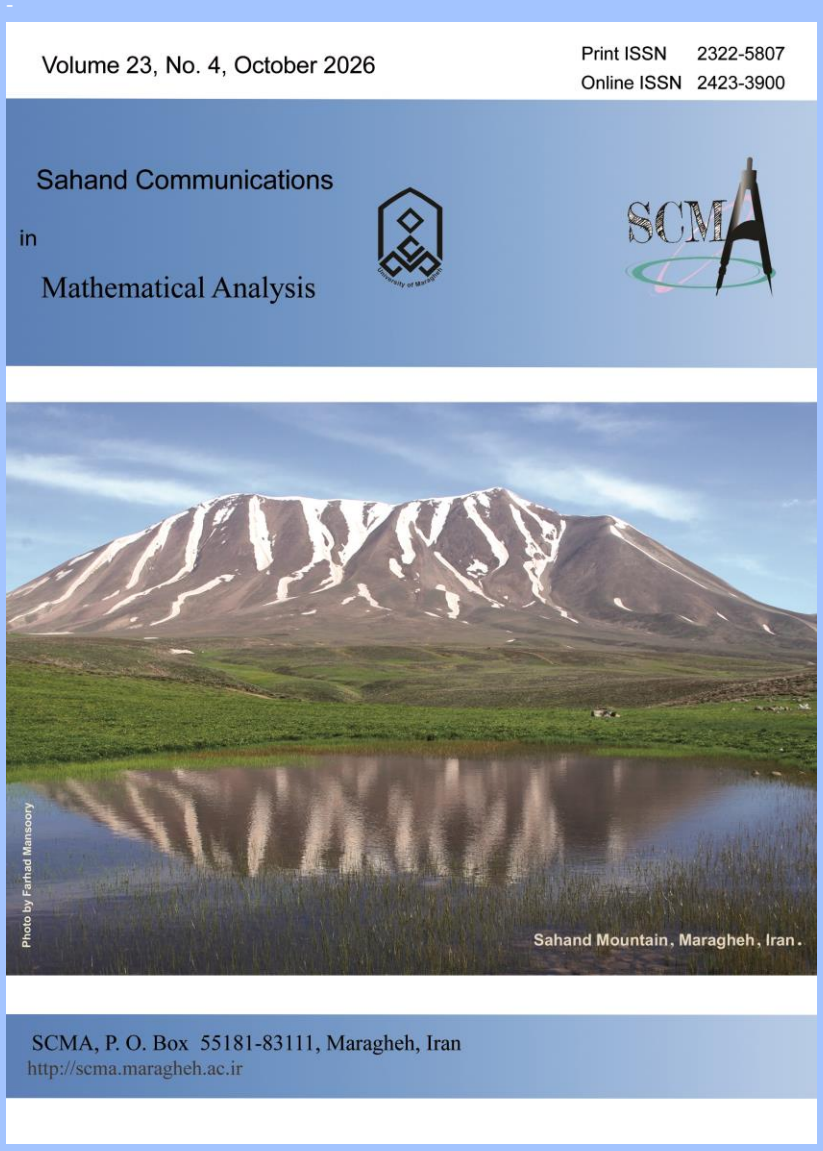
The Radon-Nikodym Theory for Banach Spaces Involving Variable Exponents

Mykola Ivanovich Yaremenko

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 4
Pages: 167-201

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2026.2077696.2413



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 4 (2026), 167-201
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2077696.2413

The Radon-Nikodym Theory for Banach Spaces Involving Variable Exponents

Mykola Ivanovich Yaremenko

Abstract. This paper establishes a comprehensive Radon-Nikodym theory for Banach space-valued functions within the framework of variable exponent Lebesgue spaces. We begin by developing a generalized Bochner integration theory for functions with respect to vector measures, leveraging a continuous bilinear mapping $\Lambda : X \times Y \to Z$. A central result is the establishment of a fundamental duality: the space of bounded linear operators on the space of totally measurable functions is isometrically isomorphic to the space of vector measures with finite semi-variation.

We then extend this framework to variable exponent Bochner spaces $L^{p(\cdot)}(E, X, \Sigma, \mu)$. We prove that if $p \in P(E)$, $1 < p_m \leq p_S < \infty$, $p(\cdot) + q(\cdot) = p(\cdot)\, q(\cdot)$ and μ is a scalar σ-finite measure on Σ and a linear mapping $U : L^{p(\cdot)}(E, \Sigma, \mu) \to Z$ has finite norm $\|U_D\|_{L^{p(\cdot)}} = \mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$ for all D. Then there exists a function ψ mapping from the set E to the space $LB(X, Z)$ such that:

1) for each μ-measurable set $D \subset E$, we obtain $1_D \|\psi\| \in L^{q(\cdot)}(E, \Sigma, \mu)$ and $\|\psi 1_D\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|U_D\|_{L^{p(\cdot)}}$;
2) for each function $f \in L^{p(\cdot)}(E, \Sigma, \mu)$ and each $z^* \in Z^*$, the identity $\langle U(f), z^* \rangle = \int_D \langle \psi(x)(f(x)), z^* \rangle \, d\mu(x)$, holds for all μ-measurable set $D \subset E$.

1. Introduction

We develop an extension of the scalar Radon-Nikodym theorem for Banach space-valued functions, situated within the theory of variable

2020 *Mathematics Subject Classification.* 46G10, 46G12, 28A15, 28A99.
Key words and phrases. Radon-Nikodym theorem, Banach space, Conditional expectation, Measure, Vector measure, σ-algebra, Radon-Nikodym property, Bochner integral, Radon measure.
Received: 2025-11-19, Accepted: 2026-02-16.

exponent spaces. This research contributes to the broader generalization of the standard theories of Lebesgue integration, L^p spaces and the Radon-Nikodym property.

The foundational work on these problems can be traced to [1, 5–8, 20, 21] and the field remains active, as evidenced by recent studies in [2, 4, 12, 15, 17, 23]. The study of variable exponent Lebesgue spaces is motivated by their extensive applications in various fields, such as physical modeling and image processing, making them a prominent subject in contemporary analysis [9, 18, 19, 24] and references therein.

In the second Section, we collect some basic information about Lebesgue-Bochner integrals. We show that the Radon-Nikodym property of a vector-valued measure $\eta : \Sigma \to X$ follows from absolute continuity of η with respect to the scalar measure μ. The classical measure constructions integration schemes were described in [1] and more recent approaches can be found in [23] and [21].

The third and fourth Sections are dedicated to the generalization of Bochner integration with a chosen bilinear mapping $\Lambda : X \times Y \to Z$, where X, Y and Z are given separable reflexive Banach spaces. (This bilinear form will be simply denoted by $\Lambda(a, b) = ab$ for every $a \in X, b \in Y$.) We define the integral of an η-measurable function $f : E \to Y$ with respect to the σ-finite vector-valued measure $\eta : \Sigma \to X$ with finite variation $|\eta|$; the integral $\int_E f\, d\eta$ is an element of the Banach space Z. We studied the properties of such integrals.

In the fourth Section, we describe the properties of totally measurable functions and show the existence of the isomorphism between the space $\widetilde{LB}(M_Y(\Sigma), Z)$ and the set of all measures $\eta : \Sigma \to LB(Y, Z)$ with finite variation. The space $\widetilde{LB}(M_Y(\Sigma), Z)$ consists of all linear bounded operators U such that there is a linear positive functional Φ defined on $M_R(\Sigma)$ for which $\|U(f)\|_Z \leq \Phi(\|f\|_Y)$ holds for all $f \in M_Y(\Sigma)$. We also proved that the operator norm (or suitable variation norm) satisfies $\|U_D\| = |\eta|(D) = \inf_\Phi \Phi_D$.

In the fifth Section, we defined the generalized Bochner-Lebesgue spaces for the case of variable exponent. Assuming $p \in P(E)$, $1 < p_m \leq p_S < \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in E$, we proved the generalized Holder inequality

$$\left\| \int_E f(x)\, g(x)\, d\mu(x) \right\|_Z \leq \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}},$$

for all $f \in L^{p(\cdot)}(E, \Sigma, \mu)$ and $g \in L^{q(\cdot)}(E, \Sigma, \mu)$. Preliminary definitions on variable exponent Lebesgue spaces can be found in [9, 18, 19, 24] and references therein.

The sixth Section studies linear operation on the variable exponent Bochner-Lebesgue spaces $L^{p(\cdot)}(E,\Sigma,\mu)$.

We present the main results in the subsequent three sections: 7, 8 and 9.

The seventh Section studies the connection between the characteristic of a given Banach space related to the Radon-Nikodym theorem and the dentability of its subsets; initial information can be found in [5–8, 20, 23].

The eighth Section investigates general conditions under which variable exponent spaces $L^{p(\cdot)}(E,\Sigma,\mu)$ have the Radon-Nikodym property. We show that, if $p \in P(E)$, $1 < p_m \leq p_S < \infty$, then the space $L^{p(\cdot)}(E,X,\Sigma,\mu)$ possesses the Radon-Nikodym property if and only if the space X has the Radon-Nikodym property.

In the ninth Section, we present and prove the main results of the paper that assume $p \in P(E)$, $1 < p_m \leq p_S < \infty$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and assume μ is a scalar σ-finite measure on Σ and presume a linear mapping $U : L^{p(\cdot)}(E,\Sigma,\mu) \to Z$ has finite norm $\|U_D\|_{L^{p(\cdot)}} = \mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$ for all D. Then there exists a function ψ mapping E into $LB(X,Z)$ such that:

(i) for each μ-measurable set $D \subset E$, we obtain $1_D\|\psi\| \in L^{q(\cdot)}(E,\Sigma,\mu)$ and $\|\psi 1_D\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|U_D\|_{L^{p(\cdot)}}$;

(ii) for each function $f \in L^{p(\cdot)}(E,\Sigma,\mu)$ and each $z^* \in Z^*$, the identity

$$\langle U(f), z^* \rangle = \int_D \langle \psi(x)(f(x)), z^* \rangle \, d\mu(x),$$

holds for all μ-measurable set $D \subset E$.

2. Prerequisite, the Classical Results on the Radon-Nikodym Property for the Set Function with Finite Variation

Let E be a locally compact space and let Σ denote a σ-algebra of subsets of E. Suppose μ is a positive scalar Radon measure defined on Σ. Additionally, let X be a separable reflexive Banach space equipped with the norm $\|\cdot\|_X$.

Definition 2.1. Let $\Pi(D)$ represent the collection of all partitions

$$\pi_k = \Pi(D) = \{B_1, \ldots, B_k, k \in \mathbb{N}\},$$

where the sets B_i are pairwise disjoint ($B_i \cap B_j = \emptyset$ for $i \neq j$) and satisfy $\bigcup_{i=1}^{k} B_i = D$ for a measurable set $D \in \Sigma$. The variation of the measure μ is then defined as the set function

$$\mathrm{Var}(\mu)(D) = |\mu|(D) = \sup_{\Pi(D)} \sum_{B \in \Pi(D)} \mu(B),$$

for every $D \in \Sigma$.

Definition 2.2. A function $\chi : E \to X$ is called simple if it can be written as a finite linear combination of characteristic functions of disjoint measurable sets:

$$\chi(x) = \sum_{i=1}^{k} a_i \mathbf{1}_{B_i}(x),$$

where $\mathbf{1}_{B_i}$ denotes the characteristic function of B_i and each a_i belongs to X for $i = 1, \ldots, k$.

Definition 2.3. A set function $\eta : \Sigma \to X$ which is σ-additive is termed a vector measure on Σ.

Definition 2.4. For any $D \in \Sigma$, the variation of a vector measure η is the scalar set function given by

$$\operatorname{Var}(\eta)(D) = |\eta|(D) = \sup_{\Pi(D)} \sum_{B \in \Pi(D)} \|\eta(B)\|_X.$$

Lemma 2.5. *The variation of a vector measure is a positive, monotone, scalar set function; however, it may be unbounded.*

Definition 2.6. The vector measure η is said to be of bounded variation whenever $\operatorname{Var}(\eta)(E)$ is finite.

Lemma 2.7. *A vector measure η is of bounded variation if and only if $\operatorname{Var}(\eta)(D) < \infty$ holds for every $D \in \Sigma$.*

Lemma 2.8. *If $\operatorname{Var}(\eta)(D) < \infty$, then $\operatorname{Var}(\eta)$ itself is a measure and consequently $(D, \Sigma, \operatorname{Var}(\eta))$ forms a measure space.*

Definition 2.9 (Bochner integral)**.** Let $\chi : E \to X$ be a simple function given by

$$\chi(x) = \sum_{i=1}^{k} a_i \mathbf{1}_{B_i}(x),$$

with $\mathbf{1}_{B_i}$ the characteristic function of B_i and $a_i \in X$ for all i. Assume $\mu(B_i) < \infty$ whenever $a_i \neq 0$; then the integral of χ is defined as

$$\int_E \chi(x)\, d\mu(x) = \sum_{i=1}^{k} a_i\, \mu(B_i).$$

For a Bochner measurable function $f : E \to X$, its integral is defined by the limit

$$\int_E f(x)\, d\mu(x) = \lim_{k \to \infty} \int_E \chi_k(x)\, d\mu(x),$$

where $\{\chi_k\}$ is a sequence of simple functions $\chi_k : E \to X$ satisfying

$$\lim_{k\to\infty} \int_E \|f(x) - \chi_k(x)\|_X \, d\mu(x) = 0,$$

and this limit characterizes the Bochner integrability of f.

Lemma 2.10. *Suppose a vector measure η is defined by*

$$\eta(D) = \int_D f \, d\mu,$$

with f Bochner integrable. Then the variation $\mathrm{Var}\,(\eta)$ *of η is given by*

$$\mathrm{Var}\,(\eta)\,(D) = \int_D \|f\|_X \, d\mu. \tag{2.1}$$

Proof. Assume that $f : E \to X$ is Bochner integrable with respect to μ. For any partition $\{B_1, \ldots, B_k\}$ of $D \in \Sigma$ we have

$$\begin{aligned}\sum_{i=1}^{k} \|\eta(B_i)\|_X &= \sum_{i=1}^{k} \left\| \int_{B_i} f \, d\mu \right\|_X \\ &\le \sum_{i=1}^{k} \int_{B_i} \|f\|_X \, d\mu \\ &= \int_D \|f\|_X \, d\mu.\end{aligned}$$

Now let $\{\chi_k\}$ be a sequence of simple functions approximating f. For each $\varepsilon > 0$ there exists k_0 such that

$$\int_E \|f - \chi_k\|_X \, d\mu < \varepsilon, \quad \text{for all } k > k_0.$$

Define vector measures η_k by

$$\eta_k(E) = \int_E \chi_k \, d\mu = \sum_{i=1}^{k} a_i \, \mu(B_i).$$

Then for all $k > k_0$ we obtain

$$|\mathrm{Var}(\eta)(E) - \mathrm{Var}(\eta_k)(E)| < \varepsilon.$$

Moreover,

$$\begin{aligned}\mathrm{Var}(\eta_k)(E) &= \sum_{i=1}^{k} \|a_i\|_X \, \mu(B_i) \\ &= \int_E \|\chi_k\|_X \, d\mu.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\operatorname{Var}(\eta)(E) &\le \operatorname{Var}(\eta_k)(E) + \varepsilon \\ &= \int_E \|\chi_k\|_X \, d\mu + \varepsilon \\ &\le \int_E \|f\|_X \, d\mu + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired equality follows. □

Definition 2.11. A vector measure $\eta : \Sigma \to X$ is said to possess the Radon–Nikodym property if it is of bounded variation and absolutely continuous with respect to some finite positive measure μ and there exists a μ-Bochner integrable function f such that

$$\eta(D) = \int_D f \, d\mu, \quad \text{for all } D \in \Sigma.$$

A Banach space X has the Radon–Nikodym property if it enjoys this property with respect to every finite measure.

Lemma 2.12. *Let $D \in \Sigma$ and let $f : E \to X$ be μ-Bochner integrable. Then the vector measure defined by*

$$\eta(D) = \int_D f \, d\mu$$

has the Radon–Nikodym property.

Proof. Because f is μ-integrable, the variation $\operatorname{Var}(\eta)(E)$ is finite and given by (2.1), so η indeed satisfies the Radon–Nikodym property. □

3. Generalization of Bochner's Integral

Consider separable reflexive Banach spaces denoted by X, Y and Z, together with a σ-finite measure space (E, Σ, μ). Suppose $\eta : \Sigma \to X$ is a vector measure and let $\Lambda : X \times Y \to Z$ be a continuous bilinear map satisfying the norm inequality

$$\|\Lambda(a, b)\|_Z \le \|a\|_X \|b\|_Y$$

for every $a \in X$ and $b \in Y$. To simplify notation, we shall denote the bilinear operation simply by juxtaposition, i.e., we write $\Lambda(a, b) = ab$.

Definition 3.1. Let $\chi : E \to Y$ be a simple function expressed as

$$\chi(x) = \sum_{i=1}^{k} a_i \mathbf{1}_{B_i}(x),$$

where $\mathbf{1}_{B_i}$ denotes the indicator function of pairwise disjoint sets $B_i \in \Sigma$ and each a_i belongs to Y. The general Bochner integral of χ with respect to η is then defined by

$$\int_E \chi(x)\, d\eta(x) = \sum_{i=1}^{k} \Lambda(\eta(B_i), a_i) \equiv \sum_{i=1}^{k} a_i \eta(B_i) \in Z.$$

The following lemma provides a fundamental estimate for this integral.

Lemma 3.2. *If $\chi : E \to Y$ is any simple function, then the inequality*

$$\left\| \int_E \chi \, d\eta \right\|_Z \leq \int_E \|\chi\|_Y \; d|\eta|$$

holds true, where $|\eta|$ denotes the total variation measure of η.

Proof. Write the simple function χ as

$$\chi(x) = \sum_{i=1}^{k} a_i \mathbf{1}_{B_i}(x),$$

with disjoint sets $B_i \in \Sigma$ and coefficients $a_i \in Y$. The norm of χ is then

$$\|\chi\|_Y = \sum_{i=1}^{k} \|a_i\|_Y \mathbf{1}_{B_i},$$

so that integration against $|\eta|$ yields

$$\int_E \|\chi\|_Y \, d|\eta| = \sum_{i=1}^{k} \|a_i\|_Y |\eta|(B_i).$$

Applying the triangle inequality and the definition of the integral for simple functions we obtain

$$\begin{aligned} \left\| \int_E \chi \, d\eta \right\|_Z &= \left\| \sum_{i=1}^{k} a_i \eta(B_i) \right\|_Z \\ &\leq \sum_{i=1}^{k} \|a_i\|_Y \, \|\eta(B_i)\|_X \\ &= \sum_{i=1}^{k} \|a_i\|_Y |\eta|(B_i) \\ &= \int_E \|\chi\|_Y \, d|\eta|. \end{aligned}$$

□

For brevity we often write $\mu = |\eta|$, so that the estimate becomes

$$\left\| \int_E \chi \, d\eta \right\|_Z \leq \int_E \|\chi\|_Y \, d\mu.$$

For every step function $\chi : E \to Y$ we introduce the positive functional

$$N(\chi) = \int_E \|\chi\|_Y \, d\mu.$$

The positive function N is a semi-norm on the space $S(E, Y, \mu)$ consisting of all simple (step) functions $\chi : E \to Y$. The topology generated by N is called the topology of convergence in the mean (or L^1 topology). Straightforwardly, we have the inequality

$$\left\| \int_E \chi \, d\eta \right\|_Z \leq N(\chi).$$

Definition 3.3. A sequence $\{\chi_k\}_k \subset S(E, Y, \mu)$ is called a Cauchy (fundamental) sequence with respect to the semi-norm N if

$$\lim_{k,m\to\infty} N(\chi_k - \chi_m) = \lim_{k,m\to\infty} \int_E \|\chi_k - \chi_m\|_Y d\mu = 0.$$

We have the following standard properties of fundamental sequences in $S(E, Y, \mu)$.

Statement. Let $\{\chi_k\} \subset S(E, Y, \mu)$ be a fundamental in N sequence of step functions. Then, first, there exists a subsequence $\{\chi_k\}$ convergent almost everywhere to a function $f : E \to Y$; second, for all $\varepsilon > 0$ there exists a set $D(\varepsilon) \in \Sigma$, $\mu(D(\varepsilon)) < \varepsilon$ such that the subsequence $\{\chi_k\}$ uniformly converges to $f : E \to Y$ on $E \backslash D(\varepsilon) \in \Sigma$.

Statement. Let $\{\chi_k\} \subset S(E, Y, \mu)$ be a fundamental in N sequence of step functions and let $\{\chi_k\}$ converges almost everywhere to 0 then $\{\chi_k\}$ converges to 0 in mean.

Definition 3.4. A mapping $f : E \to Y$ is called vector integrable relative to a vector measure η whenever one can find a sequence of simple functions $\{\chi_k\}$ satisfying:

(i) $\{\chi_k\}$ forms a Cauchy (fundamental) sequence in the semi-norm N;
(ii) $\{\chi_k\}$ converges to f in the μ-almost everywhere sense.

The integral of f is subsequently defined via the limit:

$$\int_E f \, d\eta = \lim_{k\to\infty} \int_E \chi_k \, d\eta.$$

The consistency of this definition follows from statements 1 and 2, as it does not depend on the particular choice of the approximating sequence of simple functions.

The vector integral inherits the fundamental properties of the classical Lebesgue integral, including linearity and the fact that functions which coincide almost everywhere have identical integrals. Consequently, we may formulate the following definition.

Definition 3.5. The space $L^1(E,Y,\eta)$ of η-integrable functions is a vector space under the usual pointwise operations. The Bochner integral with respect to η is a linear mapping from $L^1(E,Y,\eta)$ to Z, meaning

$$\int_E (\alpha f+\beta g)d\eta=\alpha\int_E fd\eta+\beta\int_E gd\eta$$

holds for all scalars α,β and all $f,g\in L^1(E,Y,\eta)$.

This definition is consistent since two elements of $L^1(E,Y,\eta)$ are equal η-almost everywhere if and only if their integrals are equal so that the equality

$$\int_E fd\eta=\int_E gd\eta$$

holds for all $f,g\in L^1(E,Y,\eta)$ such that $f\overset{a.e}{=}{}^{\eta} g$.

Statement. Let $f\in L^1(E,Y,\eta)$ then $\|f\|_Y\in L^1(E,Y,\mu=\mathrm{Var}(\eta))$ and

$$\left\|\int_D fd\eta\right\|_Z\leq\int_D\|f\|_Y d\mu$$

for all sets $D\in\Sigma$.

Proof. Let sequence $\{\chi_k\}\subset S(E,Y,\mu)$ converge almost everywhere to element $f\in L^1(E,Y,\eta)$. Then, $\|\chi_k\|_Y$ converges almost everywhere to $\|f\|_Y$ and

$$\left\|\int_D \chi_k d\eta\right\|_Z\leq\int_D\|\chi_k\|_Y d\mu$$

for all $D\in\Sigma$.

Let $\{\chi_k\in S(E,Y,\eta),\chi_k:E\to Y\}$ be a sequence of simple functions converging η-almost everywhere to f, then the identity

$$\nu(D)=\int_D fd\eta=\lim_{k\to\infty}\int_D\chi_k d\eta$$

defines measure on $\Sigma(\mu)$. □

Statement. Let $f \in L^1(E, Y, \eta)$ and $\{D_k\} \subset \Sigma$ be a sequence of disjoint subsets. Then,

$$\int_{\bigcup_k D_k} f d\eta = \sum_{k=1,2,\ldots} \int_{D_k} f d\eta.$$

Statement. Let $A : Y \to Z$ be a continuous linear operator and $\tilde{\mu}$ a scalar measure. The operator A commutes with the Bochner integral; that is, for any $f \in L^1(E, Y, \tilde{\mu})$ and any $D \in \Sigma$, we have

$$\int_D A(f) d\tilde{\mu} = A\left(\int_D f d\tilde{\mu}\right).$$

Proof. Let $\{\chi_k\} \subset S(E, Y, \tilde{\mu})$ be sequence converged almost everywhere to $f \in L^1(E, Y, \tilde{\mu})$. Then, we deduce $A(\chi_k) \xrightarrow[k\to\infty]{a.e.} A(f)$, which proves statement 5. □

Statement. Let $f \in L^1(E, Y, \eta)$ then $\int_E \|f(x)\|_Y d\mu(x) = 0$ if and only if $f(x) = 0$ μ-almost everywhere and $\int_E f(x) d\eta(x) = 0$.

Proof. We assume $f(x) \overset{\mu}{=} \overset{a.e.}{0}$ and choose a sequence $\{\chi_k\} \subset S(E, Y, \tilde{\mu})$ given by $\chi_k(x) = 0$ for all $k \in N$. So, we have

$$\int_E f(x) d\eta(x) = \lim_{k\to\in} 0 = \int_E \chi_k(x) d\eta(x) = 0$$

so

$$\int_E \|f(x)\|_Y d\mu(x) = 0.$$

Now, we show the reverse. We assume

$$\int_E \|f(x)\|_Y d\mu(x) = 0.$$

We define the sets

$$D_k = \left\{x \in E : \|f(x)\|_Y \geq \frac{1}{k}, k \in N\right\}$$

and we obtain

$$0 = \int_E \|f(x)\|_Y d\mu(x) \geq \int_{D_k} \|f(x)\|_Y d\mu(x) \geq \frac{1}{k}\mu(D_k) \geq 0.$$

Therefore, we conclude that $f(x) = 0$ μ-almost everywhere since

$$D = \{x \in E : f(x) \neq 0\} = \bigcup_k D_k.$$ □

Theorem 3.6. *The vector space $S(E, Y, \mu)$ of all simple functions $\chi : E \to Y$ is dense in the space $L^1(E, Y, \eta)$ of all η-integrable functions.*

Theorem 3.7. *Let a sequence $\{f_k\} \subset L^1(E,Y,\mu)$ converge in mean to a function $f \in L^1(E,Y,\mu)$, then there is a subsequence $\{f_k\}$ converging μ-almost everywhere to function f.*

Proof. We assume sequence $\{f_k\} \subset L^1(E,Y,\mu)$ converges in mean to $f \in L^1(E,Y,\mu)$, then there is a number $k(j)$ such that

$$\int_E \left\|f - f_{k(j)}\right\|_Y d\mu(x) < \frac{1}{2^{2j}}.$$

We denote sets

$$D_j = \left\{x \in E : \int_E \left\|f - f_{k(j)}\right\|_Y d\mu \geq \frac{1}{2^j}, \quad j \in N\right\}.$$

Since $\mu(D_j) < \frac{1}{2^j}$ we obtain

$$\int_E \left\|f - f_{k(j)}\right\|_Y d\mu \geq \int_{D_j} \left\|f - f_{k(j)}\right\|_Y d\mu \geq \frac{1}{2^j}\mu(D_j).$$

We have

$$\mu\left(\bigcap_{j=1,\ldots}\bigcup_{i=j,\ldots} D_i\right) \leq \mu\left(\bigcup_{i=j,\ldots} D_i\right) \leq \sum_{i=j,\ldots}\mu(D_i) \leq \frac{1}{2^j}$$

for all $j \geq 1$ so we deduce $\mu\left(\bigcap_{j=1,\ldots}\bigcup_{i=j,\ldots} D_i\right) = 0$. Thus, the subsequence $\left\{f_{k(j)}\right\}$ converges to f on $E\backslash\bigcap_{j=1,\ldots}\bigcup_{i=j,\ldots} D_i$ and the set on which $\left\{f_{k(j)}\right\}$ does not converge has measure zero. □

We have generalizations of the standard Lebesgue-dominated theorem in the following forms.

Theorem 3.8 (Lebesgue-dominated)**.** *Let $\{f_k\} \subset L^1(E,Y,\eta)$ converge η-almost everywhere to a function $f : E \to Y$. Let there be a positive μ-integrable function h such that*

$$\|f_k(x)\|_Y \leq h(x)$$

μ-almost everywhere for all k. Then,

$$\int_E f d\eta = \lim_{k\to\infty}\int_E f_k d\eta.$$

Proof. We assume that $\{f_k\} \subset L^1(E,Y,\eta)$ converges μ-almost everywhere to a function $f : E \to Y$. Then, we have

$$\limsup_{k,m\to\infty}\|f_k - f_m\|_Y \overset{\eta-a.e.}{=} 0$$

so

$$\limsup_{k,m\to\infty}\int_E \|f_k - f_m\|_Y d\eta \leq \int_E \limsup_{k,m\to\infty}\|f_k - f_m\|_Y d\eta = 0.$$

Thus, the sequence $\{f_k\}$ is fundamental in $L^1(E, Y, \eta)$ and the limit f is η-integrable and

$$\int_E f_k d\eta \overset{k\to\infty}{\longrightarrow} \int_E f d\eta. \qquad \square$$

4. Integration of Totally Measurable Functions

The classical theory of vector integration, particularly for Bochner-integrable functions, requires the notion of strong measurability and is fundamentally linked to the separability of the range space. However, a broader and often more tractable class of functions exists within the framework of totally measurable functions. These functions, characterizable as uniform limits of simple functions, provide a natural and robust setting for developing a comprehensive integration theory for vector-valued functions with respect to operator-valued measures.

This section is dedicated to constructing such a theory. We begin by defining the space $M_Y(\Sigma)$ of totally measurable functions and its associated supremum norm. The cornerstone of our approach is the definition of the integral for these functions via a limit of integrals of approximating simple functions, a process whose consistency we rigorously establish. The central result of this section, Theorem 4.5, establishes a fundamental and powerful duality: the space of bounded linear operators on $M_Y(\Sigma)$ is isometrically isomorphic to the space of vector measures with finite semi-variation. This isomorphism identifies an operator U with a measure η such that the action of U on a function f is precisely given by the integral of f with respect to η.

The framework developed here is not only elegant but also essential for the subsequent analysis of the Radon-Nikodym property in variable exponent spaces, as it provides the necessary tools to handle the interaction between function space geometry and vector integration.

Definition 4.1. A function $f : E \to Y$ is said to be totally measurable with respect to Σ if there is a set $D \in \Sigma$ such that f vanishes on $E\backslash D$ and if there is a sequence $\{\chi_k\} \subset S(E, Y, \mu)$ of step functions converging uniformly to f.

The set of all totally measurable functions $f : E \to Y$ is denoted by $\mathrm{M}_Y(\Sigma)$. In the set $\mathrm{M}_Y(\Sigma)$, we introduce the norm of uniform convergence by

$$\|f\|_{\mathrm{M}_Y(\Sigma)} = \sup_{x\in E} \|f(x)\|_Y .$$

Definition 4.2. Let $D \in \Sigma$ be a measurable set and let $\chi : E \to Y$ be a simple (step) function. Then, the semi-variation

$$\widetilde{\operatorname{var}}(\eta)(D) = \sup_{\substack{\chi \in S(E,Y,\eta) \\ \|\chi\| \leq 1}} \left\| \int_D \chi d\eta \right\|_Z \leq \infty.$$

We will assume that the semi-variation $\widetilde{\operatorname{var}}(\eta)$ of measure η is finite.

Definition 4.3. Let $f : E \to Y$, $f \in \mathrm{M}_Y(\Sigma)$ be a totally measurable function, then the integral of f is defined by the identity

$$\int_E f d\eta = \lim_{k\to\infty} \int_E \chi_k d\eta$$

for an arbitrary sequence $\{\chi_k\}$ of step functions $\chi_k : E \to Y$.

To show the consistency of the last definition, we prove the statement.

Statement. Let $\{\chi_k\}, \{\zeta_k\} \subset S(E,Y,\mu)$ be a pair of sequences of step functions that vanish outside a given set $D \in \Sigma$ and uniformly converge to the same limit. Then

$$\lim_{k\to\infty} \int_E \chi_k d\eta = \lim_{k\to\infty} \int_E \zeta_k d\eta.$$

Proof. We assume step function sequences $\{\chi_k\}$ and $\{\zeta_k\}$ uniformly converge to zero. Then, we have

$$\left\| \int_D (\chi_k - \zeta_k)\, d\eta \right\|_Z \leq \|\chi_k - \zeta_k\|_Y \, \widetilde{\operatorname{var}}(D)$$

thus

$$\lim_{k\to\infty} \int_D (\chi_k - \zeta_k)\, d\eta = 0$$

By definition, we have that all classical properties of the integral are preserved for the integral on $\mathrm{M}_Y(\Sigma)$.

Statement. Let $A : Y \to Z$ be a continuous linear operator and $f \in \mathrm{M}_Y(\Sigma)$. Then, the identity

$$\int_D A(f)\, d\mu = A\left(\int_D f d\mu \right)$$

holds for all $f \in \mathrm{M}_Y(\Sigma)$ and for all $D \in \Sigma$.

Definition 4.4. For each linear operator $U : \mathrm{M}_Y(\Sigma) \to Z$ and each set $D \in \Sigma$, we define the restriction U_D of U to subspace $\mathrm{M}_Y(\Sigma, D)$ and we define norms

$$\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} = \sup_{f\neq 0} \frac{\|U(f)\|_Z}{\|f\|_{\mathrm{M}_Y(\Sigma,D)}}$$

and

$$[\![U_D]\!] = \sup_{\chi \neq 0} \sum_{i=1}^{k} \frac{\|U(a_i 1_{B_i})\|_Z}{\|\chi\|_{M_Y(\Sigma,D)}},$$

where $\chi(x) = \sum_{i=1,\dots,k} a_i 1_{B_i}(x)$ and $B_i \in \Sigma$ are disjoint sets $B_i \in \Sigma, \quad B_i \subset D$.

Theorem 4.5. *Let $LB(\mathrm{M}_Y(\Sigma), Z)$ be the space of all linear bounded operators such that $U : \mathrm{M}_Y(\Sigma) \to Z$, $\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} < \infty$ for all $D \in \Sigma$. Then, for all $f \in \mathrm{M}_Y(\Sigma)$, the identity*

$$U(f) = \int_E f d\eta$$

defines an isomorphism between $LB(\mathrm{M}_Y(\Sigma), Z)$ and the space of all measures $\eta : \Sigma \to X$ with finite semi-variation $\widetilde{\mathrm{var}}(\eta) < \infty$. For each $D \in \Sigma$, we obtain $\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} = \widetilde{\mathrm{var}}(\eta)(D)$ and $[\![U_D]\!] = \mathrm{Var}(\eta)(D)$.

Proof. Let $\eta : \Sigma \to X$ be a vector-value additive set function with semi-variation $\widetilde{\mathrm{var}}(\eta) < \infty$. We define a linear operator $U : \mathrm{M}_Y(\Sigma) \to Z$ by

$$U(f) = \int_E f d\eta.$$

Straightforwardly, we have $\eta \to U$ is an injection hence from $U = 0$ follows $\eta \to U = 0$. Also, we have $\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} = \widetilde{\mathrm{var}}(\eta)(D)$ for all $D \in \Sigma$.

Now, assume that $U : \mathrm{M}_Y(\Sigma) \to Z$ is a linear bounded operator, which have property: $\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} < \infty$ for each set $D \in \Sigma$. We have

$$\eta(D)\, a = U(a 1_D(x))$$

for all $D \in \Sigma$ and $a \in Y$. Linear mapping $\eta(D) : a \to U(a 1_D(x))$ is continuous and satisfies the inequality

$$\|\eta(D)\, a\|_Z = \|U(a 1_D(x))\|_Z \leq \|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} \|a\|_Y$$

so

$$\|\eta(D)\|_{\mathrm{M}_Y(\Sigma,D)\to Z} \leq \|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z}.$$

Consider a simple function $\chi(x) = \sum_{i=1}^{k} a_i 1_{B_i}(x)$, defined with respect to disjoint sets $B_i \in \Sigma$ and coefficients $a_i \in Y$. We then obtain the following:

$$\int_E \chi d\eta = \sum_i a_i \eta(B_i) = \sum_i U(a_i 1_{B_i}) = U(\chi)$$

and

$$\begin{aligned}\widetilde{\mathrm{var}}(\eta)(D) &= \sup_{\substack{\chi \in S(D,Y) \\ \|\chi\| \leq 1}} \left\| \int_D \chi d\eta \right\|_Z \\ &= \sup_{\substack{\chi \in S(D,Y) \\ \|\chi\| \leq 1}} \|U(\chi)\|_Z \\ &= \|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z}.\end{aligned}$$

Next, we assume $f \in \mathrm{M}_Y(\Sigma, D)$. We find a sequence $\{\chi_k\} \subset S(E, Y, \mu)$ that uniformly converge to element $f \in \mathrm{M}_Y(\Sigma, D)$. Then,

$$U(f) = \lim_{k\to\infty} U(\chi_k) = \lim_{k\to\infty} \int_E \chi_k d\eta = \int_E f d\eta.$$

We assume $\|a_i\|_Y \leq 1$ for all i then we have

$$\begin{aligned}\sum_i \|U(a_i 1_{B_i})\|_Z &= \sum_i \|a_i \eta(B_i)\|_Z \\ &\leq \sum_i \|\eta(B_i)\|_{\mathrm{M}_Y(\Sigma,D)\to Z} \\ &\leq \mathrm{Var}(\eta)(D)\end{aligned}$$

so

$$[\![U_D]\!] \leq \mathrm{Var}(\eta)(D)$$

for all $D \in \Sigma$.

Now, we are going to show that $\mathrm{Var}(\eta)(D) \leq [\![U_D]\!]$. Let $D \in \Sigma$ and let $\{B_i \subset D, \quad i = 1, 2, \ldots, k\} \subset \Sigma$ be a collection of disjoint sets. For all $\varepsilon > 0$ there is $a_i \in Y$, $\|a_i\|_Y \leq 1$ such that

$$\|\eta(B_i)\|_{\mathrm{M}_Y(\Sigma,D)\to Z} \leq \|a_i \eta(B_i)\|_Z + \frac{\varepsilon}{k}.$$

Since $\|f\|_{\mathrm{M}_Y(\Sigma)} \leq 1$ we obtain

$$\begin{aligned}\sum_i \|\eta(B_i)\|_Z - \varepsilon &\leq \sum_i \|a_i \eta(B_i)\|_Z \\ &= \sum_i \|U(a_i 1_{B_i})\|_Z \\ &\leq [\![U_D]\!].\end{aligned}$$

Thus, we conclude $\mathrm{Var}(\eta)(D) - \varepsilon \leq [\![U_D]\!]$ and $[\![U_D]\!] = \mathrm{Var}(\eta)(D)$. □

Remark 4.6. Let $U \in LB(\mathrm{M}_Y(\Sigma), Z)$ be a linear operator such that $\|U_D\|_{\mathrm{M}_Y(\Sigma,D)\to Z} < \infty$ for all $D \in \Sigma$. Then, there is a linear operator

$U' : \mathrm{M}_R(D) \to LB(X,Z)$ such that $\|U'_D\|_{\mathrm{M}_R(D)\to Z} < \infty$ for all $D \in \Sigma$ and

$$U(a\phi) = aU'(\phi)$$

for all $a \in Y$ and $\phi \in \mathrm{M}_R(\Sigma)$.

Theorem 4.7. *Let $\widetilde{LB}(\mathrm{M}_Y(\Sigma), Z)$ be the space of all linear bounded operators such that there is a linear positive functional Φ defined on $\mathrm{M}_R(\Sigma)$ and*

$$\|U(f)\|_Z \le \Phi(\|f\|_Y)$$

for all $f \in \mathrm{M}_Y(\Sigma)$. Let $f \in \mathrm{M}_Y(\Sigma)$ then the identity

$$U(f) = \int_E f d\eta$$

establishes an isomorphism between $\widetilde{LB}(\mathrm{M}_Y(\Sigma), Z)$ and set of all measures $\eta : \Sigma \to LB(Y,Z)$ with finite variation; and there is the equality

$$[\![U_D]\!] = \mathrm{Var}(\eta)(D) = \inf_{\Phi} \Phi_D.$$

Proof. First, we assume η has finite variation, then we have

$$\|U(f)\|_Z = \left\| \int_E f d\eta \right\|_Z \le \int_E \|f\|_Y \, d\|\eta\|_X$$

for all $f \in \mathrm{M}_Y(\Sigma)$.

Second, we show the reverse. Let $D \in \Sigma$ and $a \in Y$. Let there be a linear positive set function ϕ such that $\|U(a1_D)\|_Z \le \phi(D)\|a\|$ then we obtain

$$\begin{aligned} \|\eta(D)a\|_Z &= \left\| \int_E a1_D d\eta \right\|_Z \\ &= \|U(a1_D)\|_Z \\ &\le \int_E \|a\|_Y 1_D d\phi \\ &\le \phi(D)\|a\|_Y \\ &< \infty. \end{aligned}$$

Thus, we deduce $\mathrm{Var}(\eta) \le \phi$ for arbitrary linear positive set function ϕ, thus $\mathrm{Var}(\eta)$ is the least dominant of $\|U\|$. □

Lemma 4.8. *There exists an isomorphism between $\widetilde{LB}(M_Y(\Sigma), Z)$ and $LB(M_R(\Sigma), LB(Y,Z))$ defined by*

$$U(\varphi\xi) = V(\varphi)\xi$$

for all $U \in \widetilde{LB}(M_Y(\Sigma), Z)$, $V \in LB(M_R(\Sigma), LB(Y,Z))$ and all $\xi \in Y$.

If the measures μ_U and μ_V are equal (i.e., $\mu_U = \mu_V$), then there exists a measure $\eta : \Sigma \to LB(Y,Z)$ such that

$$U(f) = \int_E f \, d\eta$$

for all $f \in M_Y(\Sigma)$ and

$$V(\varphi) = \int_E \varphi \, d\eta$$

for all $\varphi \in M_R(\Sigma)$.

Lemma 4.9. *A set additive function $\eta : \Sigma \to LB(Y,Z)$ with finite variation is σ-additive if and only if for all decreasing sequences $\{\varphi_k\} \subset \mathrm{M}_R(\Sigma)$ such that $\varphi_k(x) \overset{k\to\infty}{\longrightarrow} 0 \quad \forall x \in E$ and the limit*

$$U(\varphi_k) = \int_E \varphi_k d\eta \overset{k\to\infty}{\longrightarrow} 0$$

holds for all $D \in \Sigma$.

Proof. If a function $\eta : \Sigma \to LB(Y,Z)$ is σ-additive, then its total variation $\mathrm{Var}(\eta)$ is also σ-additive. Let $\{\varphi_k\} \subset \mathrm{M}_R(\Sigma)$ be decreasing sequence such that $\varphi_k(x) \overset{k\to\infty}{\longrightarrow} 0 \quad \forall x \in E$ then

$$\|U(\varphi_k)\|_Z = \left\| \int_E \varphi_k d\eta \right\|_Z \leq \int_E \varphi_k d\mathrm{Var}(\eta).$$

Now, if $U(\varphi_k) = \int_E \varphi_k d\eta \overset{k\to\infty}{\longrightarrow} 0$ for each decreasing sequence $\{\varphi_k\} \subset \mathrm{M}_R(\Sigma)$ such that $\varphi_k(x) \overset{k\to\infty}{\longrightarrow} 0 \quad \forall x \in E$ then $\eta(D_k) \overset{k\to\infty}{\longrightarrow} 0$ for every decreasing sequence $\{D_k\}$ of disjoint sets D_k. Thus, η is σ-additive. □

5. Variable Exponent Lebesgue Spaces

In this Section, we introduce and investigate the properties of the Banach functional space $L^{p(\cdot)}(E,\Sigma,\mu)$ consisting of all vector-valued functions integrable with variable exponent with respect to the continuous, bounded bilinear form $\Lambda : Y \times X \to Z$.

We denote a set $P(E)$ consisting of all measurable functions $p : E \to (1,\infty)$. The elements of $P(E)$ are called exponent function. We denote

$$p_m = p_m(E) = ess\inf_{x\in E} p(x)$$

and

$$p_S = p_S(E) = ess\sup_{x\in E} p(x).$$

For fixed exponent function $p : E \to (1, \infty)$, the conjugate exponent q is defined by the identity

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$$

for all $x \in E$.

We define the variable exponent Banach-Lebesgue space $L^{p(\cdot)}(E, \Sigma, \mu)$ as a Banach space consisting of all μ-measurable functions $f : E \to Y$ such that

$$\rho(f) = \int_E \|f(x)\|_Y^{p(x)} \, d\mu(x) < \infty$$

and the norm of $L^{p(\cdot)}(E, \Sigma, \mu)$ is define by

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{L^{p(\cdot)}(E,\Sigma,\mu)} = \inf_{\lambda>0} \left\{ \rho\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

for all $f \in L^{p(\cdot)}(E, \Sigma, \mu)$.

Lemma 5.1. *If $p \in P(E)$, $p_S < \infty$ and $\|f\|_{L^{p(\cdot)}} > 0$ then $\rho\left(\frac{f}{\|f\|_{L^{p(\cdot)}}}\right) = 1$ for all $f \in L^{p(\cdot)}(E, \Sigma, \mu)$ such that $f \neq 0$.*

Proof. We can select a decreasing sequence $\{\lambda_k\}$ such that $\lambda_k \to \|f\|_{L^{p(\cdot)}}$ as $k \to \infty$. Applying Fatou's lemma to this sequence yields the estimate:

$$\rho\left(\frac{f}{\|f\|_{L^{p(\cdot)}}}\right) \leq \liminf_{k\to\infty} \rho\left(\frac{f}{\lambda_k}\right) \leq 1.$$

For all $0 < \lambda < \|f\|_{L^{p(\cdot)}}$, we have

$$\rho\left(\frac{f}{\lambda}\right) \leq \left(\frac{\|f\|_{L^{p(\cdot)}}}{\lambda}\right)^{p_S} \rho\left(\frac{f}{\|f\|_{L^{p(\cdot)}}}\right).$$

Firstly, we can find a number λ for which $\rho(f/\lambda) < 1$. Secondly, according to the very definition of the norm, we must have $\rho(f/\lambda) \geq 1$. This contradiction establishes the lemma. □

We now assume that X, Y and Z are separable reflexive Banach spaces. Let (E, Σ, μ) be a σ-finite scalar measure space and suppose $\eta : \Sigma \to X$ is a vector measure. We consider a continuous bilinear mapping $\Lambda : X \times Y \to Z$ satisfying the condition

$$\|a\|_X = \sup_{\|b\|_Y \leq 1} \|\Lambda(a, b)\|_Z$$

for every $a \in X$.

Lemma 5.2 (generalized Holder inequality)**.** *Let $p \in P(E)$, $1 < p_m \leq p_S < \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in E$. Let X, Y and Z be separable reflexive Banach spaces and let $\Lambda : Y \times X \to Z$ be a continuous, bounded bilinear mapping. (We will use notation $\Lambda(a,b) = ab$ for all $a \in X, \quad b \in Y$). Let $f : E \to Y$, $f \in L^{p(\cdot)}(E, \Sigma, \mu)$ and $g : E \to X$, $g \in L^{q(\cdot)}(E, \Sigma, \mu)$. Then, the inequality*

$$\begin{aligned}\left\| \int_E f(x)\, g(x)\, d\mu(x) \right\|_Z &\leq \int_E \|f(x)\|_Y \|g(x)\|_X \, d\mu(x) \\ &\leq k_{Hol} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}\end{aligned}$$

holds for all $f \in L^{p(\cdot)}(E, \Sigma, \mu)$ and $g \in L^{q(\cdot)}(E, \Sigma, \mu)$, where generalized Holder constant equals

$$k_{Hol} = 1 + \frac{1}{p_m} - \frac{1}{p_S}.$$

Proof. The boundedness of the form Λ yields

$$\left\| \int_E f(x)\, g(x)\, d\mu(x) \right\|_Z \leq \int_E \|f(x)\|_Y \|g(x)\|_X \, d\mu(x).$$

Assume $\|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}} > 0$, by Young's inequality, we estimate

$$\begin{aligned}\int_E \frac{\|f(x)\|_Y \|g(x)\|_X}{\|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}} d\mu(x) &\leq \int_E \frac{1}{p(x)} \left(\frac{\|f(x)\|_Y}{\|f\|_{L^{p(\cdot)}}} \right)^{p(x)} d\mu(x) \\ &\quad + \int_E \frac{1}{q(x)} \left(\frac{\|g(x)\|_X}{\|g\|_{L^{q(\cdot)}}} \right)^{q(x)} d\mu(x) \\ &\leq \frac{1}{p_m} \int_E \left(\frac{\|f(x)\|_Y}{\|f\|_{L^{p(\cdot)}}} \right)^{p(x)} d\mu(x) \\ &\quad + \frac{1}{q_m} \int_E \left(\frac{\|g(x)\|_X}{\|g\|_{L^{q(\cdot)}}} \right)^{q(x)} d\mu(x) \\ &\leq 1 + \frac{1}{p_m} - \frac{1}{p_S}\end{aligned}$$

since $\frac{1}{q_m} = 1 - \frac{1}{p_S}$. □

Lemma 5.3 (generalized Holder inequality for the measure)**.** *Let $p \in P(E)$ with $p_S < \infty$ and suppose that $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ holds for every $x \in E$. Assume X, Y and Z are separable reflexive Banach spaces and let $\Lambda : X \times Y \to Z$ be a continuous bilinear mapping that is bounded (we use the shorthand $\Lambda(a,b) = ab$ for $a \in X$, $b \in Y$). Take a function $f \in L^{p(\cdot)}(E, \Sigma, \mu)$. Furthermore, let $\eta : \Sigma \to X$ be a vector measure with finite variation whose total variation $|\eta|$ is absolutely continuous*

with respect to the scalar measure μ. Then for any $D \in \Sigma$ we obtain the estimate

$$\left\| \int_E f(x)\, d\eta(x) \right\|_Z \leq k_{\mathrm{Hol}} \|f\|_{L^{p(\cdot)}} \operatorname{Var}_{L^{q(\cdot)}}(\eta)(E)$$

where the generalized Hölder constant is given by

$$k_{\mathrm{Hol}} = 1 + \frac{1}{p_m} - \frac{1}{p_S},$$

and we have introduced the notation

$$\operatorname{Var}_{L^{q(\cdot)}}(\eta)(E) = \sup \sum_i \|a_i\|_Y \, \|\eta(B_i)\|_X,$$

with the supremum taken over all step functions $\chi : E \to Y$ of the form $\chi(x) = \sum_{i=1}^k a_i \mathbf{1}_{B_i}(x)$ satisfying $\|\chi\|_{L^{p(\cdot)}} \leq 1$. Here each a_i belongs to Y and the $\mathbf{1}_{B_i}$ denote characteristic functions of pairwise disjoint measurable sets $B_i \in \Sigma$.

Proof. We assume that $f : E \to Y$, $f \in L^{p(\cdot)}(E, \Sigma, \mu)$. There is a sequence $\{f_k\} \subset L^{p(\cdot)}(E, \Sigma, \mu)$ of step functions converging to f μ-almost everywhere and in the $L^{p(\cdot)}$-norm. Employing the absolute continuity of vector measure η relative to the scalar measure μ, the sequence $\{f_k\}$ converges to the function f η-almost everywhere. Thus, we have

$$\int_E \|f_k - f_j\|_Y \, d\left(\operatorname{Var}(\eta)\right) \leq k_{Hol} \|f_k - f_j\|_{L^{p(\cdot)}(E,\Sigma,\mu)} \operatorname{Var}_{L^{q(\cdot)}}(\eta)(E),$$

so $\{f_k\}$ is fundamental in $f \in L^1(E, \Sigma, \eta)$. For each k, we obtain

$$\left\| \int_E f_k d\eta \right\|_Z \leq \int_E \|f_k\|_Y \, d\|\eta\|_X \leq k_{Hol} \|f_k\|_{L^{p(\cdot)}} \operatorname{Var}_{L^{q(\cdot)}}(\eta)(E).$$

Taking the limit as k approaches infinity, we have

$$\left\| \int_E f d\eta \right\|_Z \leq \int_E \|f\|_Y \, d\|\eta\|_X \leq k_{Hol} \|f\|_{L^{p(\cdot)}} \operatorname{Var}_{L^{q(\cdot)}}(\eta)(E).$$

□

6. Linear Operators on $L^{p(\cdot)}(E, \Sigma, \mu)$

Assume that X, Y and Z are separable reflexive Banach spaces. We denote the space of all step functions vanishing outside D by $\Theta_Y(D, \mu)$. The space $\Theta_Y(D, \mu)$ is dense in $L^{p(\cdot)}(D, \Sigma, \mu)$. We assume that $U : L^{p(\cdot)}(D, \Sigma, \mu) \to Z$ is a linear mapping and U_D is the restriction of U to $L^{p(\cdot)}(D, \Sigma, \mu)$. We define the operator norms

$$\|U_D\|_{L^{p(\cdot)}} = \sup_{\|f\|_{L^{p(\cdot)}} \leq 1} \|U(f)\|_Z$$

and

$$\|U_D\|_{\Theta_Y} = \sup \sum_i \|U(1_{B_i} a_i)\|_Z$$

where the supremum is taken over all simple functions $\chi : E \to Y$ of the form $\chi(x) = \sum_{i=1}^k a_i 1_{B_i}(x)$ with $a_i \in Y$ and B_i disjoint measurable sets, such that $\|\chi\|_{L^{p(\cdot)}} \leq 1$.

Since the space $\Theta_Y(D, \mu)$ is dense in $L^{p(\cdot)}(D, \Sigma, \mu)$, we conclude

$$\|U_D\|_{L^{p(\cdot)}} = \sup \sum_i \|U(1_{B_i} a_i)\|_Z,$$

where the supremum is taken over all simple functions $\chi : E \to Y$ of the form $\chi(x) = \sum_{i=1}^k a_i 1_{B_i}(x)$, defined using disjoint measurable sets B_i and coefficients $a_i \in Y$, subject to the condition $\|\chi\|_{L^{p(\cdot)}} \leq 1$.

Theorem 6.1. *Let $p \in P(E)$, $p_S < \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in E$. Then, there exists an isomorphism between the set of linear mappings $U : L^{p(\cdot)}(D, \Sigma, \mu) \to Z$ $\|U_D\|_{L^{p(\cdot)}} < \infty$ for all $D \in \Sigma$ and the set of measures $\eta : \Sigma \to LB(X, Z)$ with $\mathrm{Var}_{L^{q(\cdot)}}(\eta)(D) < \infty$ for all $D \in \Sigma$ defined by*

$$U(f) = \int_E f d\eta$$

for all $f \in L^{p(\cdot)}(D, \Sigma, \mu)$.

Let U and η be in correspondence, then for each μ-measurable set $D \in \Sigma$, there is identity

$$\mathrm{Var}_{L^{q(\cdot)}}(\eta)(D) = \|U_D\|_{L^{p(\cdot)}}.$$

Proof. Let $\eta : \Sigma \to LB(X, Z)$ be a measure with property that its variation $\mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$ is finite for all $D \in \Sigma$. We define a linear mapping $U : L^{p(\cdot)}(D, \Sigma, \mu) \to Z$ by

$$U(f) = \int_E f \, d\eta$$

for all $f \in L^{p(\cdot)}(D, \Sigma, \mu)$.

Let $D \in \Sigma$ and consider a simple function $\chi(x) = \sum_{i=1}^k a_i 1_{B_i}(x)$ defined using disjoint measurable sets B_i and coefficients $a_i \in Y$.

Then, there is the equality for the norm of U_D:

$$\|U_D\|_Z = \left\| \sum_i a_i \eta(B_i) \right\|_Z$$

and the inequality:

$$\sum_i \|U(1_{B_i} a_i)\|_Z = \sum_i \|a_i \eta(B_i)\|_Z \leq \sum_i \|a_i\|_Y \|\eta(B_i)\|_X.$$

We then take the supremum over all such simple functions χ with disjoint measurable sets B_i such that $\|\chi\|_{L^{p(\cdot)}} \leq 1$. Thus, we conclude that $\|U_D\|_{L^{p(\cdot)}} = \mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$.

Now, we show the reverse. We assume $U : L^{p(\cdot)}(D, \Sigma, \mu) \to Z$ is a linear mapping with $\|U_D\|_{L^{p(\cdot)}} < \infty$ for all $D \in \Sigma$. For each $D \in \Sigma$, we define the mapping $\eta(D) : X \to Z$ by the identity

$$\eta(D)\, a = U(1_B a)$$

for all $a \in Y$. Since

$$\|\eta(D)\, a\|_Z = \|U(1_B a)\|_Z \leq \|U_D\|_{L^{p(\cdot)}} \|1_B a\|_{L^{p(\cdot)}},$$

we have $\eta(D) \in LB(X, Z)$. We calculate

$$U(f) = \sum_i U(1_{B_i} a_i) = \sum_i a_i \eta(B_i) = \int_E f d\eta.$$

We take the supremum over all step functions χ with disjoint measurable sets B_i and $\|\chi\|_{L^{p(\cdot)}} \leq 1$, therefore, we deduce $\|U_D\|_{L^{p(\cdot)}} = \mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$. □

7. Radon-Nikodym Theory and Dentable Sets

A Banach space X has the Radon-Nikodym property if every X-valued measure of finite variation that is absolutely continuous with respect to a scalar measure μ can be expressed as the integral of a Bochner integrable function. Not every Banach space possesses this property; for instance, c_0 and $L^1[0,1]$ are classic examples of spaces failing the Radon-Nikodym property. We investigate the Radon-Nikodym property within the context of generalized variable exponent Bochner spaces $L^{p(\cdot)}(E, X, \Sigma, \mu)$. Our main objective is to explore how the geometric property of dentability in the underlying Banach space X governs the validity of a Radon-Nikodym-type theorem in these more complex function spaces.

Definition 7.1. A Banach space X has the Radon-Nikodym property if, for any finite positive measure space (E, Σ, μ) and any X-valued measure η that is μ-continuous and has finite total variation ($|\eta|(E) < \infty$), there exists a Bochner integrable function $f : E \to X$ such that the measure can be represented by the integral:

$$\eta(D) = \int_D f d\mu, \quad \text{for all } D \in \Sigma.$$

Definition 7.2. A bounded set A of a Banach space X is said to be dentable if for any given $\varepsilon > 0$ there exists an element $a(\varepsilon) \in A$ such that $a(\varepsilon) \notin clos(hull(A \backslash B_\varepsilon(a(\varepsilon))))$, where ball $B_\varepsilon(a(\varepsilon)) =$

$\{b \in X : \|a(\varepsilon) - b\|_X \leq \varepsilon\}$ and $clos\,(hull\,(A \backslash B_\varepsilon\,(a\,(\varepsilon))))$ means the closure of the convex hull of $A \backslash B_\varepsilon\,(a\,(\varepsilon))$.

If this definition holds with the same $a\,(\varepsilon) \in A$ for all $\varepsilon > 0$ then $a(\varepsilon)$ is said to be a denting point of the set $A \subset X$.

We recall the statement of the Rieffel dentability theorem.

Theorem 7.3 (Rieffel)**.** *Let $\eta : \Sigma \to X$ be a countably additive vector-valued measure with finite total variation* $\mathrm{Var}\,(\eta)$*, such that* $\mathrm{Var}\,(\eta)$ *is absolutely continuous with respect to μ. Then the measure η is Bochner differentiable with respect to μ if and only if every bounded subset of X is a dentable set.*

Proof. We consider a finite positive measure space

$$([0,1)\,, \Sigma_{Borel}, \mu_{Lebeg}) = ([0,1)\,, \Sigma, \mu)\,.$$

We will follow the arguments of Diestel and Uhl [5]. We assume that there is a bounded non-dentable set $D \subset X$. We select $\varepsilon > 0$ such that for all $a \in D$, $a \in clos\,(hull\,(D \backslash B_\varepsilon\,(a)))$. We have to show that there exist sequences of countable partitions $\pi_k = \Pi\,(D) = \{B_1, \ldots, B_k, k \in N\}$ of the interval $[0,1)$ into the half-open intervals and a sequence $\{f_k\}$ of at most countably valued functions on $[0,1)$, which satisfies the following conditions:

1) every function f_k is given as $f_k = \sum_{C \in \pi_k} a_C 1_C$ where $a_C \in D$ for all $C \in \pi_k$;
2) the partition π_{k+1} is a refinement of π_k such that each interval of π_k is a union of intervals of π_{k+1};
3) the σ-algebra of Borel sets of $[0,1)$ is the smallest σ-algebra containing $\bigcup_k \pi_k$;
4) $\|f_k(\tau) - f_{k+1}(\tau)\|_X \geq \varepsilon$ for all k and all $\tau \in [0,1)$;
5) $\left\|\int_C (f_m(\tau) - f_k(\tau))\, d\mu(\tau)\right\|_X < \frac{\mu(C)}{2^k}$ for all $C \in \pi_k$ and all $m \geq k$.

Now, we verify the conditions 1-5. For $\varepsilon > 0$ and each $\delta > 0$, there exists, for every $a \in D$, a pair of sequences of numbers $\beta_k\,(a, \delta) > 0$ such that $\sum_k \beta_k\,(a, \delta) = 1$ and of elements $\{a_k\,(a, \delta)\} \subset D \backslash B_\varepsilon\,(a)$ such that

$$\left\| a - \sum_k \beta_k\,(a, \delta)\, a_k\,(a, \delta) \right\|_X < \delta.$$

We assume that π_k and $f_k = \sum_{C \in \pi_k} a_C 1_C$ have been defined for $a_C \in D$ for all $C \in \pi_k$. We have

$$\left\| a_C - \sum_k \beta_k \left(a_C, 2^{-(k+1)}\right) a_k \left(a_C, 2^{-(k+1)}\right) \right\|_X < 2^{-(k+1)}$$

where we take $\delta = 2^{-(k+1)}$. We assume $C = [\lambda_1, \lambda_2)$ and obtain

$$\gamma_k = (\lambda_2 - \lambda_1) \sum_{m=1,\dots,k} \beta_m \left(a_C, 2^{-(k+1)}\right)$$

and

$$\gamma_0 = 0.$$

We denote $I_m = [\lambda_1 + \gamma_{m-1}, \lambda_1 + \gamma_m)$ and

$$f_{k+1} 1_C = \sum_{m=1,2,\dots} a_m \left(a_C, 2^{-(k+1)}\right) 1_{I_m}.$$

Proceeding as above for each $C \in \pi_k$, we obtain 4 since

$$\|f_k(\tau) - f_{k+1}(\tau)\|_X \geq \varepsilon$$

for all $\tau \in [0,1)$.

We consider the countable partition π_{k+1} of $[0,1)$ constructed of half-open intervals of all intervals I_k as above, thus 1 and 2 are verified.

Finally, since $\beta_m \left(a_C, 2^{-(k+1)}\right) < 2^{-(k+1)}$ for all k and $C \in \pi_k$, we obtain 3.

We associate a martingale with $\{f_k\}$ by

$$\eta(C) = \lim_{k\to\infty} \int_C f_k(\tau)\, d\mu(\tau)$$

for all $C \in \bigcup_k \pi_k$. We define $\psi_k = \sum_{C\in\pi_k} \frac{\eta(C)}{\mu(C)} 1_C$. Let B_k be a σ-algebra of Borel sets generated by π_k (do not mix with partitions sets). The pair (ψ_k, B_k) is a martingale on $L^1([0,1), X)$. We obtain that there is a positive constant c such that

$$\left\| \frac{\eta(C)}{\mu(C)} \right\|_X = \frac{\left\| \lim_{k\to\infty} \int_C f_k(\tau)\, d\mu(\tau) \right\|_X}{\mu(C)} \leq c$$

for all $C \in \bigcup_k \pi_k$. Since $\sup_{\substack{k\in N,\\ \tau\in[0,1)}} \|\psi_k(\tau)\|_X \leq c$, the martingale (ψ_k, B_k) is $L^1([0,1), X)$-bounded and uniformly integrable. We calculate

$$\begin{aligned}
\int_{[0,1)} \|f_k(\tau) - \psi_k(\tau)\|_X\, d\mu(\tau) &= \sum_{C\in\pi_k} \|a_C \mu(C) - \eta(C)\|_X \\
&= \lim_{m\to\infty} \sum_{C\in\pi_k} \left\| \int_C (f_k(\tau) - f_m(\tau)) d\mu(\tau) \right\|_X \\
&\leq \sum_{C\in\pi_k} \frac{\mu(C)}{2^k}
\end{aligned}$$

$$\leq \frac{1}{2^k}.$$

Thus, the sequence $\{f_k - \psi_k\}$ is fundamental in $L^1([0,1), X)$. However, from 4, we conclude that the sequence $\{f_k\}$ cannot be fundamental in $L^1([0,1), X)$. Therefore, (ψ_k, B_k) is a nonconvergent uniformly integrable martingale bounded in $L^1([0,1), X)$. By the martingale mean convergence theorem, the Banach space X does not have the Radon-Nikodym property. □

Thus, we obtain that if the Banach space X has the Radon-Nikodym property then all bounded subsets of X are dentable.

Straightforwardly, we obtain that if a bounded set A of a Banach space X is dentable then for all $\varepsilon > 0$ there is an element $a(\varepsilon) \in A$ such that $a(\varepsilon) \neq \sum_{k=1,2,\ldots} \beta_k a_k$ for all $\sum_{k=1,2,\ldots} \beta_k = 1$, $\beta_k > 0$, $a_k \in A$ and

$$\|a - a_k\|_X \leq \varepsilon$$

for all $k \in N$. We will call these conditions countable dentabillity .

Next, we will show that if all bounded subsets of a Banach space X are dentable then the Banach space X possesses the Radon-Nikodym property.

We assume that (E, Σ, μ) is a finite measure space and $\eta : \Sigma \to X$ is a μ-continuous vector measure with finite variation. There exists a disjoint countable partition $\{B_k\}_k \subset \Sigma$ of E (i.e., $\bigcup_k B_k = E$) such that

$$\left\| \frac{\eta(B)}{\mu(B)} \right\|_X < \infty$$

holds for all $B \subseteq B_k$ for any fixed $k \in \mathbb{N}$. There are $k \in N$ such that $\mu(B \cap B_k) > 0$. So, there is $\tilde{B} \subseteq B$, $\tilde{B} \in \Sigma$, $\mu\left(\tilde{B}\right) > 0$ such that the set

$$\left\{ \frac{\eta(D)}{\mu(D)} : D \subseteq \tilde{B}, \quad \mu(D) > 0 \right\}$$

is bounded in X.

For all $\varepsilon > 0$, we can find a set $A \subseteq \tilde{B} \subseteq B$, $\mu(A) > 0$ such that

$$\frac{\eta(A)}{\mu(A)} = \sum_{k=1,2,\ldots} \beta_k \frac{\eta(K_k)}{\mu(K_k)}$$

for $\sum_{k=1,2,\ldots} \beta_k = 1$, $\beta_k > 0$ and $K_k \subseteq A$, then there exists a number k_0 such that

$$\left\| \frac{\eta(A)}{\mu(A)} - \frac{\eta(K_{k_0})}{\mu(K_{k_0})} \right\|_X \leq \varepsilon.$$

Proposition 7.4 (Radon-Nikodym)**.** *Let (E, Σ, μ) be a finite measure space and $\eta : \Sigma \to X$ is a μ-continuous vector measure with finite*

variation. Let for all $\varepsilon > 0$ *and all* $B \in \Sigma$, $\mu(B) > 0$, *there is* $A \subseteq B$, $A \in \Sigma$, $\mu(A) > 0$ *such that the diameter of the set*

$$\left\{ \frac{\eta(D)}{\mu(D)} : D \in \Sigma, \quad D \subseteq A, \mu(D) > 0 \right\}$$

is less than ε. *Then, there exists a function* $f \in L^1(E, \Sigma, \mu)$ *such that*

$$\eta(D) = \int_D f \, d\mu, \quad \forall D \in \Sigma.$$

The proof of this proposition is straightforward by exhaustion arguments.

If the supremum

$$\sup \left\{ \left\| \frac{\eta(A)}{\mu(A)} - \frac{\eta(D)}{\mu(D)} \right\|_X : D \subseteq A \right\} \leq \varepsilon$$

then the Radon-Nikodym proposition shows that the Banach space X possesses the Radon-Nikodym property. Alternatively, we can find the smallest integer m_1 such that there is a set $A_1 \subseteq A$ with $\mu(A_1) \geq \frac{1}{m_1}$ and

$$\left\| \frac{\eta(A_1)}{\mu(A_1)} - \frac{\eta(A)}{\mu(A)} \right\|_X > \varepsilon,$$

and

$$\frac{\eta(A)}{\mu(A)} = \frac{\eta(A_1)}{\mu(A_1)} \frac{\mu(A_1)}{\mu(A)} + \frac{\eta(A \backslash A_1)}{\mu(A \backslash A_1)} \frac{\mu(A \backslash A_1)}{\mu(A)}.$$

Next, if

$$\sup \left\{ \left\| \frac{\eta(A)}{\mu(A)} - \frac{\eta(D)}{\mu(D)} \right\|_X : D \subseteq A \backslash A_1 \right\} \leq \varepsilon$$

then the Radon-Nikodym proposition concludes the proof. Alternatively, we can find the smallest integer $m_2 > m_1$ such that there is a set $A_2 \subseteq A \backslash A_1$ with $\mu(A_2) \geq \frac{1}{m_2}$ and

$$\left\| \frac{\eta(A_2)}{\mu(A_2)} - \frac{\eta(A)}{\mu(A)} \right\|_X > \varepsilon,$$

and

$$\frac{\eta(A)}{\mu(A)} = \frac{\eta(A_1)}{\mu(A_1)} \frac{\mu(A_1)}{\mu(A)} + \frac{\eta(A_2)}{\mu(A_2)} \frac{\mu(A_2)}{\mu(A)} + \frac{\eta(A \backslash A_1 \backslash A_2)}{\mu(A \backslash A_1 \backslash A_2)} \frac{\mu(A \backslash A_1 \backslash A_2)}{\mu(A)}.$$

And, we proceed in this manner. In the case, that the process stops at a certain step, then the statement is established by the Radon-Nikodym proposition.

If the process does not stop, we obtain two infinite sequences $\{A_k\} \subset A$, $\mu(A_k) > 0$ and $\{m_k\}$, which satisfy the following conditions:

1) $\left\| \frac{\eta(A_k)}{\mu(A_k)} - \frac{\eta(A)}{\mu(A)} \right\|_X > \varepsilon$ for all $k \in N$;

2) let $D \subseteq A \setminus \bigcup_{k=1,\ldots,i} A_k$, $D \in \Sigma$ and $\left\| \frac{\eta(D)}{\mu(D)} - \frac{\eta(A)}{\mu(A)} \right\|_X > \varepsilon$ then $\mu(D) < (m_i - 1)^{-1}$;

3)

$$\frac{\eta(A)}{\mu(A)} = \sum_{k=1}^{i} \frac{\eta(A_k)}{\mu(A_k)} \frac{\mu(A_k)}{\mu(A)} + \frac{\eta\left(A \setminus \bigcup_{k=1}^{i} A_k\right)}{\mu\left(A \setminus \bigcup_{k=1}^{i} A_k\right)} \frac{\mu\left(A \setminus \bigcup_{k=1}^{i} A_k\right)}{\mu(A)}$$

for all $i \in N$.

The set

$$\lim_{i \to \infty} \frac{\eta\left(A \setminus \bigcup_{k=1,\ldots,i} A_k\right)}{\mu\left(A \setminus \bigcup_{k=1,\ldots,i} A_k\right)}$$

is bounded in X. The assumption of countable dentabillity yields that

$$\lim_{i \to \infty} \mu\left(A \setminus \bigcup_{k=1,\ldots,i} A_k\right) = \mu\left(A \setminus \bigcup_{k=1,\ldots,} A_k\right) \neq 0,$$

so, we deduce that $\mu\left(A \setminus \bigcup_{k=1,\ldots,} A_k\right) > 0$. We state that the conditions of the Radon-Nikodym proposition are satisfied thus the Banach space X possesses the Radon-Nikodym property. Indeed, assume that there is $D \subseteq A \setminus \bigcup_{k=1,\ldots,} A_k$ such that $\mu(D) > 0$ and

$$\left\| \frac{\eta(D)}{\mu(D)} - \frac{\eta(A)}{\mu(A)} \right\|_X > \varepsilon$$

then $D \subseteq A \setminus \bigcup_{k=1,\ldots,i} A_k$ for all $i \in N$ so $\mu(D) < (m_i - 1)^{-1}$ for all $i \in N$ by 2. However, $\mu(A_i) \geq m_i^{-1}$ for all $i \in N$ and sequence $\{A_k\}$ is disjoint. Therefore, we have

$$\sum_{i=1,2,\ldots} m_i^{-1} \leq \sum_{i=1,2,\ldots} \mu(A_i) < \infty$$

and

$$\mu(D) \leq \lim_{i \to \infty} (m_i - 1)^{-1} = 0.$$

Thus, we obtain

$$\left\| \frac{\eta(D)}{\mu(D)} - \frac{\eta(A)}{\mu(A)} \right\|_X \leq \varepsilon$$

for all $D \subseteq A \setminus \bigcup_k A_k$ with $\mu(D) > 0$. The proof is completed.

8. Radon-Nikodym Property for Variable Exponent Spaces $L^{p(\cdot)}(E,\Sigma,\mu)$

In this Section, we consider the Radon-Nikodym property of $L^{p(\cdot)}(E,X,\Sigma,\mu)$ with $p\in P(E)$, $1<p_m\leq p_S<\infty$.

Theorem 8.1. *Let (E,Σ,μ) be a finite positive measure space without atoms and let X be a Banach space. Let $p\in P(E)$, $1<p_m\leq p_S<\infty$, then the space $L^{p(\cdot)}(E,X,\Sigma,\mu)$ has the Radon-Nikodym property if and only if the space X has the Radon-Nikodym property.*

Proof. It is obvious that if $L^{p(\cdot)}(E,X,\Sigma,\mu)$ possesses the Radon-Nikodym property then X possesses the Radon-Nikodym property.

We show that assuming X possesses the Radon-Nikodym property then $L^{p(\cdot)}(E,X,\Sigma,\mu)$ has the Radon-Nikodym property. We assume (Ω,Ξ,λ) is a finite measure space and a λ-continuous vector-valued measure $\eta:\Xi\to L^{p(\cdot)}(E,X,\Sigma,\mu)$ with finite variation $\operatorname{Var}(\eta)(X)<\infty$. We can presume that

$$\|\eta(D)\|_{L^{p(\cdot)}(E,X,\Sigma,\mu)}\leq\lambda(D)$$

for all $D\in\Xi$. Let $\pi=\Pi(\Omega)=\{B_1,\ldots,B_k,B_i\in\Xi,i=1,\ldots,k,k\in N\}$ be a finite partition of Ω and let P be a finite partition of E.

We define a function $\psi_{\pi,P}:\Omega\times E\to R$ by

$$\psi_{\pi,P}(\omega,x)=\sum_{D\in\pi}\sum_{I\in P}\frac{\int_I\eta(D)\,d\mu}{\lambda(D)\,\mu(I)}1_I(x)\,1_D(\omega).$$

Let $B_{\pi,P}=\{D\times I\subseteq\Omega\times E:D\in\pi,I\in P\}$ then the pair $(\psi_{\pi,P},B_{\pi,P})$ is a martingale in $L^{p(\cdot)}(\Omega\times E,\lambda\times\mu)$.

Since X has the Radon-Nikodym property, the martingale $(\psi_{\pi,P},B_{\pi,P})$ is convergent in $L^{p(\cdot)}(\Omega\times E,X,\lambda\times\mu)$ if it is bounded in $L^{p(\cdot)}(\Omega\times E,X,\lambda\times\mu)$. By general Holder inequality, we estimate

$$\begin{aligned}
&\left\|\int_I\eta(D)\,d\mu\right\|_X\\
&\quad=\left\|\int_I\eta(D)\,1_I(x)\,d\mu\right\|_X\\
&\quad\leq k_{Hol}\left\|\int_I\eta(D)\,d\mu\right\|_{L^{p(\cdot)}(E,X,\mu)}\|1_I(x)\|_{L^{p(\cdot)}(E,X,\mu)}\\
&\quad\leq k_{Hol}\left\|\int_I\eta(D)\,d\mu\right\|_{L^{p(\cdot)}(E,X,\mu)}\max\left\{\mu(I)^{1-p_m^{-1}},\mu(I)^{1-p_S^{-1}}\right\},
\end{aligned}$$

where $p(\cdot)+q(\cdot)=p(\cdot)\,q(\cdot)$. We have

$$\rho_{L^{p(\cdot)}(\Omega\times E,X,\lambda\times\mu)}(\psi_{\pi,P})$$

$$
\begin{aligned}
&= \sum_{D\in\pi}\sum_{I\in P}\left(\frac{\left\|\int_I \eta(D)d\mu\right\|_X}{\lambda(D)\mu(I)}1_I(x)1_D(\omega)\right)^{p(x)} \\
&\leq \sum_{D\in\pi}\sum_{I\in P}\frac{\rho_{L^{p(\cdot)}(E,X,\mu)}(\eta(D)1_I)}{\min\left\{\lambda(D)^{p_m},\lambda(D)^{p_S}\right\}}\lambda(D) \\
&\quad \times \max\left\{\mu(I)^{\frac{1+p_m}{q_S-p_m}},\mu(I)^{\frac{1+p_S}{q_m-p_S}}\right\} \\
&= \sum_{D\in\pi}\sum_{I\in P}\rho_{L^{p(\cdot)}(E,X,\mu)}(\eta(D)1_I)\max\left\{\lambda(D)^{p_m-1},\lambda(D)^{p_S-1}\right\}.
\end{aligned}
$$

Set function $\rho_{L^{p(\cdot)}(E,X,\mu)}\left(\eta\left(D\right)1_I\right)$ is additive function of $I\in P$. Since $\rho_{L^{p(\cdot)}(E,X,\mu)}\left(\eta\left(D\right)\right)\leq\lambda\left(D\right)$ for all $D\in\Xi$, we have

$$
\begin{aligned}
\rho_{L^{p(\cdot)}(\Omega\times E,X,\lambda\times\mu)}\left(\psi_{\pi,P}\right) &= \sum_{D\in\pi}\left(\frac{\left\|\int_I\eta\left(D\right)d\mu\right\|_X}{\lambda\left(D\right)}\right)^{p(x)}\lambda\left(D\right) \\
&\leq \sum_{D\in\pi}\lambda\left(D\right).
\end{aligned}
$$

Thus, there exists a limit

$$\lim\psi_{\pi,P}=\psi$$

in the topology of $L^{p(\cdot)}\left(\Omega\times E,X,\lambda\times\mu\right)$-norm.

For all $C\in\Xi$, we obtain

$$
\begin{aligned}
\int_C\psi\left(\omega,\cdot\right)d\lambda &= \lim_{\pi,P}\int_C\sum_{D\in\pi}\sum_{I\in P}\frac{\int_I\eta\left(D\right)d\mu}{\lambda\left(D\right)\mu\left(I\right)}1_I\left(x\right)1_D\left(\omega\right)d\lambda \\
&= \lim_{\pi}\int_C\sum_{D\in\pi}\lim_{P}\left(\sum_{I\in P}\frac{\int_I\eta\left(D\right)d\mu}{\mu\left(I\right)}1_I\left(x\right)\right)\frac{1_D\left(\omega\right)}{\lambda\left(D\right)}d\lambda \\
&= \lim_{\pi}\int_C\sum_{D\in\pi}\frac{\eta\left(D\right)}{\lambda\left(D\right)}1_D\left(\omega\right)d\lambda.
\end{aligned}
$$

since the martingale $\sum_{I\in P}\frac{\int_I\eta(D)1_I(x)d\mu}{\mu(I)}$ converges to $\eta\left(D\right)$ in $L^{p(\cdot)}\left(E,X,\mu\right)$. Thus, since

$$\lim_{\pi}\int_C\sum_{D\in\pi}\frac{\eta\left(D\right)}{\lambda\left(D\right)}1_D\left(\omega\right)d\lambda=\eta\left(C\right),$$

we conclude $\eta(C)=\int_C\psi(\omega,\cdot)d\lambda,\forall C\in\Xi$. □

To complete our investigation, we prove the following lemma.

Lemma 8.2. *Let B be a sub σ-algebra of Σ. Then, the conditional expectation $E(f|B)$ exists for all $f \in L^1(\mu, X)$ and for all $f \in L^{p(\cdot)}(E, \Sigma, X, \mu)$, $p \in P(E)$, $1 < p_m \leq p_S < \infty$, we have*

$$\|E(f|B)\|_{L^{p(\cdot)}(E,\Sigma,X,\mu)} \leq \|f\|_{L^{p(\cdot)}(E,\Sigma,X,\mu)}.$$

Proof. We define a function by $\chi(x) = \sum_{i=1}^{k} a_i 1_{B_i}(x)$, where 1_{B_i} denotes the indicator function of disjoint sets $B_i \in \Sigma$ and elements $a_i \in X$ for all $i = 1, \ldots, k$. So, the conditional expectation $E(\chi|B)$ is given by

$$E(\chi|B) = \sum_{i=1,\ldots,k} a_i E(1_{B_i}|B).$$

We have

$$\begin{aligned}
\rho_{L^{p(\cdot)}(E,\Sigma,X,\mu)}(E(f|B)) &= \int_E \|E(f|B)\|_X^{p(x)} d\mu(x) \\
&\leq \int_E E\left(\sum_{i=1,\ldots,k} \|a_i\|_X 1_{B_i}(x)|B\right)^{p(x)} d\mu(x) \\
&\leq \rho_{L^{p(\cdot)}(E,\Sigma,X,\mu)}\left(\sum_{i=1,\ldots,k} \|a_i\|_X 1_{B_i}(x)\right) \\
&= \rho_{L^{p(\cdot)}(E,\Sigma,X,\mu)}(f).
\end{aligned}$$

The operator $E(f|B)$ is extended to a contractive linear extension to the whole space $L^{p(\cdot)}(E, \Sigma, X, \mu)$. That extension of $E(\cdot|B)$ to the whole space $L^{p(\cdot)}(E, \Sigma, X, \mu)$ is a linear contractive projection on $L^{p(\cdot)}(E, \Sigma, X, \mu)$. □

Thus, we have proven the following theorem.

Theorem 8.3. *The conditional expectation $E(f|B)$ is a linear contractive projection on $L^{p(\cdot)}(E, \Sigma, X, \mu)$ with $p \in P(E)$, $1 < p_m \leq p_S < \infty$.*

9. Radon-Nikodym Theory for General Variable Exponent Lebesgue Spaces

The classical Radon-Nikodym theorem provides a density for a scalar measure absolutely continuous with respect to another. Extending this fundamental result to the context of vector and operator-valued measures is a central challenge in functional analysis, with profound implications for the geometry of Banach spaces and the theory of integration. This section presents a progressive generalization of the Radon-Nikodym theorem, culminating in a powerful result within the modern framework of variable exponent Lebesgue spaces.

Our analysis is founded on the following core result. Throughout this section, we work within the framework of a finite measure space (E, Σ, μ). The finiteness of μ is a crucial assumption that enables the application of certain convergence arguments and ensures the well-definedness of key constructs in the proof.

Theorem 12 presents the main result of this section. It translates the abstract measure-theoretic results into the concrete setting of variable exponent spaces $L^{p(\cdot)}$. This theorem provides a complete characterization of linear operators on these spaces, showing that any such operator of finite norm can be represented by an integral against an operator-valued function ψ, whose $L^{q(\cdot)}$-norm precisely captures the local behavior of the original operator. This result seamlessly connects the deep theory of vector measures with the nuanced structure of variable exponent spaces, providing an essential tool for the further analysis of their geometric properties.

Theorem 9.1. *Let $\eta : \Sigma \to LB(Y, Z)$ be a measure with finite total variation $\mu = \operatorname{Var}(\eta)$. Assume that μ is a σ-finite scalar measure. Then, there exists a function $\Phi : E \to LB(Y, Z)$ (we suppress the η dependence in notation) such that:*

(i) $\|\Phi(x)\|_{Y\to Z} = 1$ μ-a.e.;

(ii) for every $z^* \in Z^*$, the function $x \mapsto \langle \Phi(x)(f(x)), z^* \rangle$ is in $L^1(\mu)$ and the following identity relates the vector integral to the scalar integral:

$$\left\langle \int_D f \, d\eta, z^* \right\rangle = \int_D \langle \Phi(f), z^* \rangle \, d\mu$$

holds for all $f \in L^1(E, Y, \eta)$ and $D \in \Sigma$.

Proof. We first address the trivial case: if $\eta = 0$, then defining $\Phi(\eta)(\cdot) = 0$ immediately proves the theorem. We may therefore assume $\eta \neq 0$ for the remainder of the proof.

For every fixed $y \in Y$ and $z^* \in Z^*$, we define a scalar measure by

$$\eta(y, z^*)(D) = \langle \eta(D)y, z^* \rangle, \quad \text{for all } D \in \Sigma.$$

This measure satisfies the inequality

$$|\eta(y, z^*)| \leq \mu \|y\|_Y \|z^*\|_{Z^*}.$$

By the Radon-Nikodym theorem, there exists a scalar, locally μ-integrable function ϕ_{y,z^*} such that $d\eta(y, z^*) = \phi_{y,z^*} d\mu$ and $d|\eta(y, z^*)| = |\phi_{y,z^*}| d\mu$.

Now, for all $x \in E$ and $y \in Y$, the map

$$\phi_y(x) : z^* \mapsto \phi_{y,z^*}(x)$$

defines a continuous linear functional on Z^* satisfying $|\phi_y(x)| \leq \|y\|_Y$.

Furthermore, for every fixed $x \in E$, the mapping

$$\Phi(\eta)(x)(\cdot) : y \mapsto \phi_y(x)$$

is a linear continuous operator from Y to Z with norm $\|\Phi(\eta)(x)\|_{Y\to Z} \leq 1$.

By construction, for all $x \in E$, $y \in Y$ and $z^* \in Z^*$, we have

$$\langle \Phi(\eta)(x)y, z^*\rangle = \phi_{y,z^*}(x).$$

Consequently, for any $D \in \Sigma$ and any simple function $\chi : E \to Y$,

$$\langle \eta(D)y, z^*\rangle = \int_D \langle \Phi(\eta)(x)y, z^*\rangle d\mu(x)$$

and

$$\left\langle \int_D \chi d\eta, z^* \right\rangle = \int_D \langle \Phi(\eta)(x)\chi(x), z^*\rangle d\mu(x).$$

To extend this to a general function $f \in L^1(E, Y, \eta)$, let $\{\chi_k\}$ be a sequence of simple functions converging in mean and μ-almost everywhere to f. There exists $k_0 \in \mathbb{N}$ such that for all $k, m \geq k_0$,

$$|\langle \Phi(\eta)\chi_k, z^*\rangle - \langle \Phi(\eta)\chi_m, z^*\rangle| \leq \|\chi_k - \chi_m\|_Y \|z^*\|_{Z^*}.$$

This implies that $\{\langle \Phi(\eta)\chi_k, z^*\rangle\}$ is a Cauchy sequence in $L^1(E, \Sigma, \mu)$ and hence it converges μ-almost everywhere to $\langle \Phi(\eta)f, z^*\rangle$.

Therefore, for all $z^* \in Z^*$,

$$\lim_{k\to\infty} \int_E \langle \Phi(\eta)\chi_k, z^*\rangle d\mu = \int_E \langle \Phi(\eta)f, z^*\rangle d\mu,$$
$$\lim_{k\to\infty} \left\langle \int_E \chi_k d\eta, z^* \right\rangle = \left\langle \int_E f d\eta, z^* \right\rangle.$$

Since $\lim_{k\to\infty} \int_E \chi_k d\eta = \int_E f d\eta$ and

$$\left\langle \int_E \chi_k d\eta, z^* \right\rangle = \int_E \langle \Phi(\eta)\chi_k, z^*\rangle d\mu,$$

we conclude that

$$\left\langle \int_E f d\eta, z^* \right\rangle = \int_E \langle \Phi(\eta)f, z^*\rangle d\mu.$$

Finally, we establish the norm bound. From the inequality

$$\langle \eta(D)y, z^*\rangle \leq \int_D \|y\|_Y \|z^*\|_{Z^*} \|\Phi(\eta)(x)\|_{Y\to Z} d\mu(x),$$

we deduce that $\|\eta(D)\|_{Y\to Z} \leq \int_D \|\Phi(\eta)(x)\|_{Y\to Z} d\mu(x)$. Defining $\varpi = \|\Phi(\eta)\|_{Y\to Z}\mu$, we have

$$\|\eta(D)\|_{Y\to Z} \leq \varpi(D).$$

Given that μ is the least positive measure satisfying

$$\|\eta(D)\|_{Y\to Z} \le \mu(D), \quad \forall D \in \Sigma,$$

it follows that $\|\Phi(\eta)(x)\|_{Y\to Z} = 1$ μ-almost everywhere. □

As a consequence of this theorem, we obtain the following theorem.

Theorem 9.2. *Consider a scalar σ-finite measure μ. Let $\eta : \Sigma \to LB(Y,Z)$ be a vector measure with finite total variation $\nu = \mathrm{Var}\,(\eta)$, which is absolutely continuous with respect to μ. Then a function $\Upsilon : E \to LB(Y,Z)$ (suppressing the η dependence in notation) exists for which the following properties hold:*

(i) the following identity holds for the scalar integral with respect to the variation measure ν:

$$\int_D \phi \, d\nu = \int_D \|\Upsilon(x)\|_{Y\to Z} \phi \, d\mu$$

for all functions $\phi \in L^1(\nu)$;

(ii) the identity relating the vector and scalar integrals holds:

$$\left\langle \int_D f \, d\eta, z^* \right\rangle = \int_D \langle \Upsilon(f), z^* \rangle \, d\mu$$

for all $f \in L^1(E,Y,\eta)$, all $z^* \in Z^*$ and all $D \in \Sigma$.

Employing this theorem, we prove the statement of the next theorem.

Theorem 9.3. *Let $p \in P(E)$ such that $1 < p_m \le p_S < \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in E$. Let μ be a scalar σ-finite measure on Σ. Let $U : L^{p(\cdot)}(E,\Sigma,\mu) \to Z$ be a linear mapping with finite norm $\|U_D\|_{L^{p(\cdot)}} = \mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$ for all D. Then, there exists a function $\psi : E \to LB(X,Z)$ such that:*

(i) for each μ-measurable set $D \subset E$, we have $1_D\|\psi\| \in L^{q(\cdot)}(E,\Sigma,\mu)$ and $\|\psi 1_D\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|U_D\|_{L^{p(\cdot)}}$;

(ii) for each function $f \in L^{p(\cdot)}(E,\Sigma,\mu)$ and $z^* \in Z^*$, we have

$$\langle U(f), z^* \rangle = \int_D \langle \psi(x)(f(x)), z^* \rangle \, d\mu, \quad \forall D \subset E \text{ measurable.}$$

Proof. Consider a measure $\eta : \Sigma \to LB(Y,Z)$ such that its variation $\mathrm{Var}_{L^{q(\cdot)}}(\eta)(D)$ is finite for every $D \in \Sigma$. Define the linear mapping $U : L^{p(\cdot)}(E,\Sigma,\eta) \to Z$ via the Bochner integral:

$$U(f) = \int_E f \, d\eta, \quad f \in L^{p(\cdot)}(E,\Sigma,\eta).$$

Since measure η has finite variation $\mathrm{Var}\,(\eta)$ and is absolutely continuous with respect to the scalar measure μ so that we have $\mathrm{Var}\,(\eta) = \varphi\mu$.

For each μ-measurable set $D \subset E$ we obtain $1_D\varphi \in L^{q(\cdot)}(E,\Sigma,\mu)$ and $\|\varphi 1_D\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|U_D\|_{L^{p(\cdot)}}$.

By the previous theorem, we have the existence of $\Upsilon(\eta)(\cdot) : E \to LB(Y,Z)$ corresponding to measures η and μ. We obtain $\operatorname{Var}(\eta) = \|\Upsilon(\eta)\|_{Y\to Z}\operatorname{Var}(\mu)$, here $\|\Upsilon(\eta)(x)\|_{Y\to Z} = \varphi(x)$ μ-almost everywhere. For all μ-measurable set $D \subset E$ such that $\|U_D\|_{L^{p(\cdot)}} < \infty$, then $1_D\|\Upsilon(\eta)\|_{Y\to Z} \in L^{q(\cdot)}(E,\Sigma,\mu)$ and $\|1_D\Upsilon(\eta)\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|\varphi\|_{L^{q(\cdot)}(E,\Sigma,\mu)} = \|U_D\|_{L^{p(\cdot)}}$. □

References

1. N. Bourbaki, *Intégration, Chapitre 6*, Actualités Scientifiques et Industrielles, No. 1281, Hermann, Paris, 1959.
2. F. Botelho and R.J. Fleming, *The dual of the space of bounded operators on a Banach space*, Concr. Oper., 8 (2021), pp. 48–59.
3. S.D. Chatterji, *The Radon–Nikodym property*, in Probability in Banach Spaces II, A. Beck, Ed., Lecture Notes in Math., vol. 709, Springer, Berlin, 1979, pp. 18–24.
4. Y.S. Choi, S.K. Kim, and H.J. Lee, *On norm-attaining mappings on Banach spaces*, in Geometry of Banach Spaces and Related Fields, G. Godefroy et al., Eds., Proc. Sympos. Pure Math., vol. 106, Amer. Math. Soc., Providence, RI, 2024, pp. 43–66.
5. J. Diestel and J.J. Uhl, *Vector Measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, RI, 1977.
6. R.G. Douglas, *Contractive projections on an L^p space*, Pacific J. Math., 15 (1965), pp. 443–462.
7. N. Dunford and A.P. Morse, *Remarks on the preceding paper of James A. Clarkson*, Trans. Amer. Math. Soc., 40 (1936), pp. 415–420.
8. N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc., 44 (1938), pp. 305–356.
9. X. Fan and D. Zhao, *On the spaces $L^{p(x)}$ and $W^{1,p(x)}$*, J. Math. Anal. Appl., 263 (2001), pp. 424–446.
10. W. Feldman, *A factorization for orthogonally additive operators on Banach lattices*, J. Math. Anal. Appl., 472 (2019), pp. 238–245.
11. T.X. Guo, *Representation theorems of the dual of Lebesgue–Bochner function spaces*, Sci. China Ser. A, 43 (2000), pp. 234–243.
12. A. Kamińska, H.J. Lee, and H.J. Tag, *Daugavet and diameter two properties in Orlicz–Lorentz spaces*, J. Math. Anal. Appl., 529 (2024), 127323.
13. S.G. Kim, *The norming sets of $L(2\mathbb{R}^2_h)$*, Acta Sci. Math. (Szeged), 89 (2023), pp. 61–79.

14. S.K. Kim, H.J. Lee, M. Martín, and J. Merí, *On a second numerical index for Banach spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 150 (2020), pp. 1003–1051.
15. J. Li, *On null-continuity of monotone measures*, Mathematics, 8 (2020), 205.
16. H.J. Lee and T. Hyung-Joon, *Remark on the Daugavet property for complex Banach spaces*, Demonstr. Math., 57 (2024), 20240004.
17. M. Martín and A. Rueda Zoca, *Daugavet property in projective symmetric tensor products of Banach spaces*, Banach J. Math. Anal., 16 (2022), Paper No. 35, 32 pp.
18. H. Sun, A. Chang, Y. Zhang, and W. Chen, *A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications*, Fract. Calc. Appl. Anal., 22 (2019), pp. 27–59.
19. C. Sun and J. Liu, *Reconstruction of the space-dependent source from partial Neumann data for slow diffusion system*, Acta Math. Appl. Sin. (Engl. Ser.), 36 (2020), pp. 166–182.
20. G.E.F. Thomas, *The Lebesgue–Nikodym theorem for vector-valued Radon measures*, Mem. Amer. Math. Soc., 139 (1974), no. 660, vi+101 pp.
21. Z. Wang and G.J. Klir, *Fuzzy measures defined by fuzzy integral and their absolute continuity*, J. Math. Anal. Appl., 203 (1996), pp. 150–165.
22. Z. Wang, G.J. Klir, and W. Wang, *Monotone set functions defined by Choquet integral*, Fuzzy Sets Syst., 81 (1996), pp. 241–250.
23. Z. Wang and G.J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.
24. L.L. Wang, Y.H. Fan, and W.G. Ge, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal., 71 (2009), pp. 4259–4270.
25. M.Z. Wu, X.L. Zeng, and S.E. Zhao, *On L-convex compactness in random locally convex modules*, J. Math. Anal. Appl., 515 (2022), 126404.
26. X. Zhang and M. Liu, *A characterization for a complete random normed module to be mean ergodic*, Acta Math. Sin. (Engl. Ser.), 33 (2017), pp. 899–910.

Department of Mathematics, National Technical University of Ukraine, "Igor Sikorsky Kyiv Polytechnic Institute" Kyiv, Ukraine, 37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056.

Email address: Math.kiev@gmail.com.