\right)\right]$ then $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\right)$ or $\text{int}_r\left(G^A\right)$. By not effecting generality let $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\right)$. Then $\exists H_1^A\in\mathbb{S}\left(\overline{X}^A\right)$, such that $\mathcal{T}\left(H_1^A\right)\geq r$ and $\tilde{\epsilon}\tilde{\in}H_1^A\tilde{\subseteq}F^A$. Now since, $F^A\tilde{\subseteq}\left(F^A\Cup G^A\right)$ implies that $\tilde{\epsilon}\tilde{\in}H_1^A\tilde{\subseteq}F^A\tilde{\subseteq}\left(F^A\Cup G^A\right)$, then $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\Cup G^A\right)$. Therefore, $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\Cup G^A\right)$.

(iv) Let $\tilde{\epsilon}\tilde{\in}\left[\text{int}_r\left(F^A\right)\Cap\text{int}_r\left(G^A\right)\right]$ then $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\right)$ and $\text{int}_r\left(G^A\right)$. implies that $\exists H_1^A\in\mathbb{S}\left(\overline{X}^A\right)$, such that $\mathcal{T}\left(H_1^A\right)\geq r$ and $\tilde{\epsilon}\tilde{\in}H_1^A\tilde{\subseteq}F^A$. Also, $\exists H_2^A\in\mathbb{S}\left(\overline{X}^A\right)$, such that $\mathcal{T}\left(H_2^A\right)\geq r$ and $\tilde{\epsilon}\tilde{\in}H_2^A\tilde{\subseteq}G^A$, implies that $\tilde{\epsilon}\tilde{\in}\left(H_1^A\Cup H_2^A\right)\tilde{\subseteq}\left(F^A\Cup G^A\right)$.

Since, $\mathcal{T}\left(H_1^A\Cap H_2^A\right)\geq\left[\mathcal{T}\left(H_1^A\right)\wedge\mathcal{T}\left(H_2^A\right)\right]=r$, then we get $\tilde{\epsilon}\tilde{\in}\text{int}_r\left(F^A\Cap G^A\right)$. Hence,

$$\left[\text{int}_r\left(F^A\right)\Cap\text{int}_r\left(G^A\right)\right]\tilde{\subseteq}\text{int}_r\left(F^A\Cap G^A\right).$$

Conversely, $\left(F^A\Cap G^A\right)\tilde{\subseteq}F^A$, then by Proposition 3.23.(ii) we possess $\text{int}_r\left(F^A\Cap G^A\right)\tilde{\subseteq}\text{int}_r\left(F^A\right)$. Similarly,$\left(F^A\Cap G^A\right)\tilde{\subseteq}G^A$, then using Proposition 3.23.(ii) we have

$$\text{int}_r\left(F^A\Cap G^A\right)\tilde{\subseteq}\text{int}_r\left(G^A\right).$$

So, $\text{int}_r\left(F^A\Cap G^A\right)\tilde{\subseteq}\left[\text{int}_r\left(F^A\right)\Cap\text{int}_r\left(G^A\right)\right]$. Therefore,

$$\text{int}_r\left(F^A\right)\Cap\text{int}_r\left(G^A\right)=\text{int}_r\left(F^A\Cap G^A\right).$$ □

Proposition 3.24. *Let $\left(\overline{X}^A,\mathcal{T}\right)$ be an FSTS. Then a soft set F^A of $\mathbb{S}\left(\overline{X}^A\right)$ is r-level open soft set iff every $\tilde{\epsilon}\tilde{\in}F^A$ is an r-level interior soft element of F^A.*

Proof. Similar as in Proposition 3.14. □

Proposition 3.25. *Let $\left(\overline{X}^A,\mathcal{T}\right)$ be an FSTS and a soft set F^A of $\mathbb{S}\left(\overline{X}^A\right)$. Then $\text{int}_r\left(F^A\right)$ is the elementary union of all r-level open soft set contained in F^A.*

It is the largest r-level open soft set in $\mathbb{S}\left(\overline{X}^A\right)$ contained in F^A.

Proof. Let $\mathcal{O}$ be a collection of all r-level open soft sets contained in F^A. Let $V^A=\Cup_{G^A\in\mathcal{O}}G^A$, then

$$\begin{aligned}\mathcal{T}\left(V^A\right)&=\mathcal{T}\left(\Cup_{G^A\in\mathcal{O}}G^A\right)\\&\geq\wedge_{G^A\in\mathcal{O}}\mathcal{T}\left(G^A\right)\\&\geq r.\end{aligned}$$

So, $\mathcal{T}\left(V^A\right) \geq r$ i.e. V^A is an r-level open soft set. Now let $\tilde{\epsilon}\tilde{\in}V^A$ then $\exists G^A \in \mathcal{O}$ such that $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}F^A$ and since $\mathcal{T}\left(G^A\right) \geq r$, $\forall G^A \in \mathcal{O}$. So, $\tilde{\epsilon}$ is an r-level interior soft element of F^A. Hence, $V^A\tilde{\subseteq}\text{int}_r\left(F^A\right)$.

Conversly let, if $\text{int}_r\left(F^A\right) = \phi$ implies that $\text{int}_r\left(F^A\right)\tilde{\subseteq}V^A$. If $\tilde{\epsilon} \in \text{int}_r\left(F^A\right)$ then $\exists H^A \in \mathbb{S}\left(\overline{X}^A\right)$ such that $\mathcal{T}\left(H^A\right) \geq r$ and $\tilde{\epsilon}\tilde{\in}H^A\tilde{\subseteq}F^A$.

Since,$\mathcal{T}\left(H^A\right) \geq r$ and $H^A\tilde{\subseteq}F^A$. So, $H^A \in \mathcal{O}$. Then, we have $\tilde{\epsilon}\tilde{\in}\mathbb{U}_{G^A\in\mathcal{O}}$ $G^A = V^A$. Thus, $\text{int}_r\left(F^A\right)\tilde{\subseteq}V^A$ and hence implies that $\text{int}_r\left(F^A\right) = \mathbb{U}_{G^A\in\mathcal{O}}G^A$. To show $\text{int}_r\left(F^A\right)$ is the largest r-level open soft set in $\mathbb{S}\left(\overline{X}^A\right)$, let K^A be an r-level open soft set such that $\text{int}_r\left(F^A\right)\tilde{\subseteq}K^A$. Since $K^A \in \mathcal{O}, K^A\tilde{\subseteq}\ \mathbb{U}_{G^A\in\mathcal{O}}\ G^A = \text{int}_r\left(F^A\right)$. Hence $\text{int}_r\left(F^A\right)$ is the largest r-level open soft set in $\mathbb{S}\left(X^A\right)$. □

Definition 3.26. Let $\left(\overline{X}^A, \mathcal{T}\right)$ be an FSTS. A soft set $F^A \in \mathbb{S}\left(\overline{X}^A\right)$ is known as r-level closed soft set on condition that $\left(F^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$ and $\mathcal{T}\left[\left(F^A\right)^c\right] \geq r$.

Proposition 3.27. *Let $\left(\overline{X}^A, \mathcal{T}\right)$ is an FSTS. Thus for every $r \in I$,*

(i) *$\overline{\phi}^A$ and $\overline{X}^A$ are r-level closed soft sets.*
(ii) *elementary intersection of arbitrary family of r-level closed soft sets is r-level closed soft set.*

Proof. (ii) Let $\left\{Y_i^A\right\}_{i\in\Delta}$ be a collection of r-level closed soft sets in $\left(\overline{X}^A, \mathcal{T}\right)$. Thus, $\left(Y_i^A\right)^c \in \mathcal{S}\left(X^A\right)$ and $\mathcal{T}\left[\left(Y_i^A\right)^C\right] \geq r, \forall i \in \Delta$. Our task is to establish that $\tilde{\cap}\left\{Y_i : i \in \Delta\right\} = Y^A$ is r-level closed soft set. If $Y^A = \overline{\phi}^A$, then Y^A is an r-level closed soft set in $\left(\overline{X}^A, \mathcal{T}\right)$. If $Y^A \neq \overline{\phi}^A$, then $\tilde{\cap}\left\{Y_i^A : i \in \Delta\right\} = \tilde{\cap}\left\{Y_i^A : i \in \Delta\right\}$. Now,

$$\begin{aligned}\left(Y^A\right)^c &= \left[\tilde{\cap}\left\{Y_i^A : i \in \Delta\right\}\right]^c \\ &= \tilde{\cup}\left\{\left(Y_i^A\right)^c : i \in \Delta\right\} \\ &= \mathbb{U}\left\{\left(Y_i^A\right)^{\mathbf{C}} : i \in \Delta\right\}.\end{aligned}$$

So,$\left(Y^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$ and

$$\begin{aligned}\mathcal{T}\left[\left(Y^A\right)^{\mathbf{C}}\right] &= \mathcal{T}\left[\left(Y^A\right)^c\right] \\ &= \mathcal{T}\left[\mathbb{U}\left\{\left(Y_i^A\right)^{\mathbf{C}} : i \in \Delta\right\}\right] \\ &\geq \wedge_{i\in\Delta}\mathcal{T}\left[\left(Y_i^A\right)^{\mathbf{C}}\right]\end{aligned}$$

$$\geq r.$$

Therefore, Y^A is an r-level closed soft set in $\left(\overline{X}^A, \mathcal{T}\right)$. □

Remark 3.28. In general, we know that finite union of closed sets is closed. But here we noticed that elementary union of finite number of r-level soft closed sets is not an r-level soft closed set usually, An illustration of this is given below.

Example 3.29. The GOSS $\mathcal{T}$ on X and soft sets $U_4^A, U_5^A, U_6^A, U_7^A, U_8^A$ are all defined in the Example 3.8.

Since, $\left(U_6^A\right)^c = U_5^A \in \mathbb{S}\left(\overline{X}^A\right)$ and $\mathcal{T}\left(\left(U_6^A\right)^{\mathbf{C}}\right) = \mathcal{T}\left(U_5^A\right) = 0.2$, follows that U_6^A is a $0.2-$level closed soft set.

Also since, $\left(U_8^A\right)^c = U_7^A \in \mathbb{S}\left(\overline{X}^A\right)$ and $\mathcal{T}\left(\left(U_8^A\right)^{\mathbf{C}}\right) = \mathcal{T}\left(U_7^A\right) = 0.5 \geq 0.2$ and hence U_8^A is 0.2-level closed soft set.

But $U_6^A \Cup U_8^A = U_4^A$ is not a 0.2-level closed soft set since $U_6^A \Cup U_8^A = U_4^A$ and $\left(U_4^A\right)^c(a) = \{x\}; \left(U_4^A\right)^c(b) = \phi$ implies that $\left(U_4^A\right)^c \notin \mathbb{S}\left(\overline{X}^A\right)$.

Proposition 3.30. *Let $\left\{Y_i^A \in \mathbb{S}\left(\overline{X}^A\right)\right\}_{i=1}^n$ be any finite family of r-level soft closed sets in $\left(\overline{X}^A, \mathcal{T}\right)$. If $\tilde{\cap}_{i=1}^n \left(Y_i^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$, then their elementary union $\Cup_{i=1}^n Y_i^A$ is also an r-level soft closed set.*

Proof. Let $\left\{Y_i^A\right\}_{i=1}^n$ be any finite family of r-level soft closed sets in $\left(\overline{X}^A, \mathcal{T}\right)$. So, $\left(Y_i^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$ and $\mathcal{T}\left(\left(Y_i^A\right)^c\right) \geq r, \forall i = 1, 2, \ldots, n$. We have to show that $\Cup_{i=1}^n Y_i^A = Y^A$ (say) is r-level soft closed set. Now,

$$\begin{aligned}\left[Y_i^A\right]^c &= \left[\Cup_{i=1}^n Y_i\right]^c \\ &= \left[\tilde{\cup}_{i=1}^n Y_i\right]^c \\ &= \tilde{\cap}_{i=1}^n \left(Y_i^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right).\end{aligned}$$

Also, $\mathcal{T}\left(Y^A\right) = \mathcal{T}\left[\Cup_{i=1}^n Y_i^A\right] \geq \bigwedge_{i=1}^n \mathcal{T}\left(Y_i^A\right) \geq r$. Therefore Y^A is an r-level soft closed set. □

Theorem 3.31. *Let $F^A \in \mathbb{S}\left(\overline{X}^A\right)$ such that $\left(F^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$. Then F^A is an r-level closed soft set if for any $\tilde{\epsilon}\tilde{\in}F^A$ and $\forall G^A \in \mathbb{S}\left(\overline{X}^A\right)$ containing $\tilde{\epsilon}$ with $\mathcal{T}\left(G^A\right) \geq r$ implies that $\exists \alpha \in A$ such that $F(\alpha) \cap [G(\alpha) \setminus \tilde{\epsilon}(\alpha)] \neq \phi$.*

Proof. Let $F^A \in \mathbb{S}\left(\overline{X}^A\right)$ such that $\left(F^A\right)^c \in \mathbb{S}\left(\overline{X}^A\right)$. If $\left(F^A\right)^c = \overline{\phi}^A$, then $F^A = \overline{X}^A$ is an r-level soft closed set. So let $\left(F^A\right)^c \left(\neq \overline{\phi}^A\right) \in$

$\mathbb{S}\left(\overline{X}^A\right)$. Then $\left(F^A\right)^{\mathbf{C}} \in \mathbb{S}\left(\overline{X}^A\right)$ and hence $\left(F^A\right)^{\mathbf{C}}$ contains soft elements. Let $\tilde{\epsilon}\tilde{\in}\left(F^A\right)^{\mathbf{C}}$ be arbitrary. Then $\tilde{\epsilon}\tilde{\notin}\left(F^A\right)$ and hence by the given condition $\exists G^A \in \mathbb{S}\left(\overline{X}^A\right)$ containing $\tilde{\epsilon}$ with $\mathcal{T}\left(G^A\right) \geq r$ such that $F(\alpha) \cap [G(\alpha) \setminus \tilde{\epsilon}(\alpha)] = \phi, \forall \alpha \in A$. Again since $\tilde{\epsilon}\tilde{\in}\left(F^A\right)^{\mathbf{C}}$, follows that $\tilde{\epsilon}(\alpha) \in F^{\mathbf{C}}(\alpha), \forall \alpha \in A$ and hence $\tilde{\epsilon}(\alpha) \notin F(\alpha), \forall \alpha \in A$. Thus, $F(\alpha) \cap G(\alpha) = \phi, \forall \alpha \in A$ and hence $G(\alpha) \subseteq F^{\mathbf{C}}(\alpha), \forall \alpha \in A$. So, $G^A\tilde{\subseteq}\left(F^A\right)^{\mathbf{C}}$ and hence, $\tilde{\epsilon}\tilde{\in}G^A\tilde{\subseteq}\left(F^A\right)^{\mathbf{C}}$. Thus, $\left(F^A\right)^{\mathbf{C}}$ is an r-level neighbourhood of soft element $\tilde{\epsilon}$. Since $\tilde{\epsilon}\tilde{\in}\left(F^A\right)^{\mathbf{C}}$ is arbitrary, follows that $\left(F^A\right)^{\mathbf{C}}$ is an r-level neighbourhood of all of its soft elements and hence by the Proposition 3.14 we get $\mathcal{T}\left[\left(F^A\right)^{\mathbf{C}}\right] \geq r$.

Hence, F^A is an r-level closed soft set. □

Definition 3.32. Let $\left(\overline{X}^A, \mathcal{T}\right)$ is an FSTS and $F^A \in \mathbb{S}\left(\overline{X}^A\right)$. Then the r-level soft closure of F^A, represented by $\mathcal{CL}_r\left(F^A\right)$ and is defined by the elementary intersection of all r-level closed soft sets containing F^A.

Clearly, $\mathcal{CL}_r\left(F^A\right)$ is the smallest r-level closed soft set containing F^A.

Proposition 3.33. *Let* $\left(\overline{X}^A, \mathcal{T}\right)$ *is an FSTS and* $F^A, G^A \in \mathbb{S}\left(\overline{X}^A\right)$. *Then*

(i) $\mathcal{CL}_r\left(\overline{\phi}^A\right) = \overline{\phi}^A$ *and* $\mathcal{CL}_r\left(\overline{X}^A\right) = \overline{X}^A$;
(ii) $F^A\tilde{\subseteq}\mathcal{CL}_r\left(F^A\right)$;
(iii) F^A *is an r-level soft closed set iff* $F^A = \mathcal{CL}_r\left(F^A\right)$;
(iv) $\mathcal{CL}_r\left(\mathcal{CL}_r\left(F^A\right)\right) = \mathcal{CL}_r\left(F^A\right)$;
(v) *if* $F^A\tilde{\subseteq}G^A$ *implies that* $\mathcal{CL}_r\left(F^A\right)\tilde{\subseteq}\mathcal{CL}_r\left(G^A\right)$;
(vi) $\left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right]\tilde{\subseteq}\mathcal{CL}_r\left(F^A \Cup G^A\right)$;
equality holds if $\left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right]^c \in \mathbb{S}\left(\overline{X}^A\right)$.
(vii) $\mathcal{CL}_r\left(F^A \Cap G^A\right) \tilde{\subseteq} \left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right]$.

Proof. (iii) Let F^A be an r-level closed soft set. Then F^A is an r-level closed soft set consisting F^A and hence

$$F^A \in \left\{G^A : G^A \text{ is an r-level closed set, containing } F^A\right\}$$

Thus, $\mathcal{CL}_r\left(F^A\right) = \Cap\left\{G^A : G^A \text{ is an r-level closed set, } F^A\tilde{\subseteq}G^A\right\}$ i.e, $\mathcal{CL}_r\left(F^A\right)\tilde{\subseteq}F^A$. Also by (ii), $F^A\tilde{\subseteq}\mathcal{CL}_r\left(F^A\right)$. Therefore, $\mathcal{CL}_r\left(F^A\right) = F^A$. Conversly, let $\mathcal{CL}_r\left(F^A\right) = F^A$. Since, $\mathcal{CL}_r\left(F^A\right)$ is elementary intersection of all r-level closed soft set consisting F^A and arbitrary elementary intersection of r-level closed soft

set is r-level closed soft set, follows that $\mathcal{CL}_r\left(F^A\right) = F^A$ is an r-level closed soft set.

(iv) Proofs follows from (iii)

(v) Suppose $F^A \tilde{\subseteq} G^A$. Let $\mathcal{F}, \mathcal{G}$ be the collections of all r-level closed soft sets containing F^A, G^A respectively. Then clearly $\mathcal{G} \subseteq \mathcal{F}$ and hence

$$\begin{aligned}\mathcal{CL}_r\left(F^A\right) &= \Cap_{H^A \in \mathcal{F}} H^A \tilde{\subseteq} \Cap_{H^A \in \mathcal{G}} H^A \\ &= \mathcal{CL}_r\left(G^A\right).\end{aligned}$$

(vi) Since, $F^A \tilde{\subseteq} \left(F^A \Cup G^A\right)$ and $G^A \tilde{\subseteq} \left(F^A \Cup G^A\right)$, and hence by (v) $\mathcal{CL}_r\left(F^A\right) \tilde{\subseteq} \mathcal{CL}_r\left(F^A \Cup G^A\right)$ and $\mathcal{CL}_r\left(G^A\right) \tilde{\subseteq} \mathcal{CL}_r\left(F^A \Cup G^A\right)$. Thus, $\left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right] \tilde{\subseteq} \mathcal{CL}_r\left(F^A \Cup G^A\right)$.

Now, $F^A \tilde{\subseteq} \mathcal{CL}_r\left(F^A\right)$ and $F^A \tilde{\subseteq} \mathcal{CL}_r\left(F^A\right)$ so

$$F^A \Cup G^A \tilde{\subseteq} \mathcal{CL}_r\left(F^A \Cup G^A\right).$$

Again,

$$\begin{aligned}\mathcal{T}\left[\left(\mathcal{CL}_r\left(F^A \Cup G^A\right)\right)^c\right] &= \mathcal{T}\left[\mathcal{CL}_r\left(F^A\right)^c \Cup \mathcal{CL}_r\left(G^A\right)^c\right] \\ &\geq \mathcal{T}\left[\mathcal{CL}_r\left(F^A\right)\right]^c \bigwedge \mathcal{T}\left[\mathcal{CL}_r\left(G^A\right)^c\right] \\ &\geq r\end{aligned}$$

Since, $\left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right]^c \in \mathbb{S}\left(X^A\right)$. This implies the soft set,$\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)$ is an r-level closed soft set containing $F^A \Cup G^A$ and $\mathcal{CL}_r\left(F^A \Cup G^A\right)$ is an intersection of all r-level closed soft set containing $F^A \Cup G^A$. So,

$$\mathcal{CL}_r\left(F^A \Cup G^A\right) \tilde{\subseteq} \mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right).$$

(vii) Since, $\left(F^A \Cap G^A\right) \tilde{\subseteq} \left(F^A\right)$ and $\left(F^A \Cap G^A\right) \tilde{\subseteq} \left(G^A\right)$ and hence by (v) $\mathcal{CL}_r\left(F^A \Cap G^A\right) \tilde{\subseteq} \mathcal{CL}_r\left(F^A\right)$ and $\mathcal{CL}_r\left(F^A \Cap G^A\right) \tilde{\subseteq} \mathcal{CL}_r\left(G^A\right)$. Thus $\mathcal{CL}_r\left[\left(F^A\right) \Cap \left(G^A\right)\right] \tilde{\subseteq} \left[\mathcal{CL}_r\left(F^A\right) \Cap \mathcal{CL}_r\left(F^A\right)\right]$. □

Remark 3.34. $\mathcal{CL}_r\left[\left(F^A\right) \Cap \left(G^A\right)\right] = \left[\mathcal{CL}_r\left(F^A\right) \Cap \mathcal{CL}_r\left(F^A\right)\right]$ is not true in general, which is shown by the following example:

Example 3.35. Here is an example where,

$$\left[\mathcal{CL}_r\left(F^A\right) \Cup \mathcal{CL}_r\left(G^A\right)\right] \neq \mathcal{CL}_r\left(F^A \Cup G^A\right).$$

Consider $X = \{m, n, p\}, A = \{\alpha_1, \alpha_2\}$ and $\mathcal{T} : \mathbb{S}\left(\overline{X}^A\right) \to I$ is define by $\mathcal{T}\left(\overline{\phi}^A\right) = \mathcal{T}\left(\overline{X}^A\right) = 1, \mathcal{T}\left(F^A\right) = 0.5, \mathcal{T}\left(G^A\right) = 0.5$ where, $F(\alpha_1) = \{m, p\}, F(\alpha_2) = \{n, p\}$ and $G(\alpha_1) = \{m, n\}, F(\alpha_2) = \{m\}$. Then $\mathcal{T}$ is a GOSS over X. Let $U^A, V^A \in \mathbb{S}\left(\overline{X}^A\right)$ given that $U(\alpha_1) =$

$\{n\}, U(\alpha_2) = \{m\}$ and $V(\alpha_1) = \{p\}, V(\alpha_2) = \{n, p\}$. Then U^A, V^A both are 0.5-level closed sets in $\left(\overline{X}^A, \mathcal{T}\right)$. So, $\mathcal{CL}_{0.5}\left(U^A\right) = U^A$, and $\mathcal{CL}_{0.5}\left(V^A\right) = V^A$. Now, $\mathcal{CL}_{0.5}\left(U^A\right) \Cup \mathcal{CL}_{0.5}\left(V^A\right) = \{\{n, p\}, \{m, n, p\}\}$, but $\mathcal{CL}_{0.5}\left(U^A \Cup V^A\right) = \overline{X}^A$. Therefore, from this example we get

$$\left[\mathcal{CL}_{0.5}\left(U^A\right) \Cup \mathcal{CL}_{0.5}\left(V^A\right)\right] \neq \mathcal{CL}_{0.5}\left(U^A \Cup V^A\right).$$

4. Elementary Soft Function and Its Properties

In this part, we have formulated elementary soft function, sgp map, FS open map etc. and their properties are discussed.

Definition 4.1. Suppose X, Y being non empty sets and A, B being parameters sets, $\{f_\alpha : X \to Y, \alpha \in A\}$ be a family of functions and $\varphi : A \to B$ be a bijective function. Then $f_\varphi : \overline{X}^A \to \overline{Y}^B$, defined by $\left[f_\varphi(\tilde{x})\right](\beta) = f_{\varphi^{-1}(\beta)}\left[\tilde{x}\left(\varphi^{-1}(\beta)\right)\right], \forall \tilde{x} \tilde{\in} \overline{X}^A, \forall \beta \in B$ is called an elementary soft function. It is clear to see that,

(i) $f_\varphi\left(F^A\right) = SS\left\{f_\varphi\left(SE\left(F^A\right)\right)\right\}, \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$ and

(ii) $f_\varphi^{-1}\left(G^B\right) = SS\left\{f_\varphi^{-1}\left(SE\left(G^B\right)\right)\right\}, \forall G^B \in \mathbb{S}\left(\overline{Y}^B\right)$.

Definition 4.2. Let $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ be an elementry soft function, where $\{f_\alpha : X \to Y, \alpha \in A\}$ be the corresponding family of functions . Then f_φ is described as

(i) injective if $\tilde{x} \neq \tilde{y}$ implies that $f_\varphi(\tilde{x}) \neq f_\varphi(\tilde{y})$,

(ii) surjective if $f_\varphi\left(\overline{X}^A\right) = \overline{Y}^B$, and (iii)

(iii) bijective if f_φ is injective and surjective.

Proposition 4.3. *Suppose $\left(\overline{X}^A, \mathcal{T}\right)$ and $\left(\overline{Y}^B, \mathcal{U}\right)$ are FSTS and $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ be an elementry soft function, where the corresponding family of functions is $\{f_\alpha : X \to Y, \alpha \in A\}$. Then*

(i) $F^A \tilde{\subseteq} f_\varphi^{-1}\left(f_\varphi\left(F^A\right)\right), \forall F^A \tilde{\subseteq} \left(\overline{X}, A\right)$, *equality hold if f_φ is injective.*

(ii) $f_\varphi\left(f_\varphi^{-1}\left(G^B\right)\right) \tilde{\subseteq} G^B, \forall G^B \tilde{\subseteq} \left(\overline{Y}, A\right)$, *equality hold if f_φ is surjective.*

Definition 4.4. Let $\left(\overline{X}^A, \mathcal{T}\right)$ and $\left(\overline{Y}^B, \mathcal{U}\right)$ be two FSTSs. Then the elementary soft mapping $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ is called a soft gradation preserving map (shortly sgp-map) if

$$\mathcal{T}\left[f_\varphi^{-1}\left(G^B\right)\right] \geq \mathcal{U}\left(G^B\right), \quad \forall G^B \in \mathbb{S}\left(\overline{Y}^B\right).$$

Proposition 4.5. *Let* $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ *be an elementry soft function associated with the family of functions* $\{f_\alpha : X \to Y, \alpha \in A\}$. *Then the following conditions are equivalent:*

(i) $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ *is an sgp-map.*

(ii) $f_\varphi : \left(\overline{X}^A, \mathcal{T}_r\right) \to \left(\overline{Y}^B, \mathcal{U}_r\right)$ *is soft continuous,* $\forall r \in [0,1]$.

(iii) *for every* $W^B \in \mathbb{S}\left(\overline{Y}^B\right)$ *with* $\mathcal{U}\left(W^B\right) \geq r$ *implies that* $\mathcal{T}\left[f_\varphi^{-1}\left(W^B\right)\right] \geq r$.

(iv) *for each* $\tilde{\epsilon_0} \tilde{\in} \overline{X}^A$ *and for all* $W^B \tilde{\subseteq} \overline{Y}^B$ *containing* $f_\varphi\left(\tilde{\epsilon_0}\right)$ *with* $\mathcal{U}\left(W^B\right) \geq r, \exists U^A \tilde{\subseteq} \overline{X}^A$ *containing* $\tilde{\epsilon_0}$ *with* $\mathcal{T}\left(U^A\right) \geq r$ *and* $f_\varphi\left(U^A\right) \tilde{\subseteq} W^B$.

Proof. First consider $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ is an sgp-map. Let $r \in [0,1]$ and $G^B \in \mathcal{U}_r$. Then $\mathcal{U}\left(G^B\right) \geq r$. Since f_φ is an sgp-map, follows that $\mathcal{T}\left[f_\varphi^{-1}\left(G^B\right)\right] \geq \mathcal{U}\left(G^B\right) \geq r$ and hence $f_\varphi^{-1}\left(G^B\right) \in \mathcal{T}_r$. Therefore, $f_\varphi : \left(\overline{X}^A, \mathcal{T}_r\right) \to \left(\overline{Y}^B, \mathcal{U}_r\right)$ is soft continuous and (i) implies (ii).

Also (ii) implies (iii) is obvious.

Next let the condition (iii) holds. Let $\tilde{\epsilon_0}$ be any soft element of $\overline{X}^A$ and W^B be any soft subset of $\overline{Y}^B$ containing $f_\varphi\left(\tilde{\epsilon_0}\right)$ with $\mathcal{U}\left(W^B\right) \geq r$. Then by the given condition, $\mathcal{T}\left[f_\varphi^{-1}\left(W^B\right)\right] \geq r$ and hence $f_\varphi^{-1}\left(W^B\right) \in \mathcal{T}_r$. Let $U^A = f_\varphi^{-1}\left(W^B\right)$. Then $U^A \tilde{\subseteq} \overline{X}^A, \tilde{\epsilon_0} \tilde{\in} f_\varphi^{-1}\left(W^B\right) = U^A$ with $\mathcal{T}\left(U^A\right) \geq r$ and $f_\varphi\left(U^A\right) = f_\varphi\left(f_\varphi^{-1}\left(W^B\right)\right) \tilde{\subseteq} W^B$ by Proposition 4.3. Thus (iii) implies (iv).

Finally let the condition (iv) holds. Let W^B be any member of $\mathbb{S}\left(\overline{Y}^B\right)$ and $\mathcal{U}\left(W^B\right) = r$. If $W^B = \overline{\phi}^B$, then $\mathcal{T}\left(f_\varphi^{-1}\left(W^B\right)\right) = \mathcal{T}\left(\overline{\phi}^A\right) = 1 \geq r = \mathcal{U}\left(W^B\right)$ and hence the condition (i) is true. So let $W^B \neq \overline{\phi}^B$. Then $f_\varphi^{-1}\left(W^B\right) \neq \overline{\phi}^A$. Let $\tilde{\epsilon_0} \tilde{\in} f_\varphi^{-1}\left(W^B\right)$. Then $f\left(\tilde{\epsilon_0}\right) \tilde{\in} W^B$ and hence by the given condition, $\exists U^A$ containing $\tilde{\epsilon_0}$ with $\mathcal{T}\left(U^A\right) \geq r$ and $f_\varphi\left(U^A\right) \tilde{\subseteq} W^B$.

Let $\mathcal{F} = \left\{U^A : \tilde{\epsilon_0} \tilde{\in} U^A \in \mathbb{S}\left(\overline{X}^A\right), \mathcal{T}\left(U^A\right) \geq r \text{ and } f_\varphi\left(U^A\right) \tilde{\subseteq} W^B\right\}$. Then clearly $\mathcal{F}$ is non-empty. We now show that $f_\varphi^{-1}\left(W^B\right) = \Cup_{U^A \in \mathcal{F}} U^A$. Since $f_\varphi\left(U^A\right) \tilde{\subseteq} W^B, \forall U^A \in \mathcal{F}$ and hence

$$U^A \tilde{\subseteq} f_\varphi^{-1}\left(f_\varphi\left(U^A\right)\right) \tilde{\subseteq} f_\varphi^{-1}\left(w^B\right), \quad \forall U^A \in \mathcal{F}.$$

Thus,

$$\Cup_{U^A \in \mathcal{F}} U^A \tilde{\subseteq} f_\varphi^{-1}\left(W^B\right). \tag{4.1}$$

Again let, $\tilde{\epsilon}\tilde{\in}f_{\varphi}^{-1}\left(W^{B}\right)$, then $f\left(\tilde{\epsilon}\right)\tilde{\in}W^{B}$ and hence by(iv), $\exists U^{A}\in\mathbb{S}\left(\overline{X}^{A}\right)$ containing $\tilde{\epsilon_{0}}$ with $\mathcal{T}\left(U^{A}\right)\geq r$ and $f_{\varphi}\left(U^{A}\right)\tilde{\subseteq}W^{B}$. So, $U^{A}\in\mathcal{F}$and$\tilde{\epsilon_{0}}\tilde{\in}U^{A}$ and hence $\tilde{\epsilon_{0}}\tilde{\in}\Cup_{U^{A}\in\mathcal{F}}U^{A}$. Thus

$$f_{\varphi}^{-1}\left(W^{B}\right)\tilde{\subseteq}\Cup_{U^{A}\in\mathcal{F}}U^{A} \tag{4.2}$$

Therefore, $f_{\varphi}^{-1}\left(W^{B}\right)=\Cup_{U^{A}\in\mu}U^{A}$. Now,

$$\begin{aligned}\mathcal{T}\left[f_{\varphi}^{-1}\left(W^{B}\right)\right] &= \mathcal{T}\left[\Cup_{U^{A}\in\mu}U^{A}\right]\\ &\geq \wedge_{U^{A}\in\mu}\mathcal{T}\left(U^{A}\right)\\ &\geq r\\ &= \mathcal{U}\left(W^{B}\right).\end{aligned}$$

Thus, $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ is an sgp-map and (iv) implies (i). Therefore, the above conditions are equivalent. □

Proposition 4.6. *Let $\mathcal{T},\mathcal{U}$ be two GOSSs on $\overline{X}^{A},\overline{Y}^{B}$ defined in Theorem 3.5 by the families of soft topological spaces $\left\{\left(\overline{X}^{A},\mathcal{T}_{r}\right):r\in[0,1]\right\}$, $\left\{\left(\overline{Y}^{B},\mathcal{U}_{r}\right):r\in[0,1]\right\}$ respectively. Then the elementary soft mapping $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ is an sgp map iff the elementary soft mapping $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}_{r}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}_{r}\right)$ is soft continuous $\forall r\in[0,1]$.*

Proof. First let the elementary soft mapping $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ is an sgp map. Let r be any real number in $[0,1]$ and $G^{B}\in\mathcal{U}_{r}$ .Then $\mathcal{U}\left(G^{B}\right)\geq r$. Since $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ is an sgp map, follows that $\mathcal{T}\left[f_{\varphi}^{-1}\left(G^{B}\right)\right]\geq\mathcal{U}\left(G^{B}\right)\geq r$ and hence $f_{\varphi}^{-1}\left(G^{B}\right)\in\mathcal{T}_{r}$. Thus, $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}_{r}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}_{r}\right)$ is soft continuous $\forall r\in[0,1]$.

Conversly, let $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}_{r}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}_{r}\right)$ is soft continuous map $\forall r\in[0,1]$. Let $G^{B}\in\mathbb{S}\left(Y^{B}\right)$ and $\mathcal{U}\left(G^{B}\right)=r$ for some $r\in[0,1]$. Then $G^{B}\in\mathcal{U}_{r}$. Since $f_{\varphi}:(X,A,\mathcal{T}_{r})\rightarrow(Y,B,\mathcal{U}_{r})$ is soft continuous, follows that $f_{\varphi}^{-1}\left(G^{B}\right)\in\mathcal{T}_{r}$ and hence $\mathcal{T}\left[f_{\varphi}^{-1}\left(G^{B}\right)\right]\geq r=\mathcal{U}\left(G^{B}\right)$.

Therefore, $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ is an sgp map. □

Proposition 4.7. *Let $\left(\overline{X}^{A},\mathcal{T}\right),\left(\overline{Y}^{B},\mathcal{U}\right),\left(\overline{Z}^{C},\mathcal{V}\right)$ be FSTSs where $\mathcal{T},\mathcal{U}$ and $\mathcal{V}$ are GOSSs on $\overline{X}^{A},\overline{Y}^{B}$ and $\overline{Z}^{C}$ respectively. If the elementary soft mappings $f_{\varphi}:\left(\overline{X}^{A},\mathcal{T}\right)\rightarrow\left(\overline{Y}^{B},\mathcal{U}\right)$ and $g_{\psi}:\left(\overline{Y}^{B},\mathcal{U}\right)\rightarrow$*

$\left(\overline{Z}^C, \mathcal{V}\right)$ *are sgp maps, then the elementary soft mapping* $g_\psi \circ f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Z}^C, \mathcal{V}\right)$ *is an sgp map.*

Proof. Let $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$, $g_\psi : \left(\overline{Y}^B, \mathcal{U}\right) \to \left(\overline{Z}^C, \mathcal{V}\right)$ are sgp-maps and $H^C \in \mathbb{S}\left(\overline{Z}^C\right)$. Since $g_\psi : \left(\overline{Y}^B, \mathcal{U}\right) \to \left(\overline{Z}^C, \mathcal{V}\right)$ is an sgp-map, follows that

$$\mathcal{U}\left(g_\psi^{-1}\left(H^C\right)\right) \geq \mathcal{V}\left(H^C\right) \tag{4.3}$$

Again since $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ is an sgp-map and $g_\psi^{-1}\left(H^C\right) \in \mathbb{S}\left(\overline{Y}^B\right)$, implies that

$$\mathcal{T}\left[f_\varphi^{-1}\left[g_\psi^{-1}\left(H^C\right)\right]\right] \geq \mathcal{U}\left[g_\psi^{-1}\left(H\right)\right] \tag{4.4}$$

Thus, by (4.3) and (4.4), we have

$$\begin{aligned}\mathcal{T}\left[\left[g_\psi \circ f_\varphi\right]^{-1}\left(H^C\right)\right] &= \mathcal{T}\left[f_\varphi^{-1}\left[g_\psi^{-1}\left(H^C\right)\right]\right] \\ &\geq \mathcal{V}\left(H^C\right).\end{aligned}$$

Therefore, $g_\psi \circ f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Z}^C, \mathcal{V}\right)$ is an sgp map. □

Definition 4.8. Assume $\left(\overline{X}^A, \mathcal{T}\right)$ and $\left(\overline{Y}^B, \mathcal{U}\right)$ are FSTSs. An elementary soft mapping $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ is called

(i) FS-open map if $\mathcal{U}\left[f_\varphi\left(F^A\right)\right] \geq \mathcal{T}\left(F^A\right), \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$.
(ii) FS-homeomorphism if f_φ is a bijective and $f_\varphi, f_\varphi^{-1}$ are both sgp-maps.

Proposition 4.9. *Let* $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{U}\right)$ *be a bijective elementary soft mappings. The following conditions are equivalent:*

(i) f_φ *is an FS-homeomorphism.*
(ii) $f_\varphi, f_\varphi^{-1}$ *are sgp-map.*
(iii) f_φ *is FS-open map and sgp-map.*
(iv) $\mathcal{T}\left(F^A\right) = \mathcal{U}\left[f_\varphi\left(F^A\right)\right], \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$.

Proof. From the definition of FS-homeomorphism, (i) implies (ii) is obvious. Now let the condition (ii) holds and $F^A \in \mathbb{S}\left(\overline{X}^A\right)$. Since, f_φ is a bijective, follows that $\left(f_\varphi^{-1}\right)^{-1}\left(F^A\right) = f_\varphi\left(F^A\right)$. Also since f_φ^{-1} is sgp-map, implies that $\mathcal{U}\left[\left(f_\varphi^{-1}\right)^{-1}\left(F^A\right)\right] \geq \mathcal{T}\left(F^A\right)$ and hence

$\mathcal{U}\left[f_\varphi\left(F^A\right)\right] \geq \mathcal{T}\left(F^A\right)$. Thus, f_φ is an FS-open map and hence (ii) implies (iii).

Next let the condition (iii)holds and $F^A \in \mathbb{S}\left(\overline{X}^A\right)$. Since f_φ is an FS-open, follows that

$$\mathcal{U}\left[f_\varphi\left(F^A\right)\right] \geq \mathcal{T}\left(F^A\right) \tag{4.5}$$

Again since $f_\varphi\left(F^A\right) \in \mathbb{S}\left(\overline{Y}^B\right)$ and f_φ is sgp-map, follows that

$$\begin{aligned} \mathcal{T}\left(F^A\right) &= \mathcal{T}\left[f_\varphi^{-1}\left[f_\varphi\left(F^A\right)\right]\right] \\ &\geq \mathcal{U}\left[f_\varphi\left(F^A\right)\right] \end{aligned} \tag{4.6}$$

Therefore, by (4.5) and (4.6), we get that

$$\mathcal{T}\left(F^A\right) = \mathcal{U}\left[f_\varphi\left(F^A\right)\right], \quad \forall F^A \in \mathbb{S}\left(\overline{X}^A\right)$$

and hence (iii) implies (iv). Finally let the condition(iv) holds. Let $G^B \in \mathbb{S}\left(\overline{Y}^B\right)$. Then

$$\begin{aligned} \mathcal{T}\left[f_\varphi^{-1}\left(G^B\right)\right] &= \mathcal{U}\left[f_\varphi\left(f_\varphi^{-1}\left(G^B\right)\right)\right] \\ &= \mathcal{U}\left(G^B\right), \quad \forall G^B \in \mathbb{S}\left(\overline{Y}^B\right) \end{aligned}$$

and hence, f_φ is an sgp-map. Again let $F^A \in \mathbb{S}\left(\overline{X}^A\right)$. Since f_φ is bijective, follows that $f_\varphi\left[f_\varphi^{-1}\left(F^A\right)\right] = F^A$ and hence by the given condition,

$$\begin{aligned} \mathcal{U}\left[\left[f_\varphi^{-1}\right]^{-1}\left(F^A\right)\right] &= \mathcal{U}\left[f_\varphi\left(F^A\right)\right] \\ &= \mathcal{T}\left(F^A\right). \end{aligned}$$

Thus f_φ^{-1} is an sgp-map. Since, f_φ and f_φ^{-1} both are sgp map, we get f_φ is an FS-homeomorphism and hence (iv) implies (i).

Therefore, the above conditions are equivalent. □

5. Subspace Gradation of Openness of Soft Sets and Their Properties

The preceding section covers subspace gradation of openness of soft sets and some key characteristics are discussed.

Definition 5.1. Let $\mathcal{T}$ be a GOSS on $\overline{X}^A$ and $F^A\left(\neq \overline{\phi}^A\right) \in \mathbb{S}\left(\overline{X}^A\right)$. A mapping $\mathcal{T}^{F^A} : \mathbb{S}\left(F^A\right) \to I$ is said to be a subspace gradation of openness of soft sets (briefly SGOSS) of $\mathcal{T}$ induced by F^A if $\mathcal{T}^{F^A}$ satisfies the following conditions.

(i) $\mathcal{T}^{F^A}\left(G^A \Cap F^A\right) \geq \mathcal{T}\left(G^A\right), \forall G^A \in \mathbb{S}\left(\overline{X}^A\right)$.

(ii) $\mathcal{T}^{F^A}\left(F^A\right)=1, \mathcal{T}^{F^A}\left(\overline{\phi}^A\right)=1$;

(iii) If $F_i^A \in \mathbb{S}\left(F^A\right), i=1,2,\ldots,n$ then

$$\mathcal{T}^{F^A}\left(\Cap_{i=1}^n F_i^A\right) \geq \wedge_{i=1}^n \mathcal{T}^{F^A}\left(F_i^A\right);$$

(iv) If $F_i^A \in \mathbb{S}\left(F^A\right), i \in \Delta$ then $\mathcal{T}^{F^A}\left(\Cup_{i\in\Delta} F_i^A\right) \geq \wedge_{i\in\Delta} \mathcal{T}^{F^A}\left(F_i^A\right)$

Example 5.2. Let $\mathcal{T}$ be a GOSS on $\overline{X}^A$ and $F^A\left(\neq \overline{\phi}^A\right) \in \mathbb{S}\left(\overline{X}^A\right)$. A mapping $\mathcal{T}^{F^A}: \mathbb{S}\left(F^A\right) \to I$ defined by $\mathcal{T}^{F^A}\left(H^A\right)=1, \forall H^A \in \mathbb{S}\left(F^A\right)$, then $\mathcal{T}^{F^A}$ is an SGOSS of $\mathcal{T}$ induced by F^A.

Proposition 5.3. *Let $\mathcal{T}$ be a GOSS on $\overline{X}^A$ and $F^A\left(\neq \overline{\phi}^A\right) \in \mathbb{S}\left(\overline{X}^A\right)$. Then the mapping $\mathcal{T}^{F^A}: \mathbb{S}\left(F^A\right) \to I$ defined by*

$$\mathcal{T}^{F^A}\left(H^A\right)=\bigvee\left[\mathcal{T}\left(G^A\right): G^A \Cap F^A = H^A, G^A \in \mathbb{S}\left(\overline{X}^A\right)\right],$$

$\forall H^A \in \mathbb{S}\left(F^A\right)$ is an SGOSS on F^A.

Proof. (i) Let $G^A \in \mathbb{S}\left(\overline{X}^A\right)$. Then clearly from the definition $\mathcal{T}^{F^A}\left(G^A \Cap F^A\right) \geq \mathcal{T}\left(G^A\right)$.

(ii) Since $\overline{X}^A \Cap F^A = F^A, \overline{\phi}^A \Cap F^A = \phi^A$ and $\mathcal{T}\left(\overline{X}^A\right) = \mathcal{T}\left(\overline{\phi}^A\right) = 1$, follows that $\mathcal{T}^{F^A}\left(F^A\right) = \mathcal{T}^{F^A}\left(\phi^A\right) = 1$.

(iii) Let $F_i^A \in \mathbb{S}\left(F^A\right), \forall i = 1$ to n and $H^A = \Cap_{i=1}^n F_i^A$. If $H^A = \overline{\phi}^A$ then $\mathcal{T}^{F^A}\left(H^A\right)=1$ and hence,

$$\mathcal{T}^{F^A}\left(\Cap_{i=1}^n F_i^A\right) \geq \bigwedge_{i=1}^n \mathcal{T}^{F^A}\left(F_i^A\right).$$

So, let $H^A \neq \overline{\phi}^A$. Now construct

$$\beta_i = \left\{G^A \in \mathbb{S}\left(\overline{X}^A\right) : G^A \Cap F^A = F_i^A\right\}$$

and let G_i^A be any arbitrary member of $\beta_i, 1 \leq i \leq n$. Then $G_i^A \Cap F^A = F_i^A$. So,

$$\begin{aligned} H^A &= \Cap_{i=1}^n F_i^A \\ &= \Cap_{i=1}^n \left(G_i^A \Cap F^A\right) \\ &= \left[\Cap_{i=1}^n G_i^A\right] \Cap F^A \\ &= \Cap_{i=1}^n F_i^A \end{aligned}$$

and hence

$$\mathcal{T}^{F^A}\left(H^A\right) \geq \mathcal{T}\left[\Cap_{i=1}^{n} G_i^A\right] \geq \bigwedge_{i=1}^{n} \mathcal{T}\left(G_i^A\right).$$

Since $G_i^A \in \beta_i$ is arbitrary, follows that

$$\begin{aligned}\mathcal{T}^{F^A}\left(\Cap_{i=1}^{n} F_i^A\right) &= \mathcal{T}^{F^A}\left(H^A\right) \\ &\geq \bigwedge_{i=1}^{n}\left[\bigvee_{G_i^A \in \beta_i} \mathcal{T}\left(G_i^A\right)\right] \\ &= \bigwedge_{i=1}^{n} \mathcal{T}^{F^A}\left(F_i^A\right).\end{aligned}$$

(iv) Let $\mathcal{F} = \left\{F_i^A : i \in \Delta\right\}$ be an arbitrary subfamily of $\mathbb{S}\left(F^A\right)$ and $H^A = \Cup_{i\in\Delta} F_i^A$ and $\beta_i = \left\{G^A \in \mathbb{S}\left(\overline{X}^A\right) : G^A \Cap F^A = F_i^A\right\}$, $i \in \Delta$ and G_i^A be any arbitrary member of $\beta_i, i \in \Delta$. Now if $\overline{\phi}^A \notin \mathcal{F}$, then

$$\begin{aligned}H^A &= \Cup_{i\in\Delta} F_i^A \\ &= \Cup_{i\in\Delta}\left[G_i^A \Cap F^A\right] \\ &= \left[\Cup_{i\in\Delta} G_i^A\right] \Cap F^A\end{aligned}$$

and as per the description of $\mathcal{T}^{F^A}$, we get

$$\mathcal{T}^{F^A}\left(H^A\right) \geq \mathcal{T}\left[\Cup_{i\in\Delta} G_i^A\right] \geq \bigwedge_{i\in\Delta} \mathcal{T}\left(G_i^A\right).$$

This is true for any $G_i^A \in \beta_i$ and hence

$$\begin{aligned}\mathcal{T}^{F^A}\left[\Cup_{i\in\Delta} F_i^A\right] &= \mathcal{T}^{F^A}\left(H^A\right) \\ &\geq \bigwedge_{i\in\Delta}\left[\bigvee_{G_i^A \in \beta_i} \mathcal{T}\left(G_i^A\right)\right] \\ &= \bigwedge_{i\in\Delta} \mathcal{T}^{F^A}\left(F_i^A\right).\end{aligned}$$

Again if $\overline{\phi}^A \in \mathcal{F}$ and assume without restricting generality that $F_1^A = \overline{\phi}^A$. Then,

$$\begin{aligned}H^A &= \left[\overline{\phi}^A\right] \Cup \left[\Cup_{i\in\Delta-\{1\}} F_i^A\right] \\ &= \left[\overline{\phi}^A \Cap F^A\right] \Cup \left[\Cup_{i\in\Delta-\{1\}}\left[G_i^A \Cap F^A\right]\right]\end{aligned}$$

and hence,

$$\begin{aligned}\mathcal{T}^{F^A}\left(H^A\right) &\geq \left[\mathcal{T}\left(\overline{\phi}^A\right) \wedge \mathcal{T}\left[\Cup_{i\in\Delta-\{1\}} G_i^A\right]\right] \\ &\geq \left[1 \wedge \left[\bigwedge_{i\in\Delta-\{1\}} \mathcal{T}\left(G_i^A\right)\right]\right] \\ &= \bigwedge_{i\in\Delta-\{1\}} \mathcal{T}\left(G_i^A\right).\end{aligned}$$

This is true for any $G_i^A \in \beta_i$ and hence

$$\begin{aligned}\mathcal{T}^{F^A}\left[\Cup_{i\in\Delta-\{1\}} F_i^A\right] &\geq \bigwedge_{i\in\Delta-\{1\}} \left[\bigvee_{G_i^A\in\beta_i} \mathcal{T}\left(G_i^A\right)\right] \\ &= \bigwedge_{i\in\Delta-\{1\}} \mathcal{T}^{F^A}\left(F_i^A\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{T}^{F^A}\left[\Cup_{i\in\Delta} F_i^A\right] &\geq \mathcal{T}^{F^A}\left(F_1^A\right) \bigwedge \left[\bigwedge_{i\in\Delta-\{1\}} \mathcal{T}^{F^A}\left(F_i^A\right)\right] \\ &= \bigwedge_{i\in\Delta} \mathcal{T}^{F^A}\left(F_i^A\right).\end{aligned}$$

Therefore, $\mathcal{T}^{F^A}$ is an SGOSS of $\mathcal{T}$ induced by F^A. □

Remark 5.4. If $\mathcal{T}^{F^A}$ is an SGOSS of $\mathcal{T}$ induced by F^A and G^A be an r-level soft open set in $\mathcal{T}$, then $\mathcal{T}\left(G^A\right) \geq r$ and hence $\mathcal{T}^{F^A}\left(G^A \Cap F^A\right) \geq \mathcal{T}\left(G^A\right) \geq r$. So, $G^A \Cap F^A$ is an r-level soft open set in $\mathcal{T}^{F^A}$.

Proposition 5.5. *Consider $\mathcal{T}^{F^A}$ be an SGOSS, defined in Proposition 5.3 and $\mathcal{T}_1^{F^A}$ be any SGOSS of $\mathcal{T}$ induced by the soft set F^A. Then $\mathcal{T}_1^{F^A} \geq \mathcal{T}^{F^A}$.*

Proof. Let H^A be any member of $\mathbb{S}\left(F^A\right)$ and

$$\mathcal{F} = \left\{G^A \in \mathbb{S}\left(\overline{X}^A\right) : G^A \Cap F^A = H^A\right\}.$$

Clearly, $\mathcal{F} \neq \phi$. Then by the Proposition 5.3, we have

$$\mathcal{T}^{F^A}\left(H^A\right) = \bigvee_{G^A\in\mathcal{F}} \mathcal{T}\left(G^A\right)$$

and hence

$$\mathcal{T}^{F^A}\left(H^A\right) \leq \mathcal{T}\left(G^A\right), \quad \forall G^A \in \mathcal{F} \tag{5.1}$$

Also, form the definition of SGOSS, we get

$$\begin{aligned}\mathcal{T}_1^{F^A}\left(H^A\right) &= \mathcal{T}_1^{F^A}\left(G^A \Cap F^A\right)\\ &\geq \mathcal{T}\left(G^A\right), \quad \forall G^A \in \mathcal{F}\end{aligned} \tag{5.2}$$

Thus, $\mathcal{T}_1^{F^A}\left(H^A\right) \geq \mathcal{T}\left(G^A\right) \geq \mathcal{T}^{F^A}\left(H^A\right), \forall H^A \in \mathbb{S}\left(F^A\right)$. [by (5.1) and (5.2).] Therefore, $\mathcal{T}_1^{F^A} \geq \mathcal{T}^{F^A}$. □

Proposition 5.6. *Suppose $\mathcal{T}$ and $\mathcal{V}$ are GOSSs on $\overline{X}^A$ and $\overline{Y}^B$ respectively, also $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{V}\right)$ be a bijective FS-open map. If $\mathcal{V}^{G^B}$ is an SGOSS of $\mathcal{V}$ induced by $G^B\left(\neq \overline{\Phi}^B\right) \in \mathbb{S}\left(\overline{Y}^B\right)$ then $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]} : \mathbb{S}\left[f_\varphi^{-1}G^B\right] \to I$ defined by $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F^A\right) = \mathcal{V}^{G^B}\left[f_\varphi F^A\right]$ is an SGOSS of $\mathcal{T}$ induced by $f_\varphi^{-1}\left(G^B\right)$.*

Proof. (i) Let $F^A \in \mathbb{S}\left(\overline{X}^A\right)$. Then

$$\begin{aligned}&\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F^A \Cap f_\varphi^{-1}G^B\right)\\ &= \mathcal{V}^{G^B}\left[f_\varphi\left(F^A \Cap f_\varphi^{-1}G^B\right)\right]\\ &= \mathcal{V}^{G^B}\left[f_\varphi\left(F^A\right) \Cap f_\varphi\left(f_\varphi^{-1}G^B\right)\right]\\ &= \mathcal{V}^{G^B}\left[f_\varphi\left(F^A\right) \Cap G^B\right], \text{since, } f_\varphi \text{ is bijective}\\ &\geq \mathcal{V}\left[f_\varphi\left(F^A\right)\right], \text{since, } \mathcal{V}^{G^B} \text{is SGOSS of } \mathcal{V} \text{ induced by } G^B\\ &\geq \mathcal{T}\left(F^A\right), \text{since, } f_\varphi \text{ is FS-open.}\end{aligned}$$

(ii) $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left[f_\varphi^{-1}G^B\right] = \mathcal{V}^{G^B}\left[f_\varphi\left(f_\varphi^{-1}G^B\right)\right] = \mathcal{V}^{G^B}\left(G^B\right) = 1$ and $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left[\overline{\phi}^A\right] = \mathcal{V}^{G^B}\left[f_\varphi\left(f_\varphi^{-1}\left[\overline{\phi}^B\right]\right)\right] = \mathcal{V}^{G^B}\left(\overline{\phi}^B\right) = 1$.

(iii) Let, $F_i^A \in \mathbb{S}\left[f_\varphi^{-1}G^B\right], i = 1, 2, \ldots, n$ and $F^A = \Cap_{i=1}^n F_i^A$.
If $F^A = \overline{\phi}^A$, Then $f_\varphi\left(F^A\right) = \overline{\phi}^B$ and hence

$$\begin{aligned}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(\Cap_{i=1}^n F_i^A\right) &= \mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left[\overline{\phi}^A\right]\\ &= \mathcal{V}^{G^B}\left(\overline{\phi}^B\right)\\ &= 1\\ &\geq \wedge_{i=1}^n \mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F_i^A\right);\end{aligned}$$

If $F^A \neq \overline{\phi}^A$ then $f_\varphi\left(F^A\right) = f_\varphi\left[\Cap_{i=1}^n F_i^A\right] = \Cap_{i=1}^n f_\varphi\left[F_i^A\right]$ and hence,

$$\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F^A\right) = \mathcal{V}^{G^B}\left[f_\varphi\left(F^A\right)\right]$$

$$\begin{aligned}&= \mathcal{V}^{G^B}\left[\Cap_{i=1}^{n}\left[f_\varphi F_i^A\right]\right]\\ &\geq \wedge_{i=1}^{n}\mathcal{V}^{G^B}\left[f_\varphi F_i^A\right]\\ &= \wedge_{i=1}^{n}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F_i^A\right).\end{aligned}$$

Thus, $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(\Cap_{i=1}^{n}F_i^A\right) \geq \wedge_{i=1}^{n}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F_i^A\right)$.

(iv) Let, $\left\{F_i^A : i \in \Delta\right\}$ be any subcollection of $\mathbb{S}\left[f_\varphi^{-1}G^B\right]$ and $f_\varphi\left(F^A\right) = f_\varphi\left[\cup_{i\in\Delta}F_i^A\right] = \cup_{i\in\Delta}f_\varphi\left[F_i^A\right]$ and hence,

$$\begin{aligned}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F^A\right) &= \mathcal{V}^{G^B}\left[f_\varphi\left(F^A\right)\right]\\ &= \mathcal{V}^{G^B}\left[\left(\cup_{i\in\Delta}f_\varphi F_i^A\right)\right]\\ &\geq \wedge_{i\in\Delta}\mathcal{V}^{G^B}\left[f_\varphi F_i^A\right]\\ &= \wedge_{i\in\Delta}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F_i^A\right)\end{aligned}$$

Thus, $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(\cup_{i\in\Delta}F_i^A\right) \geq \wedge_{i\in\Delta}\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}\left(F_i^A\right)$.

Therefore, $\mathcal{T}^{\left[f_\varphi^{-1}G^B\right]}$ is an SGOSS of $\mathcal{T}$ induced by $f_\varphi^{-1}G^B$. □

Proposition 5.7. *Let $f_\varphi : \left(\overline{X}^A, \mathcal{T}\right) \to \left(\overline{Y}^B, \mathcal{V}\right)$ be a bijective sgp-map. If $\mathcal{T}^{F^A}$ is an SGOSS of $\mathcal{T}$ induced by $F^A\left(\neq \overline{\Phi}^A\right) \in \mathbb{S}\left(\overline{X}^A\right)$ then $\mathcal{V}^{\left[f_\varphi F^A\right]} : \mathbb{S}\left[f_\varphi F^A\right] \to I$ defined by $\mathcal{V}^{\left[f_\varphi F^A\right]}\left(G^B\right) = \mathcal{T}^{F^A}\left[f_\varphi^{-1}G^B\right]$ is an SGOSS of $\mathcal{V}$ induced by $f_\varphi\left(F^A\right)$.*

Proof. The proof is similar as the Proposition 5.6. □

6. Conclusion:

Within this article, we have fuzzified the soft set and determined the gradation of openness of soft sets using soft elements and elementary soft operations. After that, a detailed fundamental topological aspects include neighbourhood system of a soft element, interior, limiting elements, continuity, openness, closedness etc. are studied. Also, for any real number $r \in [0, 1]$, the relations between fuzzy soft topology and r-level soft topology are discussed. Further we have introduced elementary soft function, soft preserving map, FS-open map, FS- homeomorphism and their relationships are established. Finally, the definition of subspace gradation of openness of soft sets (SGOSS) are given and the behaviours of SGOSS on elementary soft functional image and pre image of soft sets are mentioned. We think this work has a great importance to the researchers to study the features of separation axioms, completeness, connectedness, topological group structures etc. in fuzzy soft settings.

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[1] Department of Mathematics, Faculty of Science, Technology and Vocational studies, Kazi Nazrul University, P.O. Kalla, Asansol-713340, West Bengal, India.

Email address: sk.nazmul_math@yahoo.in

[2] Department of Mathematics, Faculty of Science, Technology and Vocational studies, Kazi Nazrul University, P.O. Kalla, Asansol-713340, West Bengal, India.

Email address: barnalidhang6@gmail.com