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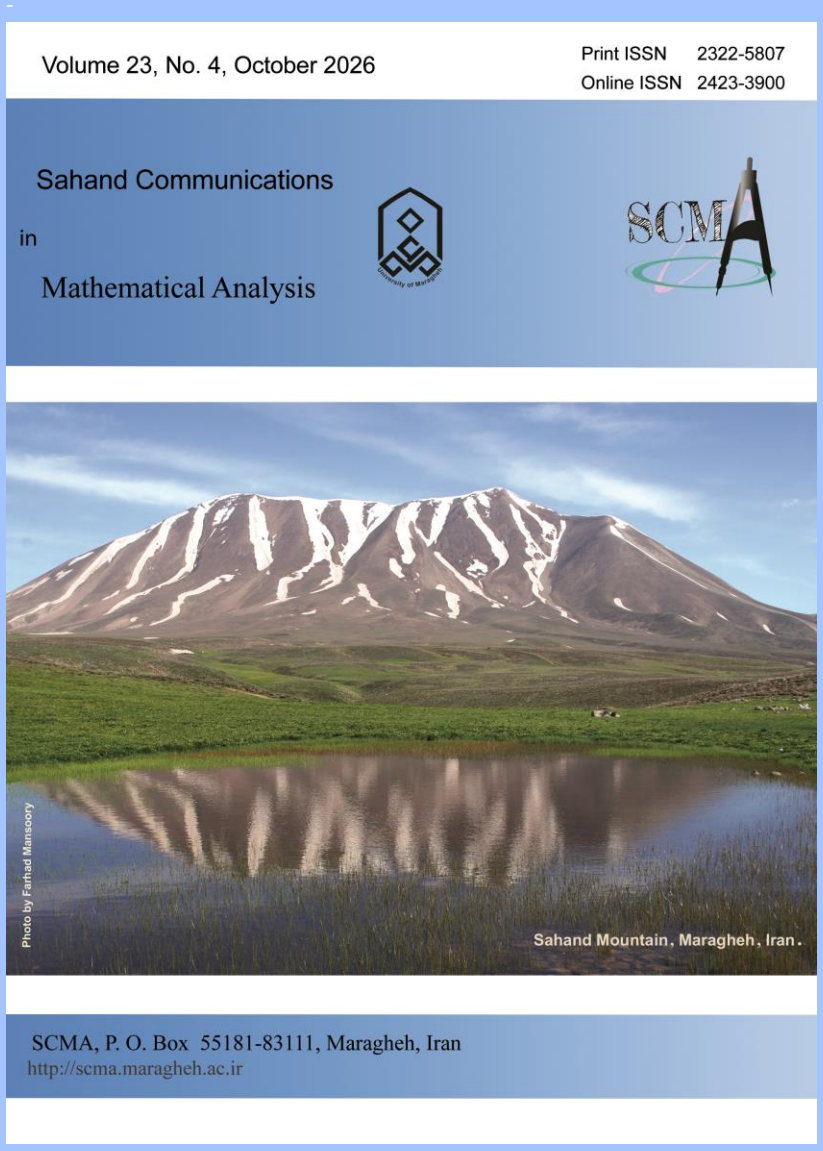

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P-Superquadratic Stochastic Processes via Fractional Stochastic Calculus with Applications to Divergence Measures

Dawood Khan[1], Saad Ihsan Butt[2*] and Ghulam Jallani[3]

Abstract. The notion of P-superquadratic stochastic processes, a new extension and improvement of conventional superquadratic stochastic processes, is presented and thoroughly developed in this work. In the context of mean-square stochastic calculus, we methodically examine their fundamental structural characteristics and use them to develop sophisticated versions of Jensen's inequality and (Hermite-Hadamard) $\mathbb{H}.\mathbb{H}$-type inequalities. The stochastic (Riemann-Liouville) $\mathbb{RL}$ fractional integrals are used to further extend these inequalities to their fractional counterparts, offering a more comprehensive analytical toolset for fractional stochastic analysis. Theoretical conclusions are supported by thorough tabular representations and graphical visualisations that are built from a variety of instructive cases. We also formulate new classes of stochastic divergence measures to show that the proposed framework is applicable in information theory. We give direct access to the commands used to create all graphs and tables, as well as the times at which each computation was executed, enabling repeatability and computational transparency.

1. Introduction

The mathematical concept of convexity holds a central position in both engineering and mathematics, as it underpins a broad spectrum of theoretical and applied phenomena. At the heart of convex analysis

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lie convex sets and convex functions, which serve as indispensable tools in simplifying highly complex systems and optimizing intricate mathematical models. Due to their well-behaved structure, particularly the existence of a unique global minimum problems involving convex functions become far more tractable, making them pivotal in the field of optimization. Other than its purely mathematical interest, convexity has profound meanings in many areas such as control theory, economics and systems engineering. In control theory, for example, convexity ensures that system designs are stable and robust under diverse operating conditions. In economics, convexity also plays a role in the description of consumer preferences and analysis of market behavior, hence enabling efficient decision-making and resource allocation. Interestingly, convexity is not only a theoretical or academic concept but one with real-world application. Engineers and practitioners encounter convex structures on a day-to-day basis in trying to optimize system performance to achieve greater efficiency and reliability in operations. Perhaps the most aesthetically pleasing application of convexity is seen in integral inequalities, where convex functions play a central role within this application. Integral inequalities are integral to mathematical analysis, especially in terms of approximating or bounding integral values. Among the most important of these is the $\mathbb{H}.\mathbb{H}$ inequality that provides the estimates of the average value of a convex function over a given interval [21, 45, 46]. This interrelation between convexity and inequalities of the integral type makes the subject a rich and powerful area of contemporary mathematics research.

Fractional calculus has emerged as a remarkably powerful and widely sought-after mathematical tool, capturing the interest not only of physicists and engineers, but also perhaps unexpectedly computer scientists [26]. By generalizing the framework of classical calculus to include derivatives and integrals of arbitrary (non-integer) order, fractional calculus offers a sophisticated foundation for modeling systems with memory and hereditary properties. This distinctive capability has significantly contributed to the recent surge in its popularity and applicability across various scientific and engineering domains. Beyond its practical uses in modeling and simulation, fractional calculus has profoundly influenced the evolution of the theory of inequality, particularly through the formulation of fractional integral inequalities. Over the past decade, substantial research efforts have been devoted to developing and refining such inequalities. A notable milestone in this trajectory was the seminal work by Sarikaya et al. in 2013 [39], where they established a fractional analogue of the $\mathbb{H}.\mathbb{H}$ inequality within the framework of $\mathbb{RL}$ fractional integrals. This generalization of the classical $\mathbb{H}.\mathbb{H}$ inequality not only

extended the theoretical boundaries of convex analysis but also laid the groundwork for further exploration in the field of fractional inequalities, offering new insights into the bounding behavior of convex functions under fractional integration.

Following the groundbreaking presentation of the $\mathbb{H}.\mathbb{H}$ inequalities for $\mathbb{RL}$ fractional integrals, there has been significant progress in extending $\mathbb{H}.\mathbb{H}$-type inequalities across various definitions of fractional integrals. Researchers have explored these inequalities in the context of numerous fractional operators, including the generalized proportional fractional integrals [35], Sarikaya fractional integrals [24], Katugampola fractional integrals [19], k-fractional integrals [47], ψ-$\mathbb{RL}$-fractional integrals [32], $(k-p)$-fractional integrals [43], generalized $\mathbb{RL}$-fractional integrals [22] and conformable fractional integrals [15]. Alongside these developments, a wide array of inequalities of fractional order has also been established, reflecting the growing depth and diversity of the field. These include fractal Hadamard-Mercer-type inequalities [17], Simpson-type inequalities [34], Euler-Maclaurin-type inequalities [25], Bullen-type inequalities [48], and Ostrowski-type inequalities [38], among others.

For readers seeking a comprehensive overview of recent advancements in fractional integral inequalities, several up-to-date contributions are recommended, such as those found in [7, 49] and the references therein.

In probability theory and statistics, a stochastic process is defined as a collection of random variables indexed by time or space, representing the evolution of a system under uncertainty. They specify how a system evolves in what appears to be a random way, often over discrete periods of time or space. "Stochastic" is an adjective applied to processes occurring at irregular or random rates. Stochastic processes can be applied to model random change within systems over time. For instance, there are many random parameters that may result in fluctuations of bacterial growth, but long-term forecasting is not impossible. Similar to thermal noise in electrical systems causing random variations in voltage or current, such seemingly random activity can be modeled with simple stochastic processes and these can provide insight into the dynamics of the system.

Stochastic processes are mathematical tools that physicists and mathematicians have long used to describe a wide range of events, from stock market volatility to randomly moving gas particles or population shifts in ecosystems. Whenever there is unpredictability, these processes and other elements like molecular interactions may frequently be modelled using stochastic models, which have extensive applications in engineering and research. They represent systems based on the mobility of different molecules in physics and they mimic population dynamics or the

development of illnesses in biology. The importance of stochastic processes lies in their ability to systematically explain systems that behave randomly. They are essential in fields like computer science, data transfer, cryptography and signal processing, where randomness or specific outcomes at each stage are crucial. As a result, they are vital for research and solving real world problems. By modeling these processes numerically, scientists and engineers can forecast future trends, optimize systems and develop solutions that account for environmental uncertainties. Other key applications include secure communication through cryptography, data compression methods and system state forecasting.

Convex stochastic processes, which expand on the notion of classical convex functions to include stochastic processes systems including randomness were first proposed by Nikodem [36] in 1980. Additionally, he discovered that these stochastic processes have several characteristics with classical convex functions. The Jensen convex mapping, which is essential to the theory of inequalities, is one useful tool in this area. Several important inequalities may be derived from this mapping and their applicability to stochastic processes was demonstrated by Skowronski [42] in 1992. Expanding on these concepts, Kotrys [29] showed that, in particular, for convex stochastic processes, the upper and lower bounds of the $\mathbb{H}.\mathbb{H}$ inequality may be found using integral operators. Kotrys's study demonstrated how the $\mathbb{H}.\mathbb{H}$ inequality, a crucial finding in convex analysis, might be modified for use with stochastic processes. In recent years, a large number of research on various forms of convexity for stochastic processes have been published in the literature. Okur et al. [37] used the idea of p-convexity, which is a particular kind of convexity, to construct the $\mathbb{H}.\mathbb{H}$ inequality for stochastic processes. Erhan [40] made a further contribution by building these inequalities with the use of another version of convexity called s-convexity. Budak et al. [16] generalised the work of other academics by presenting these inequalities using the concept of h-convexity. As mentioned in the sources [11, 23, 33], there have been many research investigating different kinds of convexity for individuals who are interested in the most recent developments in stochastic processes.

Superquadraticity provides tighter and better bounds than general convexity, which greatly improves the theory of integral inequalities. Such an improvement is highly desired in applied mathematics applications, where better approximation translates into higher modelling accuracy and in optimisation problems, where the quality of the boundary estimate determines the optimal solution. Therefore, superquadraticity expands the theory foundation of inequality structures and increases

their practical applicability. It successfully provides a strong analytical framework that supports the creation and application of integral inequalities in a vast array of applied and mathematical problems.

The idea of superquadraticity, which extends the classical notion of convexity, was initially introduced by Abramovich et al. [1]. This foundational work laid the groundwork for a broader analytical structure capable of yielding sharper inequalities than those derived from standard convex functions. Later, Abramovich and co-authors [2] provided a formal definition of superquadratic functions along with key theoretical insights that deepened the understanding of this function class. Building upon this foundation, Li and Chen [30] advanced the theory by examining the fractional perspective of the inequalities of $\mathbb{H}.\mathbb{H}$-type using $\mathbb{RL}$ fractional integrals, marking a significant extension of superquadraticity into the realm of fractional calculus. Further developments were made by Alomari et al. [6], who initiated the notion of h-superquadraticity, investigated their structural properties and established a new subclass within the broader family. In a significant contribution to interval analysis and generalized function theory, Khan and Butt [27] introduced cr-order interval-valued superquadratic functions along with their fractional analogs, highlighting their utility through various applications. Extending this line of research, Khan et al. [28] proposed the (P, m)-superquadratic function, a more general form that incorporates examples, key properties, integral inequalities, and practical applications, thereby enriching the functional landscape of superquadraticity. Continuing their exploration of information-theoretic applications, Butt and Khan [18] developed information inequalities for superquadratic functions using the tools of interval calculus, connecting functional inequalities with measures of information. A foundational theoretical advance was made by Banić et al. [10], who refined classical inequalities and solidified the theoretical underpinnings of superquadratic functions, paving the way for future research. The works cited in [8, 9] make substantial contributions to the development of superquadratic stochastic analysis and its applications in information theory. In [8], the authors introduce and advance the theory of superquadratic stochastic processes by establishing their fractional counterparts and demonstrating their usefulness through information-theoretic measures. This study provides a strong foundation for further advancements in fractional stochastic calculus. Similarly, the contribution in [9] extends this framework by incorporating fractional inclusion analysis based on the center–radius total order relation. The authors present a comprehensive structure for interval-type stochastic generalizations and further link their results to applications in information theory. Together, these two papers form

a coherent and meaningful progression in the study of superquadratic stochastic processes and their fractional and interval-valued extensions.

Following advances in convexity, fractional calculus, stochastic processes and superquadraticity, we complete the missing link in research: the investigation of P-superquadratic stochastic processes, their inequalities and information theory applications. Following the expansion of convex functions, superquadratic functions provide more accuracy and a broader range of applications in mathematical inequalities. Therefore, compared to previous results on convex stochastic processes, our findings represent a significant advance. For the first time, we provide detailed instances to illustrate our theoretical conclusions, backed up with detailed graphical representations. We also carry out our analysis to information theory applications, broadening the boundaries of theoretical understanding and operational applicability.

The paragraph that follows provides an explanation of the paper's structure:

Following the review of the required background knowledge and pertinent data regarding convexity, superquadraticity, convex stochastic processes and their inequalities in Sections 1 and 2, we use 2D and 3D graphical representations to analyze the stochastic version of superquadraticity and their features. More importantly, in Section 3, we develop new Jensen's and $\mathbb{H}.\mathbb{H}$-type inequalities. Then, in Section 4, Inequalities of $\mathbb{H}.\mathbb{H}$-type are obtained in fractional form. To evaluate the effectiveness of the results, examples and graphic descriptions of the findings are also taken into account. How the findings may be used to information theory is covered in Part 5. In the final section, part 6, a clear and concise conclusion is drawn, accompanied by an overview of potential future developments inspired by the current work.

2. Preliminaries

The terms of significance for fractional integrals, convexity and superquadraticity are defined in this section along with the associated inequalities.

Definition 2.1 ([21])**.** Let the function $\Psi : [\kappa_o, \omega_o] \subset \Re \longrightarrow \Re$ be a convex then

$$\Psi(\beta \mathrm{y}_1 + (1-\beta)\mathrm{y}_2) \leq \beta\Psi(\mathrm{y}_1) + (1-\beta)\Psi(\mathrm{y}_2),$$

is valid for every y_1, $\mathrm{y}_2 \in [\kappa_o, \omega_o]$ and $\beta \in [0,1]$.

The $\mathbb{H}.\mathbb{H}$ inequality is a fundamental result that provides both lower and upper bounds for mean values and stands among the earliest inequalities derived using convex functions. It serves as a key tool in numerical integration, particularly for estimating errors in the trapezoidal

and midpoint rules. Several further developments and related findings connected to this inequality can be found in [3, 44]. This classical result is formally stated as follows.

Theorem 2.2 ([21])**.** $\mathbb{H}.\mathbb{H}$*'s inequality states that if the function* $\Psi : \mathfrak{J} \subseteq \Re \longrightarrow \Re$*, is convex in* $\mathfrak{I} \subseteq \Re$*, where* $\omega_o > \kappa_o$*. The following statement satisfies:*

(2.1)
$$\Psi\left(\frac{\kappa_o+\omega_o}{2}\right) \leq \frac{1}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi(\mathrm{y})d\mathrm{y} \leq \frac{\Psi(\kappa_o)+\Psi(\omega_o)}{2}, \quad \forall \kappa_o, \omega_o \in \mathfrak{I}$$

Definition 2.3. $\mathbb{RL}$-fractional integrals with order $\beta \geq 0$ with $\kappa_o \geq 0$ are defined as

$$\mathfrak{I}^{\beta}_{\kappa_o{}^+}\Psi(\mathrm{y}) = \frac{1}{\Gamma(\beta)}\int_{\kappa_o}^{\mathrm{y}}(\mathrm{y}-\mathfrak{s})^{\beta-1}\Psi(\mathfrak{s})d\mathfrak{s}, \quad (\mathrm{y} > \kappa_o),$$
$$\mathfrak{I}^{\beta}_{\omega_o{}^-}\Psi(\mathrm{y}) = \frac{1}{\Gamma(\beta)}\int_{\mathrm{y}}^{\omega_o}(\mathfrak{s}-\mathrm{y})^{\beta-1}\Psi(\mathfrak{s})d\mathfrak{s}, \quad (\mathrm{y} < \omega_o).$$

The notations $\mathfrak{I}^{\beta}_{\kappa_o{}^+}\Psi(\mathrm{y})$ and $\mathfrak{I}^{\beta}_{\omega_o{}^-}\Psi(\mathrm{y})$ are the left and right sided operators. Where Γ is a stated as gamma function and given by

$$\Gamma(\beta) = \int_0^{\infty}\mathfrak{s}^{\beta-1}e^{-\mathfrak{s}}d\mathfrak{s}.$$

Definition 2.4. A function $\Psi : [0,\infty) \longrightarrow \Re$ is superquadratic, if

$$\Psi(\mathfrak{s}) \geq \Psi(\mathrm{y}) + \mathfrak{C}_{\mathrm{y}}(\mathfrak{s}-\mathrm{y}) + \Psi(|\mathfrak{s}-\mathrm{y}|), \tag{2.2}$$

holds for every $\mathfrak{s} \geq 0$, $\mathfrak{C}_{\mathrm{y}} \in \Re$, where $\mathrm{y} \geq 0$ and $\mathfrak{C}_{\mathrm{y}}$ is a constant.

Remark 2.5. If the inequality (2.2) is reversed then Ψ is called subquadratic.

As an example, we may use $\Psi(\mathrm{y}) = \mathrm{y}^{\mathfrak{k}}$. The function Ψ is superquadratic for $\mathrm{y} \geq 0$ and $\mathfrak{k} \geq 2$, but subquadratic for $0 \leq \mathfrak{k} < 2$. Let $\mathfrak{C}_{\mathrm{y}} = \mathfrak{k}\mathrm{y}^{\mathfrak{q}_o-1}$ in this instance. At $\mathfrak{k} = 2$, equality is also preserved in (2.2).

More precisely, any randomly chosen superquadratic function meets the three supplementary conditions outlined in Lemma 2.6:

Lemma 2.6. *Let* $\Psi : [0,\infty) \to \Re$ *be a superquadratic function, then*

- $\Psi(0) \leq 0$.
- *Provided that* Ψ *is differentiable at* $\mathrm{y} > 0$ *and satisfies* $\Psi(0) = \Psi'(0) = 0$*, it follows that* $\Psi'(\mathrm{y}) = \mathfrak{C}_{\mathrm{y}}$.
- *If* Ψ *is positive and* $\Psi(0) = \Psi'(0) = 0$ *on* $(0,\infty)$*, then* Ψ *is convex.*

Two major inequalities that contribute to the growth and development of superquadraticity are inequalities of Jensen's and $\mathbb{H}.\mathbb{H}$'s types, which are the most significant and frequently applied results.

Theorem 2.7 ([2])**.** *Let* $\Psi\colon J \subset [0,\infty[\longrightarrow \Re$ *be a superquadratic then*

$$\sum_{i=1}^{\mathfrak{n}} \beta_i \Psi(\mathrm{y}_i) \geq \Psi(\overline{\mathrm{y}}) + \sum_{i=1}^{\mathfrak{n}} \beta_i \Psi(|\mathrm{y}_i - \overline{\mathrm{y}}|),$$

holds for all $\beta_i \in (0,1)$ *and* $\mathrm{y}_i \geq 0$*, where* $\overline{\mathrm{y}} = \sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i$ *and* $\sum_{i=1}^{\mathfrak{n}} \beta_i = 1$.

Banić et al. [10] contributed to the field of superquadraticity by formulating the following $\mathbb{H}.\mathbb{H}$-type inequalities.

Theorem 2.8. *Let* $\Psi\colon J \subset [0,\infty[\longrightarrow \Re$ *be a superquadratic on* $J=[\kappa_o, \omega_o]$ *then*

$$\begin{aligned}
&\Psi\left(\frac{\kappa_o+\omega_o}{2}\right) + \frac{1}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o} \Psi\left(\left|\mathrm{y} - \frac{\kappa_o+\omega_o}{2}\right|\right) d\mathrm{y} \\
&\quad \leq \frac{1}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o} \Psi(\mathrm{y}) d\mathrm{y} \\
&\quad \leq \frac{\Psi(\kappa_o)+\Psi(\omega_o)}{2} \\
&\qquad - \frac{1}{(\omega_o-\kappa_o)^2}\int_{\kappa_o}^{\omega_o} \left[(\omega_o-\mathrm{y})\Psi(\mathrm{y}-\kappa_o) + (\mathrm{y}-\kappa_o)\Psi(\omega_o-\mathrm{y})\right] d\mathrm{y}.
\end{aligned}$$

The line of support concept of superquadraticity is defined in 2.4. In the following, we offer an additional definition.

Definition 2.9 ([1])**.** Let $\Psi\colon J \subset [0,\infty[\longrightarrow \Re$ be a superquadratic function then

(2.3)
$$\begin{aligned}
\Psi((1-\beta)\mathrm{y}_1 + \beta\mathrm{y}_2) \leq (1-\beta)\Psi(\mathrm{y}_1) + \beta\Psi(\mathrm{y}_2) - \beta\Psi((1-\beta)|\mathrm{y}_1 - \mathrm{y}_2|) \\
- (1-\beta)\Psi(\beta|\mathrm{y}_1 - \mathrm{y}_2|),
\end{aligned}$$

holds for every $\beta \in (0,1)$ and $\mathrm{y}_1, \mathrm{y}_2 \in J$.

Remark 2.10. Subquadraticity of the function Ψ corresponds to the reversal of the inequality sign in (2.3).

Theorem 2.11 ([30])**.** *Let* $\Psi\colon J \subset [0,\infty[\longrightarrow \Re$ *be a superquadratic function on* $J=[\kappa_o, \omega_o]$ *then*

$$\begin{aligned}
&\Psi\left(\frac{\kappa_o+\omega_o}{2}\right) + \frac{\beta}{2(\omega_o-\kappa_o)^{\beta}}\int_{\kappa_o}^{\omega_o} \Psi\left(\left|\frac{\kappa_o+\omega_o}{2} - \mathrm{y}\right|\right) \\
&\qquad \times \left((\omega_o-\mathrm{y})^{\beta-1} + (\mathrm{y}-\kappa_o)^{\beta-1}\right) d\mathrm{y}
\end{aligned}$$

$$\leq \frac{\Gamma(1+\beta)}{2(\omega_o-\kappa_o)^{\beta}}\left(\mathfrak{J}^{\beta}_{\kappa_o{}^{+}}\Psi(\omega_o)+\mathfrak{J}^{\beta}_{\omega_o{}^{-}}\Psi(\kappa_o)\right)$$

$$\leq \frac{\Psi(\kappa_o)+\Psi(\omega_o)}{2}-\frac{\beta}{2(\omega_o-\kappa_o)^{\beta}}\int_{\kappa_o}^{\omega_o}\left(\left(\frac{\mathrm{y}-\kappa_o}{\omega_o-\kappa_o}\right)\Psi(\omega_o-\mathrm{y})\right.$$

$$\left.+\left(\frac{\omega_o-\mathrm{y}}{\omega_o-\kappa_o}\right)\Psi(\mathrm{y}-\kappa_o)\right)\left((\omega_o-\mathrm{y})^{\beta-1}+(\mathrm{y}-\kappa_o)^{\beta-1}\right)d\mathrm{y},$$

where $\mathfrak{J}^{\beta}_{\kappa_o{}^{+}}$ *and* $\mathfrak{J}^{\beta}_{\omega_o{}^{-}}$ *are given in Definition 2.3.*

Definition 2.12 ([28])**.** A function $\Psi : J \subseteq \Re \to \Re$ is called P-superquadratic on $J=[0,\mathfrak{k}]$ for $\mathfrak{k}>0$; if

$$\Psi\left(\beta\kappa_o+(1-\beta)\omega_o\right)\leq \Psi(\kappa_o)-\Psi((1-\beta)|\kappa_o-\omega_o|)$$
$$+\Psi(\omega_o)-\Psi(\beta|\kappa_o-\omega_o|),$$

holds for any $\kappa_o,\omega_o\in J$ and $\beta\in[0,1]$. Ψ is defined as a P-subquadratic function whenever its negation $-\Psi$ satisfies the P-superquadratic condition.

Definition 2.13 ([36])**.** A mapping $\Psi:\prod\to\Re$ is considered a random variable if it is $\mathfrak{A}$-measurable. It is defined on the probability space $(\prod,\mathfrak{A},\mathbb{P})$, where $\prod$ denotes the set of all possible outcomes, $\mathfrak{A}$ is a σ-algebra representing the collection of measurable events (subsets of $\prod$) and $\mathbb{P}$ is a probability measure that assigns probabilities to these events.

Example 2.14. Imagine a coin toss in which $\prod=\{H=Head,T=Tail\}$, $\mathfrak{A}$ includes complete subsets of $\prod$ and $\mathbb{P}$ assigns a probability of $\frac{1}{2}$ to each conceivable circumstance. So, $\Psi(T)=1$ and $\Psi(H)=0$ would be the definitions of a random variable Ψ.

Definition 2.15 ([36])**.** If for all $\mathrm{y}\in J$, $\Psi(\mathrm{y},\cdot)$ is a random variable, then a map $\Psi:J\times\prod\to\Re$ is called a stochastic process.

Example 2.16. Think about taking daily readings of the temperature at noon for a week. $\prod$ can represent all possible weather conditions in this instance, $J=\{1,2,\ldots,7\}$ represents each day of the week, and $\Psi(\mathrm{y},\omega)$ gives the temperature on day y, provided that ω is met. As a random variable, $\Psi(\mathrm{y},\cdot)$ for each fixed day y shows how the temperature varies under different meteorological conditions. Temperature variation over time, as influenced by changes in y, is modeled by the stochastic process $\Psi(\mathrm{y},\cdot)$.

Definition 2.17 ([36])**.** A stochastic process $\Psi:J\times\prod\to\Re$ is called to be continuous on $J\subset\Re$, if for every $\mathrm{y}\in J$, we have

$$\mathbb{P}-\lim_{\mathrm{y}\to\mathrm{k}}\Psi(\mathrm{y},\cdot)=\Psi(\mathrm{k},\cdot).$$

The notation $\mathbb{P}$- lim is used to represent convergence in probability within a probability space.

Definition 2.18 ([36])**.** A stochastic process $\Psi : J \times \prod \to \Re$ is called to be mean-square continuous on $J \subset \Re$, if for every $\mathrm{y} \in J$, we have

$$\lim_{\mathrm{y}\to\mathrm{k}} E\left[(\Psi(\mathrm{y},\cdot) - \Psi(\mathrm{k},\cdot))^2\right] = 0.$$

Where the random variable's expectation is represented by $E[\Psi(\mathrm{y},\cdot)]$.

Remark 2.19. Probability continuity is obviously shown by mean-square continuity, while the contrary is not true.

Definition 2.20 ([36])**.** A stochastic process $\Psi : J \times \prod \to \Re$ is called to be mean-square differentiable on $J \subset \Re$ at k, if a random variable $\Psi' : J \times \prod \to \Re$ is given. This means that:

$$\Psi'(\mathrm{y},\cdot) = \mathbb{P} - \lim_{\mathrm{y}\to\mathrm{k}} \frac{\Psi(\mathrm{y},\cdot) - \Psi(\mathrm{k},\cdot)}{\mathrm{y}-\mathrm{k}},$$

Definition 2.21 ([36])**.** A stochastic process $\Psi : J \times \prod \to \Re$, is called to be mean-square integrable on $J \subset \Re$, if for every $\mathrm{y} \in J$, with $E[\Psi(\mathrm{y},\cdot)] < \infty$, $[\mathrm{y}_1, \mathrm{y}_2] \subseteq J$, $\mathrm{y}_1 = \mathfrak{k}_0 < \mathfrak{k}_1 < \mathfrak{k}_2 ... < \mathfrak{k}_n = \mathrm{y}_2$, partitions $[\mathrm{y}_1, \mathrm{y}_2]$ and $\mathrm{y}_i \in [\mathfrak{k}_{i-1}, \mathfrak{k}_i], \forall i = 1, \ldots, n$ then

$$\lim_{n\to\infty} E\left[\left(\sum_{i=1}^{n} \Psi(\mathrm{y}_i,\cdot)(\mathfrak{k}_i - \mathfrak{k}_{i-1}) - S(\cdot)\right)^2\right] = 0.$$

It indicates that the stochastic process's mean-square integral is $S : J \times \prod \to \Re$ and it may be expressed as

$$S(\cdot) = \int_{\mathrm{y}_1}^{\mathrm{y}_2} \Psi(\mathrm{y},\cdot)d\mathrm{y}.$$

Remark 2.22. The mean-square continuity of the stochastic process Ψ is sufficient to ensure the existence of the mean-square integral.

Remark 2.23. It asserts that if $\Psi(\mathrm{y},\cdot) \leq S(\mathrm{y},\cdot)$ for all $\mathrm{y} \in [\mathrm{y}_1, \mathrm{y}_2]$, then the following holds:

$$\int_{\mathrm{y}_1}^{\mathrm{y}_2} \Psi(\mathrm{y},\cdot)d\mathrm{y} \leq \int_{\mathrm{y}_1}^{\mathrm{y}_2} S(\mathrm{y},\cdot)d\mathrm{y}.$$

Lemma 2.24 ([29])**.** *Let a stochastic process $\Psi : J \times \prod \to \Re$ be defined by $\Psi(\mathrm{y},\cdot) = \mathrm{A}(\cdot)\mathrm{y} + \mathrm{B}(\cdot)$, where $\mathrm{A}, \mathrm{B} : \prod \to \Re$ are such that $E(\mathrm{A}^2) < \infty$ and $E(\mathrm{B}^2) < \infty$. If $[\mathrm{y}_1, \mathrm{y}_2] \subset J$, then the following holds:*

$$\int_{\mathrm{y}_1}^{\mathrm{y}_2} \Psi(\mathrm{y},\cdot)d\mathrm{y} = \mathrm{A}(\cdot)\frac{{\mathrm{y}_2}^2 - {\mathrm{y}_1}^2}{2} + \mathrm{B}(\cdot)(\mathrm{y}_2 - \mathrm{y}_1),$$

where A *and* B *denote random variables.*

Definition 2.25 ([29])**.** We define a process $\Psi : J\times\prod \to \Re$ to be convex stochastic on $J \subset \Re$ if, for every $\beta \in [0,1]$, the following condition is satisfied:

$$\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot) \leq \beta\Psi(\kappa_o, \cdot) + (1-\beta)\Psi(\omega_o, \cdot), \tag{2.4}$$

for any $\kappa_o, \omega_o \in J$.

If $\beta = \frac{1}{2}$ is chosen in (2.4). In that case, the process Ψ is referred to as Jensen convex. Furthermore, if Ψ exhibits convexity, then its negation $-\Psi$ is regarded as a concave stochastic process. For more interesting characteristics, the reader is referred to [42].

Lemma 2.26 ([42])**.** *If* $\mathrm{y}_1, \mathrm{y}_2 \in J_o$ *and* $\mathrm{y}_1 < \mathrm{y}_2$*, then the following inequalities hold.*

$$\Psi'_-(\mathrm{y}_1, \cdot) \leq \Psi'_+(\mathrm{y}_1, \cdot) \leq \Psi'_-(\mathrm{y}_2, \cdot) \leq \Psi'_+(\mathrm{y}_2, \cdot).$$

Where Ψ'_+ *and* Ψ'_-*, are the right and left derivatives of* Ψ*.*

Theorem 2.27 ($\mathbb{H}.\mathbb{H}$-type inequalities for convex stochastic processes [42])**.** *It states that if a convex stochastic process* $\Psi : J \times \prod \to \Re$ *is mean-square continuous and satisfies Jensen's convexity on* J*, then the following inequalities holds*

$$\Psi\left(\frac{\mathrm{y}_1 + \mathrm{y}_2}{2}, \cdot\right) \leq \frac{1}{\mathrm{y}_2 - \mathrm{y}_1}\int_{\mathrm{y}_1}^{\mathrm{y}_2} \Psi(\mathrm{y}, \cdot)d\mathrm{y} \leq \frac{\Psi(\mathrm{y}_1, \cdot) + \Psi(\mathrm{y}_2, \cdot)}{2},$$

for every $\mathrm{y}_1, \mathrm{y}_2 \in J$

Definition 2.28 (Mean-square Stochastic $\mathbb{RL}$ Fractional Integrals [23])**.** Let $\Psi : J \times \prod \to \Re$ be a stochastic process satisfying the conditions stated in Definition 2.21. Then, the mean-square stochastic $\mathbb{RL}$ fractional integrals of order $\beta > 0$, denoted by $I^{\beta}_{\kappa_o+}$ and $I^{\beta}_{\omega_o-}$, are defined for Ψ as follows.

$$I^{\beta}_{\kappa_o+}[\Psi](\mathrm{y}) = \frac{1}{\Gamma(\beta)}\int_{\kappa_o}^{\mathrm{y}} (\mathrm{y} - \mathfrak{s})^{\beta-1}\Psi(\mathfrak{s}, \cdot)d\mathfrak{s}, \quad \mathrm{y} > \kappa_o,$$

$$I^{\beta}_{\omega_o-}[\Psi](\mathrm{y}) = \frac{1}{\Gamma(\beta)}\int_{\mathrm{y}}^{\omega_o} (\mathfrak{s} - \mathrm{y})^{\beta-1}\Psi(\mathfrak{s}, \cdot)d\mathfrak{s}, \quad \mathrm{y} < \omega_o.$$

3. Mean-Square Integral Inequalities for P-Superquadratic Stochastic Models

In this section, we begin by introducing the definition of a P- superquadratic stochastic process, which generalizes the concept of an P-convex stochastic process. This generalization allows for the development of more refined results. Based on this definition, we investigate its properties and establish associated integral inequalities.

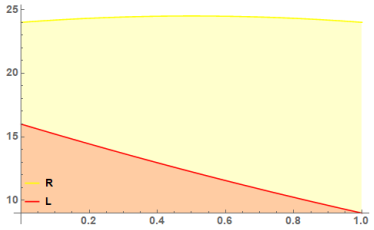
Definition 3.1. Let $\Psi : J \times \prod \to \Re$ be a P-superquadratic stochastic process over $J = [0, \mathfrak{k}]$ then

$$\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot) \leq [\Psi(\kappa_o, \cdot) - \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot)] + [\Psi(\omega_o, \cdot) - \Psi(\beta|\omega_o - \kappa_o|, .)], \tag{3.1}$$

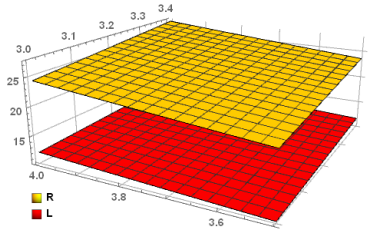
holds for every $\beta \in [0, 1]$ and $\kappa_o, \omega_o \in J$.

If $\beta = \frac{1}{2}$ is chosen in (3.1), then the process Ψ is termed as Jensen-P-superquadratic. If the process Ψ is classified as P-superquadratic, then $-\Psi$ is then termed as P-subquadratic stochastic process.

Example 3.2. A process $\Psi : [2, \infty] \times \prod \to \Re$ defined by $\Psi(\mathrm{y}, \cdot) = \mathrm{y}^{\mathrm{k}}$ for $\mathrm{k} \geq 2$ is a P-superquadratic stochastic process. The graphs in Figure 1 illustrate the validity of Definition 3.1 via the process Ψ considering different values of the parameters that go into it. Let R and L denote the right and left sides of the expression 3.1 respectively.



(A) κ_o and ω_o are fixed, while β varies such that κ_o=3, ω_o=4 and $\beta \in [0, 1]$.



(B) κ_o and ω_o are varies, while β is fixed such that $\beta = 0.5$, $\kappa_o \in [3, 3.4]$ and $\omega_o \in [3, 3.4]$.

FIGURE 1. Graphical illustration of Definition 3.1.

Next, we investigate the core properties that define and characterize P-superquadratic stochastic processes.

Proposition 3.3. *If $\Psi_1, \Psi_2 : [0, \mathfrak{k}] \times \prod \to \Re$ are P-superquadratic stochastic processes, then $\Psi_1 + \Psi_2$ and $\mathrm{r}\Psi$, where $\mathrm{r} > 0$ are P-superquadratic stochastic processes.*

Proof. Let Ψ_1 and Ψ_2 be P-superquadratic stochastic processes i.e., $\forall\ \kappa_o, \omega_o \in [0, \mathfrak{k}]$ and $\beta \in [0, 1]$, we have

$$\Psi_1(\beta\kappa_o + (1-\beta)\omega_o, \cdot) \leq (\Psi_1(\kappa_o, \cdot) - \Psi_1((1-\beta)|\kappa_o - \omega_o|, \cdot)) + (\Psi_1(\omega_o, \cdot) - \Psi_1(\beta|\kappa_o - \omega_o|, \cdot)),$$

and

$$\Psi_2(\beta\kappa_o + (1-\beta)\omega_o, \cdot) \leq (\Psi_2(\kappa_o, \cdot) - \Psi_2((1-\beta)|\kappa_o - \omega_o|, \cdot))$$

$$+ \left(\Psi_2(\omega_o,\cdot) - \Psi_2(\beta|\kappa_o-\omega_o|,\cdot)\right).$$

Now

$$\begin{aligned}
&\left(\Psi_1+\Psi_2\right)\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\\
&\quad=\Psi_1\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)+\Psi_2\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\\
&\quad\leq\left(\Psi_1(\kappa_o,\cdot)-\Psi_1((1-\beta)|\kappa_o-\omega_o|,\cdot)\right)\\
&\qquad+\left(\Psi_1(\omega_o,\cdot)-\Psi_1(\beta|\kappa_o-\omega_o|,\cdot)\right)\\
&\qquad+\left(\Psi_2(\kappa_o,\cdot)-\Psi_2((1-\beta)|\kappa_o-\omega_o|,\cdot)\right)\\
&\qquad+\mathrm{m}\left(\Psi_2(\omega_o,\cdot)-\Psi_2(\beta|\kappa_o-\omega_o|,\cdot)\right)\\
&\quad=\left((\Psi_1+\Psi_2)(\kappa_o,\cdot)-(\Psi_1+\Psi_2)((1-\beta)|\kappa_o-\omega_o|,\cdot)\right)\\
&\qquad+\left((\Psi_1+\Psi_2)(\omega_o,\cdot)-(\Psi_1+\Psi_2)(\beta|\kappa_o-\omega_o|,\cdot)\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left(\Psi_1+\Psi_2\right)\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\\
&\quad\leq\left((\Psi_1+\Psi_2)(\kappa_o,\cdot)-(\Psi_1+\Psi_2)((1-\beta)|\kappa_o-\omega_o|,\cdot)\right)\\
&\qquad+\left((\Psi_1+\Psi_2)(\omega_o,\cdot)-(\Psi_1+\Psi_2)(\beta|\kappa_o-\omega_o|,\cdot)\right).
\end{aligned}\tag{3.2}$$

Similarly

$$\begin{aligned}
&\mathrm{r}\Psi\left(\beta\kappa_o+\mathrm{m}(1-\beta)\omega_o,\cdot\right)\\
&\quad\leq\mathrm{r}\left(\Psi(\kappa_o,\cdot)-\Psi((1-\beta)|\kappa_o-\omega_o|,\cdot)+\Psi(\omega_o,\cdot)-\Psi(\beta|\kappa_o-\omega_o|,\cdot)\right)\\
&\quad=\mathrm{r}\Psi(\kappa_o,\cdot)-\mathrm{r}\Psi((1-\beta)|\kappa_o-\omega_o|,\cdot)+\mathrm{r}\Psi(\omega_o,\cdot)-\mathrm{r}\Psi(\beta|\kappa_o-\omega_o|,\cdot).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathrm{r}\Psi\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\leq\ &\mathrm{r}\Psi(\kappa_o,\cdot)-\mathrm{r}\Psi((1-\beta)|\kappa_o-\omega_o|,\cdot)\\
&+\mathrm{r}\Psi(\omega_o,\cdot)-\mathrm{r}\Psi(\beta|\kappa_o-\omega_o|,\cdot).
\end{aligned}\tag{3.3}$$

Therefore, (3.2) and (3.3) jointly imply that both $\Psi_1+\Psi_2$ and $\mathrm{r}\Psi$ satisfy the definition of P-superquadratic stochastic processes. □

Proposition 3.4. *Let $\Psi_1,\Psi_2:[0,\mathfrak{k}]\times\prod\to\Re$ be the P-superquadratic stochastic processes, then $\Psi(\mathrm{y},\cdot)=max\{\Psi_1(\mathrm{y},\cdot),\Psi_2(\mathrm{y},\cdot)\}$ is a P- superquadratic stochastic process.*

Proof. Let $\kappa_o,\omega_o\in[0,\mathfrak{k}]$ and $\beta\in[0,1]$, one derives that

$$\begin{aligned}
&\Psi\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\\
&\quad=\max\left\{\Psi_1\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right),\Psi_2\left(\beta\kappa_o+(1-\beta)\omega_o,\cdot\right)\right\}\\
&\quad\leq\max\Big\{\left[\Psi_1(\kappa_o,\cdot)-\Psi_1((1-\beta)|\kappa_o-\omega_o|,\cdot)\right]\\
&\qquad+\left[\Psi_1(\omega_o,\cdot)-\Psi_1(\beta|\kappa_o-\omega_o|,\cdot)\right],\left[\Psi_2(\kappa_o,\cdot)-\Psi_2((1-\beta)|\kappa_o-\omega_o|,\cdot)\right]\\
&\qquad+\left[\Psi_2(\omega_o,\cdot)-\Psi_2(\beta|\kappa_o-\omega_o|,\cdot)\right]\Big\}
\end{aligned}$$

$$\leq \Big[\max\{\Psi_1(\kappa_o,\cdot),\Psi_2(\kappa_o,\cdot)\} - \max\{\Psi_1((1-\beta)|\kappa_o-\omega_o|,\cdot) , \Psi_2((1-\beta)|\kappa_o-\omega_o|,\cdot)\}\Big] + \Big[\max\{\Psi_1(\omega_o,\cdot),\Psi_2(\omega_o,\cdot)\} - \max\{\Psi_1(\beta|\kappa_o-\omega_o|,\cdot),\Psi_2(\beta|\kappa_o-\omega_o|,\cdot)\}\Big]$$
$$= [\Psi(\kappa_o,\cdot) - \Psi((1-\beta)|\kappa_o-\omega_o|,\cdot)] + [\Psi(\omega_o.\cdot) - \Psi(\beta|\kappa_o-\omega_o|,\cdot)].$$

Consequently, $\Psi(\mathrm{y},\cdot)$ is a P-superquadratic stochastic process. □

Theorem 3.5. *Let* $\Psi : [0,\mathfrak{k}]\times\prod\to\Re$ *be a P-superquadratic stochastic process and* $\mathrm{y}_1<\mathrm{y}_2<\mathrm{y}_3$ *for all* $\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3\in[0,\mathfrak{k}]$*, then*

$$\begin{vmatrix}\Psi(\mathrm{y}_2,\cdot) & 0\\ 0 & 1\end{vmatrix} \leq \begin{vmatrix}\Psi(\mathrm{y}_1,\cdot) & 1\\ \Psi(\mathrm{y}_2-\mathrm{y}_1,\cdot) & 1\end{vmatrix} + \begin{vmatrix}\Psi(\mathrm{y}_3,\cdot) & 1\\ \Psi(\mathrm{y}_3-\mathrm{y}_2,\cdot) & 1\end{vmatrix} \tag{3.4}$$

holds.

Proof. Let the stochastic process Ψ is a P-superquadratic, then

$$\Psi\left(\beta\mathrm{y}+(1-\beta)\mathrm{y}_o,\cdot\right) \leq \left(\Psi(\mathrm{y},\cdot) - \Psi((1-\beta)|\mathrm{y}-\mathrm{y}_o|,\cdot)\right) + \left(\Psi(\mathrm{y}_o,\cdot) - \Psi(\beta|\mathrm{y}-\mathrm{y}_o|,\cdot)\right). \tag{3.5}$$

Setting

$$\mathrm{y}_2 = \beta\mathrm{y}+(1-\beta)\mathrm{y}_o. \tag{3.6}$$

Next putting $\mathrm{y}=\mathrm{y}_1$ and $\mathrm{y}_o=\mathrm{y}_3$ in (3.6), we get

$$\mathrm{y}_2 = \beta\mathrm{y}_1+(1-\beta)\mathrm{y}_3. \quad\Rightarrow\quad \beta = \frac{\mathrm{y}_3-\mathrm{y}_2}{\mathrm{y}_3-\mathrm{y}_1}. \tag{3.7}$$

Substituting $\mathrm{y}=\mathrm{y}_1$, $\mathrm{y}_o=\mathrm{y}_3$ and $\beta=\frac{\mathrm{y}_3-\mathrm{y}_2}{\mathrm{y}_3-\mathrm{y}_1}$ in (3.5), we get

$$\Psi\left(\mathrm{y}_2,\cdot\right) \leq [\Psi(\mathrm{y}_1,\cdot) - \Psi(\mathrm{y}_2-\mathrm{y}_1,\cdot)] + [\Psi(\mathrm{y}_3,\cdot) - \Psi(\mathrm{y}_3-\mathrm{y}_2,\cdot)].$$

It implies that

$$[\Psi(\mathrm{y}_1,\cdot) - \Psi(\mathrm{y}_2-\mathrm{y}_1,\cdot)] + [\Psi(\mathrm{y}_3,\cdot) - \Psi(\mathrm{y}_3-\mathrm{y}_2,\cdot)] - \Psi\left(\mathrm{y}_2,\cdot\right) \geq 0. \tag{3.8}$$

Inequality (3.8) can be written as follows

$$\begin{vmatrix}\Psi(\mathrm{y}_1,\cdot) & 1\\ \Psi(\mathrm{y}_2-\mathrm{y}_1,\cdot) & 1\end{vmatrix} + \begin{vmatrix}\Psi(\mathrm{y}_3,\cdot) & 1\\ \Psi(\mathrm{y}_3-\mathrm{y}_2,\cdot) & 1\end{vmatrix} \geq \begin{vmatrix}\Psi(\mathrm{y}_2,\cdot) & 0\\ 0 & 1\end{vmatrix}. \tag{3.9}$$

Hence, this concludes the proof. □

Next, we provide the proof of an inequality of Jensen's type for P-superquadratic stochastic processes.

Theorem 3.6. *Let $\Psi : [0, \mathfrak{k}] \times \prod \to \Re$ be a P-superquadratic stochastic process then*

$$\Psi\left(\sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i, \cdot\right) \leq \sum_{i=1}^{\mathfrak{n}} \Psi(\mathrm{y}_i, \cdot) - \sum_{i=1}^{\mathfrak{n}} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i\right|, \cdot\right), \tag{3.10}$$

holds $\forall \mathrm{y}_1, \ldots, \mathrm{y}_n \in [0, \mathfrak{k}], \beta_1, \ldots, \beta_{\mathfrak{n}} > 0$, such that $\sum_{i=1}^{\mathfrak{n}} \beta_i = 1$.

Proof. Assume that Ψ is P-superquadratic stochastic process. If $\mathfrak{n} = 2$, then we have

$$\Psi\left(\beta_1 \mathrm{y}_1 + \beta_2 \mathrm{y}_2, \cdot\right) \leq \Psi(\mathrm{y}_2, \cdot) - \Psi(\beta_1 |\mathrm{y}_2 - \mathrm{y}_1|, \cdot) + \Psi(\mathrm{y}_1, \cdot) - \Psi(\beta_2 |\mathrm{y}_2 - \mathrm{y}_1|, \cdot).$$

Assuming the validity of (3.10) for $\mathfrak{n} - 1$, it follows that:

$$\Psi\left(\sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i, \cdot\right) \leq \sum_{i=1}^{\mathfrak{n}-1} \Psi(\mathrm{y}_i, \cdot) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right). \tag{3.11}$$

Now let us prove that (3.10) is valid for $\mathfrak{n}$.

$$\Psi\left(\sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i, \cdot\right) = \Psi\left(\beta_n \mathrm{y}_n + \beta \sum_{i=1}^{\mathfrak{n}-1} \frac{\beta_i \mathrm{y}_i}{\beta}, \cdot\right).$$

Where $\beta = \sum_{i=1}^{\mathfrak{n}-1} \beta_i$, then it implies that

$$\begin{aligned} \Psi\left(\sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i, \cdot\right) \leq{}& \Psi(\mathrm{y}_n, \cdot) + \Psi\left(\sum_{i=1}^{\mathfrak{n}-1} \left(\frac{\beta_i \mathrm{y}_i}{\beta}\right), \cdot\right) \\ &- \Psi\left(\beta \left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1} \left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\ &- \Psi\left(\beta_n \left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1} \left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right). \end{aligned} \tag{3.12}$$

Using (3.11) in (3.12), we obtain

$$\begin{aligned} \Psi\left(\sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i, \cdot\right) \leq{}& \Psi(\mathrm{y}_n, \cdot) + \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \\ &- \Psi\left(\beta \left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1} \left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\ &- \Psi\left(\beta_n \left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1} \left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\ ={}& \Psi\left(\mathrm{y}_n, \cdot\right) + \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \end{aligned}$$

$$
\begin{aligned}
&- \Psi\left(\beta\left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\
&- \Psi\left(\beta_n\left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\
&\leq \sum_{i=1}^{\mathfrak{n}} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \\
&- \Psi\left(\beta\left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\beta_i \mathrm{y}_i}{\beta}\right)\right|, \cdot\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \\
&- \Psi\left(\left|\mathrm{y}_n - \beta_n \mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\beta_i \mathrm{y}_i\right)\right|, \cdot\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \\
&- \Psi\left(\left|\mathrm{y}_n - \beta_n \mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\beta_i \mathrm{y}_i\right)\right|, \cdot\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}-1} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}-1} \beta_i \mathrm{y}_i\right|, \cdot\right) \\
&- \Psi\left(\left|\mathrm{y}_n - \sum_{i=1}^{\mathfrak{n}}\left(\beta_i \mathrm{y}_i\right)\right|, \cdot\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \Psi\left(\mathrm{y}_i, \cdot\right) - \sum_{i=1}^{\mathfrak{n}} \Psi\left(\left|\mathrm{y}_i - \sum_{i=1}^{\mathfrak{n}} \beta_i \mathrm{y}_i\right|, \cdot\right).
\end{aligned}
$$

Hence the proof. □

Theorem 3.7. *The following statements are true for all* κ_o, ω_o, $\mathfrak{c}_o \in [0, \mathfrak{k}]$ *and* $\beta \in [0, 1]$

(i) $\Psi \in S_{\mathrm{P}}[0, \mathfrak{k}]$, *where* $S_{\mathrm{P}}[0, \mathfrak{k}]$=*Set of P-superquadratic stochastic processes.*

(ii) *The process* Ψ *holds the result given below:*

$$
\begin{aligned}
\Psi\left(\beta_1 \kappa_o + (1-\beta_1)\omega_o, \cdot\right) \leq \Psi(\kappa_o, \cdot) - \Psi((1-\beta_1)|\kappa_o - \omega_o|, \cdot) \\
+ \left(\Psi(\omega_o, \cdot) - \Psi(\beta_1|\kappa_o - \omega_o|, \cdot)\right).
\end{aligned}
$$

(iii) *For $\beta_1+\beta_2=1$, the process Ψ holds the inequality given below:*

$$\Psi\left(\beta_1\kappa_o+\beta_2\omega_o,\cdot\right)\leq\Psi(\kappa_o,\cdot)-\Psi(\beta_2|\kappa_o-\omega_o|,\cdot)+\left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right).$$

(iv) *For $\kappa_o\leq\mathfrak{c}_o\leq\omega_o$ the inequality given below holds:*

$$\begin{vmatrix}1 & \Psi(\kappa_o,\cdot)-\Psi\left((1-\beta)|\kappa_o-\omega_o|,\cdot\right) & 1\\ -1 & \Psi(\omega_o,\cdot)-\Psi(\beta|\kappa_o-\omega_o|,\cdot) & 0\\ 0 & \Psi(\mathfrak{c}_o,\cdot) & 1\end{vmatrix}\geq 0.$$

Proof. It is straight forward that (1)$\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4). Since $\kappa_o\leq\mathfrak{c}_o\leq\omega_o$ there exists a scalar $\beta_1\in[0,1]$ such that $\mathfrak{c}_o=\beta_1\kappa_o+\beta_2\omega_o=\beta_1\kappa_o+(1-\beta_1)\omega_o$. Consequently, the following results can be established.

$$\begin{aligned}\Psi(\mathfrak{c}_o,\cdot)&=\Psi(\beta_1\kappa_o+\beta_2\omega_o,\cdot)\\&=\Psi(\beta_1\kappa_o+(1-\beta_1)\omega_o,\cdot)\\&=\Psi\left(\beta_1\kappa_o+(1-\beta_1)\omega_o,\cdot\right)\\&\leq\left(\Psi(\kappa_o,\cdot)-\Psi((1-\beta_1)|\kappa_o-\omega_o|,\cdot)\right)\\&\quad+\left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right).\end{aligned}$$

Since, it follows that

$$\begin{aligned}&\begin{vmatrix}1 & \Psi(\kappa_o,\cdot)-\Psi((1-\beta_1)|\kappa_o-\omega_o|,\cdot) & 1\\ -1 & \left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right) & 0\\ 0 & \Psi(\mathfrak{c}_o,\cdot) & 1\end{vmatrix}\\&=\Psi(\kappa_o,\cdot)-\Psi((1-\beta_1)|\kappa_o-\omega_o|,\cdot)\\&\quad+\left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right)-\Psi(\mathfrak{c}_o,\cdot)\\&\geq 0.\end{aligned}$$

Hence, (4) is satisfied

(4)$\Rightarrow$(1). In view of the assumption

$$\Psi(\kappa_o,\cdot)-\Psi((1-\beta_1)|\kappa_o-\omega_o|,\cdot)+\left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right)-\Psi(\mathfrak{c}_o,\cdot)\geq 0,$$

it can be applied to

$$\Psi(\mathfrak{c}_o,\cdot)=\Psi\left(\beta_1\kappa_o+(1-\beta_1)\omega_o,\cdot\right)\leq\left(\Psi(\kappa_o,\cdot)-\Psi((1-\beta_1)|\kappa_o-\omega_o|),\cdot\right)+\left(\Psi(\omega_o,\cdot)-\Psi(\beta_1|\kappa_o-\omega_o|,\cdot)\right).$$

Hence the proof. □

Next, we offer the $\mathbb{H}.\mathbb{H}$'s inequalities for P-superquadratic stochastic processes.

Theorem 3.8. *Let $\Psi : J \times \prod \to \Re$ be a P-superquadratic stochastic process. If $\Psi \in L_{\mathrm{y}}[\kappa_o, \omega_o]$ with $\kappa_o < \omega_o$, then*

$$\begin{aligned}(3.13)\qquad \frac{1}{2}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) &+ \frac{1}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi\left(\left|\frac{\kappa_o+\omega_o}{2}-\mathrm{y}\right|,\cdot\right)d\mathrm{y}\\ &\leq \frac{1}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi(\mathrm{y},\cdot)d\mathrm{y}\\ &\leq \Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)\\ &\quad -\frac{1}{\omega_o-\kappa_o}\left\{\int_{\kappa_o}^{\omega_o}\left(\Psi(\mathrm{y}-\kappa_o,\cdot)+\Psi(\omega_o-\mathrm{y},\cdot)\right)d\mathrm{y}\right\},\end{aligned}$$

holds $\forall\ \kappa_o, \omega_o \in J$. Where $J = [0, \mathfrak{k}]$.

Proof. Since Ψ is a P-superquadratic stochastic process therefore we have

$$(3.14)\qquad \Psi\left(\frac{\mathrm{y}+\mathrm{y}_o}{2},\cdot\right) \leq \Psi(\mathrm{y},\cdot)+\Psi(\mathrm{y}_o,\cdot)-2\Psi\left(\frac{1}{2}|\mathrm{y}-\mathrm{y}_o|,\cdot\right).$$

Replacing y by $\beta\kappa_o + (1-\beta)\omega_o$ and y_o by $\beta\omega_o + (1-\beta)\kappa_o$ in (3.14), we get.

(3.15)

$$\begin{aligned}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) &\leq \Psi(\beta\kappa_o+(1-\beta)\omega_o,\cdot)+\Psi(\beta\omega_o+(1-\beta)\kappa_o,\cdot)\\ &\quad -2\Psi\left(\left|(\kappa_o\beta+\omega_o(1-\beta))-\frac{\kappa_o+\omega_o}{2}\right|,\cdot\right).\end{aligned}$$

Integrating (3.15) w.r.t β over $[0,1]$, we have

$$\begin{aligned}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) &\leq \int_0^1\Psi(\beta\kappa_o+(1-\beta)\omega_o,\cdot)d\beta\\ &\quad +\int_0^1\Psi\left(\beta\omega_o+(1-\beta)\kappa_o\right)d\beta\\ &\quad -2\int_0^1\Psi\left(\left|(\kappa_o\beta+\omega_o(1-\beta))-\frac{(\kappa_o+\omega_o)}{2}\right|,\cdot\right)d\beta.\end{aligned}$$

It implies that

$$\begin{aligned}(3.16)\qquad \Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) &\leq \frac{2}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi(\mathrm{y},\cdot)d\mathrm{y}\\ &\quad -\frac{2}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi\left(\left|\mathrm{y}-\frac{(\kappa_o+\omega_o)}{2}\right|,\cdot\right)d\mathrm{y}.\end{aligned}$$

Thus

$$(3.17)\qquad \Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right)+\frac{2}{\omega_o-\kappa_o}\int_{\kappa_o}^{\omega_o}\Psi\left(\left|\mathrm{y}-\frac{(\kappa_o+\omega_o)}{2}\right|,\cdot\right)d\mathrm{y}$$

$$\leq \frac{2}{\omega_o - \kappa_o} \int_{\kappa_o}^{\omega_o} \Psi(\mathrm{y}, \cdot) d\mathrm{y}.$$

Since Ψ is an P-superquadratic stochastic process, it follows that

(3.18)
$$\begin{aligned} &\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot) + \Psi\left(\beta\omega_o + (1-\beta)\kappa_o, \cdot\right) \\ &\quad \leq 2\left(\Psi(\kappa_o, \cdot) - \Psi((1-\beta)|\kappa_o - \omega_o|, \cdot) + \Psi\left(\omega_o, \cdot\right) - \Psi\left(\beta|\kappa_o - \omega_o|, \cdot\right)\right). \end{aligned}$$

integrating (3.18) w.r.t β over $[0,1]$ and after simple calculation and change of variables, we get

(3.19)
$$\begin{aligned} \frac{1}{\omega_o - \kappa_o} \int_{\kappa_o}^{\omega_o} \Psi(\mathrm{y}, \cdot) d\mathrm{y} &\leq \Psi(\kappa_o, \cdot) + \Psi(\omega_o, \cdot) \\ &\quad - \frac{1}{\omega_o - \kappa_o} \left\{ \int_{\kappa_o}^{\omega_o} \left(\Psi\left(\mathrm{y} - \kappa_o\right), \cdot\right) + \Psi(\omega_o - \mathrm{y}, \cdot) d\mathrm{y} \right\}. \end{aligned}$$

Combining (3.17) and (3.19), we get what we want. □

Corollary 3.9. *When the inequality in* (3.13) *is reversed, it gives rise to an inequality of* $\mathbb{H}.\mathbb{H}$*-type characterizing P-subquadratic stochastic processes.*

4. $\mathbb{H}.\mathbb{H}$ Type Inequalities for P-Superquadratic Stochastic Processes: A Fractional Calculus Approach

This section is devoted to deriving fractional analogues of the $\mathbb{H}.\mathbb{H}$ inequalities via $\mathbb{RL}$ fractional operators.

Theorem 4.1. *Let* $\Psi : J \times \prod \to \Re$ *be a P-superquadratic stochastic process. If* $\Psi \in I_{\mathrm{y}}[\kappa_o, \omega_o]$ *with* $\kappa_o < \omega_o$*, then*

(4.1)
$$\begin{aligned} &\frac{\Psi\left(\frac{\mathrm{u}_o + \omega_o}{2}, \cdot\right)}{2} + \frac{2\alpha}{(\omega_o - \kappa_o)^\alpha} \int_{\kappa_o}^{\frac{\kappa_o + \omega_o}{2}} (\omega_o - \mathrm{y})^{\alpha - 1} \Psi\left(\left|\frac{\kappa_o + \omega_o}{2} - \mathrm{y}\right|, \cdot\right) d\mathrm{y} \\ &\quad \leq \frac{\Gamma(1+\alpha)}{2(\omega_o - \kappa_o)^\alpha} \left[\mathfrak{J}^\alpha_{\kappa_o^+} \Psi(\omega_o, \cdot) + \mathfrak{J}^\alpha_{\omega_o^-} \Psi(\kappa_o, \cdot)\right] \leq \Psi(\kappa_o, \cdot) + \Psi(\omega_o, \cdot) \\ &\qquad - \frac{\alpha}{(\omega_o - \kappa_o)^\alpha} \left\{ \int_{\kappa_o}^{\omega_o} (\omega_o - \mathrm{y})^{\alpha - 1} \left(\Psi(|\mathrm{y} - \kappa_o|, \cdot) + \Psi(|\omega_o - \mathrm{y}|, \cdot)\right) d\mathrm{y} \right\}, \end{aligned}$$

holds $\forall \alpha \in (0,1)$.

Proof. Let the process Ψ, satisfying the conditions of P-superquadratic stochastic process, suggests that

$$\Psi\left(\frac{\mathrm{y} + \mathrm{y}_o}{2}, \cdot\right) \leq \Psi(\mathrm{y}, \cdot) + \Psi(\mathrm{y}_o, \cdot) - 2\Psi\left(\frac{|\mathrm{y} - \mathrm{y}_o|}{2}, \cdot\right), \tag{4.2}$$

Replacing y with $\beta\kappa_o + (1-\beta)\omega_o$ and y_o with $\beta\omega_o + (1-\beta)\kappa_o$ in (4.2), we obtain the following

$$\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) \leq \Psi(\beta\kappa_o + (1-\beta)\omega_o,\cdot) + \Psi\left(\beta\omega_o + (1-\beta)\kappa_o,\cdot\right) - 2\Psi\left(\left|(\kappa_o\beta + \omega_o(1-\beta)) - \frac{\kappa_o+\omega_o}{2}\right|,\cdot\right), \tag{4.3}$$

Multiplying (4.3) by $\beta^{\alpha-1}$ and then integrating it via β over $[0,1]$, we have

$$\begin{aligned}\frac{1}{\alpha}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) \leq{}& \int_0^1 \beta^{\alpha-1}\Psi(\beta\kappa_o + (1-\beta)\omega_o,\cdot)d\beta \\ &+ \int_0^1 \beta^{\alpha-1}\Psi(\beta\omega_o + (1-\beta)\kappa_o,\cdot)d\beta \\ &- 2\int_0^1 \beta^{\alpha-1}\Psi\left(\left|(\kappa_o\beta + \omega_o(1-\beta)) - \frac{\kappa_o+\omega_o}{2}\right|,\cdot\right)d\beta.\end{aligned}$$

It implies that

$$\begin{aligned}\frac{1}{\alpha}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) \leq{}& \frac{1}{(\omega_o-\kappa_o)^\alpha}\left\{\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}\right. \\ &\left.+ \int_{\kappa_o}^{\omega_o}(\mathrm{y}-\kappa_o)^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}\right\} \\ &- \frac{2}{(\omega_o-\kappa_o)^\alpha}\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\Psi\left(\left|\mathrm{y} - \frac{\kappa_o+\omega_o}{2}\right|,\cdot\right)d\mathrm{y}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{\alpha}\Psi\left(\frac{\kappa_o+\omega_o}{2},\cdot\right) + \frac{2}{(\omega_o-\kappa_o)^\alpha}\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\Psi\left(\left|\mathrm{y} - \frac{\kappa_o+\omega_o}{2}\right|,\cdot\right)d\mathrm{y} \\ &\quad\leq \frac{\Gamma(\alpha)}{(\omega_o-\kappa_o)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_o+}\Psi(\omega_o,\cdot) + \mathfrak{I}^\alpha_{\omega_o-}\Psi(\kappa_o,\cdot)\right].\end{aligned} \tag{4.4}$$

Again as Ψ is a P-superquadratic stochastic process, we have

$$\begin{aligned}&\Psi(\beta\kappa_o + (1-\beta)\omega_o,\cdot) + \Psi(\beta\omega_o + (1-\beta)\kappa_o,\cdot) \\ &\quad\leq 2\left(\Psi(\kappa_o,\cdot) - \Psi\left((1-\beta)|\kappa_o-\omega_o|,\cdot\right) + \Psi\left(\omega_o,\cdot\right) - \Psi\left(\beta|\kappa_o-\omega_o|,\cdot\right)\right).\end{aligned} \tag{4.5}$$

Multiplying (4.5) by $\beta^{\alpha-1}$ and then integrating it via β over $[0,1]$, we have

$$\int_0^1 \beta^{\alpha-1}\Psi(\beta\kappa_o + (1-\beta)\omega_o,\cdot)d\beta + \int_0^1 \beta^{\alpha-1}\Psi\left(\beta\omega_o + (1-\beta)\kappa_o,\cdot\right)d\beta$$

$$\leq 2\Bigg[\int_0^1 \beta^{\alpha-1}\left(\Psi(\kappa_o,\cdot)-\Psi\left(\left(1-\beta\right)|\kappa_o-\omega_o|,\cdot\right)\right)d\beta$$
$$+\int_0^1 \beta^{\alpha-1}\left(\Psi\left(\omega_o,\cdot\right)-\Psi\left(\beta|\kappa_o-\omega_o|,\cdot\right)\right)d\beta\Bigg].$$

It implies that

(4.6)
$$\frac{1}{(\omega_o-\kappa_o)^\alpha}\left\{\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}+\int_{\kappa_o}^{\omega_o}\left(\mathrm{y}-\kappa_o\right)^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}\right\}$$
$$\leq \frac{2[\Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)]}{\alpha}-\frac{2}{(\omega_o-\kappa_o)^\alpha}$$
$$\times\left\{\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)\right)d\mathrm{y}\right\}.$$

Thus

(4.7)
$$\frac{\Gamma(\alpha)}{(\omega_o-\kappa_o)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_o^+}\Psi(\omega_o,\cdot)+\mathfrak{I}^\alpha_{\omega_o^-}\Psi(\kappa_o,\cdot)\right]$$
$$\leq \frac{2[\Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)]}{\alpha}$$
$$-\frac{2}{(\omega_o-\kappa_o)^\alpha}\left\{\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)\right)d\mathrm{y}\right\}.$$

The proof is completed by merging inequalities (4.4) and (4.7). □

Corollary 4.2. *The fractional inequalities of* $\mathbb{H}.\mathbb{H}$*-type for P-subquadratic stochastic processes are obtained by applying* $\mathbb{RL}$ *fractional integral operators, provided that the inequalities in* (4.1) *are reversed.*

Corollary 4.3. *The* $\mathbb{H}.\mathbb{H}$ *type inequalities for P-superquadratic stochastic process are obtained by taking* $\alpha=1$ *in* (4.1).

Theorem 4.4. *Let* $\Psi: J\times\prod\to\Re$ *be a P-superquadratic stochastic process. If* $\Psi\in J_{\mathrm{y}}[\kappa_o,\omega_o]$ *with* $\kappa_o<\omega_o$*, then*

$$\frac{\Gamma(1+\alpha)}{2(\omega_o-\kappa_o)^\alpha}\left\{\left(\mathfrak{I}^\alpha_{\kappa_o^+}\Psi(\omega_o,\cdot)+\mathfrak{I}^\alpha_{\omega_o^-}\Psi(\kappa_o,\cdot)\right)\right\}$$
$$\leq\left(\Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)\right)$$
$$-\frac{\alpha}{2(\omega_o-\kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\left(\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)+\Psi(|(\mathrm{y}-\kappa_o)|,\cdot)\right)d\beta\right)$$
$$-\frac{\alpha}{2(\omega_o-\kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}\left(\mathrm{y}-\kappa_o\right)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)\right)d\beta\right).$$

Proof. Suppose that the process Ψ exhibits P-superquadraticity, then

$$\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot) \leq \Psi(\kappa_o, \cdot) - \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot) + \Psi(\omega_o, \cdot) - \Psi(\beta|\omega_o - \kappa_o|, \cdot), \tag{4.8}$$

and

$$\Psi((1-\beta)\kappa_o + \beta\omega_o, \cdot) \leq \Psi(\kappa_o, \cdot) - \Psi(\beta|\omega_o - \kappa_o|, \cdot) + \Psi(\omega_o, \cdot) - \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot). \tag{4.9}$$

Adding inequalities (4.8) and (4.9), we arrive at the following result:

(4.10)

$$\begin{aligned}
&\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot) + \Psi((1-\beta)\kappa_o + \beta\omega_o, \cdot) \\
&\quad \leq \Psi(\kappa_o, \cdot) - \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot) + (\Psi(\omega_o, \cdot) - \Psi(\beta|\omega_o - \kappa_o|, \cdot)) \\
&\qquad + \Psi(\kappa_o, \cdot) - \Psi(\beta|\omega_o - \kappa_o|, \cdot) + (\Psi(\omega_o, \cdot) - \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot)) \\
&\quad = 2\Psi(\kappa_o, \cdot) + 2\Psi(\omega_o, \cdot) - 2\left(\Psi(\beta|\omega_o - \kappa_o|, \cdot) + \Psi((1-\beta)|\omega_o - \kappa_o|, \cdot)\right).
\end{aligned}$$

The inequality (4.10) is multiplied by $\beta^{\alpha-1}$ and integrating via β over [0, 1], so we then attain that

$$\begin{aligned}
&\int_0^1 \beta^{\alpha-1}\Psi(\beta\kappa_o + (1-\beta)\omega_o, \cdot)d\beta + \int_0^1 \beta^{\alpha-1}\Psi((1-\beta)\kappa_o + \beta\omega_o, \cdot)d\beta \\
&\quad \leq 2\int_0^1 \beta^{\alpha-1}\left(\Psi(\kappa_o, \cdot) + \Psi(\omega_o, \cdot)\right)d\beta - 2\Bigg(\int_0^1 \beta^{\alpha-1}\Psi(\beta|\omega_o - \kappa_o|, \cdot)d\beta \\
&\qquad + \int_0^1 \beta^{\alpha-1}\Psi((1-\beta)|\omega_o - \kappa_o|, \cdot)d\beta\Bigg).
\end{aligned}$$

It implies that

$$\begin{aligned}
&\frac{\alpha}{(\omega_o - \kappa_o)^\alpha}\left\{\int_{\kappa_o}^{\omega_o}(\omega_o - \mathrm{y})^{\alpha-1}\Psi(\mathrm{y})d\mathrm{y} + \int_{\kappa_o}^{\omega_o}(\mathrm{y} - \kappa_o)^{\alpha-1}\Psi(\mathrm{y}, \cdot)d\mathrm{y}\right\} \\
&\quad \leq 2\Psi(\kappa_o, \cdot) + 2\Psi(\omega_o, \cdot) \\
&\qquad - \frac{2\alpha}{(\omega_o - \kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}(\omega_o - \mathrm{y})^{\alpha-1}\left(\Psi\left(|\omega_o - \mathrm{y}|, \cdot\right) + \Psi\left(|\mathrm{y} - \kappa_o|, \cdot\right)\right)d\beta\right).
\end{aligned}$$

Thus

(4.11)

$$\begin{aligned}
&\frac{\Gamma(1+\alpha)}{2(\omega_o - \kappa_o)^\alpha}\left\{\mathfrak{I}^\alpha_{\kappa_o+}\Psi(\omega_o, \cdot) + \mathfrak{I}^\alpha_{\omega_o-}\Psi(\kappa_o, \cdot)\right\} \\
&\quad \leq \left(\Psi(\kappa_o, \cdot) + \Psi(\omega_o, \cdot)\right) \\
&\qquad - \frac{\alpha}{(\omega_o - \kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}(\omega_o - \mathrm{y})^{\alpha-1}\left(\Psi\left(|\omega_o - \mathrm{y}|, \cdot\right) + \Psi\left(|\mathrm{y} - \kappa_o|, \cdot\right)\right)d\beta\right).
\end{aligned}$$

Likewise there are

$$\Psi(\beta\omega_o+(1-\beta)\kappa_o,\cdot)\leq \Psi(\omega_o,\cdot)-\Psi((1-\beta)|\omega_o-\kappa_o|,\cdot) + \left(\Psi(\kappa_o,\cdot)-\Psi(\beta|\omega_o-\kappa_o|,\cdot)\right), \tag{4.12}$$

and

$$\Psi((1-\beta)\omega_o+\beta\kappa_o,\cdot)\leq \Psi(\omega_o,\cdot)-\Psi(\beta|\omega_o-\kappa_o|,\cdot) + \left(\Psi(\kappa_o,\cdot)-\Psi((1-\beta)|\omega_o-\kappa_o|,\cdot)\right). \tag{4.13}$$

Adding inequalities (4.12) and (4.13), we derive:

$$\begin{aligned}&\Psi(\beta\omega_o+(1-\beta)\kappa_o,\cdot)+\Psi((1-\beta)\omega_o+\beta\kappa_o,\cdot)\\ &\quad\leq 2\Psi(\omega_o,\cdot)+2\Psi(\kappa_o,\cdot)-2\left(\Psi(\beta|\omega_o-\kappa_o|,\cdot)+\Psi((1-\beta)|\omega_o-\kappa_o|,\cdot)\right).\end{aligned} \tag{4.14}$$

The inequality (4.14) is multiplied by $\beta^{\alpha-1}$ and integrating via β over $[0,\,1]$, we then attain that

$$\begin{aligned}&\int_0^1\beta^{\alpha-1}\Psi(\beta\omega_o+(1-\beta)\kappa_o,\cdot)d\beta+\int_0^1\beta^{\alpha-1}\Psi((1-\beta)\omega_o+\beta\kappa_o,\cdot)d\beta\\ &\quad\leq 2\int_0^1\beta^{\alpha-1}\left(\Psi(\omega_o,\cdot)+\Psi(\kappa_o,\cdot)\right)d\beta-2\Bigg(\int_0^1\beta^{\alpha-1}\Psi(\beta|\omega_o-\kappa_o|,\cdot)d\beta\\ &\qquad+\int_0^1\beta^{\alpha-1}\Psi((1-\beta)|\omega_o-\kappa_o|,\cdot)d\beta\Bigg).\end{aligned}$$

It implies that

$$\begin{aligned}&\frac{\alpha}{(\omega_o-\kappa_o)^\alpha}\left\{\int_{\kappa_o}^{\omega_o}(\mathrm{y}-\kappa_o)^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}+\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\Psi(\mathrm{y},\cdot)d\mathrm{y}\right\}\\ &\quad\leq\left(2\Psi(\omega_o,\cdot)+2\Psi(\kappa_o,\cdot)\right)\\ &\qquad-\frac{2\alpha}{(\omega_o-\kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}(\mathrm{y}-\kappa_o)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)\right)d\beta\right).\end{aligned}$$

Thus

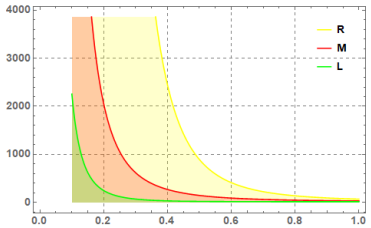
$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{2(\omega_o-\kappa_o)^\alpha}\left\{\mathfrak{I}^\alpha_{\omega_o^-}\Psi(\kappa_o,\cdot)+\mathfrak{I}^\alpha_{\kappa_o^+}\Psi(\omega_o,\cdot)\right\}\\ &\quad\leq\left(\Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)\right)\\ &\qquad-\frac{\alpha}{(\omega_o-\kappa_o)^\alpha}\left(\int_{\kappa_o}^{\omega_o}(\mathrm{y}-\kappa_o)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi\left(|\omega_o-\mathrm{y}|,\cdot\right)\right)d\beta\right).\end{aligned} \tag{4.15}$$

The required result is obtained by summing (4.11) and (4.15). □

Example 4.5. Consider the same process as given in Example 3.2, then we have the Table 2a and Graph 2b given by Figure 2, shows the right, middle and left terms, which are denoted by R, M and L respectively for the Theorem 4.1.

α	L	M	R
0.1	2255.42	16266.3	2.09×10^6
0.2	247.638	2047.5	68121.4
0.3	70.7146	618.102	9492.71
0.4	33.5577	270.156	2441.21
0.5	22.4043	146.25	892
0.6	18.3731	91.4815	411.64
0.7	16.7615	63.6261	224.643
0.8	16.0863	47.9883	138.905
0.9	15.8024	38.5408	94.4187
1.0	15.6875	32.5	69.00

(A) Numerical description of Theorem 4.1.



(B) Graphical description of Theorem 4.1.

FIGURE 2. Graphical and numerical description of $\Psi(y) = y^3$ via Theorem (3.8) for $\kappa_o = 2, \omega_o = 3$ and $\alpha \in [0, 1.0]$.

5. Implications for Information Theory

Distinguishing between two probability distributions is a fundamental task across various disciplines, including statistics, information theory and machine. Divergence measures serve as quantitative tools for assessing the dissimilarity between probability distributions. Lin [31], in 1991, proposed a novel class of divergence measures derived from Shannon entropy, a core principle in information theory used to quantify uncertainty and the informational content within a probability distribution. Lin's divergence offered a principled, information-theoretic approach to measuring distributional discrepancy. Building upon this work, Shioya and Da-te [41] extended Lin's method in 1995 by introducing the $\mathbb{H}.\mathbb{H}$ Ψ-divergence. Their formulation leveraged the $\mathbb{H}.\mathbb{H}$ inequality, a classical mathematical result associated with convex functions, to establish a more general and flexible framework. This advancement significantly broadened the applicability of divergence measures, enabling deeper insights and novel applications in the comparative analysis of probability distributions.

For clarity and focus, we restrict ourselves to the definitions that are directly relevant to the results that follow.

Definition 5.1 (Csiszár Ψ-divergence[20])**.**

$$D_{\Psi}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\mathfrak{q}_o(\mathrm{y})\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)d\mu(\mathrm{y}),\quad \mathfrak{p}_o,\mathfrak{q}_o\in\mathbb{P}.$$

In this context, $\mathbb{P}$ represents the collection of all probability densities with respect to a σ-finite measure μ, $\mho$ is assumed to be a nonempty set and Ψ is a convex function defined over $(0,+\infty)$.

$$\mathbb{P}=\left\{\mathfrak{p}_o|\mathfrak{p}_o:\mho\to\Re,\mathfrak{p}_o(\mathrm{y})\geq 0,\int_{\mho}\mathfrak{p}_o(\mathrm{y})d\mu(\mathrm{y})=1\right\}.$$

Remark 5.2. Taking the function Ψ as an P-superquadratic function yields the Csiszár Ψ-divergence for P-superquadratic function.

Definition 5.3 ($\mathbb{H}.\mathbb{H}$ Ψ-divergence [12])**.**

$$D^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\mathfrak{q}_o(\mathrm{y})\frac{\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\Psi(\mathrm{y})d\mathrm{y}}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)}d\mu(\mathrm{y}).\tag{5.1}$$

Definition 5.4 ($\mathbb{RL}$ fractional $\mathbb{H}.\mathbb{H}$ Ψ-divergence [4])**.**

(5.2)

$${}^{\alpha}D^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\mathfrak{q}_o(\mathrm{y})\frac{\left[(\mathfrak{I}^{\alpha}_{1^+}\Psi)\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)+(\mathfrak{I}^{\alpha}_{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)^-}\Psi(1))\right]}{2\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}d\mu(\mathrm{y}).$$

Where the Definition 2.3 provides the fractional operators used in (5.2).

Remark 5.5. We obtain (5.1), if $\alpha=1$ is set in (5.2).

Definition 5.6 ($\mathbb{RL}$ fractional $\mathbb{H}.\mathbb{H}$ Ψ-divergence [4])**.**

(5.3)

$${}^{\alpha}D^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\mathfrak{q}_o(\mathrm{y})\frac{\left[(\mathfrak{I}^{\alpha}_{1^+}\Psi)\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)+(\mathfrak{I}^{\alpha}_{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)^-}\Psi(1))\right]}{2\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}d\mu(\mathrm{y}).$$

The fractional operators appearing in (5.3) are defined in Definition 2.3.

Definition 5.7 (Stochastic divergence [5])**.** Let $\Psi:J\times\prod\to\Re$ be a convex stochastic process defined on $\mathrm{J}\subseteq(0,\infty)$, satisfying the condition $\Psi(1,\cdot)=0$. Then, the stochastic divergence associated with $\mathfrak{p}_o,\mathfrak{q}_o\in\mathbb{P}$ is defined as follows:

$$SD_{\Psi}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\mathfrak{q}_o(\mathrm{y})\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},.\right)d\mu(\mathrm{y}),\quad \mathfrak{p}_o,\mathfrak{q}_o\in\mathbb{P}.\tag{5.4}$$

Remark 5.8. When the convex stochastic process in (5.4) is substituted with an P-superquadratic stochastic process, the resulting expression defines the stochastic divergence for P-superquadratic stochastic processes.

Definition 5.9 (Stochastic $\mathbb{H}.\mathbb{H}$-divergence [5])**.** The stochastic $\mathbb{H}.\mathbb{H}$-divergence for $\mathfrak{p}_o, \mathfrak{q}_o \in \mathbb{P}$ for a convex stochastic process $\Psi : J \times \prod \to \Re$ on $\mathrm{J} \subseteq (0, \infty)$ is stated as

$$SD^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o) = \int_{\mho} \mathfrak{q}_o(\mathrm{y}) \frac{\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi(\mathrm{y}, .)dy}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})} - 1\right)} d\mu(\mathrm{y}), \quad \mathfrak{p}_o, \mathfrak{q}_o \in \mathbb{P}. \tag{5.5}$$

Remark 5.10. We find stochastic $\mathbb{H}.\mathbb{H}$-divergence for P-superquadratic stochastic process when P-superquadratic stochastic process is substituted for convex stochastic process in (5.5).

Definition 5.11 ($\mathbb{RL}$ fractional stochastic $\mathbb{H}.\mathbb{H}$-divergence [5])**.** A convex stochastic process $\Psi : J \times \prod \to \Re$, defined on $\mathrm{J} \subseteq (0, \infty)$ and satisfies $\Psi(1, \cdot) = 0$, provides the following definition for the stochastic divergence corresponding to $\mathfrak{p}_o, \mathfrak{q}_o \in \mathbb{P}$.

$$\begin{aligned} &{}^{\alpha}SD^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o) \\ &= \Gamma(\alpha+1) \int_{\mho} \frac{\mathfrak{q}_o(\mathrm{y}) \left[\mathfrak{J}^{\alpha}_{1^+}[\Psi]\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right) + \mathfrak{J}^{\alpha}_{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}^-}[\Psi](1) \right]}{2\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})} - 1\right)^{\alpha}} d\mu(\mathrm{y}). \end{aligned} \tag{5.6}$$

The fractional integral operators appearing in (5.6) are those provided in Definition 2.28.

Remark 5.12. By replacing the convex stochastic process in (5.6) with an P-superquadratic stochastic process, we derive the $\mathbb{RL}$ fractional stochastic $\mathbb{H}.\mathbb{H}$ divergence corresponding to the P-superquadratic case.

Here, we present the proofs of the results pertaining to fractional version of stochastic $\mathbb{H}.\mathbb{H}$-divergence and non-fractional stochastic $\mathbb{H}.\mathbb{H}$-divergence for P-superquadratic stochastic processes.

Theorem 5.13. *Let $\Psi : J \times \prod \to \Re$, on $\mathrm{J} \subseteq (0, \infty)$, is a P-superquadratic stochastic process and $\Psi(1, .) = 0$, then*

$$SD_{\Psi}\left(\frac{1}{2}\mathfrak{q}_o + \frac{1}{2}\mathfrak{p}_o||\mathfrak{q}_o\right) + \eta_1 \leq SD^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o) \leq 2SD_{\Psi}(\mathfrak{p}_o||\mathfrak{q}_o) - \eta_2.$$

$\forall\ \mathfrak{p}_o, \mathfrak{q}_o \in \mathbb{P}$, *where*

$$SD^{\Psi}_{HH}(\mathfrak{p}_o||\mathfrak{q}_o) = \int_{\mho} \frac{\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})} - 1\right)} \left(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi\left(\mathrm{y}, \cdot\right) d\mathrm{y} \right) d\mu(\mathrm{y}),$$

$$\eta_1 = \int_{\mho} \frac{\mathfrak{q}_o(\mathrm{y})(2)}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)} \left(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi\left(\left| \frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})} - \mathrm{y} \right|, \cdot \right) d\mathrm{y} \right) d\mu(\mathrm{y}),$$

$$\eta_2 = \left\{ \int_{\mho} \frac{2\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)} \left(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \left(\Psi\left(|\mathrm{y}-1|,\cdot\right) \right.\right.\right.$$
$$\left.\left.\left. + \Psi\left(\left| \left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})} - \mathrm{y} \right) \right|, \cdot \right) \right) d\mathrm{y} \right) d\mu(\mathrm{y}) \right\}.$$

Proof. Consider the $\mathbb{H}.\mathbb{H}$'s inequalities for P-superquadratic stochastic processes, from Theorem 3.8:

$$\begin{aligned} (5.7) \qquad & \Psi\left(\frac{\kappa_o+\omega_o}{2}, \cdot \right) + \frac{2}{\omega_o-\kappa_o} \int_{\kappa_o}^{\omega_o} \Psi\left(\left| \frac{(\kappa_o+\omega_o)-2\mathrm{y}}{2} \right|, \cdot \right) d\mathrm{y} \\ & \leq \frac{2}{\omega_o-\kappa_o} \int_{\kappa_o}^{\omega_o} \Psi(\mathrm{y},\cdot) d\mathrm{y} \\ & \leq 2\left[\Psi(\kappa_o,\cdot) + \Psi(\omega_o,\cdot) \right] \\ & \quad - \frac{2}{\omega_o-\kappa_o} \left\{ \int_{\kappa_o}^{\omega_o} \left(\Psi(|\mathrm{y}-\kappa_o|,\cdot) + \Psi\left(|\omega_o-\mathrm{y}|,\cdot\right) \right) d\mathrm{y} \right\}. \end{aligned}$$

Now setting $\kappa_o = 1$ and $\omega_o = \frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}$, in (5.7), we obtain

$$\begin{aligned} & \Psi\left(\frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})}, \cdot \right) + \frac{2}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)} \int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi\left(\left| \frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})} - \mathrm{y} \right|, \cdot \right) d\mathrm{y} \\ & \leq \frac{2}{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1} \int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi(\mathrm{y},\cdot) d\mathrm{y} \\ & \leq 2\left[\Psi(1,\cdot) + \Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}, \cdot \right) \right] \\ & \quad - \frac{1}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)} \left\{ \int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \left(\Psi\left(|(\mathrm{y}-1)|,\cdot\right) + \Psi\left(\left| \frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})} - \mathrm{y} \right|, \cdot \right) \right) d\mathrm{y} \right\}. \end{aligned}$$

Since $\Psi(1,\cdot) = 0$, it follows that:

(5.8)

$$\begin{aligned} & \Psi\left(\frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})}, \cdot \right) + \frac{2}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)} \int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi\left(\left| \frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})} - \mathrm{y} \right|, \cdot \right) d\mathrm{y} \\ & \leq \frac{2}{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1} \int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}} \Psi(\mathrm{y},\cdot) d\mathrm{y} \end{aligned}$$

$$\leq 2\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)$$
$$-\frac{1}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)}\left\{\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\left(\Psi\left(\left|(\mathrm{y}-1)\right|,\cdot\right)+\Psi\left(\left|\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right|,\cdot\right)\right)d\mathrm{y}\right\}.$$

Multiplying both sides of (5.8) by $\mathfrak{q}_o(\mathrm{y})\geq 0$, for all $\mathrm{y}\in\mho$ and integrating the resulting expression over $\mho$ yields:

(5.9)

$$\int_{\mho}\mathfrak{q}_o(\mathrm{y})\Psi\left(\frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})},\cdot\right)d\mu(\mathrm{y})$$
$$+\int_{\mho}\frac{2\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)}\left(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\Psi\left(\left|\frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right|,\cdot\right)d\mathrm{y}\right)d\mu(\mathrm{y})$$
$$\leq\int_{\mho}\frac{2\mathfrak{q}_o(\mathrm{y})}{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1}\left(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\Psi(\mathrm{y},\cdot)d\mathrm{y}\right)d\mu(\mathrm{y})$$
$$\leq 2\int_{\mho}\mathfrak{q}_o(\mathrm{y})\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)d\mu(\mathrm{y})-\int_{\mho}\frac{\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)}$$
$$\times\left\{\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\left(\Psi\left(\left|(\mathrm{y}-1)\right|,\cdot\right)+\Psi\left(\left|\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right|,\cdot\right)d\mathrm{y}\right)\right\}d\mu(\mathrm{y}).$$

By applying the definitions of stochastic divergence and stochastic $\mathbb{H}.\mathbb{H}$-divergence, we arrive at the desired result. □

Theorem 5.14. *Let* $\Psi: J\times\prod\to\Re$, *on* $\mathrm{J}\subseteq(0,\infty)$ *be a P- superquadratic stochastic process and* $\Psi(1,.)=0$ *then*

$$\frac{SD_\Psi\left(\frac{1}{2}\mathfrak{q}_o+\frac{1}{2}\mathfrak{p}_o||\mathfrak{q}_o\right)}{\Gamma(1+\alpha)}+\frac{4\alpha\tau_1}{\Gamma(1+\alpha)}\leq 2\ {}^{\alpha}SD_{HH}^{\Psi}(\mathfrak{p}_o||\mathfrak{q}_o)$$
$$\leq\frac{(1+\alpha)SD_\Psi(\mathfrak{p}_o||\mathfrak{q}_o)}{\Gamma(1+\alpha)}-\frac{\alpha\tau_2}{\Gamma(1+\alpha)}.$$

$\forall\ \mathfrak{p}_o,\mathfrak{q}_o\in\mathbb{P}$ *and* $\alpha>0$, *where*

$${}^{\alpha}SD_{HH}^{\Psi}(\mathfrak{p}_o||\mathfrak{q}_o)=\int_{\mho}\frac{\mathfrak{q}_o(\mathrm{y})\left[\mathfrak{I}_{1^+}^{\alpha}\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)+\mathfrak{I}_{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-}^{\alpha}\Psi\left(1,\cdot\right)\right]}{2\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}d\mu(\mathrm{y}),$$

$$\tau_1=\int_{\mho}\left(\frac{\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\int_1^{\frac{\mathfrak{q}_o(\mathrm{y})+\mathfrak{p}_o(\mathrm{y})}{2\mathfrak{q}_o(\mathrm{y})}}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\right.$$

$$\times\Psi\left(\left|\frac{\left(1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)-\mathrm{y}}{2}\right|,\cdot\right)d\mathrm{y}\Bigg)d\mu(\mathrm{y}),$$

$$\tau_2=2\int_{\mho}\frac{\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\Bigg(\int_1^{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}$$

$$\times\left(\Psi\left(\mathrm{y}-1|,\cdot\right)+\Psi\left(\left|\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right|,\cdot\right)\right)\Bigg)d\mu(\mathrm{y}).$$

Proof. Consider the fractional $\mathbb{H}.\mathbb{H}$-type inequalities for P-superquadratic stochastic processes, established via $\mathbb{RL}$ integral operators of order $\alpha>0$, as presented in Theorem 4.1:

(5.10)

$$\begin{aligned}
&\Psi\left(\frac{\mathrm{u}_o+\omega_o}{2},\cdot\right)+\frac{4\alpha}{(\omega_o-\kappa_o)^{\alpha}}\int_{\kappa_o}^{\frac{\kappa_o+\omega_o}{2}}(\omega_o-\mathrm{y})^{\alpha-1}\Psi\left(\left|\frac{(\kappa_o+\omega_o)-2\mathrm{y}}{2}\right|,\cdot\right)d\mathrm{y}\\
&\leq\frac{\Gamma(1+\alpha)}{(\omega_o-\kappa_o)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\kappa_o+}\Psi(\omega_o,\cdot)+\mathfrak{I}^{\alpha}_{\omega_o-}\Psi\left(\kappa_o,\cdot\right)\right]\\
&\leq 2\left[\Psi(\kappa_o,\cdot)+\Psi(\omega_o,\cdot)\right]\\
&\quad-\frac{2\alpha}{(\omega_o-\kappa_o)^{\alpha}}\left\{\int_{\kappa_o}^{\omega_o}(\omega_o-\mathrm{y})^{\alpha-1}\left(\Psi\left(|\mathrm{y}-\kappa_o|,\cdot\right)+\Psi(\omega_o-\mathrm{y},\cdot)\right)d\mathrm{y}\right\},
\end{aligned}$$

Now setting $\kappa_o=1$ and $\omega_o=\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}$, in (5.10), we obtain

$$\begin{aligned}
&\Psi\left(\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2},\cdot\right)+\frac{4\alpha}{(1-\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})})^{\alpha}}\int_1^{\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2}}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\\
&\quad\times\Psi\left(\left|\frac{\left(1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)-2\mathrm{y}}{2}\right|,\cdot\right)d\mathrm{y}\\
&\leq\frac{\Gamma(1+\alpha)}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{1+}\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)+\mathfrak{I}^{\alpha}_{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-}\Psi\left(1,\cdot\right)\right]\\
&\leq 2\left[\Psi(1,\cdot)+\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)\right]-\frac{2\alpha}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\\
&\quad\times\left\{\int_1^{\omega_o}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-1|,\cdot\right)+\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y},\cdot\right)\right)d\mathrm{y}\right\},
\end{aligned}$$

Given that $\Psi(1,\cdot)=0$, it follows that:

(5.11)

$$\begin{aligned}
&\Psi\left(\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2},\cdot\right)+\frac{4\alpha}{\left(1-\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)^{\alpha}}\int_1^{\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2}}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\\
&\quad\times\Psi\left(\left|\frac{(1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})})-2\mathrm{y}}{2}\right|,\cdot\right)d\mathrm{y}\\
&\leq\frac{\Gamma(1+\alpha)}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{1^+}\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)+\mathfrak{I}^{\alpha}_{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-}\Psi\left(1,\cdot\right)\right]\\
&\leq 2\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)-\frac{2\alpha}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\\
&\quad\times\left\{\int_1^{\omega_o}\left(\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-1|,\cdot\right)+\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y},\cdot\right)\right)\right)d\mathrm{y}\right\},
\end{aligned}$$

By multiplying both sides of (5.11) by $\mathfrak{q}_o(\mathrm{y})\geq 0$, where $\mathrm{y}\in\mho$ and integrating the resulting expression over $\mho$, we derive the following:

$$\begin{aligned}
&\int_{\mho}\mathfrak{q}_o(\mathrm{y})\Psi\left(\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2},\cdot\right)d\mu(\mathrm{y})\\
&\quad+\int_{\mho}\left(\frac{4\alpha\mathfrak{q}_o(\mathrm{y})}{\left(1-\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}\right)^{\alpha}}\int_1^{\frac{1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}}{2}}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\right.\\
&\quad\left.\times\Psi\left(\left|\frac{(1+\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})})-2\mathrm{y}}{2}\right|,\cdot\right)d\mathrm{y}\right)d\mu(\mathrm{y})\\
&\leq\int_{\mho}\frac{\Gamma(1+\alpha)\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{1^+}\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)+\mathfrak{I}^{\alpha}_{\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-}\Psi\left(1,\cdot\right)\right]d\mu(\mathrm{y})\\
&\leq 2\int_{\mho}\mathfrak{q}_o(\mathrm{y})\left[\Psi(1,\cdot)+\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})},\cdot\right)\right]d\mu(\mathrm{y})-\int_{\mho}\left(\frac{2\alpha\mathfrak{q}_o(\mathrm{y})}{\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-1\right)^{\alpha}}\right.\\
&\quad\left.\times\left\{\int_1^{\omega_o}\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y}\right)^{\alpha-1}\left(\Psi\left(|\mathrm{y}-1|,\cdot\right)+\Psi\left(\frac{\mathfrak{p}_o(\mathrm{y})}{\mathfrak{q}_o(\mathrm{y})}-\mathrm{y},\cdot\right)\right)d\mathrm{y}\right\}\right)d\mu(\mathrm{y}),
\end{aligned}$$

Employing the definitions of stochastic divergence, stochastic $\mathbb{H}.\mathbb{H}$- divergence and $\mathbb{RL}$ fractional stochastic $\mathbb{H}.\mathbb{H}$- divergence, we obtain the required result. □

Remark 5.15. By setting $\alpha = 1$ in Theorem 5.14, we recover the result stated in Theorem 5.13.

5.1. **Assumptions Underlying the Study and Their Limitations.** This study is built upon a carefully designed set of assumptions that ensure both mathematical rigor and practical relevance. The newly introduced class of P-superquadratic stochastic processes is examined within the framework of mean-square continuity, a standard yet robust condition that guarantees the existence of the stochastic $\mathbb{RL}$ fractional integrals employed in our derivations. All processes and random variables are assumed to be square-integrable, thereby reinforcing the reliability and consistency of the theoretical outcomes. The development of refined $\mathbb{H}.\mathbb{H}$-type inequalities is grounded in the concept of generalized convexity via the m-superquadratic condition, which notably expands the analytical horizon beyond traditional convexity-based methods. By utilizing $\mathbb{RL}$ operators in the context of fractional extensions, the paper paves the way for new directions in stochastic fractional calculus and establishes a solid foundation for future research involving alternative fractional operators such as Caputo or Hadamard. The framework's practical utility is demonstrated through its application to information theory, particularly via the construction of novel stochastic divergence measures, highlighting its interdisciplinary significance. To complement the theoretical developments, the paper presents illustrative numerical examples and detailed visualizations, with transparent documentation of computation times. Taken together, these contributions form a robust foundation for continued research and practical application, positioning this work as a meaningful addition to the evolving field of stochastic analysis and its applications.

5.2. **Complexity and Implementation.** To support the theoretical developments and demonstrate the practical feasibility of our proposed framework for P-superquadratic stochastic processes, we provide empirical benchmarks for all symbolic computations and graphical outputs. The symbolic routines used to generate figures and numerical data for the Jensen-type and $\mathbb{H}.\mathbb{H}$ inequalities, as well as their fractional $\mathbb{RL}$ analogues were executed using Mathematica 11.3 on a standard computing environment (Intel Core i7 processor with 16 GB RAM). For representative tasks, the observed wall-clock times ranged from approximately 0.1657486 to 4.95325 seconds, depending on the complexity of the symbolic integration and plotting involved. Despite the symbolic nature

of the methods, these results confirm that the computations remain efficient and comfortably within practical limits. To ensure full reproducibility and computational transparency, we provide a publicly accessible GitHub repository containing the complete Mathematica notebook: https://github.com/dawoodpk1947/P-superquadratic-stochastic-processes.git. Additionally, detailed timing results can be accessed via the same link, (accessed on 10 August 2025), allowing users to explore and replicate the computational procedures step by step.

6. Conclusion

In this study, we introduced the concept of P-superquadratic stochastic processes, offering a novel and rigorous generalization of classical superquadratic frameworks within the setting of mean-square stochastic calculus. By systematically examining their structural properties, we derived strengthened versions of Jensen's inequality and $\mathbb{H}.\mathbb{H}$ inequalities and further extended these results to their fractional counterparts using stochastic $\mathbb{RL}$ fractional integrals. These contributions significantly broaden the analytical tools available for fractional stochastic analysis.

The theoretical advancements were complemented by carefully designed graphical illustrations and tabulated examples, affirming both the practicality and the scope of the proposed inequalities. The applicability of the framework was further demonstrated through innovative applications in information theory, particularly in the construction of new classes of stochastic divergence measures.

To promote reproducibility and computational transparency, all symbolic routines, graphical outputs and execution timings have been thoroughly documented and made publicly accessible. The analytical techniques and fractional tools developed in this paper provide a foundation for exploring broader classes of generalized convexity. In particular, these methods may be effectively extended to establish new stochastic fractional integral inequalities for (h, m)-convex [13] and (h, m, s)-convex modified functions [14]. Such an investigation would further enrich the theory of stochastic fractional calculus and open new avenues for applications in generalized convexity frameworks.

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