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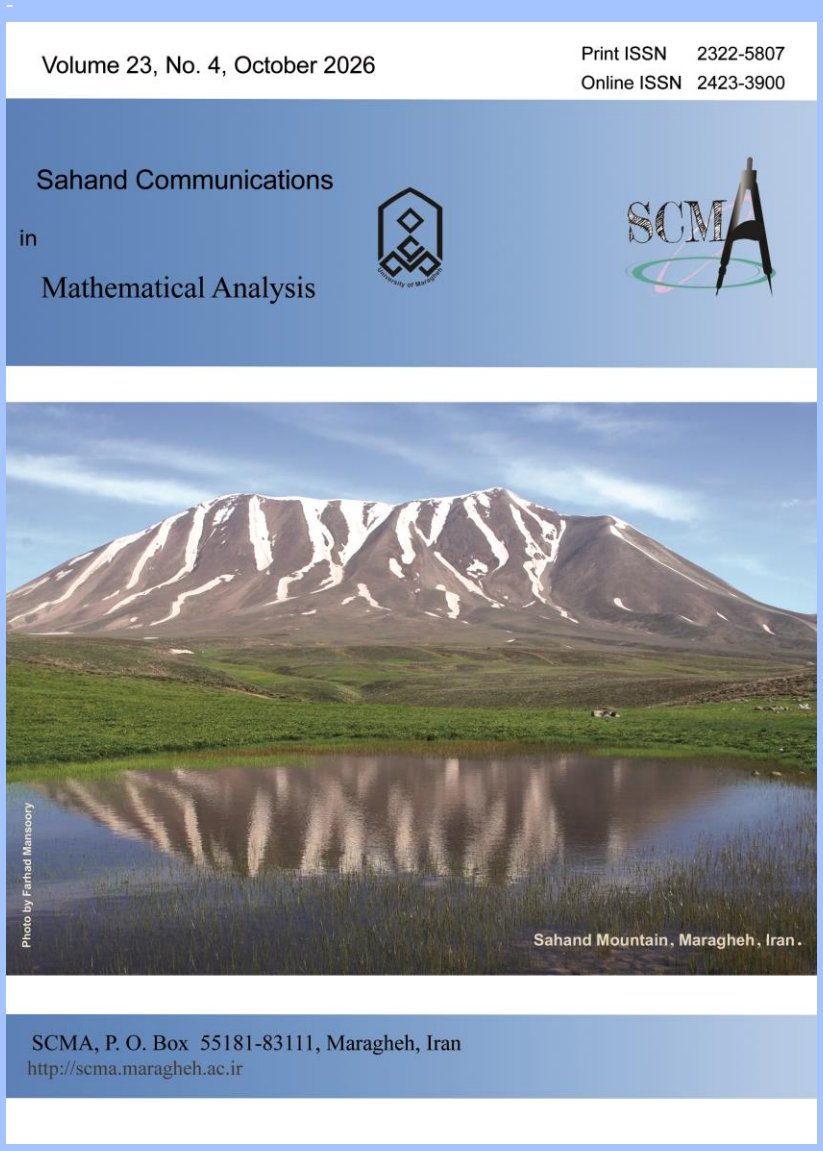

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Fuzzy Continuous and Fuzzy Bounded Linear Operators on Type-2 Fuzzy Normed Linear Spaces

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Abstract. In the following work we define various types of fuzzy continuity over type-2 fuzzy normed linear spaces (type-2 FNLS) with examples and examine their co-relations. Three types of fuzzy boundedness have also been studied and it has shown that the collection of all strongly fuzzy bounded linear transformations forms a type-2 FNLS. Finally we prove that the set of all level fuzzy bounded linear functionals over a type-2 FNLS is a type-2 fuzzy Banach space.

1. Introduction

The founding paper on fuzzy sets [21], written by Lotfi Zadeh, is about 60 years old. Since then it has affected many areas of Mathematics such as many-valued logics, topology, algebra, number theory, functional analysis, differential equations, category theory, statistical convergence, optimization, data processing, image processing and operations research [1, 2, 4–7, 10, 12, 14, 19].

Towards the fuzzification of functional analysis C. Felbin [6] in 1992 presented the idea of fuzzy norm by associating a fuzzy real number to an element and satisfying some conditions. After that Cheng and Mordeson [2] in 1994 defined a fuzzy norm by giving a grade to a norm of an elememt by comparing the norm to a real number. Bag and Samanta [1] in 2003 proposed a new definition of fuzzy norm given by Cheng and Mordeson by slightly changing it and obtained a decomposition theorem

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from it. Zadeh following the legacy of his own further introduced the concept of type-2 fuzzy set [22] in 1975 to process more fuzzinezss. After that type-2 fuzzy sets have been applied to various fields of mathematics [8, 9, 11, 13, 15–18, 20].

Taking note from the previous work done, in [3], Chiney, Biswas and Samanta introduced the idea of type-2 fuzzy normed linear space for the first time. In [3], it has been proved that a type-2 FNLS can be decomposed into a family of Felbin type fuzzy normed linear spaces and its converse is also true; relation between finite dimensionality and completeness in type-2 FNLS has been studied.

This paper is a continuation of our previous work done. In the present work we define various types of continuity in type-2 FNLSs and examine relations between them with justified examples. Also we have studied three types of boundedness in type-2 FNLSs and their co-relations. It is shown that the set of all strongly fuzzy bounded linear operators from a type-2 FNLS to another type-2 FNLS forms a real vector space. We also introduce a type -2 fuzzy norm of a strongly fuzzy bounded linear operator. At last it is examined that the type -2 FNLS formed by the set of all level fuzzy bounded linear functionals is complete.

2. Preliminaries

Definition 2.1 ([10])**.** A fuzzy real number is a fuzzy set on $\mathbb{R}$, that is, a mapping $\eta : \mathbb{R} \to I = [0,1]$ which is normal, convex, and upper semicontinuous. A fuzzy real number η is said to be non-negative if $\eta(t) = 0$ for all $t < 0$.

The collection of all non-negative fuzzy real numbers is denoted by $\mathbb{R}^+(I)$. For each $\alpha \in (0,1]$, the α-level set of η, defined by

$$[\eta]_\alpha = \{t \in \mathbb{R} : \eta(t) \geq \alpha\},$$

is a closed and bounded interval, denoted by $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$.

For any $r \in \mathbb{R}$, $\bar{r}$ denotes the fuzzy real number defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{when } t = r, \\ 0, & \text{when } t \neq r. \end{cases}$$

We define

$$\mathcal{I} = \left\{\zeta \in \mathbb{R}^+(I) : \bar{0} \preceq \zeta \preceq \bar{1}\right\} \quad \text{and} \quad \mathcal{I}_0 = \left\{\zeta \in \mathcal{I} : \bar{0} \prec \zeta \prec \bar{1}\right\}.$$

The definitions of convergence of fuzzy real numbers, definition of arithmetic operations on fuzzy real numbers, definition of partial ordering of a fuzzy real number and some propositions are taken from [10].

Proposition 2.2 ([10])**.** *Let* $\zeta, \xi \in \mathbb{R}^+(I)$ *and* $[\zeta]_\alpha = [a_1^\alpha, b_1^\alpha]$ *and* $[\xi]_\alpha = [a_2^\alpha, b_2^\alpha]$*. Then*

$$[\zeta \oplus \xi]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha],$$

$$[\zeta \ominus \xi]_\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha],$$
$$[\zeta \odot \xi]_\alpha = [a_1^\alpha a_2^\alpha, b_1^\alpha b_2^\alpha],$$
$$[\bar{1} \oslash \xi]_\alpha = \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}\right], \quad a_2^\alpha > 0.$$

Remark 2.3 ([4])**.** Congxin and Cong considered the fuzzy real number as in [10] with the property that $[\eta]_0 = cl\{x \in \mathbb{R}, \eta(x) > 0\}$ is compact.

Inequalities among the members of $\mathbb{R}^+(I)$ are defined as

$(i)\ \eta \preceq \zeta$ if $\eta^-(\alpha) \leq \zeta^-(\alpha)$ and $\eta^+(\alpha) \leq \zeta^+(\alpha), \forall \alpha \in [0,1]$,

$(ii)\ \eta \succeq \zeta$ if $\eta^-(\alpha) \geq \zeta^-(\alpha)$ and $\eta^+(\alpha) \geq \zeta^+(\alpha), \forall \alpha \in [0,1]$,

$(iii)\ \eta \succ \zeta$ if there is $\epsilon > 0$ such that $\eta \oplus \bar{\epsilon} \leq \zeta$.

Theorem 2.4 ([4])**.** *Assume $\{u_t : t \in \Omega\}$ is a bounded set of fuzzy real numbers. Then the supremum and infimum exist and are determined by two pairs of usual functions of λ on $[0,1]$, namely*

$$\left(u_{s,\Omega}^-(\lambda),\ u_{s,\Omega}^+(\lambda)\right) \quad \text{and} \quad \left(u_{I,\Omega}^-(\lambda),\ u_{I,\Omega}^+(\lambda)\right).$$

These functions are defined as follows:

$$u_{s,\Omega}^-(\lambda) = \begin{cases} u_s^-(\lambda), & \lambda \in (0,1], \\ u_s^-(0+0), & \lambda = 0, \end{cases}$$

$$u_{s,\Omega}^+(\lambda) = \begin{cases} u_s^+(\lambda)), & \lambda \in [0,1] \setminus \{\lambda_m^s\}, \\ u_s^+(\lambda_m^s - 0), & \lambda = \lambda_m^s, \quad m = 1,2,\ldots, \end{cases}$$

$$u_{I,\Omega}^-(\lambda) = \begin{cases} u_I^-(\lambda), & \lambda \in [0,1] \setminus \{\lambda_m'\}, \\ u_I^-(\lambda_m' - 0), & \lambda = \lambda_m', \quad m = 1,2,\ldots, \end{cases}$$

$$u_{I,\Omega}^+(\lambda) = \begin{cases} u_I^+(\lambda), & \lambda \in (0,1], \\ u_I^+(0+0), & \lambda = 0. \end{cases}$$

Here,

$$u_s^-(\lambda) = \sup_{t\in\Omega}(u_t)_\lambda^-, \qquad u_s^+(\lambda) = \sup_{t\in\Omega}(u_t)_\lambda^+,$$
$$u_I^-(\lambda) = \inf_{t\in\Omega}(u_t)_\lambda^-, \qquad u_I^+(\lambda) = \inf_{t\in\Omega}(u_t)_\lambda^+.$$

The sets $\{\lambda_m^s\}$ and $\{\lambda_m'\}$ denote all points of discontinuity of $u_s^+(\lambda)$ and $u_t^-(\lambda)$ on $[0,1]$, respectively.

Definition 2.5 ([3])**.** Suppose $\mathcal{U}$ be a NLS over $\mathbb{R}$. A fuzzy subset $\mathcal{N} : \mathcal{U} \times \mathbb{R}^+(I) \to \mathcal{I}$ is called a type-2 fuzzy norm on $\mathcal{U}$ if $\forall u, v \in \mathcal{U}, k \in \mathbb{R}$ and $\mu \in \mathbb{R}^+(I)$

(C1) When $\mu = \bar{0}, \mathcal{N}(u,\mu) = \bar{0}$;
(C2) $(\forall \mu \succ \bar{0}, \mathcal{N}(u,\mu) = \bar{1})$ iff $u = \underline{0}$;
(C3) $(\forall \mu \in \mathbb{R}^+(I), \mu \succ \bar{0}, \mathcal{N}(ku,\mu) = \mathcal{N}\left(u, \frac{1}{|k|}\mu\right)$ if $k \neq 0$;
(C4) $\forall \mu_1, \mu_2 \in \mathbb{R}^+(I)$ and $u, v \in U$,

$$\mathcal{N}(u+v, \mu_1 \oplus \mu_2) \succeq \min\{\mathcal{N}(u,\mu_1), \mathcal{N}(v,\mu_2)\};$$

(C5) As a function $\mathcal{N}(u,\cdot)$ is increasing in $\mathbb{R}^+(I)$ that is if $\mu_1, \mu_2 \in \mathbb{R}^+(I)$ with $\mu_2 \succeq \mu_1$ then

$$\mathcal{N}(u,\mu_2) \succeq \mathcal{N}(u,\mu_1) \text{ and } \lim_{\mu \to \overline{\infty}} \mathcal{N}(u,\mu) = \bar{1}.$$

The pair $(\mathcal{U}, \mathcal{N})$ is called type-2 fuzzy normed linear space (type-2 FNLS).

Theorem 2.6 ([3]). *Suppose $(\mathcal{U}, \mathcal{N})$ be a type-2 FNLS. Next we take that*

(C6) $\wedge\{\mu \succ \bar{0} : \mathcal{N}(u,\mu) \succ \bar{0}\} = \bar{0} \quad \Rightarrow \quad u = \underline{0}$.

and define

$$\|u\|^{\zeta} = \wedge\{\mu \succ \bar{0} : \mathcal{N}(u,\mu) \succeq \zeta\}, \quad \forall \zeta \in \mathcal{I}_0.$$

Then $\left\{\|\cdot\|^{\zeta} : \zeta \in \mathcal{I}_0\right\}$ is a family of Felbin norms on $\mathcal{U}$ such that $\zeta_2 \succeq \zeta_1$ $(\zeta_1, \zeta_2 \in \mathcal{I}_0) \Rightarrow \|u\|^{\zeta_2} \succeq \|u\|^{\zeta_1}$.

Theorem 2.7 ([3]). *Let $\left\{\|u\|^{\zeta} : \zeta \in \mathcal{I}_0\right\}$ be a family of fuzzy norms which are of Felbin type on a linear space $\mathcal{U}$ such that $\zeta_2 \succeq \zeta_1$ $(\zeta_1, \zeta_2 \in \mathcal{I}_0)$ $\Rightarrow \|u\|^{\zeta_2} \succeq \|u\|^{\zeta_1}$ and $\vee\left\{\zeta : \|u\|^{\zeta} \preceq \mu\right\} = \bar{1}$, $\forall \mu \succ \bar{0} \Rightarrow \|u\|^{\zeta_0} = \bar{0}$ for some $\zeta_0 \in \mathcal{I}_0$. Now a function defined by $\mathcal{N}' : \mathcal{U} \times \mathbb{R}^+(I) \to \mathcal{I}$ as*

$$\mathcal{N}'(u,\mu) = \vee\left\{\zeta : \|u\|^{\zeta} \preceq \mu\right\}, \qquad \text{when } (u,\mu) \neq (\underline{0}, \bar{0})$$
$$= \bar{0}, \qquad \text{when } (u,\mu) = (\underline{0}, \bar{0}) \text{ or } \left\{\zeta : \|u\|^{\zeta} \preceq \mu\right\} = \phi.$$

Then $\mathcal{N}'$ is a type-2 fuzzy norm on $\mathcal{U}$.

Definition 2.8 ([3]). A sequence $\{u_n\}$ in a type-2 FNLS $(\mathcal{U}, \mathcal{N})$ is called convergent if $\exists\, u \in \mathcal{U}$ such that $\lim \mathcal{N}(u_n - u, \mu) = \bar{1}$, $\forall \mu \succ \bar{0}$. Then the limit of the sequence is said to be u and it is expressed by $\lim u_n = u$.

Note 2.9. This means that for a pre-assigned $\zeta \in \mathcal{I}_0$ and $\mu \succ \bar{0}$ $\exists$ a $k = k(\zeta, \mu)$ such that $\mathcal{N}(u_n - u, \eta) \succ \zeta$, $\forall n \geq k$.

Definition 2.10 ([3]). In a type-2 FNLS $(\mathcal{U}, \mathcal{N})$ a sequence $\{u_n\}$ is called Cauchy if $\lim \mathcal{N}(u_n - u_{n+p}, \mu) = \bar{1}, \quad \forall \mu \succ \bar{0}$ and $p = 1, 2, 3, \ldots$.

Note 2.11. This means that for a pre-assigned $\zeta \in \mathcal{I}_0$ and $\mu \succ \bar{0}$ $\exists$ a $k = k(\zeta, \mu)$ s.t. $\mathcal{N}(u_n - u_{n+p}, \mu) \succ \zeta$, $\forall n \geq k$ and $p = 1, 2, 3, \ldots$.

3. Fuzzy Continuous Functions on Type-2 FNLS

In this section by NLS, FC, SFC, SEFC we refer to as normed linear space, fuzzy continuous, strongly fuzzy continuous, sequentially fuzzy continuous respectively.

Definition 3.1. If $(\mathcal{U},\mathcal{N}_1)$ and $(\mathcal{V},\mathcal{N}_2)$ are two type-2 FNLSs then we call a function $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ FC at $u_0\in\mathcal{U}$ if for any $\mu_1\succ\bar{0},\zeta_0\in\mathcal{I}_0,\exists\,\mu_2=\mu_2(\mu_1,\zeta_0)\succ\bar{0},\zeta_0'=\zeta_0'(\mu_1,\zeta_0)\in\mathcal{I}_0$ such that

$$\mathcal{N}_1(u-u_0,\mu_2)\succ\zeta_0' \quad\Rightarrow\quad \mathcal{N}_2(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_1)\succ\zeta_0,\quad \forall u\in\mathcal{U}.$$

FC of $\mathcal{T}$ at every point of $\mathcal{U}$ will imply that $\mathcal{T}$ is FC on $\mathcal{U}$.

Definition 3.2. Let $\mathcal{T}:(\mathcal{U},\mathcal{N}_1)\to(\mathcal{V},\mathcal{N}_2)$ be a mapping. Then $\mathcal{T}$ is called SFC at $u_0\in\mathcal{U}$ if for any $\mu_1\succ\bar{0},\exists\mu_2\succ\bar{0}$ such that,

$$\mathcal{N}_2(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_1)\succeq\mathcal{N}_1(u-u_0,\mu_2),\quad \forall u\in\mathcal{U}.$$

SFC of $\mathcal{T}$ at every point of $\mathcal{U}$ will imply that $\mathcal{T}$ is SFC on $\mathcal{U}$.

Definition 3.3. A mapping $\mathcal{T}:(\mathcal{U},\mathcal{N}_1)\to(\mathcal{V},\mathcal{N}_2)$ is called SEFC at a point $u_0\in\mathcal{U}$ if for a given sequence $\{u_n\}$ in $\mathcal{U}$ with $u_n\to u_0$ will imply that $\mathcal{T}u_n\to\mathcal{T}u_0$ in $\mathcal{V}$ i.e., $\forall\mu\succ\bar{0}$

$$\lim\mathcal{N}_1(u_n-u_0,\mu)=\bar{1}\quad\Rightarrow\quad\lim\mathcal{N}_2(\mathcal{T}(u_n)-\mathcal{T}(u_0),\mu)=\bar{1}.$$

SEFC of $\mathcal{T}$ at every point of $\mathcal{U}$ will imply that $\mathcal{T}$ is SEFC on $\mathcal{U}$.

Theorem 3.4. *A mapping $\mathcal{T}:(\mathcal{U},\mathcal{N}_1)\to(\mathcal{V},\mathcal{N}_2)$ is SFC on $\mathcal{U}$ will imply that $\mathcal{T}$ is FC on $\mathcal{U}$.*

Proof. Let $u_0\in\mathcal{U}$ and $\mathcal{T}$ be SFC on $\mathcal{U}$. Choose $\mu_1\succ\bar{0}$. Then $\exists\,\mu_2\succ\bar{0}$ such that for all $u\in\mathcal{U}$,

$$\mathcal{N}_2(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_1)\succ\mathcal{N}_1(u-u_0,\mu_2). \tag{3.1}$$

Assume $\exists$ a $\zeta_0\in\mathcal{I}_0$ such that $\forall\,u\in\mathcal{U}$, $\mathcal{N}_1(u-u_0,\mu_2)\succ\zeta_0$.

Then relation (3.1) will imply that

$$\mathcal{N}_2(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_1)\succ\zeta_0,\quad\forall u\in\mathcal{U}.$$

Thus for a given $\mu_1\succ\bar{0}$ and $\zeta_0\in\mathcal{I}_0$, $\exists\mu_2=\mu_2(\zeta_0,\mu_2)$ and $\zeta_0\in\mathcal{I}_0$ such that

$$\mathcal{N}_1(u-u_0,\mu_2)\succ\zeta_0\quad\Rightarrow\quad\mathcal{N}_2(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_1)\succ\zeta_0,\quad\forall u\in\mathcal{U}.$$

Hence $\mathcal{T}$ is FC at $u_0\in\mathcal{U}$. Since $u_0\in\mathcal{U}$ is arbitrary, $\mathcal{T}$ is FC on $\mathcal{U}$. □

Theorem 3.5. *If a mapping $\mathcal{T}:(\mathcal{U},\mathcal{N}_1)\to(\mathcal{V},\mathcal{N}_2)$ is FC on $\mathcal{U}$, then $\mathcal{T}$ is SEFC on $\mathcal{U}$.*

Proof. Let $u_0 \in \mathcal{U}$ and $\mathcal{T}$ be FC at u_0. Also suppose that $\{u_n\}$ is a sequence in $\mathcal{U}$ such that $u_n \to u_0$.

Choose $\mu_1 \succ \bar{0}$ and $\zeta_0 \in \mathcal{I}_0$.

Since $\mathcal{T}$ is FC at u_0, $\exists\, \mu_2 = \mu_2(\mu_1, \zeta_0) \succ \bar{0}$ and $\zeta_0' = \zeta_0'(\mu_1, \zeta_0) \in \mathcal{I}_0$ such that,

(3.2)
$$\mathcal{N}_1(u - u_0, \mu_2) \succ \zeta_0' \quad \Rightarrow \quad \mathcal{N}_2(\mathcal{T}(u) - \mathcal{T}(u_0), \mu_1) \succ \zeta_0, \quad \forall u \in \mathcal{U}.$$

Now since $u_n \to u_0$ in $\mathcal{U}$ there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{N}_1(u_n - u_0, \mu_2) \succ \zeta_0', \quad \forall\, n \geq k_0.$$

So using equation (3.2) we have,

$$\mathcal{N}_2(\mathcal{T}(u_n) - \mathcal{T}(u_0), \mu_1) \succ \zeta_0, \quad \forall n \geq k_0.$$

Therefore, $\lim \mathcal{N}_2(\mathcal{T}(u_n) - \mathcal{T}(u_0), \mu_1) = \bar{1}$.

Since $\mu_1 \succ \bar{0}$ is arbitrary, we have $\mathcal{T}(u_n) \to \mathcal{T}(u_0)$ in $(\mathcal{V}, \mathcal{N}_2)$. □

Theorem 3.6. *Let $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ be a mapping. If $\mathcal{T}$ is SFC on $\mathcal{U}$, then it is SEFC on $\mathcal{U}$.*

Proof. Combining Theorems 3.4 and 3.5 we can get this result. □

The reverse of the preceding theorem is not true can be demonstrated by the following example.

Example 3.7. Let $(\mathcal{U} = \mathbb{R}, \|\cdot\|)$ be the usual NLS. Let $\mu \succ \bar{0}$ be an arbitrary fuzzy real number. Then we define μ_0 by

$$\mu_0 = \overline{\frac{a_0 + b_0}{2}}$$

where by $[a_0, b_0]$ we denote the closure of the support of fuzzy real number μ. Take $m > 0$ be a real number. Suppose

$$\mathcal{N}_1(u, \mu) = \begin{cases} \mu_0 \oslash \left(\mu_0 \oplus \overline{|u|}\right), & \text{if } \mu \succ \bar{0}, \\ \bar{0}, & \text{if } \mu = \bar{0}, \end{cases}$$

and

$$\mathcal{N}_2(u, \mu) = \begin{cases} \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right), & \text{if } \mu \succ \bar{0}, \\ \bar{0}, & \text{if } \mu = \bar{0}. \end{cases}$$

Then $(\mathbb{R}, \mathcal{N}_1)$ and $(\mathbb{R}, \mathcal{N}_2)$ are two type-2 FNLSs.

Let us take a mapping $\mathcal{T}(u) = \frac{u^4}{1+u^2}$. If $\{u_n\}$ be a sequence in $\mathcal{U}$ such that $u_n \to u_0$, then $\forall \mu \succ \bar{0}$

(3.3)
$$\lim \mathcal{N}_1(u_n - u_0, \mu) = \bar{1}$$
$$\Rightarrow \quad \lim \left\{ \left(\overline{\frac{a_0 + b_0}{2}} \right) \oslash \left(\overline{\frac{a_0 + b_0}{2}} \oplus \overline{|u_n - u_0|} \right) \right\} = \bar{1}$$

$$\Rightarrow \quad \lim \frac{\frac{a_0+b_0}{2}}{\frac{a_0+b_0}{2}+|u_n-u_0|}=1$$
$$\Rightarrow \quad \lim |u_n-u_0|=0.$$

Now

$$\mathcal{N}_2\left(\mathcal{T}(u_n)-\mathcal{T}(u_0),\mu\right)$$
$$=\left\{\left(\overline{\frac{a_0+b_0}{2}}\right)\oslash\left(\overline{\frac{a_0+b_0}{2}}\oplus\overline{m\left|\frac{u_n^4}{1+u_n^2}-\frac{u_0^4}{1+u_0^2}\right|}\right)\right\}$$

Again from equation (3.3) we have $\lim\left(m\left|\frac{u_n^4}{1+u_n^2}-\frac{u_0^4}{1+u_0^2}\right|\right)=0$. Therefore,

$$\lim \mathcal{N}_2\left(\mathcal{T}(u_n)-\mathcal{T}(u_0),\mu\right)=\bar{1},\quad \forall \mu\succ\bar{0}.$$

So, $\mathcal{T}(u_n)\to\mathcal{T}(u_0)$ in $(\mathbb{R},\mathcal{N}_2)$. Hence $\mathcal{T}:(\mathbb{R},\mathcal{N}_1)\to(\mathbb{R},\mathcal{N}_2)$ is SEFC.

Let $\mu_2\succ\bar{0}$ be given. Then for any $\mu_1\succ\bar{0}$

$$\mathcal{N}_2\left(\mathcal{T}(u)-\mathcal{T}(u_0),\mu_2\right)\succeq\mathcal{N}_1\left(u-u_0,\mu_1\right),\quad \forall u\in\mathcal{U}$$
$$\Rightarrow \quad \mathcal{N}_2\left(\frac{u^4}{1+u^2}-\frac{u_0^4}{1+u_0^2},\mu_2\right)\succeq\mathcal{N}\left(u-u_0,\mu_1\right)$$
$$\Rightarrow \quad \left\{\left(\overline{\frac{a_2+b_2}{2}}\right)\oslash\left(\overline{\frac{a_2+b_2}{2}}\oplus\overline{m|\frac{u^4}{1+u^2}-\frac{u_0^4}{1+u_0^2}|}\right)\right\}$$
$$\succeq\left\{\left(\overline{\frac{a_1+b_1}{2}}\right)\oslash\left(\overline{\frac{a_1+b_1}{2}}\oplus\overline{|u-u_0|}\right)\right\}$$

[where by $[a_1,b_1]$, $[a_2,b_2]$ we respectively denote the closure of the support of fuzzy real numbers μ_1, μ_2]

$$\Rightarrow \quad \frac{\frac{a_2+b_2}{2}}{\frac{a_2+b_2}{2}+m\left|\frac{u^4}{1+u^2}-\frac{u_0^4}{1+u_0^2}\right|}\geq\frac{\frac{a_1+b_1}{2}}{\frac{a_1+b_1}{2}+|u-u_0|}$$
$$\Rightarrow \quad \frac{r_2}{r_2+m\left|\frac{u^4}{1+u^2}-\frac{u_0^4}{1+u_0^2}\right|}\geq\frac{r_1}{r_1+|u-u_0|},$$
$$\text{where}\quad r_1=\frac{a_1+b_1}{2}\quad\text{and}\quad r_2=\frac{a_2+b_2}{2}$$
$$\Rightarrow \quad \frac{r_2\left|1+u^2\right|\left|1+u_0^2\right|}{r_2\left|1+u^2\right|\left|1+u_0^2\right|+m\left|u^4+u^4u_0^2-u_0^4-u_0^4u^2\right|}\geq\frac{r_1}{r_1+|u-u_0|}$$
$$\Rightarrow \quad \frac{r_2\left|1+u^2\right|\left|1+u_0^2\right|}{r_2\left|1+u^2\right|\left|1+u_0^2\right|+m\left|u-u_0\right|\left|u+u_0\right|\left|u^2+u_0^2+u^2u_0^2\right|}$$
$$\geq\frac{r_1}{r_1+|u-u_0|}$$

$$\Rightarrow \quad mr_1 |u-u_0| |u+u_0| \left|u^2+u_0^2+u^2u_0^2\right| \leq r_2 \left|1+u^2\right| \left|1+u_0^2\right| |u-u_0| \tag{3.4}$$

$$\Rightarrow \quad r_1 \leq \frac{r_2 \left|1+u^2\right| \left|1+u_0^2\right| |u-u_0|}{m |u-u_0| |u+u_0| \left|u^2+u_0^2+u^2u_0^2\right|}.$$

If $\mathcal{T}$ is to be SFC on $\mathcal{U}$, then inequality (3.4) is to be satisfied $\forall u\,(\neq u_0) \in \mathcal{U}$.

Since $\inf_{u(\neq u_0)\in U} \frac{r_2\left|1+u^2\right|\left|1+u_0^2\right||u-u_0|}{m|u-u_0||u+u_0|\left|u^2+u_0^2+u^2u_0^2\right|} = 0$, from the inquality (3.4) we get,

$$\begin{aligned} & r_1 = 0 \\ \Rightarrow \quad & \frac{a_1+b_1}{2} = 0 \\ \Rightarrow \quad & a_1+b_1 = 0 \\ \Rightarrow \quad & a_1 = b_1 = 0 \ [\text{Since } a_1, b_1 \geq 0] \\ \Rightarrow \quad & \mu_1 = \bar{0}, \end{aligned}$$

which is not possible. Therefore $\mathcal{T}$ is not SFC.

Note 3.8. A mapping $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ is said to satisfy additive property if $\mathcal{T}(u+v) = \mathcal{T}(u) + \mathcal{T}(v)$ for all $u, v \in \mathcal{U}$.

Theorem 3.9. *Let $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ satisfy the additive property. If $\mathcal{T}$ is SEFC at a point $u_0 \in \mathcal{U}$, then $\mathcal{T}$ is SEFC on $\mathcal{U}$.*

Proof. Take $u \in \mathcal{U}$ be arbitrary. Let $\{u_n\}$ be a sequence in $\mathcal{U}$ such that $u_n \to u$. Then

$$\begin{aligned} & \lim \mathcal{N}_1 (u_n - u, \mu) = \bar{1}, \quad \forall \mu \succ \bar{0} \\ \Rightarrow \quad & \lim \mathcal{N}_1 ((u_n - u + u_0) - u_0), \mu) = \bar{1}, \quad \forall \mu \succ \bar{0}. \\ \Rightarrow \quad & \lim \mathcal{N}_2 (\mathcal{T}(u_n - u + u_0) - \mathcal{T}(u_0), \mu) = \bar{1}, \quad \forall \mu \succ \bar{0}, \\ & \quad [\text{since } \mathcal{T} \text{ is SEFC at } u_0] \\ \Rightarrow \quad & \lim \mathcal{N}_2 (\mathcal{T}(u_n) - \mathcal{T}(u) + \mathcal{T}(u_0) - \mathcal{T}(u_0), \mu) = \bar{1}, \quad \forall \mu \succ \bar{0} \\ \Rightarrow \quad & \lim \mathcal{N}_2 (\mathcal{T}(u_n) - \mathcal{T}(u), \mu) = \bar{1}, \quad \forall \mu \succ \bar{0}. \end{aligned}$$

Thus $\mathcal{T}(u_n) \to \mathcal{T}(u)$ in $\mathcal{V}$. Hence $\mathcal{T}$ is SEFC on $\mathcal{U}$. □

Theorem 3.10. *Let $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ satisfy the additive property. If $\mathcal{T}$ is SFC at a point $u_0 \in \mathcal{U}$, then it is SFC everywhere on $\mathcal{U}$.*

Proof. Since $\mathcal{T}$ is SFC at $u_0 \in \mathcal{U}$, then for each $\mu_1 \succ \bar{0}$, $\exists\, \mu_2 \succ \bar{0}$ such that

$$\forall u \in \mathcal{U}, \quad \mathcal{N}_2 (\mathcal{T}(u) - \mathcal{T}(u_0), \mu_1) \succeq \mathcal{N}_1 (u - u_0, \mu_2).$$

Taking $v \in \mathcal{U}$ and replacing u by $u + u_0 - v$ we get,

$$\mathcal{N}_2(\mathcal{T}(u + u_0 - v) - \mathcal{T}(u_0), \mu_1) \succeq \mathcal{N}_1(u + u_0 - v - u_0, \mu_2)$$
$$\Rightarrow \quad \mathcal{N}_2(\mathcal{T}(u) - \mathcal{T}(v), \mu_1) \succeq \mathcal{N}_1(u - v, \mu_2).$$

Since $v \in \mathcal{U}$ is arbitrary, $\mathcal{T}$ is SFC on $\mathcal{U}$. □

Note 3.11. If $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ is a linear operator, then the Theorem 3.9 and Theorem 3.10 obviously hold.

4. Fuzzy Bounded Operators Over Type-2 FNLSs

In this section by SFB, WFB, FN, FB we refer to as strongly fuzzy bounded, weakly fuzzy bounded, fuzzy norm, fuzzy bounded respectively.

Definition 4.1. A linear operator $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ is said to be SFB on $\mathcal{U}$ if there exists a $m > 0$ such that

$$\forall \mu \succ \bar{0} \text{ and } \forall u \in \mathcal{U}, \quad \mathcal{N}_2(\mathcal{T}(u), \mu) \succeq \mathcal{N}_1\left(u, \frac{1}{m}\mu\right).$$

Definition 4.2. A linear operator $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{V}, \mathcal{N}_2)$ is said to be WFB on $\mathcal{U}$ if for any $\zeta_0 \in \mathcal{I}_0$ there exists $m_{\zeta_0} > 0$ such that

$$\forall\, \mu \succ \bar{0} \text{ and } \forall\, u \in \mathcal{U}, \quad \mathcal{N}_1\left(u, \frac{1}{m_{\zeta_0}}\mu\right) \succeq \zeta_0 \quad \Rightarrow \quad \mathcal{N}_2(\mathcal{T}(u), \mu) \succeq \zeta_0.$$

Definition 4.3. Let $(\mathcal{U}, \mathcal{N}_1)$ and $(\mathcal{V}, \mathcal{N}_2)$ be two type-2 FNLSs both satisfying (C6) and $\mathcal{T} : \mathcal{U} \to \mathcal{V}$ be a linear operator. Then $\mathcal{T}$ is said to be level fuzzy bounded if $\exists$ a $\xi > \bar{0}$ such that $\|\mathcal{T}(u)\|^{\zeta} \preceq \xi$, $\forall u \in \mathcal{U}$ with $\|u\|^{\zeta} \preceq \bar{1}$, $\forall \zeta \in \mathcal{I}_0$.

Now we give an example of a SFB linear operator.

Example 4.4. Let $(\mathcal{U} = \mathbb{R}, \|\cdot\|)$ be the usual NLS. Let $\mu \succ \bar{0}$ be an arbitrary fuzzy real number. Then we define μ_0 by

$$\mu_0 = \overline{\frac{a_0 + b_0}{2}}$$

where by $[a_0, b_0]$ we denote the closure of the support of fuzzy real number μ. Take $m, m_1 > 0$ be real numbers with $m > m_1$. Define

$$\mathcal{N}_1(u, \mu) = \begin{cases} \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right), & \text{if } \mu \succ \bar{0}, \\ \bar{0}, & \text{if } \mu = \bar{0}, \end{cases}$$

and

$$\mathcal{N}_2(u, \mu) = \begin{cases} \mu_0 \oslash \left(\mu_0 \oplus \overline{m_1\,|u|}\right), & \text{if } \mu \succ \bar{0}, \\ \bar{0}, & \text{if } \mu = \bar{0}. \end{cases}$$

We define $\mathcal{T} : (\mathbb{R}, \mathcal{N}_1) \to (\mathbb{R}, \mathcal{N}_2)$ by $\mathcal{T}(u) = ru$, where $r\,(\neq 0) \in \mathbb{R}$ be fixed.

We will show that $(\mathbb{R}, \mathcal{N}_1)$, $(\mathbb{R}, \mathcal{N}_2)$ are two type-2 FNLSs and $\mathcal{T}$ is a SFB linear operator.

Verification:

(C1) We have from definition $\mathcal{N}_1(u, \mu) = \bar{0}$, $\forall u \in \mathcal{U}$.

(C2) $\forall\, \mu \in \mathbb{R}^+(I)$ with $\mu \succ \bar{0}$,

$$\begin{aligned}
&\mathcal{N}_1(u, \mu) = \bar{1} \\
\Leftrightarrow\quad & \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right) = \bar{1} \\
\Leftrightarrow\quad & \left(\overline{\frac{a_0 + b_0}{2}}\right) \oslash \left(\overline{\frac{a_0 + b_0}{2}} \oplus \overline{m\,|u|}\right) = \bar{1} \\
\Leftrightarrow\quad & \frac{\frac{a_0+b_0}{2}}{\frac{a_0+b_0}{2} + m\,|u|} = 1 \\
\Leftrightarrow\quad & m\,|u| = 0 \\
\Leftrightarrow\quad & |u| = 0\,[\text{since } m > 0] \\
\Leftrightarrow\quad & u = \underline{0}
\end{aligned}$$

(C3) $\forall \mu \in \mathbb{R}^+(I)$ with $\mu \succ \bar{0}$ and $(c \neq 0) \in \mathbb{R}$, we get,

$$\mathcal{N}_1(cu, \mu) = \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|cu|}\right).$$

Again,

$$\begin{aligned}
\mathcal{N}_1\left(u, \frac{1}{c}\mu\right) &= \left(\frac{1}{c}\mu_0\right) \oslash \left(\left(\frac{1}{c}\mu_0\right) \oplus \overline{m\,|u|}\right) \\
&= \left(\overline{\frac{a_0 + b_0}{2c}}\right) \oslash \left(\left(\overline{\frac{a_0 + b_0}{2c}}\right) \oplus \overline{m\,|u|}\right) \\
&= \left(\overline{\frac{a_0 + b_0}{2}}\right) \oslash \left(\left(\overline{\frac{a_0 + b_0}{2}}\right) \oplus \overline{m\,|cu|}\right) \\
&= \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|cu|}\right)
\end{aligned}$$

(C4) Our aim is to show $\forall\, \mu, \delta \in \mathbb{R}^+(I)$ and $\forall\, u, v \in \mathcal{U}$,

$$\mathcal{N}_1(u + v, \mu \oplus \delta) \succeq \min\{\mathcal{N}_1(u, \mu), \mathcal{N}_1(v, \delta)\}.$$

If

(a) $\mu = \delta = \bar{0}$,

(b) $\mu \oplus \delta \succ \bar{0}$ and $\mu = \bar{0}, \delta \succ \bar{0}$ or $\mu \succ \bar{0}, \delta = \bar{0}$

these cases are very easy to check.

If $\mu \oplus \delta \succ \bar{0}$ and $\mu, \delta \succ \bar{0}$ we get

$$\begin{aligned}\mathcal{N}_1(u+v, \mu \oplus \delta) &= (\mu_0 \oplus \delta_0) \oslash \left(\mu_0 \oplus \delta_0 \oplus \overline{m\,|u+v|}\right) \\ &\succeq (\mu_0 \oplus \delta_0) \oslash \left(\mu_0 \oplus \delta_0 \oplus \overline{m\,|u|} \oplus \overline{m\,|v|}\right)\end{aligned}$$

Now,

$$\begin{aligned}&\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right) \succeq \delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|v|}\right) \\ \Rightarrow\quad &\left(\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right) \ominus \left(\delta_0 \oslash \left(\delta_0 \oplus \overline{m|v|}\right)\right) \succeq \bar{0} \\ \Rightarrow\quad &\left(\mu_0 \odot \overline{m|v|}\right) \ominus \left(\delta_0 \odot \overline{m\,|u|}\right) \succeq \bar{0}.\end{aligned}$$

Again,

$$\begin{aligned}&\left\{(\mu_0 \oplus \delta_0) \oslash \left(\mu_0 \oplus \delta_0 \oplus \overline{m\,|u|} \oplus \overline{m\,|v|}\right)\right\} \oslash \left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|v|}\right)\right\} \\ &\quad = \left\{\left(\mu_0 \odot \overline{m|v|}\right) \ominus \left(\delta_0 \odot \overline{m\,|u|}\right)\right\} \oslash \left\{\left(\mu_0 \oplus \delta_0 \oplus \overline{m\,|u|} \oplus \overline{m\,|v|}\right)\right. \\ &\qquad \left.\odot \left(\delta_0 \oplus \overline{m\,|v|}\right)\right\}\end{aligned}$$

So,

$$\begin{aligned}&\left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\} \succeq \left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|v|}\right)\right\} \\ \Rightarrow\quad &\left\{(\mu_0 \oplus \delta_0) \oslash \left(\mu_0 \oplus \delta_0 \oplus \overline{m\,|u+v|}\right)\right\} \succeq \left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|v|}\right)\right\}\end{aligned}$$

Similarly

$$\begin{aligned}&\left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|v|}\right)\right\} \succeq \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\} \\ \Rightarrow\quad &\left\{(\delta_0 \oplus \mu_0) \oslash \left(\delta_0 \oplus \mu_0 \oplus \overline{m\,|u+v|}\right)\right\} \succeq \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\}.\end{aligned}$$

Thus

$$\mathcal{N}_1(u+v, \mu \oplus \delta) \succeq \min\left\{\mathcal{N}_1(u, \mu), \mathcal{N}_1(v, \delta)\right\}.$$

(C5) Now we assume $\mu \prec \delta$.

When $\mu \prec \delta$ with $\mu = \bar{0}$, $\delta \succ \bar{0}$, then

$$\mathcal{N}_1(u, \mu) = \bar{0} \quad \text{and} \quad \mathcal{N}_1(u, \delta) = \left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|u|}\right)\right\} \succeq \bar{0}.$$

So $\mathcal{N}_1(u, \delta) \succeq \mathcal{N}_1(u, \mu)$.

When $\mu \prec \delta$ with $\mu \succ \bar{0}$ then $\delta \succ \bar{0}$ and so

$$\begin{aligned}&\left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|u|}\right)\right\} \ominus \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\} \\ &\quad = \left\{\left(\delta_0 \odot \overline{m\,|u|}\right) \ominus \left(\mu_0 \odot \overline{m\,|u|}\right)\right\} \oslash \left\{\left(\delta_0 \oplus \overline{m\,|u|}\right) \odot \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\} \\ &\quad \succeq \bar{0} \text{ [since } \delta_0 \succ \mu_0]\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad \left\{\delta_0 \oslash \left(\delta_0 \oplus \overline{m\,|u|}\right)\right\} \succeq \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right)\right\} \\
&\Rightarrow \quad \mathcal{N}_1\left(u, \delta\right) \succeq \mathcal{N}_1\left(u, \mu\right), \ \forall u \in \mathcal{U}.
\end{aligned}$$

Therefore $\mathcal{N}_1\left(u, \cdot\right)$ is an increasing mapping in $\mathbb{R}^+\left(I\right)$.

We see $\mu \to \overline{\infty} \Rightarrow b_0 \to \infty$ and so $\mu_0 \to \overline{\infty}$. Hence

$$\begin{aligned}
\mathcal{N}_1\left(u, \mu\right) &= \mu_0 \oslash \left(\mu_0 \oplus \overline{m\,|u|}\right) \\
&= \bar{1} \oslash \left(\bar{1} \oplus \left(\frac{\overline{m\,|u|}}{\frac{a_0+b_0}{2}}\right)\right) \\
&\succ \bar{1} \oslash \overline{(1+\epsilon)}
\end{aligned}$$

Again, $b_0 \to \infty \Rightarrow \left(\frac{\overline{m|u|}}{\frac{a_0+b_0}{2}}\right) \prec \bar{\epsilon}$ and so $\mu \to \overline{\infty} \ \Rightarrow \ \left(\frac{\overline{m|u|}}{\frac{a_0+b_0}{2}}\right) \to \bar{0}$ and hence $\mathcal{N}_1\left(u, \mu\right) \to \bar{1}$. Thus $(\mathbb{R}, \mathcal{N}_1)$ is a type-2 FNLS.

Following the above procedure we can also show that $(\mathbb{R}, \mathcal{N}_2)$ is a type-2 FNLS.

Next we have to show that $\mathcal{T}$ is SFB linear operator. It is easy to check $\mathcal{T}$ is a linear operator. Now let $N \geq |r|$.

$$\begin{aligned}
&\Rightarrow \quad mN \geq m_1\,|r| \,[\,\text{since}\, m > m_1 > 0] \\
&\Rightarrow \quad mN\,|u| \geq m_1\,|r|\,|u|\,, \quad \forall u \in \mathcal{U} = \mathbb{R} \\
&\Rightarrow \quad \left\{\mu_0 \oplus \overline{Nm\,|u|}\right\} \succeq \left\{\mu_0 \oplus \overline{m_1\,|ru|}\right\}, \ \forall \mu \succ \bar{0} \\
&\Rightarrow \quad \left\{\bar{1} \oslash \left(\mu_0 \oplus \overline{m_1\,|ru|}\right)\right\} \succeq \left\{\bar{1} \oslash \left(\mu_0 \oplus \overline{N\,|mu|}\right)\right\}, \quad \forall \mu \succ \bar{0} \\
&\Rightarrow \quad \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{m_1\,|ru|}\right)\right\} \succeq \left\{\mu_0 \oslash \left(\mu_0 \oplus \overline{N\,|mu|}\right)\right\}, \quad \forall \mu \succ \bar{0} \\
&\Rightarrow \quad \mathcal{N}_2\left(\mathcal{T}\left(u\right), \mu\right) \succeq \mathcal{N}_1\left(u, \frac{1}{N}\mu\right), \quad \forall u \in \mathcal{U}, \ \forall \mu \succ \bar{0}.
\end{aligned}$$

If $\mu = \bar{0}$ then the above relation holds for all $u \in \mathcal{U}$.

Thus $\mathcal{T}$ is SFB linear operator.

Theorem 4.5. *Let $(\mathcal{U}, \mathcal{N}_1)$ and $(\mathcal{V}, \mathcal{N}_2)$ be two type-2 FNLSs satisfying* (C6)*. Let $\mathcal{T} : \mathcal{U} \to \mathcal{V}$ be a SFB linear operator. Then there exists a constant $m > 0$ such that*

$$\left\|\mathcal{T}u\right\|^{\zeta_0} \preceq m\left\|u\right\|^{\zeta_0}, \quad \forall\, u \in \mathcal{U}, \quad \forall\, \zeta_0 \in \mathcal{I}_0.$$

Moreover, if $\left\|u\right\|^{\zeta_0} \preceq \bar{1}$, then

$$\left\|\mathcal{T}u\right\|^{\zeta_0} \preceq \bar{m}.$$

Proof. Since $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ is SFB, there exists a constant $m>0$ such that for all $\mu\succ\bar{0}$ and $u\in\mathcal{U}$,

$$\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\succeq\mathcal{N}_1\left(u,\frac{1}{m}\mu\right)\quad\Rightarrow\quad\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\succeq\mathcal{N}_1\left(mu,\mu\right). \tag{4.1}$$

Let $\zeta_0\in\mathcal{I}_0$. By definition,

$$\|\mathcal{T}u\|^{\zeta_0}=\wedge\left\{\mu\succ\bar{0}:\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\succeq\zeta_0\right\},$$

and

$$\|mu\|^{\zeta_0}=\wedge\left\{\mu\succ\bar{0}:\mathcal{N}_1\left(mu,\mu\right)\succeq\zeta_0\right\}.$$

From equation (4.1) we get

$$\left\{\mu\succ\bar{0}:\mathcal{N}_1\left(mu,\mu\right)\succeq\zeta_0\right\}\subseteq\left\{\mu\succ\bar{0}:\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\succeq\zeta_0\right\}$$
$$\Rightarrow\quad\|mu\|^{\zeta_0}\succeq\|\mathcal{T}u\|^{\zeta_0}.$$

Hence

$$\|\mathcal{T}u\|^{\zeta_0}\preceq m\|u\|^{\zeta_0},\quad\forall\,u\in\mathcal{U}.$$

If $\|u\|^{\zeta_0}\preceq\bar{1}$, then the above inequality immediately implies

$$\|\mathcal{T}u\|^{\zeta_0}\preceq\bar{m}.$$

□

Corollary 4.6. *Every SFB linear operator is a level fuzzy bounded linear operator.*

Theorem 4.7. *Let $(\mathcal{U},\mathcal{N}_1)$ and $(\mathcal{V},\mathcal{N}_2)$ be two type-2 FNLSs and $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ be a SFB linear operator. Then $\mathcal{T}$ sends each bounded set of $\mathcal{U}$ to a bounded set of $\mathcal{V}$.*

Proof. Let F be a bounded set in $(\mathcal{U},\mathcal{N}_1)$. Since $\mathcal{T}$ is SFB we can get a $m>0$ such that $\forall\mu\succ\bar{0}$ and $u\in\mathcal{U}$ we have

$$\mathcal{N}_2\left(\mathcal{T}\left(u\right),\mu\right)\succeq\mathcal{N}_1\left(u,\frac{1}{m}\mu\right). \tag{4.2}$$

Now F is bounded in $(\mathcal{U},\mathcal{N}_1)$ and so $\exists\,\zeta\in\mathcal{I}_0$ and $\mu_1\left(\zeta\right)\succ\bar{0}$ such that

$$\mathcal{N}_1\left(v,\mu_1\right)\succ\zeta,\quad\forall v\in F. \tag{4.3}$$

Let $\mu_2=m\mu_1$. Then $\mu_1=\frac{1}{m}\mu_2$ and $\mu_2\succ\bar{0}$.

Since the inequality (4.2) holds for any $\mu\succ\bar{0}$, if we replace μ by μ_2 in 4.2 we get

$$\mathcal{N}_2\left(\mathcal{T}\left(v\right),\mu_2\right)\succeq\mathcal{N}_1\left(v,\frac{1}{m}\mu_2\right) \tag{4.4}$$

Also from (4.3) we get

$$\mathcal{N}_1\left(v,\frac{1}{m}\mu_2\right)\succ\zeta,\quad\forall v\in F. \tag{4.5}$$

Combining (4.4) and (4.5) we get

$$\mathcal{N}_2\left(\mathcal{T}\left(v\right),\mu_2\right) \succeq \mathcal{N}_1\left(v,\frac{1}{m}\mu_2\right) \succ \zeta, \quad \forall v \in F$$

i.e,

$$\mathcal{N}_2\left(\mathcal{T}\left(v\right),\mu_2\right) \succ \zeta, \quad \forall v \in F.$$

Thus the image set of F i.e, $\mathcal{T}\left(F\right)$ is bounded in $(\mathcal{V},\mathcal{N}_2)$. □

Theorem 4.8. *Let $(\mathcal{U},\mathcal{N}_1)$ and $(\mathcal{V},\mathcal{N}_2)$ be two type-2 FNLSs and $\mathcal{T}: \mathcal{U} \to \mathcal{V}$ be a linear operator. If $\mathcal{T}$ is SFB then it is WFB.*

Proof. Proof is obvious. □

For the converse we consider the following example.

Example 4.9. Suppose $(\mathcal{U},\|\cdot\|)$ be a normed linear space over the field $\mathbb{R}$. For any fuzzy real number $\eta \succ \bar{0}$, let $\eta_0 = \overline{\frac{a+b}{2}}$, where $[a,b]$ presents the closure of support of the fuzzy real number η. We define two functions $\mathcal{N}_1$ and $\mathcal{N}_2$ as follows:

$\mathcal{N}_1 : \mathcal{U} \times \mathbb{R}^+\left(I\right) \to \mathcal{I}$ by

$$\mathcal{N}_1\left(u,\eta\right) = \begin{cases} \left(\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2\right), & \text{when } \eta \succ \overline{\|u\|}, \\ \bar{0}, & \text{otherwise,} \end{cases}$$

and $\mathcal{N}_2 : \mathcal{U} \times \mathbb{R}^+\left(I\right) \to \mathcal{I}$ by

$$\mathcal{N}_2\left(u,\eta\right) = \begin{cases} \eta_0 \oslash \left(\eta_0 \oplus \overline{\|u\|}\right), & \text{when } \eta \succ \bar{0} \\ \bar{0}, & \text{otherwise.} \end{cases}$$

We know that $\mathcal{N}_2$ is a type-2 fuzzy norm on $\mathcal{U}$. Next we shall show that $(\mathcal{U},\mathcal{N}_1)$ is a type-2 FNLS.

(C1) When $\eta = \bar{0}$ it is clear from the definition that $\mathcal{N}_1\left(u,\eta\right) = \bar{0}$.
(C2)

$$\begin{aligned} & \forall \eta \succ \bar{0}, \quad \mathcal{N}_1\left(u,\eta\right) = \bar{1}. \\ \Leftrightarrow \quad & \left(\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2\right) = \bar{1} \\ \Leftrightarrow \quad & \frac{\left(\frac{a+b}{2}\right)^2 - \|u\|^2}{\left(\frac{a+b}{2}\right)^2 + \|u\|^2} = 1 \\ \Leftrightarrow \quad & \|u\| = 0 \\ \Leftrightarrow \quad & u = \underline{0}. \end{aligned}$$

(C3) $\forall \eta \succ \bar{0}$ and $c\left(\neq 0\right) \in \mathbb{R}$, we get

$$\mathcal{N}_1\left(cu,\eta\right) = \left(\eta_0^2 \ominus \overline{\|cu\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|cu\|}^2\right).$$

Again,

$$\begin{aligned}\mathcal{N}_1\left(u,\frac{1}{c}\eta\right) &= \left(\frac{1}{c^2}\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\frac{1}{c^2}\eta_0^2 \oplus \overline{\|u\|}^2\right)\\ &= \left(\left(\overline{\frac{a+b}{2c}}\right)^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\left(\overline{\frac{a+b}{2c}}\right)^2 \oplus \overline{\|u\|}^2\right)\\ &= \left(\left(\overline{\frac{a+b}{2}}\right)^2 \ominus \overline{\|cu\|}^2\right) \oslash \left(\left(\overline{\frac{a+b}{2}}\right)^2 \oplus \overline{\|cu\|}^2\right)\\ &= \left(\eta_0^2 \ominus \overline{\|cu\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|cu\|}^2\right).\end{aligned}$$

(C4) Now our aim is to demonstrate that $\forall\, \eta, \xi \in \mathbb{R}^+(I)$ and $\forall\, u, v \in \mathcal{U}$,

$$\mathcal{N}_1(u+v, \eta \oplus \xi) \succeq \min\{\mathcal{N}_1(u,\eta), \mathcal{N}_1(v,\xi)\}.$$

(a) If $\eta \not\succ \overline{\|u\|}$ or $\xi \not\succ \overline{\|v\|}$ then the relation is obvious.
(b) Take $\eta \succ \overline{\|u\|}$ and $\xi \succ \overline{\|v\|}$.

Now we assume $\mathcal{N}_1(v,\xi) \succeq \mathcal{N}_1(u,\eta)$. So

$$\left(\xi_0^2 \ominus \overline{\|v\|}^2\right) \oslash \left(\xi_0^2 \oplus \overline{\|v\|}^2\right) \succeq \left(\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2\right)$$

then

$$\left\{\left(\xi_0^2 \odot \overline{\|u\|}^2\right) \ominus \left(\eta_0^2 \odot \overline{\|v\|}^2\right)\right\} \succeq \bar{0} \tag{4.6}$$

Also $\eta \oplus \xi \succ \overline{\|u\|} \oplus \overline{\|v\|} \succeq \overline{\|u+v\|}$. Now

$$\begin{aligned}&\mathcal{N}_1(u+v, \eta\oplus\xi)\\ &\quad= \left\{(\eta_0\oplus\xi_0)^2 \ominus \overline{\|u+v\|}^2\right\} \oslash \left\{(\eta_0\oplus\xi_0)^2 \oplus \overline{\|u+v\|}^2\right\}\\ &\quad\succeq \left\{(\eta_0\oplus\xi_0)^2 \ominus \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\} \oslash \left\{(\eta_0\oplus\xi_0)^2 \oplus \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\}.\end{aligned}$$

Again

$$\begin{aligned}&\left[\left\{(\eta_0\oplus\xi_0)^2 \ominus \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\} \oslash \left\{(\eta_0\oplus\xi_0)^2 \oplus \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\}\right]\\ &\qquad\ominus \left\{\left(\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2\right)\right\}\\ &\quad= \left[\left\{\overline{\|u\|}^2 \odot (\eta_0\oplus\xi_0)^2\right\} \ominus \left\{\eta_0^2 \odot \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\}\right.\\ &\qquad\left.\ominus \left\{\eta_0^2 \odot \left(\overline{\|u\|}\oplus\overline{\|v\|}\right)^2\right\} \oplus \left\{\overline{\|u\|}^2 \odot (\eta_0\oplus\xi_0)^2\right\}\right] \oslash \{\alpha\},\end{aligned}$$

$$\text{where } \alpha = \left\{ (\eta_0 \oplus \xi_0)^2 \oplus \left(\overline{\|u\|} \oplus \overline{\|v\|} \right)^2 \right\} \odot \left(\eta_0^2 \oplus \overline{\|u\|}^2 \right)$$

$$= 2 \left[\left\{ \overline{\|u\|}^2 \odot (\eta_0 \oplus \xi_0)^2 \right\} \ominus \left\{ \eta_0^2 \odot \left(\overline{\|u\|} \oplus \overline{\|v\|} \right)^2 \right\} \right] \oslash \{\alpha\}$$

$$= 2 \Big[\left(\overline{\|u\|}^2 \odot \xi_0^2 \right) \oplus \left(2\,\eta_0 \odot \xi_0 \odot \overline{\|u\|}^2 \right)$$

$$\ominus \left(\eta_0^2 \odot \overline{\|v\|}^2 \right) \ominus \left(2\,\eta_0^2 \odot \overline{\|u\|} \odot \overline{\|v\|} \right) \Big] \oslash \{\alpha\}$$

$$= 2 \Bigg[\left(\overline{\|u\|}^2 \odot \xi_0^2 \right) \ominus \left(\eta_0^2 \odot \overline{\|v\|}^2 \right) \oplus \Big[\left(2\,\eta_0 \odot \overline{\|u\|} \right) \odot \Big\{ \left(\xi_0 \odot \overline{\|u\|} \right)$$

$$\ominus \left(\eta_0 \odot \overline{\|v\|} \right) \Big\} \Big] \Bigg] \oslash \{\alpha\} \ \succeq\ \bar{0}, \quad \text{by (4.6)}.$$

Thus

$$\mathcal{N}_1 (u + v, \eta \oplus \xi) \succeq \mathcal{N}_1 (u, \eta), \text{ if } \quad \mathcal{N}_1 (v, \xi) \ \succeq\ \mathcal{N}_1 (u, \eta).$$

Similarly

$$\mathcal{N}_1 (u + v, \eta \oplus \xi) \succeq \mathcal{N}_1 (v, \xi), \text{ if } \quad \mathcal{N}_1 (u, \eta) \ \succeq\ \mathcal{N}_1 (v, \xi).$$

Hence

$$\mathcal{N}_1 (u + v, \eta \oplus \xi) \succeq \min \{\mathcal{N}_1 (u, \eta), \mathcal{N}_1 (v, \xi)\}.$$

(C5) Now $\eta \prec \xi$ and $\xi \not\succ \overline{\|u\|} \Rightarrow \mathcal{N}_1 (u, \xi) = \mathcal{N}_1 (u, \eta) = \bar{0}$ from definition. Also for $\eta \prec \xi, \eta \not\succ \overline{\|u\|}$ and $\xi \succ \overline{\|u\|}$, we have $\mathcal{N}_1 (u, \xi) \succeq \mathcal{N}_1 (u, \eta)$. Taking $\xi \ \succ\ \eta \ \succ\ \overline{\|u\|}$ we get

$$\left[\left\{ \left(\xi_0^2 \ominus \overline{\|u\|}^2 \right) \oslash \left(\xi_0^2 \oplus \overline{\|u\|}^2 \right) \right\} \ominus \left\{ \left(\eta_0^2 \ominus \overline{\|u\|}^2 \right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2 \right) \right\} \right]$$

$$= \left\{ \left(\xi_0^2 \odot \overline{\|u\|}^2 \right) \ominus \left(\eta_0^2 \odot \overline{|\|u\|\,|}^2 \right) \right\} \oslash \{\beta\},$$

$$\text{where } \beta = \left\{ \left(\xi_0^2 \oplus \overline{\|u\|}^2 \right) \odot \left(\eta_0^2 \oplus \overline{\|u\|}^2 \right) \right\}$$

$$= \left\{ \overline{\|u\|}^2 \odot \left(\xi_0^2 \ominus \eta_0^2 \right) \right\} \oslash \{\beta\} \ \succeq\ \bar{0}.$$

Thus $\mathcal{N}_1 (u, \xi) \ \succeq\ \mathcal{N}_1 (u, \eta)$. Again $\lim \mathcal{N}_1 (u, \eta) = \bar{1}, \forall u \in \mathcal{U}$. Hence $(\mathcal{U}, \mathcal{N}_1)$ is a type-2 FNLS. Now we take the linear transformation $\mathcal{T} : (\mathcal{U}, \mathcal{N}_1) \to (\mathcal{U}, \mathcal{N}_2)$ by $\mathcal{T}(u) = u$, $\forall u \in \mathcal{U}$. For $\zeta \in \mathcal{I}_0$, let closure of support of ζ be $[\zeta^-, \zeta^+]$ and $\overline{\zeta^+}$ be the fuzzy real number associated to ζ^+ as in Definition 2.1.

Define $k_\zeta = \frac{1}{1-\zeta^+}$, for all $\zeta \in \mathcal{I}_0$. Then for $\eta \succ \overline{\|u\|}$,

$$\mathcal{N}_1 \left(u, \frac{1}{k_\zeta} \eta \right) \succeq \zeta$$

$$\Leftrightarrow \quad \mathcal{N}_1\left(u, \frac{1}{k_\zeta}\eta\right) \succeq \overline{\zeta^+}$$

$$\Rightarrow \quad \left[\left\{\eta_0^2 \odot \overline{(1-\zeta^+)}^2\right\} \ominus \overline{\|u\|}^2\right] \oslash \left[\left\{\eta_0^2 \odot \overline{(1-\zeta^+)}^2\right\} \oplus \overline{\|u\|}^2\right] \succeq \overline{\zeta^+}$$

$$\Rightarrow \quad \left\{\eta_0^2 \odot \overline{(1-\zeta^+)}^2\right\} \ominus \overline{\|u\|}^2 \succeq \left\{\overline{\zeta^+} \odot \eta_0^2 \odot \overline{(1-\zeta^+)}^2\right\} \oplus \left\{\overline{\zeta^+} \odot \overline{\|u\|}^2\right\}$$

$$\Rightarrow \quad \left\{\eta_0^2 \odot \overline{(1-\zeta^+)}^2\right\} \ominus \left\{\overline{\zeta^+} \odot \overline{(1-\zeta^+)}^2 \odot \eta_0^2\right\} \succeq \overline{(1+\zeta^+)} \odot \overline{\|u\|}^2$$

$$\Rightarrow \quad \eta_0^2 \odot \overline{(1-\zeta^+)}^3 \succeq \overline{(1+\zeta^+)} \odot \overline{\|u\|}^2$$

$$\Rightarrow \quad \overline{\|u\|} \preceq \frac{\eta_0 \odot \overline{(1-\zeta^+)} \odot \overline{\sqrt{(1-\zeta^+)}}}{\overline{\sqrt{(1+\zeta^+)}}}$$

$$\Rightarrow \quad \eta_0 \oplus \overline{\|u\|} \preceq \left(\frac{\eta_0 \odot \overline{(1-\zeta^+)} \odot \overline{\sqrt{(1-\zeta^+)}}}{\overline{\sqrt{(1+\zeta^+)}}}\right) \oplus \eta_0$$

$$= \frac{\overline{(1-\zeta^+)} \odot \overline{\sqrt{(1-\zeta^+)}} \oplus \overline{\sqrt{(1+\zeta^+)}}}{\overline{\sqrt{(1+\zeta^+)}}} \odot \eta_0.$$

Also for $\eta \not\succ \overline{\|u\|}$, $\left\{\left(\eta_0^2 \ominus \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{\|u\|}^2\right)\right\} = \bar{0}$, we get,

$$\mathcal{N}_1\left(u, \frac{1}{k_\zeta}\eta\right) \succeq \zeta \Rightarrow \mathcal{N}_2\left(\mathcal{T}(u), \eta\right) \succeq \zeta, \quad \forall \zeta \in \mathcal{I}_0.$$

Thus $\mathcal{T}$ is WFB.

Again for $\eta \succ \overline{\|u\|}$, where $u \neq \underline{0}$,

$$\mathcal{N}_2\left(\mathcal{T}(u), \eta\right) \succeq \mathcal{N}_1\left(u, \frac{1}{k}\eta\right)$$

$$\Leftrightarrow \quad \left\{\eta_0 \oslash \left(\eta_0 \oplus \overline{\|u\|}\right)\right\} \succeq \left\{\left(\eta_0^2 \ominus \overline{k}^2 \odot \overline{\|u\|}^2\right) \oslash \left(\eta_0^2 \oplus \overline{k}^2 \odot \overline{\|u\|}^2\right)\right\}$$

$$\Leftrightarrow \quad \left\{\eta_0^3 \oplus \left(\overline{k}^2 \odot \eta_0 \odot \overline{\|u\|}^2\right)\right\}$$

$$\succeq \left\{\eta_0^3 \ominus \left(\overline{k}^2 \odot \eta_0 \odot \overline{\|u\|}^2\right) \oplus \left(\eta_0^2 \odot \overline{\|u\|}\right) \ominus \left(\overline{k}^2 \odot \overline{\|u\|}^3\right)\right\}$$

$$\Leftrightarrow \quad \overline{k}^2 \odot \eta_0 \odot \overline{\|u\|}^2 \succeq \left\{\left(\eta_0^2 \odot \overline{\|u\|}\right) \ominus \left(\overline{k}^2 \odot \eta_0 \odot \overline{\|u\|}^2\right) \ominus \left(\overline{k}^2 \odot \overline{\|u\|}^3\right)\right\}$$

$$\Leftrightarrow \quad \overline{k}^2 \odot \eta_0 \odot \overline{\|u\|} \succeq \left\{\eta_0^2 \ominus \left(\overline{k}^2 \odot \eta_0 \overline{\|u\|}\right) \ominus \left(\overline{k}^2 \odot \overline{\|u\|}^2\right)\right\} \text{ (as } \|u\| \neq 0)$$

$$\Leftrightarrow \quad \overline{k}^2 \odot \left\{\left(2\,\eta_0 \odot \overline{\|u\|}\right) \oplus \overline{\|u\|}^2\right\} \succeq \eta_0^2$$

$$\Leftrightarrow \quad \overline{k}^2 \succeq \left\{\eta_0^2 \oslash \left(\left(2\,\eta_0 \odot \overline{\|u\|}\right) \oplus \overline{\|u\|}^2\right)\right\}$$

$$\Leftrightarrow \quad k = \infty \text{ as } \eta \to \overline{\infty}.$$

Thus $\mathcal{T}$ is not SFB.

Theorem 4.10. *Let $(\mathcal{U},\mathcal{N}_1)$ and $(\mathcal{V},\mathcal{N}_2)$ be two type-2 FNLSs and $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ be a function satisfying the additive property. If $\mathcal{T}$ is SFB then $\mathcal{T}$ is SFC.*

Proof. Assume $\mathcal{T}$ is SFB. Then there exists a $m>0$ such that

$$\mathcal{N}_2\left(\mathcal{T}\left(u\right),\mu\right)\succeq\mathcal{N}_1\left(u,\frac{1}{m}\mu\right),\quad\forall u\in\mathcal{U},\mu\succ\bar{0}$$

$$\Rightarrow\quad\mathcal{N}_2\left(\mathcal{T}\left(u\right)-\mathcal{T}\left(\underline{0}\right),\mu\right)\succeq\mathcal{N}_1\left(u-\underline{0},\frac{1}{m}\mu\right),\quad\forall u\in\mathcal{U},\ \mu\succ\bar{0}$$

i.e,

$$\mathcal{N}_2\left(\mathcal{T}\left(u\right)-\mathcal{T}\left(\underline{0}\right),\mu\right)\succeq\mathcal{N}_1\left(u-\underline{0},\mu'\right),\quad\text{where }\mu'=\frac{1}{m}\mu\succ\bar{0}.$$

So, $\mathcal{T}$ is SFC at $\underline{0}$ and hence it is SFC on $\mathcal{U}$. □

Note 4.11. Let $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ be a linear operator. Then the Theorem 4.10 obviously holds.

Lemma 4.12. *Let $(\mathcal{U},\mathcal{N}_1)$ and $(\mathcal{V},\mathcal{N}_2)$ be two type-2 FNLSs satisfying (C6). Let $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ be a weakly fuzzy bounded (WFB) linear operator. Then, for each $\zeta_0\in\mathcal{I}_0$, there exists a constant $k_{\zeta_0}>0$ such that $\|\mathcal{T}u\|^{\zeta_0}\preceq k_{\zeta_0}\|u\|^{\zeta_0},\ \forall\, u\in\mathcal{U}$.*

Proof. Since $\mathcal{T}:\mathcal{U}\to\mathcal{V}$ is WFB, for a fixed $\zeta_0\in\mathcal{I}_0$ there exists $k_{\zeta_0}>0$ such that $\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\ \succeq\ \mathcal{N}_1\left(u,\frac{1}{k_{\zeta_0}}\mu\right),\quad\forall\,\mu\succ\bar{0},\quad\forall\,u\in\mathcal{U}.$

Consequently,

$$\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\ \succeq\ \mathcal{N}_1\left(k_{\zeta_0}u,\mu\right),\quad\forall\,\mu\succ\bar{0},\quad\forall\,u\in\mathcal{U}.\tag{4.7}$$

By definition,

$$\|\mathcal{T}u\|^{\zeta_0}\ =\ \wedge\left\{\mu\succ\bar{0}:\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\succeq\zeta_0\right\},$$

and

$$\|k_{\zeta_0}u\|^{\zeta_0}=\wedge\left\{\mu\succ\bar{0}:\mathcal{N}_1\left(k_{\zeta_0}u,\mu\right)\succeq\zeta_0\right\}.$$

From (4.7), we have

$$\left\{\mu\succ\bar{0}:\mathcal{N}_1\left(k_{\zeta_0}u,\mu\right)\ \succeq\ \zeta_0\right\}\subseteq\left\{\mu\succ\bar{0}:\mathcal{N}_2\left(\mathcal{T}u,\mu\right)\ \succeq\ \zeta_0\right\},$$

which implies

$$\|k_{\zeta_0}u\|^{\zeta_0}\ \succeq\ \|\mathcal{T}u\|^{\zeta_0}.$$

$$\therefore\quad\|\mathcal{T}u\|^{\zeta_0}\ \preceq\ k_{\zeta_0}\|u\|^{\zeta_0},\quad\forall\,u\in\mathcal{U}.$$ □

Note 4.13. Henceforth, the collection of all strongly fuzzy bounded (SFB) linear operators from a type-2 FNLS $(\mathcal{U},\mathcal{N}_1)$ into a type-2 FNLS $(\mathcal{V},\mathcal{N}_2)$ will be denoted by $\mathcal{B}\left(\mathcal{U},\mathcal{V}\right)$.

Theorem 4.14. *The set $\mathcal{B}(\mathcal{U},\mathcal{V})$ is a linear space over $\mathbb{R}$.*

Proof. Let $\mathcal{T}_1,\mathcal{T}_2\in\mathcal{B}(\mathcal{U},\mathcal{V})$. Then there exist constants $k_1,k_2>0$ such that

$$\mathcal{N}_2(\mathcal{T}_1u,\eta)\succeq\mathcal{N}_1\left(u,\frac{1}{k_1}\eta\right),\quad\forall\,\eta\succ\bar{0},\quad\forall\,u\in\mathcal{U},$$
$$\mathcal{N}_2(\mathcal{T}_2u,\eta)\succeq\mathcal{N}_1\left(u,\frac{1}{k_2}\eta\right),\quad\forall\,\eta\succ\bar{0},\quad\forall\,u\in\mathcal{U}.$$

Let $a_0,b_0\in\mathbb{R}$ and $u\in\mathcal{U}$. Then

$$\begin{aligned}\mathcal{N}_2((a_0\mathcal{T}_1+b_0\mathcal{T}_2)u,\eta)&=\mathcal{N}_2(a_0\mathcal{T}_1u+b_0\mathcal{T}_2u,\eta)\\&\succeq\min\left\{\mathcal{N}_2\left(\mathcal{T}_1(a_0u),\tfrac{1}{2}\eta\right),\mathcal{N}_2\left(\mathcal{T}_2(b_0u),\tfrac{1}{2}\eta\right)\right\}\\&\succeq\min\left\{\mathcal{N}_1\left(a_0u,\tfrac{1}{2k_1}\eta\right),\mathcal{N}_1\left(b_0u,\tfrac{1}{2k_2}\eta\right)\right\}.\end{aligned}$$

(4.8)

$$\therefore\quad\mathcal{N}_2((a_0\mathcal{T}_1+b_0\mathcal{T}_2)u,\eta)\succeq\min\left\{\mathcal{N}_1\left(u,\tfrac{1}{2|a_0|k_1}\eta\right),\mathcal{N}_1\left(u,\tfrac{1}{2|b_0|k_2}\eta\right)\right\}.$$

Let $k=\max\{2|a_0|k_1,\,2|b_0|k_2\}+1$. Then $k\geq2|a_0|k_1$ and $k\geq2|b_0|k_2$.

$$\text{So}\quad\frac{1}{2|a_0|k_1}\eta\succeq\frac{1}{k}\eta,\qquad\frac{1}{2|b_0|k_2}\eta\succeq\frac{1}{k}\eta.$$

Hence,

$$\mathcal{N}_1\left(u,\tfrac{1}{2|a_0|k_1}\eta\right)\succeq\mathcal{N}_1\left(u,\tfrac{1}{k}\eta\right),\qquad\mathcal{N}_1\left(u,\tfrac{1}{2|b_0|k_2}\eta\right)\succeq\mathcal{N}_1\left(u,\tfrac{1}{k}\eta\right).$$

Consequently,

$$\min\left\{\mathcal{N}_1\left(u,\tfrac{1}{2|a_0|k_1}\eta\right),\mathcal{N}_1\left(u,\tfrac{1}{2|b_0|k_2}\eta\right)\right\}\succeq\mathcal{N}_1\left(u,\tfrac{1}{k}\eta\right).$$

From (4.8), it follows that

$$\mathcal{N}_2((a_0\mathcal{T}_1+b_0\mathcal{T}_2)u,\eta)\succeq\mathcal{N}_1\left(u,\tfrac{1}{k}\eta\right),\quad\forall\,\eta\succ\bar{0}.$$

Thus $(a_0\mathcal{T}_1+b_0\mathcal{T}_2)\in\mathcal{B}(\mathcal{U},\mathcal{V})$, and therefore $\mathcal{B}(\mathcal{U},\mathcal{V})$ is a linear space over $\mathbb{R}$. □

Lemma 4.15. *Let $\{\zeta_i:i\in\Lambda\}$ and $\{\eta_i:i\in\Lambda\}$ be two families such that $\zeta_i,\eta_i\in\mathcal{I}_0$ for all $i\in\Lambda$. Then*

$$\vee\{\zeta_i\oplus\eta_i:i\in\Lambda\}\preceq[\vee\{\zeta_i:i\in\Lambda\}]\oplus[\vee\{\eta_i:i\in\Lambda\}].$$

Proof. For each $i\in\Lambda$, we have

$$\zeta_i\preceq\vee\{\zeta_j:j\in\Lambda\}\quad\text{and}\quad\eta_i\preceq\vee\{\eta_j:j\in\Lambda\}.$$

Hence,

$$\zeta_i\oplus\eta_i\preceq[\vee\{\zeta_j:j\in\Lambda\}]\oplus[\vee\{\eta_j:j\in\Lambda\}].$$

Taking the supremum over all $i \in \Lambda$, we obtain

$$\vee \{\zeta_i \oplus \eta_i : i \in \Lambda\} \preceq [\vee \{\zeta_i : i \in \Lambda\}] \oplus [\vee \{\eta_i : i \in \Lambda\}]. \qquad \square$$

Lemma 4.16. *Let $(\mathcal{U}, \mathcal{N}_1)$ and $(\mathcal{V}, \mathcal{N}_2)$ be two type-2 FNLSs satisfying* (C6)*, and let $\mathcal{T} : \mathcal{U} \to \mathcal{V}$ be a strongly fuzzy bounded (SFB) linear operator. For $\zeta_0 \in \mathcal{I}_0$, define*

$$\|\mathcal{T}\|_*^{\zeta_0} = \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|\mathcal{T}(u)\|^{\zeta_0} \right),$$

where $\|u\|^{\zeta_0}$ and $\|\mathcal{T}(u)\|^{\zeta_0}$ denote the Felbin-type fuzzy norms with respect to $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Then $\|\cdot\|_*^{\zeta_0}$ defines a Felbin-type fuzzy norm on $\mathcal{B}(\mathcal{U}, \mathcal{V})$ for each $\zeta_0 \in \mathcal{I}_0$.

Proof. Fix $\zeta_0 \in \mathcal{I}_0$. If $\mathcal{T} = O$, then $\mathcal{T}u = \underline{0}$ for all $u \in \mathcal{U}$, and hence $\|\mathcal{T}\|_*^{\zeta_0} = \bar{0}$. Conversely, suppose that $\|\mathcal{T}\|_*^{\zeta_0} = \bar{0}$. Then

$$\|\mathcal{T}u\|^{\zeta_0} = \bar{0}, \quad \forall\, u \in \mathcal{U} \text{ with } \|u\|^{\zeta_0} \preceq \bar{1}. \tag{4.9}$$

Let $u \in \mathcal{U}$ with $u \neq \underline{0}$ and

$$\left[\|u\|^{\zeta_0}\right]_\alpha = [a_1^\alpha, b_1^\alpha], \quad 0 < \alpha \leq 1, \text{ with } \sup_{\alpha \in (0,1]} b_1^\alpha = b > 0.$$

Then

$$\frac{1}{b}u \in \mathcal{U} \quad \text{and} \quad \left\| \frac{1}{b}u \right\|^{\zeta_0} \preceq \bar{1}.$$

By (4.9),

$$\begin{aligned} \left\|\mathcal{T}\left(\tfrac{1}{b}u\right)\right\|^{\zeta_0} = \bar{0} \quad &\Rightarrow \quad \left\|\tfrac{1}{b}\mathcal{T}(u)\right\|^{\zeta_0} = \bar{0} \\ &\Rightarrow \quad \tfrac{1}{b}\|\mathcal{T}(u)\|^{\zeta_0} = \bar{0} \\ &\Rightarrow \quad \|\mathcal{T}(u)\|^{\zeta_0} = \bar{0} \\ &\Rightarrow \quad \mathcal{T}(u) = \underline{0}. \end{aligned}$$

Since $u \neq \underline{0}$ was arbitrary, it follows that $\mathcal{T}(u) = \underline{0}$ for all $u \in \mathcal{U}$, and hence $\mathcal{T} = O$.

Next, let $c \in \mathbb{R}$. Then

$$\begin{aligned} \|c\mathcal{T}\|_*^{\zeta_0} &= \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|(c\mathcal{T})u\|^{\zeta_0} \right) \\ &= \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|c\,\mathcal{T}(u)\|^{\zeta_0} \right) \\ &= \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(|c|\ \|\mathcal{T}(u)\|^{\zeta_0} \right) \\ &= |c|\ \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|\mathcal{T}(u)\|^{\zeta_0} \right) \\ &= |c|\ \|\mathcal{T}\|_*^{\zeta_0}. \end{aligned}$$

Finally, for $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(\mathcal{U}, \mathcal{V})$,

$$\begin{aligned}\|\mathcal{T}_1 + \mathcal{T}_2\|_*^{\zeta_0} &= \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|(\mathcal{T}_1 + \mathcal{T}_2)(u)\|^{\zeta_0} \right) \\ &\preceq \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|\mathcal{T}_1(u)\|^{\zeta_0} \right) \oplus \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|\mathcal{T}_2(u)\|^{\zeta_0} \right) \\ &= \|\mathcal{T}_1\|_*^{\zeta_0} \oplus \|\mathcal{T}_2\|_*^{\zeta_0}.\end{aligned}$$

Therefore, $\|\cdot\|_*^{\zeta_0}$ satisfies all the axioms of a Felbin-type fuzzy norm on $\mathcal{B}(\mathcal{U}, \mathcal{V})$ for each $\zeta_0 \in \mathcal{I}_0$. □

Theorem 4.17. *Let $(\mathcal{U}, \mathcal{N}_1)$ and $(\mathcal{V}, \mathcal{N}_2)$ be two type-2 FNLSs satisfying* (C6)*, and let $\mathcal{T} \in \mathcal{B}(\mathcal{U}, \mathcal{V})$. For $\zeta_0 \in \mathcal{I}_0$, define*

$$\|\mathcal{T}\|_*^{\zeta_0} = \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \left(\|\mathcal{T}(u)\|^{\zeta_0} \right),$$

where $\|u\|^{\zeta_0}$ and $\|\mathcal{T}(u)\|^{\zeta_0}$ denote the Felbin-type fuzzy norms with respect to $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. For $\zeta \in \mathcal{I}_0$, set

$$\|\mathcal{T}\|^{\zeta} = \vee_{\zeta_0 \preceq \zeta} \|\mathcal{T}\|_*^{\zeta_0}.$$

Then the family $\left\{ \|\cdot\|^{\zeta} : \zeta \in \mathcal{I}_0 \right\}$ forms a family of Felbin-type fuzzy norms on $\mathcal{B}(\mathcal{U}, \mathcal{V})$ satisfying $\zeta_1 \preceq \zeta_2 \Rightarrow \|\mathcal{T}\|^{\zeta_1} \preceq \|\mathcal{T}\|^{\zeta_2}, \zeta_1, \zeta_2 \in \mathcal{I}_0$. Moreover, if $\mathcal{N} : \mathcal{B}(\mathcal{U}, \mathcal{V}) \times \mathbb{R}^+(I) \to \mathcal{I}$ is defined by

$$\begin{aligned}\mathcal{N}(\mathcal{T}, \mu) &= \vee \left\{ \zeta \in \mathcal{I}_0 : \|\mathcal{T}\|^{\zeta} \preceq \mu \right\}, \quad \text{when } (\mathcal{T}, \mu) \neq (O, \bar{0}) \\ &= \bar{0}, \quad \text{when } (\mathcal{T}, \mu) = (O, \bar{0}) \text{ or } \left\{ \zeta \in \mathcal{I}_0 : \|\mathcal{T}\|^{\zeta} \preceq \mu \right\} = \phi,\end{aligned}$$

then $\mathcal{N}$ is a type-2 fuzzy norm on $\mathcal{B}(\mathcal{U}, \mathcal{V})$.

Proof. By Lemma 4.16, $\|\mathcal{T}\|_*^{\zeta_0}$ is a Felbin-type fuzzy norm for each $\zeta_0 \in \mathcal{I}_0$.

(i) From the definition, $\|\mathcal{T}\|^{\zeta} \succeq \bar{0}$ for all $\zeta \in \mathcal{I}_0$.

(ii) Suppose $\|\mathcal{T}\|^{\zeta} = \bar{0}$. Then $\vee_{\zeta_0 \preceq \zeta} \|\mathcal{T}\|_*^{\zeta_0} = \bar{0}$, which implies $\|\mathcal{T}\|_*^{\zeta_0} = \bar{0}$ for all $\zeta_0 \preceq \zeta$. Hence $\mathcal{T} = O$.

Conversely, if $\mathcal{T} = O$, then $\|\mathcal{T}\|_*^{\zeta_0} = \bar{0}$ for all $\zeta_0 \in \mathcal{I}_0$, and therefore $\|\mathcal{T}\|^{\zeta} = \bar{0}$ for all $\zeta \in \mathcal{I}_0$.

(iii) For any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\|\lambda\mathcal{T}\|^{\zeta} &= \vee_{\zeta_0 \preceq \zeta} \|\lambda\mathcal{T}\|_*^{\zeta_0} \\ &= \vee_{\zeta_0 \preceq \zeta} |\lambda| \, \|\mathcal{T}\|_*^{\zeta_0} \\ &= |\lambda| \, \|\mathcal{T}\|^{\zeta}.\end{aligned}$$

(iv) Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(\mathcal{U}, \mathcal{V})$. Then

$$\begin{aligned}\|\mathcal{T}_1 + \mathcal{T}_2\|^{\zeta} &= \vee_{\zeta_0 \preceq \zeta} \|\mathcal{T}_1 + \mathcal{T}_2\|_*^{\zeta_0} \\ &\preceq \vee_{\zeta_0 \preceq \zeta} \left(\|\mathcal{T}_1\|_*^{\zeta_0} \oplus \|\mathcal{T}_2\|_*^{\zeta_0} \right) \\ &\preceq \left(\vee_{\zeta_0 \preceq \zeta} \|\mathcal{T}_1\|_*^{\zeta_0} \right) \oplus \left(\vee_{\zeta_0 \preceq \zeta} \|\mathcal{T}_2\|_*^{\zeta_0} \right)\end{aligned}$$

where the last inequality follows from Lemma 4.15.

Hence, $\|\mathcal{T}_1 + \mathcal{T}_2\|^{\zeta} \preceq \|\mathcal{T}_1\|^{\zeta} \oplus \|\mathcal{T}_2\|^{\zeta}, \forall\, \zeta \in \mathcal{I}_0$.

If $\zeta_1 \preceq \zeta_2$, then

$$\begin{aligned}\|\mathcal{T}\|^{\zeta_1} &= \vee_{\zeta_0 \preceq \zeta_1} \|\mathcal{T}\|_*^{\zeta_0} \\ &\preceq \vee_{\zeta_0 \preceq \zeta_2} \|\mathcal{T}\|_*^{\zeta_0} \\ &= \|\mathcal{T}\|^{\zeta_2}.\end{aligned}$$

Thus $\left\{ \|\cdot\|^{\zeta} : \zeta \in \mathcal{I}_0 \right\}$ is an increasing family of Felbin-type fuzzy norms on $\mathcal{B}(\mathcal{U}, \mathcal{V})$.

Finally, by Theorem 2.7, the mapping $\mathcal{N}$ defined as above is a type-2 fuzzy norm on $\mathcal{B}(\mathcal{U}, \mathcal{V})$. □

Note 4.18. Consider the vector space $\mathbb{R}$ over $\mathbb{R}$ endowed with $\mathcal{N}_2$, where $\mathcal{N}_2 : \mathbb{R} \times \mathbb{R}^+(I) \to \mathcal{I}$ is defined by

$$\begin{aligned}\mathcal{N}_2(u, \mu) &= \bar{1}, \quad \text{if } \|u\| \prec \mu, \\ &= \bar{0}, \quad \text{otherwise}\end{aligned}$$

Where $\|u\|$ denotes the usual Felbin norm on $\mathbb{R}$, defined as

$$\begin{aligned}\|u\|(s) &= 1, \quad \text{if } s = |u|, \\ &= 0, \quad \text{otherwise.}\end{aligned}$$

We show that $\mathcal{N}_2$ is a type-2 fuzzy norm on $\mathbb{R}$ and hence $(\mathbb{R}, \mathcal{N}_2)$ is a type-2 fuzzy normed linear space.

Moreover, it can be observed that $\mathcal{N}_2$ satisfies condition (C6), that is,

$$\wedge \{\mu \succ \bar{0} : \mathcal{N}_2(u, \mu) \succ \bar{0}\} = \bar{0} \quad \Rightarrow \quad u = 0.$$

Proof. (C1) By definition, when $\mu = \bar{0}$, we have $\mathcal{N}_2(u, \mu) = \bar{0}$.

(C2) For all $\mu \succ \bar{0}$, $\mathcal{N}_2(\underline{0}, \mu) = \bar{1}$.

Conversely, if $\mathcal{N}_2(u, \mu) = \bar{1}$ for all $\mu \succ \bar{0}$, then $\|u\| \prec \mu$ for all $\mu \succ \bar{0}$, which implies $\|u\| = \bar{0}$ and hence $u = 0$. Thus,

$$(\forall\, \mu \succ \bar{0},\, \mathcal{N}_2(u, \mu) = \bar{1}) \quad \Leftrightarrow \quad u = 0.$$

(C3) Let $c \neq 0$ and $\mu \succ \bar{0}$. If $\mathcal{N}_2(cu, \mu) = \bar{1}$, then $\|cu\| \prec \mu$, which implies $\|u\| \prec \frac{1}{|c|}\mu$, and hence $\mathcal{N}_2\left(u, \frac{1}{|c|}\mu\right) = \bar{1}$.

Conversely, if $\mathcal{N}_2\left(u, \frac{1}{|c|}\mu\right) = \bar{1}$, then $\|cu\| \prec \mu$, and so $\mathcal{N}_2(cu, \mu) = \bar{1}$. Thus,

$$\mathcal{N}_2(cu, \mu) = \mathcal{N}_2\left(u, \tfrac{1}{|c|}\mu\right).$$

(C4) Suppose $\mathcal{N}_2(u+v, \mu_1 \oplus \mu_2) = \bar{0}$. If $\|u\| \prec \mu_1$ and $\|v\| \prec \mu_2$, then

$$\|u+v\| \preceq \|u\| \oplus \|v\| \prec \mu_1 \oplus \mu_2$$

which implies $\mathcal{N}_2(u+v, \mu_1 \oplus \mu_2) = \bar{1}$, a contradiction. Hence,

$$\mathcal{N}_2(u+v, \mu_1 \oplus \mu_2) \succeq \min\{\mathcal{N}_2(u, \mu_1), \mathcal{N}_2(v, \mu_2)\}.$$

(C5) Let $\mu_2 \succeq \mu_1$. If $\mathcal{N}_2(u, \mu_2) = \bar{0}$ and $\mathcal{N}_2(u, \mu_1) = \bar{1}$, then $\|u\| \prec \mu_1 \preceq \mu_2$, which implies $\mathcal{N}_2(u, \mu_2) = \bar{1}$, a contradiction. Therefore,

$$\mu_2 \succeq \mu_1 \Rightarrow \mathcal{N}_2(u, \mu_2) \succeq \mathcal{N}_2(u, \mu_1).$$

Moreover, from the definition,

$$\lim_{\mu \to \overline{\infty}} \mathcal{N}_2(u, \mu) = \bar{1}.$$

(C6) Suppose

$$\wedge\{\mu \succ \bar{0} : \mathcal{N}_2(u, \mu) \succ \bar{0}\} = \bar{0} \quad \text{and} \quad u \neq 0.$$

Then $[\|u\|]_\alpha = [|u|, |u|]$ for all $\alpha \in (0, 1]$. Hence there exists a family $\{\mu_\lambda : \lambda \in \Lambda\}$ such that $\|u\| \prec \mu_\lambda$ for all $\lambda \in \Lambda$ and $\wedge_{\lambda \in \Lambda} \mu_\lambda = \bar{0}$. This is possible only when $u = 0$, a contradiction. Therefore, $u = 0$.

Hence, $\mathcal{N}_2$ is a type-2 fuzzy norm on $\mathbb{R}$ and $(\mathbb{R}, \mathcal{N}_2)$ is a type-2 fuzzy normed linear space. □

Definition 4.19. A strongly (or weakly) FB linear operator from a type-2 FNLS $(\mathcal{U}, \mathcal{N}_1)$ to $(\mathbb{R}, \mathcal{N}_2)$ where $\mathcal{N}_2$ is referred to as in Note 4.18 is called strongly (or weakly) FB linear functional.

Definition 4.20. A level FB linear operator from a type-2 FNLS $(\mathcal{U}, \mathcal{N}_1)$ to $(\mathbb{R}, \mathcal{N}_2)$ where $\mathcal{N}_2$ is referred to as in Note 4.18 is called level FB linear functional.

The set of all level FB linear functionals over $(\mathcal{U}, \mathcal{N}_1)$ is denoted by $\mathcal{BL}(\mathcal{U}, \mathbb{R})$.

Note 4.21. The set $\mathcal{BL}(\mathcal{U}, \mathbb{R})$ forms a linear space over $\mathbb{R}$.

To prove the next theorem we consider the fuzzy real numbers as in Remark 2.3. All the results of type-2 FNLSs are true with this definition. In [20] it has been proved that this type of fuzzy real numbers satisfies the following prperties:

- Property 1: A fuzzy real number $a \succ \bar{0} \Leftrightarrow a \succ\succ \bar{0}$.
- Property 2: A fuzzy real number $a \prec \bar{1} \Leftrightarrow a \prec\prec \bar{1}$.
- Property 3: If $a \succ \bar{0}$, then we can get a $n \in \mathbb{N}$ such that $\bar{0} \prec\prec \overline{1/n} \prec\prec a$.

Theorem 4.22. *Suppose $\mathcal{N}'$ be a type-2 FN on $\mathcal{BL}(\mathcal{U}, \mathbb{R})$ defined as in Proposition 4.17. Then $(\mathcal{BL}(\mathcal{U}, \mathbb{R}), \mathcal{N}')$ is a type-2 fuzzy Banach space (i.e. complete type-2 FNLS).*

Proof. Take $\{\mathcal{T}_p\}$ a Cauchy sequence in $(\mathcal{BL}(\mathcal{U}, \mathbb{R}), \mathcal{N}')$.

Then for each $\zeta \in \mathcal{I}_0, \eta \succ \bar{0}$,

$$\begin{aligned}
&\exists k(\zeta, \eta) \text{ such that } \mathcal{N}'(\mathcal{T}_p - \mathcal{T}_q, \eta) \succ \zeta, \quad \forall p, q \geq k(\zeta, \eta) \\
\Rightarrow \quad & \vee \left\{ \zeta_1 \in \mathcal{I}_0 : \|\mathcal{T}_p - \mathcal{T}_q\|^{\zeta_1} \preceq \eta \right\} \succ \zeta, \quad \forall p, q \geq k(\zeta, \eta) \\
\Rightarrow \quad & \exists \zeta_0 (> \zeta) \in \mathcal{I}_0 \text{ such that } \|\mathcal{T}_p - \mathcal{T}_q\|^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta) \\
\Rightarrow \quad & \vee_{\zeta_2 \preceq \zeta_0} \|\mathcal{T}_p - \mathcal{T}_q\|_*^{\zeta_2} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta) \\
\Rightarrow \quad & \|\mathcal{T}_p - \mathcal{T}_q\|_*^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta) \\
\Rightarrow \quad & \vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \|(\mathcal{T}_p - \mathcal{T}_q)(u)\|^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta)
\end{aligned}$$

(4.10)

$$\Rightarrow \quad \|\mathcal{T}_p(u) - \mathcal{T}_q(u)\|^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta), \quad \forall \|u\|^{\zeta_0} \preceq \bar{1}$$

$$\text{Let } u (\neq \underline{0}) \in \mathcal{U} \quad \text{and} \quad \left[\|u\|^{\zeta_0}\right]_\alpha = [a^\alpha, b^\alpha], \quad 0 < \alpha \leq 1.$$

Also let $\sup_{\alpha \in (0,1]} b^\alpha = b$. Then $b > 0$ and $\frac{1}{b}u \in \mathcal{U}$ with $\left\|\frac{1}{b}u\right\|^{\zeta_0} \preceq \bar{1}$.

From (4.10) we have

$$\begin{aligned}
&\left\| \mathcal{T}_p\left(\frac{1}{b}u\right) - \mathcal{T}_q\left(\frac{1}{b}u\right) \right\|^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta), \quad \forall u \in \mathcal{U} \\
\Rightarrow \quad & \frac{1}{b}\|\mathcal{T}_p(u) - \mathcal{T}_q(u)\|^{\zeta_0} \preceq \eta, \quad \forall p, q \geq k(\zeta, \eta), \quad \forall u \in \mathcal{U} \\
\Rightarrow \quad & \|\mathcal{T}_p(u) - \mathcal{T}_q(u)\|^{\zeta_0} \preceq b\eta, \quad \forall p, q \geq k(\zeta, \eta), \quad \forall u \in \mathcal{U} \\
\Rightarrow \quad & \{\mathcal{T}_p(u)\} \quad \text{is a Cauchy sequence in} \quad \left(\mathbb{R}, \|\cdot\|^{\zeta_0}\right).
\end{aligned}$$

Since $\left(\mathbb{R}, \|\cdot\|^{\zeta_0}\right)$ is complete, $\exists\, v \in \mathbb{R}$ such that $\lim \mathcal{T}_p(u) = v$.

Define $\mathcal{T} : \mathcal{U} \to \mathbb{R}$ such that $\mathcal{T}(u) = \lim \mathcal{T}_p(u) = v$, $\forall u \in \mathcal{U}$.

Since $\lim \mathcal{T}_p(cu_1 + u_2) = c \lim \mathcal{T}_p(u_1) + \lim \mathcal{T}_p(u_2)$, $\mathcal{T}$ is linear.

Again from the relation (4.10) we have,

$$\|\mathcal{T}_p(u) - \mathcal{T}(u)\|^{\zeta_0} \preceq \eta, \quad \forall p \geq k(\zeta, \eta), \quad \forall \|u\|^{\zeta_0} \preceq \bar{1}. \tag{4.11}$$

Also since $\mathcal{T}_k \in (\mathcal{BL}(\mathcal{U},\mathbb{R}),\mathcal{N}')$ $\exists$ a $\xi > \bar{0}$ such that $\|\mathcal{T}_k(u)\|^{\zeta} \preceq \xi$, $\forall u \in \mathcal{U}$ with $\|u\|^{\zeta} \preceq \bar{1}$, $\forall \zeta \in \mathcal{I}_0$.

In particular,

$$\|\mathcal{T}_k(u)\|^{\zeta_0} \preceq \xi, \quad \forall u \in \mathcal{U}, \quad \|u\|^{\zeta_0} \preceq \bar{1} \tag{4.12}$$

Now,

$$\begin{aligned}\|\mathcal{T}(u)\|^{\zeta_0} &\preceq \|\mathcal{T}_k(u) - T(u)\|^{\zeta_0} \oplus \|\mathcal{T}_k(u)\|^{\zeta_0} \\ &\preceq \eta \oplus \xi, \quad \forall u \in \mathcal{U} \text{ with } \|u\|^{\zeta_0} \preceq \bar{1}, [\text{ by (4.11) and (4.12)}].\end{aligned}$$

Therefore for each $\zeta \in \mathcal{I}_0$,

$$\begin{aligned}&\exists \zeta_0 (> \zeta) \text{ such that } \|\mathcal{T}(u)\|^{\zeta_0} \preceq \eta \oplus \xi, \quad \forall u \in \mathcal{U} \text{ with } \|u\|^{\zeta_0} \preceq \bar{1} \\ &\Rightarrow \quad \|\mathcal{T}(u)\|^{\zeta} \preceq \eta \oplus \xi, \quad \forall u \in \mathcal{U} \text{ with } \|u\|^{\zeta} \preceq \bar{1}, \forall \zeta \in \mathcal{I}_0 \\ &\Rightarrow \quad \mathcal{T} \text{ is level fuzzy bounded.}\end{aligned}$$

Again from relation (4.11) we have,

$$\begin{aligned}&\vee_{\|u\|^{\zeta_0} \preceq \bar{1}} \|(\mathcal{T}_p - \mathcal{T})(u)\|^{\zeta_0} \preceq \eta, \quad \forall p \geq k(\zeta,\eta) \\ &\Rightarrow \quad \vee_{\zeta_2 \preceq \zeta_0} \|\mathcal{T}_p - \mathcal{T}\|_*^{\zeta_2} \preceq \eta, \quad \forall p \geq k(\zeta,\eta)\end{aligned}$$

So, we have a

$$\begin{aligned}&\zeta_0 (> \zeta) \in \mathcal{I}_0 \quad \text{such that} \quad \|\mathcal{T}_p - \mathcal{T}\|^{\zeta_0} \preceq \eta, \quad \forall p \geq k(\zeta,\eta) \\ &\Rightarrow \quad \vee \left\{\zeta_1 \in \mathcal{I}_0 : \|\mathcal{T}_p - \mathcal{T}\|^{\zeta_1} \preceq \eta\right\} \succ \zeta, \quad \forall p \geq k(\zeta,\eta) \\ &\Rightarrow \quad \mathcal{N}'(\mathcal{T}_p - \mathcal{T}, \eta) \succ \zeta, \quad \forall p \geq k(\zeta,\eta).\end{aligned}$$

Therefore $\mathcal{T}_p \to \mathcal{T}$ in $\mathcal{BL}(\mathcal{U},\mathbb{R})$.

Hence $\mathcal{BL}(\mathcal{U},\mathbb{R})$ is complete. □

5. Conclusion and Future Work

This paper is a continuation of a previous work done on type-2 fuzzy normed linear spaces. Here we have introduced a variety of of fuzzy continuous functions namely strongly fuzzy continuous (SFC), fuzzy continuous(FC), sequentially fuzzy continuous(SEFC) and studied interrations between them. Also we have studied different types of fuzzy boundedness (FB) viz. SFB, WFB, level fuzzy bounded etc. Finally we observe that the set of all level fuzzy bounded linear operators forms a complete type-2 fuzzy normed linear space. We have also examined some of analogues theorems, results that can be found in functional analysis by using these concepts whereas type-2 fuzzy norm is a type-2 fuzzy set whose range set is fuzzy real numbers. It is an open question to investigate that whether the type-2 FNLS formed by the set of all SFB linear transformation is complete or not.

In future we will like to extend the four fundamental theorems viz. open mapping theorem, Hahn-Banach theorem, uniform boundedness theorem, closed graph theorem in type-2 FNLS setting. We will also work on proving some fixed point theorems in these type-2 FNLSs.

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