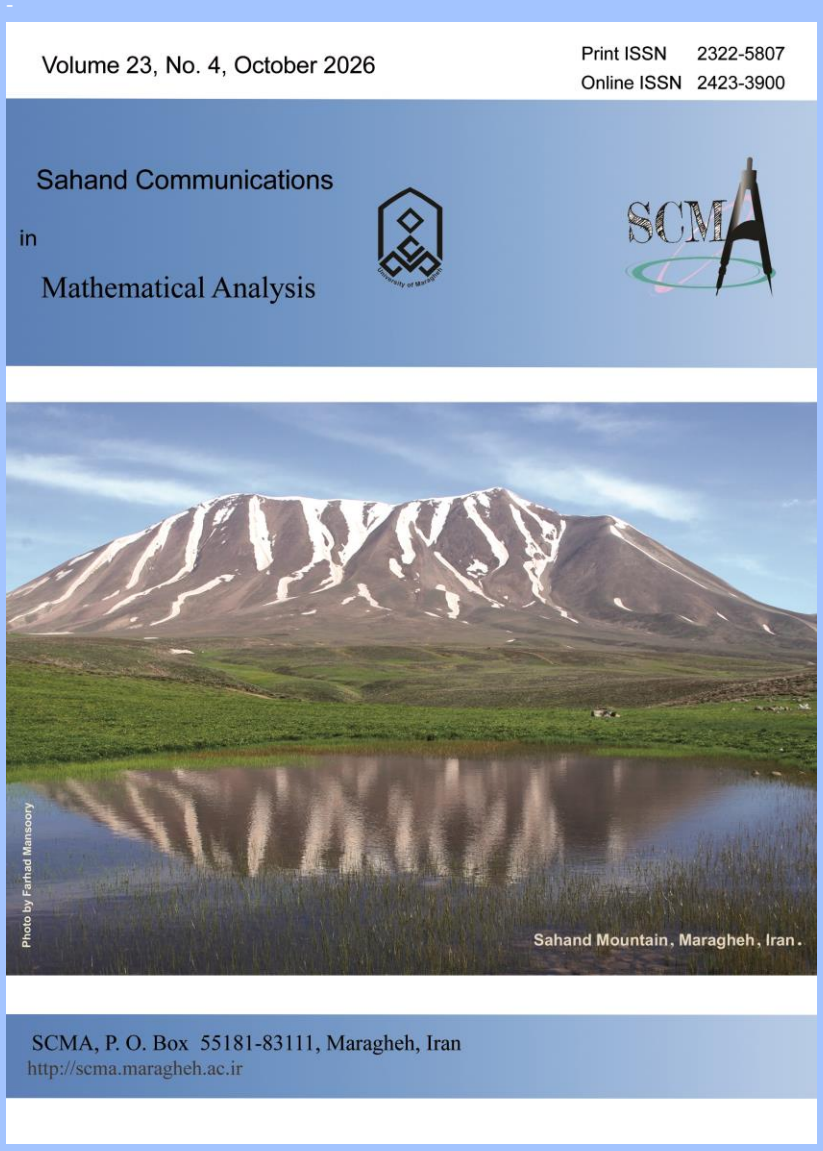
# Beyond Classical Milne's Scheme: Fractional Error Analysis Via Modified *k*-AB Integral Operators

Usama Asif and Muhammad Uzair Awan

**Sahand Communications in Mathematical Analysis**

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 4
Pages: 203-234

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2026.2082795.2478



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 4 (2026), 203-234
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2082795.2478

# Beyond Classical Milne's Scheme: Fractional Error Analysis Via Modified $k$-AB Integral Operators

Usama Asif [1] and Muhammad Uzair Awan[2*]

Abstract. The principal motive of the current study is to determine several new fractional bounds of Milne's inequality by utilizing different classes of functions. We initially start our investigation by proving an important identity for differentiable functions by using $k$-$AB$ fractional integral operator. This identity unifies many of the existing lemmas. Relying on this identity and the invexity of functions, we generate various fractional error bounds for Milne's rule. Through the utilization of bounded functions, Lipschitzian functions, and the bounded variation characteristic of functions, we construct a few other interesting relations. In favour of our essential outcomes, we deliver interesting implications to special functions, numerical integration, and graphical and numerical procedures to highlight the efficacy of results.

## 1. Introduction

Error analysis of numerical quadrature schemes is a very interesting and active field of research problems. It is a fact that to determine the error term of Simpson's rules, functions must be four times continuously differentiable. In other scenarios, classical inequality doesn't work. To fulfil these research gaps, a series of techniques have been deployed based on the nature of functions to evaluate remainder terms. Another problem is to calculate the upper and tight bounds of these error inequalities. In the following perspective, the role of convexity and its generalizations is crucial among all the other approaches in the improvement of inequalities based on their geometrical and analytical

2020 *Mathematics Subject Classification.* 26A33, 26A51, 26D10, 26D15.
*Key words and phrases.* Milne's inequality, Preinvex functions, Lipschitz function, Function of bounded variation.
Received: 2026-01-01, Accepted: 2026-05-05.
* Corresponding author.

nature. In particular, the contribution of preinvex functions are spectacular as it extends the idea of convexity. This class has marvelous implications in optimization theory and inequalities as well. Noor et al. [23] used the concept of $h$-preinvex functions and derived general trapezoidal-like inequalities. Sun [32] studied the Hermite-Hadamard-like inequalities by leveraging the preinvex functions in the setting of Yang calculus. Safdar and Attique concluded new weighted versions of Simpson's like inequalities associated with preinvex functions in [29]. Kashuri and Sarikaya [17] examined numerous inequalities via extended fractional operators and preinvex functions.

In 1998, Dragomir and Agarwal [10] explored the upper bounds of trapezoidal inequalities for first-order differentiable convex mappings, and later on, this paper proved a major progress in error analysis. The upper bounds for the error inequality of the midpoint rule of convex functions were investigated by Kirmaci [15]. Moreover, the estimates obtained through convex mappings provide us with upper bounds of classical inequalities. Dragomir et al. explored the error inequality of the Simpson $\frac{1}{3}$rd rule by using different classes of functions in [9]. Alomari et al. [1] examined the Simpson's kinds of inequalities by invoking the idea of Breckner convexity. Alomari and Dragomir [3] devoted their effort to obtain unified bounds for numerical Newton-cotes procedures associated with a generalized kernel. Javed and his coauthors [14] established a new generic equation and parametric error inequalities of Newton-Cotes schemes and applications to non-linear analysis. By utilizing the strategy of [14], several error inequalities have been investigated in the last two decades. For additional information, see [19, 25, 31, 33].

In recent years, different concepts have been applied to generalize and refine the classical concepts. The most notable approach is fractional calculus, which plays an astonishing role in engineering and applied sciences. Because it allows us to address various difficulties, such as fractional order dimension challenges and nonlinear processes, like other branches of science, its impact on the progression of inequalities is very significant. In 2103, Sarikaya [27] successfully employed the Riemann-Liouville fractional operators and convex functions to establish the fractional Hadamard's inequalities. Meftah et al. [20] utilized Yang fractional concepts to conclude bounds for Maclaurin's inequality with applications to numerical integration. Hezenci and Budak explored the Riemann-Liouville fractional analogues of Maclaurin's inequality via convex functions in [11]. Peng and Du computed the estimates of Maclaurin's type inequalities incorporated with multiplicative fractional operators in [24]. For more detail, see[5, 8, 13, 21, 22, 30].

In the last few years, the most thoroughly investigated result is Milne's inequality from different approaches; here, we disclose some recent and significant contributions related to it. Alomari [2] demonstrated the bounds of Milne's inequality by utilizing the first-order differentiable convex function and other function classes. Budak et al. [7] approximated the estimates of the fractional Milne's inequality by making use of convex functions, Lipschitzian functions, and functions of bounded variation. In 2023, Bin-Mohsin et al. [6] exploited the Milne-Mercer-like inequalities by utilizing the quantum calculus. Recently, Haider and his coauthors [12] deployed the tempered fractional operators to evaluate new bounds of Milne-like inequalities. Lakhdari et al. [18] utilized the Katugampola fractional operator to extend the previously established results linked with Milne's procedure.

The classical Milne inequality provides bounds for the remainder term of Milne rule for fourth order differentiable bounded function. In other cases, it becomes inconclusive, the authors have developed the first differentiable Milne's type inequalities to seek out this problem. Moreover, the literature shows that the existing Milne-type inequalities have been investigated under convexity framework. These inequalities become inconclusive for non-convex functions involving $k$-$AB$-fractional integral operators. To fulfil this gap, we aim to investigate the Milne-type inequalities through generalized $k$-$AB$-fractional integral identities and preinvex functions. To wrap up this project, we divide our study into three segments. The initial part is devoted to basic concepts and the background of research problems. In the subsequent part, we will establish a new general equation and several Milne-like inequalities under preinvexity and bounded variation of functions. Later on, we deliver the potential deductions, applicable, and graphical analysis to highlight the significance of findings. Also, we have presented an interesting example to showcase the significance of our proposed results. Our approach not only generalizes the existing results but also provide a unified setup to investigate the bounds of $k$-$AB$-fractional integral operators. We hope the methodologies of this article will create new insights for future developments.

## 2. Preliminaries

Now, we recall some preliminary concepts that will be useful to prove the next results. Let us start with the notion of an invex set and preinvex functions demonstrated by Weir and Mond.

**Definition 2.1** ([34, 35])**.** Consider a real-valued function $\Theta$, and assume that $\mathcal{K}$ is a subset of $\mathbb{R}$. Then, it is called an invex set if

$$x + \ell\Theta(y, x) \in \mathcal{K}, \quad x, y \in \mathcal{K}, \quad \ell \in [0, 1].$$

Note that every convex set is an invex set with $\Theta(y,x) = y - x$ but converse statement is not true.

**Example 2.2** ([34])**.** Here we give a concrete example of an invex set which is not convex. Let $K = [-3,-2] \cup [1,2]$ and let $\Theta(x,y)$ be defined as:

$$\Theta(x,y) = \begin{cases} x - y & \text{if } 1 \leq x \leq 2, 1 \leq y \leq 2; \\ x - y & \text{if } -3 \leq x \leq -2, -3 \leq y \leq -2; \\ -3 - y & \text{if } 1 \leq x \leq 2, -3 \leq y \leq -2; \\ -1 - y & \text{if } -3 \leq x \leq -2, 1 \leq y \leq 2. \end{cases}$$

Based on the invex sets, the class of pre-invex functions is defined as:

**Definition 2.3** ([34, 35])**.** Suppose $\mathcal{K}$ be an invex set with respect bi-mapping $\Theta(x,y)$. Then $\Upsilon$ is said to be preinvex function, if

$$\Upsilon(x + \ell\Theta(y,x)) \leq (1-\ell)\Upsilon(x) + \ell\Upsilon(y), \quad \ell \in [0,1].$$

Note that every convex function is preinvex with $\Theta(x,y)$ and the converse is not true. Now we give an illustrative example.

**Example 2.4** ([34, 35])**.** A function $\mathcal{Z} : (-\infty, \infty) \to \mathbb{R}$ is defined as $\mathcal{Z}(x) = -|x|$ is a preinvex under the following Definition of $\Theta$

$$\Theta(x,y) = \begin{cases} x - y, & \text{if } y \leq 0 \quad \text{and} \quad x \leq 0 \\ x - y, & \text{if } y \geq 0 \quad \text{and} \quad x \geq 0 \\ y - x, & \text{if } y > 0 \quad \text{and} \quad x < 0 \\ y - x, & \text{if } y < 0 \quad \text{and} \quad x > 0. \end{cases}$$

It is clear that $\mathcal{Z}$ is not a convex function.

Now we come up with the error inequality of Simpson's quadrature rule developed by Dragomir [9] in 1998.

**Theorem 2.5** ([9])**.** *Let $\mathcal{Z} : [\mu_1, \mu_2] \to \mathbb{R}$ be a fourth order continuously differentiable mapping on $(\mu_1, \mu_2)$ and*

$$\left\| \mathcal{Z}^{(4)} \right\|_\infty = sup_{v \in (\mu_1, \mu_2)} \left| \mathcal{Z}^{(4)} \right| < \infty,$$

*then:*

$$\left| \frac{\mu_2 - \mu_1}{6} \left[ \mathcal{Z}(\mu_1) + 4\mathcal{Z}\left( \frac{\mu_1 + \mu_2}{2} \right) + \mathcal{Z}(\mu_2) \right] - \int_{\mu_1}^{\mu_2} \mathcal{Z}(x)dx \right| \leq \frac{1}{2880} \left\| \mathcal{Z}^4 \right\|_\infty (\mu_2 - \mu_1)^5.$$

Next, we revisit the Milne's integral inequality.

**Theorem 2.6** ([2]). *Suppose that $\mathcal{Z} : [\mu_1, \mu_2] \to \mathbb{R}$ is four-times continuously differentiable mapping on $(\mu_1, \mu_2)$, and*

$$\left\|\mathcal{Z}^{(4)}\right\|_\infty = \sup_{v\in(\mu_1,\mu_2)} \left|\mathcal{Z}^{(4)}\right| < \infty,$$

*then we have*

$$\left|\frac{1}{3}\left[2\mathcal{Z}(\mu_1) - \mathcal{Z}\left(\frac{\mu_1+\mu_2}{2}\right) + 2\mathcal{Z}(\mu_2)\right] - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{Z}(v)dv\right| \leq \frac{7(\mu_2-\mu_1)^4}{23,040}\left\|\mathcal{Z}^{(4)}\right\|_\infty.$$

The Milne identity for a first order differentiable function is stated as:

**Lemma 2.7** ([7]). *Let $\mathcal{Z} : [\mu_1, \mu_2] \to \mathbb{R}$be a differentiable function and $\mathcal{Z}'$ is an integrable function, then*

$$\begin{aligned}&\frac{1}{3}\left[2\mathcal{Z}(\mu_1) - \mathcal{Z}\left(\frac{\mu_1+\mu_2}{2}\right) + 2\mathcal{Z}(\mu_2)\right] - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{Z}(\ell)\mathrm{d}\ell\\ &= \frac{\mu_2-\mu_1}{4}\left[\int_0^{\frac{1}{2}} \ell\left[\mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1 + \frac{\ell}{2}\mu_2\right) - \mathcal{Z}'\left(\frac{\ell}{2}\mu_1 + \frac{2-\ell}{2}\mu_2\right)\right]\mathrm{d}\ell\right.\\ &\quad \left. + \int_{\frac{1}{2}}^1 \left(\ell - \frac{4}{3}\right)\left[\mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1 + \frac{\ell}{2}\mu_2\right) - \mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1 + \frac{\ell}{2}\mu_2\right)\right]\mathrm{d}\ell\right].\end{aligned}$$

Now we recall the well-known Hölder's inequality.

**Theorem 2.8** ([26]). *Let $\mathcal{Z}, \mathcal{G} : [\mu_1, \mu_2] \to \mathbb{R}$ be two functions. If $|\mathcal{Z}|^p$ and $|\mathcal{G}|^q$ are integrable functions with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{\mu_1}^{\mu_2} |\mathcal{Z}(\ell)\mathcal{G}(\ell)|\mathrm{d}\ell \leq \left(\int_{\mu_1}^{\mu_2} |\mathcal{Z}(\ell)|^p \mathrm{d}\ell\right)^{\frac{1}{p}} \left(\int_{\mu_1}^{\mu_2} |\mathcal{G}(\ell)|^q \mathrm{d}\ell\right)^{\frac{1}{q}}.$$

*It can be written as:*

$$\|\mathcal{Z}(\ell)\mathcal{G}(\ell)\|_1 \leq \|\mathcal{Z}(\ell)\|_p \|\mathcal{G}(\ell)\|_q,$$

*where $\|\mathcal{Z}(\ell)\|_p = \left(\int_{\mu_1}^{\mu_2} |\mathcal{Z}(\ell)|^p \mathrm{d}\ell\right)^{\frac{1}{p}} \leq \infty$.*

Now we recall the Definitions of both left and right $k$-Atangana-Baleanu fractional operators.

**Definition 2.9** ([16]). The left and right-sided $k-$Atangana-Baleanu fractional integral operators of a real-valued function I of order $\alpha > 0$, are defined as:

$${}^{AB}_{\mu_1}I^{\alpha,k}_{\ell}\mathcal{Z}(\ell) = \frac{1-\alpha}{B(\alpha)}\mathcal{Z}(\ell) + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\ell} \mathcal{Z}(x)(\ell-x)^{\frac{\alpha}{\kappa}-1}\mathrm{d}x, \quad \mu_1 < \ell$$

and

$$ {}^{AB}I^{\alpha,k}_{\mu_2}\mathcal{Z}(\ell) = \frac{1-\alpha}{B(\alpha)}\mathcal{Z}(\ell) + \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\ell}^{\mu_2}\mathcal{Z}(x)(x-\ell)^{\frac{\alpha}{\kappa}-1}\mathrm{d}x, \quad \ell < \mu_2, $$

where $k > 0$ and $B(\alpha) > 0$ satisfies the property $B(0) = B(1) = 1$.

For $k = 1$, we obtain the AB fractional operators.

## 3. Main Results

In this section, we develop some new generalized estimates of Milne's type inequalities through the application of different function classes. First, we look at a generic fractional Milne's inequality.

**Lemma 3.1.** *Presume that $\mathcal{Z} : [\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)] \to \mathbb{R}$ be a differentiable function and $\mathcal{Z}' \in L[\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)]$, then*

(3.1)

$$ \begin{aligned} &\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right) - \mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right. \\ &\quad \left. +2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right] + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\ &\quad \times [\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))] - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\ &\quad \times \left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right. \\ &\quad \left. +{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right] \\ &= \frac{\Theta(\mu_2,\mu_1)}{4}\left[\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.\right. \\ &\quad \left.\left. -\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell + \int_{\frac{1}{2}}^1\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\right. \\ &\quad \left.\times\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right) - \mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell\right]. \end{aligned} $$

*Proof.* Considering the right-hand side of (3.1), and implementing integration by parts, we have

$$ \begin{aligned} &= \frac{\Theta(\mu_2,\mu_1)}{4}\left[\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.\right. \end{aligned} \tag{3.2} $$

$$-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\Bigg]\,\mathrm{d}\ell+\int_{\frac{1}{2}}^{1}\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)$$
$$\times\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell\Bigg]$$
$$=\frac{\Theta(\mu_2,\mu_1)}{4}\left[I_1+I_2\right],$$

where

(3.3)

$$I_1=\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell$$
$$=\frac{1}{\Theta(\mu_2,\mu_1)2^{\frac{\alpha}{\kappa}-1}}\left[\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)+\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]$$
$$-\frac{2\alpha}{k\Theta(\mu_2,\mu_1)}\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}-1}$$
$$\times\left[\mathcal{Z}\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)+\mathcal{Z}\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell.$$

And

(3.4)
$$I_2=\int_{\frac{1}{2}}^{1}\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.$$
$$\left.-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell$$
$$=\frac{2}{\Theta(\mu_2,\mu_1)}\left[-\frac{2}{3}\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)+\left(\frac{4}{3}-\left(\frac{1}{2}\right)^{\frac{\alpha}{\kappa}}\right)\right.$$
$$\left.\times\left[\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)+\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]\right]$$
$$-\frac{2\alpha}{k\Theta(\mu_2,\mu_1)}\int_{\frac{1}{2}}^{1}\ell^{\frac{\alpha}{\kappa}-1}\left[\mathcal{Z}\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.$$
$$\left.+\mathcal{Z}\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell.$$

Adding (3.3) and (3.4), we have

(3.5)

$$I_1+I_2=\frac{2}{\Theta(\mu_2,\mu_1)}\left[-\frac{2}{3}\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.$$
$$\left.+\frac{4}{3}\left[\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)+\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]\right]$$

$$- \frac{2\alpha}{k\Theta(\mu_2,\mu_1)} \int_0^1 \ell^{\frac{\alpha}{\kappa}-1} \left[ \mathcal{Z}\left(\mu_1 + \frac{\ell}{2}\Theta(\mu_2,\mu_1)\right) + \mathcal{Z}\left(\mu_1 + \frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right] \mathrm{d}\ell.$$

Subtracting $\left(\frac{2}{\Theta(\mu_2,\mu_1)}\right)^{\frac{\alpha}{\kappa}+1} \Gamma_\kappa(\alpha)(1-\alpha)(\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1)))$ in above (3.5), then

(3.6)

$$\begin{aligned}
&I_1 + I_2 - \left(\frac{2}{\Theta(\mu_2,\mu_1)}\right)^{\frac{\alpha}{\kappa}+1} \Gamma_\kappa(\alpha)(1-\alpha)(\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1))) \\
&= \frac{2}{\Theta(\mu_2,\mu_1)} \left[ -\frac{2}{3}\mathcal{Z}\left(\frac{2\mu_1 + \Theta(\mu_2,\mu_1)}{2}\right) + \frac{4}{3}\left[\mathcal{Z}\left(\frac{4\mu_1 + \Theta(\mu_2,\mu_1)}{4}\right) + \mathcal{Z}\left(\frac{4\mu_1 + 3\Theta(\mu_2,\mu_1)}{4}\right)\right]\right] - \frac{2\alpha}{k\Theta(\mu_2,\mu_1)} \int_0^1 \ell^{\frac{\alpha}{\kappa}-1} \\
&\quad \times \left[\mathcal{Z}\left(\mu_1 + \frac{\ell}{2}\Theta(\mu_2,\mu_1)\right) + \mathcal{Z}\left(\mu_1 + \frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right] \mathrm{d}\ell \\
&\quad - \left(\frac{2}{\Theta(\mu_2,\mu_1)}\right)^{\frac{\alpha}{\kappa}+1} \Gamma_\kappa(\alpha)(1-\alpha)(\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1))) \\
&= \frac{2}{\Theta(\mu_2,\mu_1)} \left[ -\frac{2}{3}\mathcal{Z}\left(\frac{2\mu_1 + \Theta(\mu_2,\mu_1)}{2}\right) + \frac{4}{3}\left[\mathcal{Z}\left(\frac{4\mu_1 + \Theta(\mu_2,\mu_1)}{4}\right) + \mathcal{Z}\left(\frac{4\mu_1 + 3\Theta(\mu_2,\mu_1)}{4}\right)\right]\right] - \left(\frac{2}{\Theta(\mu_2,\mu_1)}\right)^{\frac{\alpha}{\kappa}+1} \Gamma_\kappa(\alpha)B(\alpha) \\
&\quad \times \left[\frac{\alpha}{k\Gamma_\kappa(\alpha)B(\alpha)} \int_{\mu_1}^{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}} (u-\mu_1)^{\frac{\alpha}{\kappa}-1} \mathcal{Z}(u)\mathrm{d}u \right. \\
&\quad + \frac{\alpha}{k\Gamma_\kappa(\alpha)B(\alpha)} \int_{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}}^{\mu_1+\Theta(\mu_2,\mu_1)} (\mu_1 + \Theta(\mu_2,\mu_1) - u)^{\frac{\alpha}{\kappa}-1} \mathcal{Z}(u)\mathrm{d}u \\
&\quad \left. + \frac{1-\alpha}{B(\alpha)}(\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1)))\right] \\
&= \frac{2}{\Theta(\mu_2,\mu_1)} \left[ -\frac{2}{3}\mathcal{Z}\left(\frac{2\mu_1 + \Theta(\mu_2,\mu_1)}{2}\right) + \frac{4}{3}\left[\mathcal{Z}\left(\frac{4\mu_1 + \Theta(\mu_2,\mu_1)}{4}\right) + \mathcal{Z}\left(\frac{4\mu_1 + 3\Theta(\mu_2,\mu_1)}{4}\right)\right]\right] - \left(\frac{2}{\Theta(\mu_2,\mu_1)}\right)^{\frac{\alpha}{\kappa}+1} \Gamma_\kappa(\alpha)B(\alpha) \\
&\quad \times \left[ {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-} \mathcal{Z}(\mu_1) + {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+} \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1))\right].
\end{aligned}$$

From (3.2) and (3.6), we achieve our intend result. □

**Corollary 3.2.** *If we take $\Theta(\mu_2,\mu_1)=\mu_2-\mu_1$ in (3.1), we have the following Milne's type inequality*

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{Z}(\mu_1)-\mathcal{Z}\left(\frac{\mu_1+\mu_2}{2}\right)+2\mathcal{Z}(\mu_2)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\quad\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_2)]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\mu_2-\mu_1)^{\frac{\alpha}{\kappa}}}\\
&\quad\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}\mathcal{Z}(\mu_1)+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}\mathcal{Z}(\mu_2)\right]\\
&=\frac{\mu_2-\mu_1}{4}\left[\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1+\frac{\ell}{2}\mu_2\right)-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right]\mathrm{d}\ell\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1+\frac{\ell}{2}\mu_2\right)-\mathcal{Z}'\left(\frac{2-\ell}{2}\mu_1+\frac{\ell}{2}\mu_2\right)\right]\mathrm{d}\ell\right].
\end{aligned}$$

**Corollary 3.3.** *If we take $\alpha=k,k=1$ in (3.1), then we have the following Milne's type inequality*

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\\
&\quad\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]-\frac{1}{\Theta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)}\mathcal{Z}(x)dx\\
&=\frac{\Theta(\mu_2,\mu_1)}{4}\left[\int_0^{\frac{1}{2}}\ell\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.\right.\\
&\quad\left.\left.-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell+\int_{\frac{1}{2}}^1\left(\ell-\frac{4}{3}\right)\right.\\
&\quad\left.\times\left[\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right]\mathrm{d}\ell\right].
\end{aligned}$$

**Theorem 3.4.** *Presume that all the hypothesises of Lemma 3.1 hold and $|\mathcal{Z}'|$ is a preinvex function on $[\mu_1,\mu_{1\Theta}(\mu_2,\mu_1)]$. Then,*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.\\
&\quad\left.\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.\\
&\quad\left.\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.
\end{aligned}\tag{3.7}$$

$$\times \left[ {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1) + {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1)) \right] \Bigg|$$
$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left(\Xi_1(\alpha)+\Xi_2(\alpha)\right)\left[\left|\mathcal{Z}'(\mu_1)\right|+\left|\mathcal{Z}'(\mu_2)\right|\right].$$

*Here,*

$$\begin{cases} \Xi_1(\alpha) = \frac{\kappa}{(\alpha+k)2^{\frac{\alpha}{\kappa}+1}}. \\ \\ \Xi_2(\alpha) = \frac{\kappa}{\alpha+k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}+1}}-1\right)+\frac{2}{3}. \end{cases}$$

*Proof.* From Lemma 3.1 and preinvexity of $|\mathcal{Z}'|$, then

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right. \\
&\quad \left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\
&\quad \times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\
&\quad \times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right. \\
&\quad \left.\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right| \\
&\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left[\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\left|\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.\right. \\
&\quad \left.-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right|\mathrm{d}\ell+\int_{\frac{1}{2}}^1\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right| \\
&\quad \left.\times\left|\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right|\mathrm{d}\ell\right] \\
&\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left\{\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\left[\left(\frac{2-\ell}{2}\right)\left|\mathcal{Z}'(\mu_1)\right|+\frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|\right.\right. \\
&\quad \left.+\left(\frac{2-\ell}{2}\right)\left|\mathcal{Z}'(\mu_2)\right|+\frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|\right]\mathrm{d}\ell+\int_{\frac{1}{2}}^1\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\left[\left(\frac{2-\ell}{2}\right)\right.
\end{aligned}$$

$$\times \left|\mathcal{Z}'(\mu_1)\right| + \frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right| + \left(\frac{2-\ell}{2}\right)\left|\mathcal{Z}'(\mu_2)\right| + \frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|\Big] \mathrm{d}\ell\Big\}$$
$$= \frac{\Theta(\mu_2,\mu_1)}{4}\left(\Xi_1(\alpha)+\Xi_2(\alpha)\right)\left[\left|\mathcal{Z}'(\mu_1)\right| + \left|\mathcal{Z}'(\mu_2)\right|\right].$$

Hence, the desired result is achieved. □

**Corollary 3.5.** *Selecting $\Theta(\mu_2,\mu_1) = \mu_2 - \mu_1$ in (3.7), then*

$$\left|\frac{1}{3}\left[2\mathcal{Z}(\mu_1) - \mathcal{Z}\left(\frac{\mu_1+\mu_2}{2}\right) + 2\mathcal{Z}(\mu_2)\right] + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_2)] - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\mu_2-\mu_1)^{\frac{\alpha}{\kappa}}}$$
$$\left.\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}\mathcal{Z}(\mu_1) + {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}\mathcal{Z}(\mu_2)\right]\right|$$
$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left(\Xi_1(\alpha)+\Xi_2(\alpha)\right)\left[\left|\mathcal{Z}'(\mu_1)\right| + \left|\mathcal{Z}'(\mu_2)\right|\right],$$

*where $\Xi_1(\alpha)$ and $\Xi_2(\alpha)$ are defined in theorem 3.4.*

**Corollary 3.6.** *Selecting $\alpha = k = 1$ in (3.1), then we have the following Milne's type inequality*

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right) - \mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right] - \frac{1}{\Theta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)}\mathcal{Z}(x)dx\right|$$
$$\leq \frac{\Theta(\mu_2,\mu_1)}{12}\left[\left|\mathcal{Z}'(\mu_1)\right| + \left|\mathcal{Z}'(\mu_2)\right|\right].$$

**Theorem 3.7.** *Presume that all the hypothesises of Lemma 3.1 hold. If $|\mathcal{Z}'|^q$ is a convex on $[\mu_1, \mu_1+\Theta(\mu_2,\mu_1)]$ where $q>1$, then we achieve*

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right) - \mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right] + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))] - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\left.\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1) + {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$

$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left\{\left(\frac{\kappa}{\alpha p+k}\left(\frac{1}{2}\right)^{\frac{\alpha p+k}{\kappa}}\right)^{\frac{1}{p}}\left[\left(\frac{7\left|\mathcal{Z}'(\mu_1)\right|^q+\left|\mathcal{Z}'(\mu_2)\right|^q}{16}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{\left|\mathcal{Z}'(\mu_1)\right|^q+7\left|\mathcal{Z}'(\mu_2)\right|^q}{16}\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|^p\mathrm{d}\ell\right)^{\frac{1}{p}}$$
$$\left.\times\left[\left(\frac{5\left|\mathcal{Z}'(\mu_1)\right|^q+3\left|\mathcal{Z}'(\mu_2)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{Z}'(\mu_1)\right|^q+5\left|\mathcal{Z}'(\mu_2)\right|^q}{16}\right)^{\frac{1}{q}}\right]\right\},$$

*where* $p^{-1}+q^{-1}=1$.

*Proof.* Applying Hölder's inequality on Lemma 3.1 and by utilizing preinvexity of $|\mathcal{Z}|^q$,

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\Bigg]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\left.\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$
$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left[\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\left|\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)\right.\right.$$
$$\left.-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right|\mathrm{d}\ell+\int_{\frac{1}{2}}^{1}\left|\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\right|$$
$$\left.\times\left|\mathcal{Z}'\left(\mu_1+\frac{\ell}{2}\Theta(\mu_2,\mu_1)\right)-\mathcal{Z}'\left(\mu_1+\frac{2-\ell}{2}\Theta(\mu_2,\mu_1)\right)\right|\mathrm{d}\ell\right]$$
$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left[\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|^p\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|^p\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|^p\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}\right.$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|^{q}\mathrm{d}\ell\right)^{\frac{1}{q}}\Bigg]\\
&\leq\frac{\Theta(\mu_2,\mu_1)}{4}\Bigg[\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\Bigg]\\
&=\frac{\Theta(\mu_2,\mu_1)}{4}\Bigg\{\left(\frac{\kappa}{\alpha p+k}\left(\frac{1}{2}\right)^{\frac{\alpha p+k}{\kappa}}\right)^{\frac{1}{p}}\Bigg[\left(\frac{7\left|\mathcal{Z}'(\mu_1)\right|^{q}+\left|\mathcal{Z}'(\mu_2)\right|^{q}}{16}\right)^{\frac{1}{q}}\\
&+\left(\frac{\left|\mathcal{Z}'(\mu_1)\right|^{q}+7\left|\mathcal{Z}'(\mu_2)\right|^{q}}{16}\right)^{\frac{1}{q}}\Bigg]+\left(\int_{\frac{1}{2}}^{1}\left|\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right|^{p}\mathrm{d}\ell\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{5\left|\mathcal{Z}'(\mu_1)\right|^{q}+3\left|\mathcal{Z}'(\mu_2)\right|^{q}}{16}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{Z}'(\mu_1)\right|^{q}+5\left|\mathcal{Z}'(\mu_2)\right|^{q}}{16}\right)^{\frac{1}{q}}\right]\Bigg\}.
\end{aligned}
$$

Hence, the required result is obtained. □

**Remark 3.8.** Making different substitutions for $\alpha,\kappa$ and $\eta$ in Theorem 3.7, we generate blend of new inequalities.

**Theorem 3.9.** *Assume that all the hypothesises of the Lemma 3.1 hold. If $|\mathcal{Z}'|^{q}$ is preinvex function on $[\mu_1,\mu_1+\Theta(\mu_2,\mu_1)]$ where $q>1$, then*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.\\
&\quad+2\mathcal{Z}\left.\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_{\kappa}(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\quad\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_{\kappa}(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\quad\times\left.\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^{-}}\mathcal{Z}(\mu_1)+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^{+}}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|
\end{aligned}
$$

$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}\left\{(\Xi_1(\alpha))^{1-\frac{1}{q}}\left[(\Xi_3(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^q+\Xi_4(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^q)^{\frac{1}{q}}\right.\right.$$
$$+(\Xi_3(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^q+\Xi_4(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^q)^{\frac{1}{q}}\Big]+(\Xi_2(\alpha))^{1-\frac{1}{q}}$$
$$\times\left[(\Xi_5(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^q+\Xi_6(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^q)^{\frac{1}{q}}\right.$$
$$\left.\left.+(\Xi_5(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^q+\Xi_6(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^q)^{\frac{1}{q}}\right]\right\}.$$

*Here* $\Xi_1(\alpha)$ *and* $\Xi_2(\alpha)$ *are defined in Theorem 3.4 and*

$$\begin{cases}\Xi_3(\alpha)=\int_0^{\frac{1}{2}}\frac{\ell^{\frac{\alpha}{\kappa}+1}}{2}\mathrm{d}\ell=\frac{\kappa}{\alpha+2k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}+3}}\right),\\ \Xi_4(\alpha)=\int_0^{\frac{1}{2}}\left(\frac{2-\ell}{2}\right)\ell^{\frac{\alpha}{\kappa}}\mathrm{d}\ell=\frac{\kappa}{(\alpha+k)(\alpha+2k)}\left(\frac{3\alpha+7k}{2^{\frac{\alpha}{\kappa}+3}}\right),\\ \Xi_5(\alpha)=\int_{\frac{1}{2}}^1\frac{\ell}{2}\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\mathrm{d}\ell=\frac{\kappa}{\alpha+2k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}+3}}-\frac{1}{2}\right)+\frac{1}{4},\\ \Xi_6(\alpha)=\int_{\frac{1}{2}}^1\left(\frac{2-\ell}{2}\right)\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\mathrm{d}\ell=\frac{\kappa}{(\alpha+k)(\alpha+2k)}\left(\frac{3\alpha+7k}{2^{\frac{\alpha}{\kappa}+3}}-\frac{\alpha+3k}{2}\right)+\frac{5}{12}.\end{cases}$$

*Proof.* From lemma 3.1, power mean inequality and applying the preinvexity of $|\mathcal{Z}'|^q$. Then,

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\Big]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\left.\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$
$$\leq\frac{\Theta(\mu_2,\mu_1)}{4}\left[\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{2}}^1\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)\right|^q\mathrm{d}\ell\right)^{\frac{1}{q}}\right.$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|^{q}\mathrm{d}\ell\right)^{\frac{1}{q}}\Bigg]\\
&\leq\frac{\Theta(\mu_2,\mu_1)}{4}\\
&\times\Bigg[\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}|\ell^{\frac{\alpha}{\kappa}}|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\mathrm{d}\ell\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\left(\frac{\ell}{2}\left|\mathcal{Z}'(\mu_1)\right|^{q}+\frac{2-\ell}{2}\left|\mathcal{Z}'(\mu_2)\right|^{q}\right)\mathrm{d}\ell\right)^{\frac{1}{q}}\Bigg]\\
&=\frac{\Theta(\mu_2,\mu_1)}{4}\Big\{(\Xi_1(\alpha))^{1-\frac{1}{q}}\Big[(\Xi_3(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^{q}+\Xi_4(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^{q})^{\frac{1}{q}}\\
&+(\Xi_3(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^{q}+\Xi_4(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^{q})^{\frac{1}{q}}\Big]+(\Xi_2(\alpha))^{1-\frac{1}{q}}\\
&\times\Big[(\Xi_5(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^{q}+\Xi_6(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^{q})^{\frac{1}{q}}\\
&+(\Xi_5(\alpha)\left|\mathcal{Z}'(\mu_1)\right|^{q}+\Xi_6(\alpha)\left|\mathcal{Z}'(\mu_2)\right|^{q})^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}
$$

Hence the desired result is acquired. □

**Theorem 3.10.** *Assume that all the hypothesise of Lemma 3.1 hold. If* $m\leq\mathcal{Z}'(\ell)\leq M$ *for* $\ell\in[\mu_1,\mu_1+\Theta(\mu_2,\mu_1)]$*, then*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.\\
&\qquad\left.\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.
\end{aligned}
$$

$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.$$
$$\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\Bigg|$$
$$\leq\frac{\Theta(\mu_2,\mu_1)}{4}\left\{\frac{\kappa}{\alpha+k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}}}-1\right)+\frac{2}{3}\right\}(M-m).$$

*Proof.* By using the Lemma 3.1, we have

$$\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.$$
$$\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.$$
$$\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]$$
$$=\frac{\Theta(\mu_2,\mu_1)}{4}\left\{\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\frac{m+M}{2}\right]\mathrm{d}\ell\right.$$
$$+\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left[\frac{m+M}{2}-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right]\mathrm{d}\ell$$
$$+\int_{\frac{1}{2}}^1\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\frac{m+M}{2}\right]\mathrm{d}\ell$$
$$\left.+\int_{\frac{1}{2}}^1\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\frac{m+M}{2}-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right]\mathrm{d}\ell\right\}.$$

Implementing the absolute values on both sides, we acquire

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\right.$$

$$
\begin{aligned}
&\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.\\
&\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\Bigg|\\
\leq\ &\frac{\Theta(\mu_2,\mu_1)}{4}\left\{\int_0^{\frac{1}{2}}\left|\ell^{\frac{\alpha}{\kappa}}\left[\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\frac{m+M}{2}\right]\right|\mathrm{d}\ell\right.\\
&+\int_0^{\frac{1}{2}}\left|\ell^{\frac{\alpha}{\kappa}}\left[\frac{m+M}{2}-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right]\right|\mathrm{d}\ell\\
&+\int_{\frac{1}{2}}^1\left|\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\frac{m+M}{2}\right]\right|\mathrm{d}\ell\\
&\left.+\int_{\frac{1}{2}}^1\left|\left(\ell^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right)\left[\frac{m+M}{2}-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right]\right|\mathrm{d}\ell\right\}.
\end{aligned}
$$

It is known that $m\leq\mathcal{Z}'(\ell)\leq M$ for $\ell\in[\mu_1,\mu_1+\Theta(\mu_2,\mu_1)]$. Thus, we get

$$
\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2} \tag{3.8}
$$

and

$$
\left|\frac{m+M}{2}-\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)\right|\leq\frac{M-m}{2}. \tag{3.9}
$$

If we assume (3.8) and (3.10), then we get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.\\
&\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\\
&\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.\\
&\left.\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|
\end{aligned}
$$

$$\leq \frac{\Theta(\mu_2,\mu_1)}{4}(M-m)\left\{\int_0^{\frac{1}{2}} \ell^{\frac{\alpha}{\kappa}}\mathrm{d}\ell + \int_{\frac{1}{2}}^1 \left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\mathrm{d}\ell\right\}$$
$$= \frac{\Theta(\mu_2,\mu_1)}{4}\left\{\frac{\kappa}{\alpha+k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}}}-1\right)+\frac{2}{3}\right\}(M-m).$$

The proof of the theorem is accomplished. □

**Theorem 3.11.** *Presume that all the hypotheses of Lemma 3.1 hold. If $\mathcal{Z}'$ is an L-Lipschitzian function on $[\mu_1, \mu_1+\Theta(\mu_2,\mu_1)]$, then*

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.$$
$$\left.\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$
$$\leq \frac{(\Theta(\mu_2,\mu_1))^2}{4}L\left\{\frac{1}{6}+\frac{\kappa}{(\alpha+k)(\alpha+2k)}\left(\frac{\alpha+3k}{2^{\frac{\alpha}{\kappa}+1}}-k\right)\right\}.$$

*Proof.* Using the fact that $\mathcal{Z}'$ is an $L-$Lipschitzian function on Lemma 3.1, we have

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)\right.$$
$$\left.\left.+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$
$$\leq \frac{(\Theta(\mu_2,\mu_1))^2}{4}$$

$$\times\left\{\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|\mathrm{d}\ell\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)\left|\mathcal{Z}'\left(\frac{\ell}{2}\mu_2+\frac{2-\ell}{2}\mu_1\right)-\mathcal{Z}'\left(\frac{\ell}{2}\mu_1+\frac{2-\ell}{2}\mu_2\right)\right|\mathrm{d}\ell\right\}$$
$$\leq\frac{(\Theta(\mu_2,\mu_1))^2}{4}\left\{\int_0^{\frac{1}{2}}\ell^{\frac{\alpha}{\kappa}}L(1-\ell)\Theta(\mu_2,\mu_1)\mathrm{d}\ell\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left(\frac{4}{3}-\ell^{\frac{\alpha}{\kappa}}\right)L(1-\ell)\Theta(\mu_2,\mu_1)\mathrm{d}\ell\right\}$$
$$=\frac{(\Theta(\mu_2,\mu_1))^2}{4}L\left\{\frac{1}{6}+\frac{\kappa}{(\alpha+k)(\alpha+2k)}\left(\frac{\alpha+3k}{2^{\frac{\alpha}{\kappa}+1}}-k\right)\right\}.$$

This completes the proof. □

**Theorem 3.12.** *Consider $\mathcal{Z}:[\mu_1,\mu_2]\to\mathbb{R}$ be a function of bounded variation on $[\mu_1,\mu_1+\Theta(\mu_2,\mu_1)]$. Then, we obtain*

$$\left|\frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.$$
$$\left.+2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]+\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\times[\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))]-\frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}$$
$$\left.\times\left[{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\mathcal{Z}(\mu_1)+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]\right|$$
$$\leq\frac{1}{2}\max\left\{\left(\frac{1}{2}\right)^{\frac{\alpha}{\kappa}},\left|(1)^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|,\left|\left(\frac{1}{2}\right)^{\frac{\alpha}{\kappa}}-\frac{4}{3}\right|\right\}\bigvee_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)}(\mathcal{Z}).$$

*Here $\bigvee_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)}(\mathcal{Z})$ denotes the total variation of $\mathcal{Z}$ on $[\mu_1,\mu_1+\Theta(\mu_2,\mu_1)]$.*

*Proof.* Define the function $\Upsilon^k_\alpha(u)$ by

$\Upsilon^k_\alpha(u)$

$$= \begin{cases} \frac{2}{\Theta(\mu_2,\mu_1)}\left(\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}}, & \mu_1 \le u < \frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}, \\ \frac{2}{\Theta(\mu_2,\mu_1)}\left(\left(\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right), & \frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4} \le u < \frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}, \\ \frac{2}{\Theta(\mu_2,\mu_1)}\left(\left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right), & \frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2} \le u < \frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}, \\ \frac{2}{\Theta(\mu_2,\mu_1)}\left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}}, & \frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4} \le u \le \mu_1+\Theta(\mu_2,\mu_1). \end{cases}$$

From the definition of $\Upsilon_\alpha^k(u)$, we achieve

$$\begin{aligned} &\int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} \Upsilon_\alpha^k(u)\mathrm{d}\mathcal{Z}(u) \\ &= \frac{2}{\Theta(\mu_2,\mu_1)}\left[\int_{\mu_1}^{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}} \left(\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} \mathrm{d}\mathcal{Z}(u)\right. \\ &\quad + \int_{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}}^{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}} \left(\left(\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right)\mathrm{d}\mathcal{Z}(u) \\ &\quad - \int_{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}}^{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}} \left(\left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right)\mathrm{d}\mathcal{Z}(u) \\ &\quad \left. - \int_{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}}^{\mu_1+\Theta(\mu_2,\mu_1)} \left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} \mathrm{d}\mathcal{Z}(u)\right]. \end{aligned}$$

By applying the integration principles, we transform the above equation as:

(3.10)

$$\begin{aligned} &\frac{\Theta(\mu_2,\mu_1)}{4}\int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} \Upsilon_\alpha^k(u)\mathrm{d}\mathcal{Z}(u) \\ &= \frac{1}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right) - \mathcal{Z}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right. \\ &\quad \left. +2\mathcal{Z}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right] + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\ &\quad \times [\mathcal{Z}(\mu_1)+\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))] - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\ &\quad \times \left[{}_\kappa^{AB}I_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}^{\alpha}\mathcal{Z}(\mu_1) + {}_\kappa^{AB}I_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}^{\alpha}\mathcal{Z}(\mu_1+\Theta(\mu_2,\mu_1))\right]. \end{aligned}$$

It is known that if $g, \mathcal{Z} : [\mu_1, \mu_2] \to \mathbb{R}$ are such that $g$ is continuous on $[\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)]$ and $\mathcal{Z}$ is of bounded variation on $[\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)]$, then $\int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} \mathcal{Z} g(\ell) \mathrm{d}\ell$ exists and

$$\left| \int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} g(\ell) \mathrm{d}\mathcal{Z}(\ell) \right| \leq \sup_{\ell \in [\mu_1, \mu_1+\Theta(\mu_2,\mu_1)]} |g(\ell)| \bigvee_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} \mathcal{Z}. \tag{3.11}$$

Using (3.11) in (3.10), we have

$$\begin{aligned}
&\left| \frac{1}{3} \left[ 2\mathcal{Z}\left( \frac{4\mu_1 + \Theta(\mu_2,\mu_1)}{4} \right) - \mathcal{Z}\left( \frac{2\mu_1 + \Theta(\mu_2,\mu_1)}{2} \right) \right.\right. \\
&\quad \left. + 2\mathcal{Z}\left( \frac{4\mu_1 + 3\Theta(\mu_2,\mu_1)}{4} \right) \right] + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\
&\quad \times [\mathcal{Z}(\mu_1) + \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1))] - \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha) B(\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}} \\
&\quad \left. \times \left[ {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-} \mathcal{Z}(\mu_1) + {}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+} \mathcal{Z}(\mu_1 + \Theta(\mu_2,\mu_1)) \right] \right| \\
&= \frac{\Theta(\mu_2,\mu_1)}{4} \left| \int_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} \Upsilon^k_\alpha(u) \mathrm{d}\mathcal{Z}(u) \right| \\
&\leq \frac{1}{2} \left[ \left| \int_{\mu_1}^{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}} \left( \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} \mathrm{d}\mathcal{Z}(u) \right| \right. \\
&\quad + \left| \int_{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}}^{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}} \left( \left( \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} - \frac{4}{3} \right) \mathrm{d}\mathcal{Z}(u) \right| \\
&\quad + \left| \int_{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}}^{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}} \left( \left( 2 - \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} - \frac{4}{3} \right) \mathrm{d}\mathcal{Z}(u) \right| \\
&\quad \left. + \left| \int_{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}}^{\mu_1+\Theta(\mu_2,\mu_1)} \left( 2 - \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} \mathrm{d}\mathcal{Z}(u) \right| \right] \\
&\leq \frac{1}{2} \left\{ \sup_{u \in \left[\mu_1, \frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right]} \left| \left( \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} \right| \bigvee_{\mu_1}^{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}} (\mathcal{Z}) \right. \\
&\quad \left. + \sup_{u \in \left[\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}, \frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right]} \left| \left( \frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1) \right)^{\frac{\alpha}{\kappa}} - \frac{4}{3} \right| \right.
\end{aligned}$$

$$\times \bigvee_{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}}^{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}} (\mathcal{Z}) + \sup_{u\in\left[\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2},\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right]} \times \left|\left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right| \bigvee_{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}}^{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}} (\mathcal{Z}) + \sup_{u\in\left[\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4},\mu_1+\Theta(\mu_2,\mu_1)\right]} \left|\left(2-\frac{2}{\Theta(\mu_2,\mu_1)}(u-\mu_1)\right)^{\frac{\alpha}{\kappa}}\right| \times \bigvee_{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}}^{\mu_1+\Theta(\mu_2,\mu_1)} (\mathcal{Z}) \Bigg\}$$

$$\leq \frac{1}{2}\max\left\{\left(\frac{1}{2}\right)^{\frac{\alpha}{\kappa}}, \left|(1)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right|, \left|\left(\frac{1}{2}\right)^{\frac{\alpha}{\kappa}} - \frac{4}{3}\right|\right\} \bigvee_{\mu_1}^{\mu_1+\Theta(\mu_2,\mu_1)} (\mathcal{Z}).$$

The proof is completed. □

## 4. Visual Demonstration

In the current part of the investigation, we present some visuals to testify of our primary results, which provide the bounds for Milne's rule.

First, we demonstrate a graphical explanation of Theorem 3.4.

**Example 4.1.** For $\mathcal{Z}(x) = \frac{x^3}{3}$, $\Theta(\mu_2,\mu_1) = \mu_2 - \mu_1$ and $[\mu_1,\mu_2] = [0,2]$, then Theorem 3.4 can be written as:

- $$m_1 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} - \frac{4\left(\frac{\frac{1}{2^\alpha}-1}{\alpha+1}+\frac{2}{3}\right)}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)},$$
- $$m_2 = \frac{\alpha}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{3(\alpha+3)} + \frac{2}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{1}{3\alpha} + \frac{1}{\alpha(\alpha+1)}\right),$$
- $$m_3 = \frac{4\left(\frac{\frac{1}{2^\alpha}-1}{\alpha+1}+\frac{2}{3}\right)}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} + \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}.$$

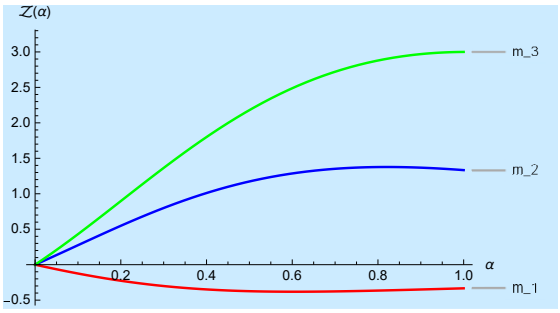



FIGURE 1. Graphical validations of Theorem 3.4

TABLE 1. Numerical comparison of Theorem 3.4

| $\alpha$ | $m_1(\alpha)$ | $m_2(\alpha)$ | $m_3(\alpha)$ |
|---|---|---|---|
| 0.2 | $-0.2267$ | 0.5484 | 0.8970 |
| 0.4 | $-0.3490$ | 1.0105 | 1.8000 |
| 0.6 | $-0.3814$ | 1.2863 | 2.4880 |
| 0.8 | $-0.3653$ | 1.3767 | 2.8792 |

The Figure 1 and computational values given in Table 1 provide the accuracy of Theorem 3.4 and bounds of mean-value integral.

**Example 4.2.** Now we check the Theorem 3.4 for $f(x) = -|x|$ defined over $[\mu_2, \mu_1] = [-2, 0]$, and $\Theta$ is defined in Example 2.4 and get the following numerical value i.e., $0 < 0.2083$.

**Example 4.3.** For $\mathcal{Z}(x) = \frac{x^3}{3}$, $\Theta(\mu_2, \mu_1) = \mu_2 - \mu_1$ and $[\mu_1, \mu_2] = [0, 2]$, then Theorem 3.7 can be written as:

- $$m_1 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1} + \alpha\right)\Gamma(\alpha)} - \frac{\left(1+\sqrt{7}\right)\sqrt{\frac{1}{(2\alpha+1)2^{2\alpha+1}}} + \left(\sqrt{3}+\sqrt{5}\right)\sqrt{-\frac{8\left(1-\frac{1}{2^{\alpha+1}}\right)}{3(\alpha+1)} + \frac{1-\frac{1}{2^{2\alpha+1}}}{2\alpha+1} + \frac{8}{9}}}{\left(\frac{1-\alpha}{\alpha+1} + \alpha\right)\Gamma(\alpha)},$$
- $$m_2 = \frac{\alpha}{\left(\frac{1-\alpha}{\alpha+1} + \alpha\right)\Gamma(\alpha)} \times \left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{3(\alpha+3)} + \frac{2}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{1}{3\alpha} + \frac{1}{\alpha(\alpha+1)}\right),$$

- $$m_3 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} + \frac{(1+\sqrt{7})\sqrt{\frac{1}{(2\alpha+1)2^{2\alpha+1}}} + (\sqrt{3}+\sqrt{5})\sqrt{-\frac{8\left(1-\frac{1}{2^{\alpha+1}}\right)}{3(\alpha+1)} + \frac{1-\frac{1}{2^{2\alpha+1}}}{2\alpha+1} + \frac{8}{9}}}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}.$$

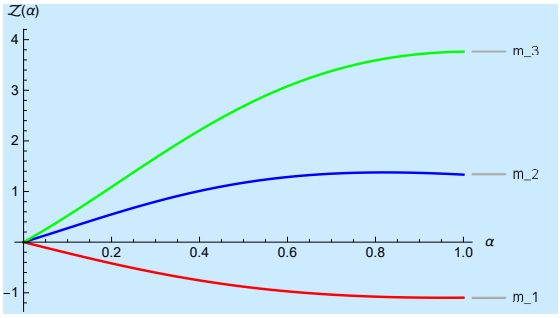



FIGURE 2. Graphical validations of Theorem 3.7

TABLE 2. Numerical comparison of Theorem 3.7

| $\alpha$ | $m_1(\alpha)$ | $m_2(\alpha)$ | $m_3(\alpha)$ |
|---|---|---|---|
| 0.2 | −0.4194 | 0.5484 | 1.0896 |
| 0.4 | −0.7561 | 1.0105 | 2.2070 |
| 0.6 | −0.9725 | 1.2863 | 3.0792 |
| 0.8 | −1.0769 | 1.3767 | 3.5908 |

The Figure 2 and computational values given in Table 2 provide the accuracy of Theorem 3.7 and bounds of mean-value integral.

**Example 4.4.** For $\mathcal{Z}(x) = \frac{x^3}{3}$, $\Theta(\mu_2, \mu_1) = \mu_2 - \mu_1$ and $[\mu_1, \mu_2] = [0, 2]$, then Theorem 3.9 can be written as:

- $$m_1 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} - \frac{1}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} \times \left[\left(\sqrt{\frac{16}{(\alpha+2)2^{\alpha+3}}} + \sqrt{\frac{16(3\alpha+7)}{((\alpha+1)(\alpha+2))2^{\alpha+3}}}\right)\sqrt{\frac{1}{(\alpha+1)2^{\alpha+1}}} + \sqrt{\frac{\frac{1}{2^{\alpha+1}}-1}{\alpha+1} + \frac{2}{3}}\left(\sqrt{\frac{16\left(\frac{1}{2^{\alpha+3}}-\frac{1}{2}\right)}{\alpha+2}+4} + \sqrt{\frac{16\left(\frac{3\alpha+7}{2^{\alpha+3}}-\frac{\alpha+3}{2}\right)}{(\alpha+1)(\alpha+2)} + \frac{20}{3}}\right)\right],$$

- $$m_2 = \frac{\alpha}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{3(\alpha+3)}\right.$$
$$\left.+\frac{2}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{3\alpha}+\frac{1}{\alpha(\alpha+1)}\right),$$

- $$m_3 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}+\frac{1}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}$$
$$\times\left[\left(\sqrt{\frac{16}{(\alpha+2)2^{\alpha+3}}}+\sqrt{\frac{16(3\alpha+7)}{((\alpha+1)(\alpha+2))2^{\alpha+3}}}\right)\sqrt{\frac{1}{(\alpha+1)2^{\alpha+1}}}\right.$$
$$\left.+\sqrt{\frac{\frac{1}{2^{\alpha+1}}-1}{\alpha+1}+\frac{2}{3}}\left(\sqrt{\frac{16\left(\frac{1}{2^{\alpha+3}}-\frac{1}{2}\right)}{\alpha+2}+4}+\sqrt{\frac{16\left(\frac{3\alpha+7}{2^{\alpha+3}}-\frac{\alpha+3}{2}\right)}{(\alpha+1)(\alpha+2)}+\frac{20}{3}}\right)\right].$$

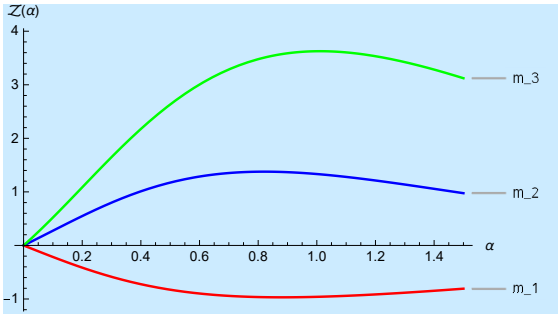



FIGURE 3. Graphical validations of Theorem 3.9

TABLE 3. Numerical comparison of Theorem 3.9

| $\alpha$ | $m_1(\alpha)$ | $m_2(\alpha)$ | $m_3(\alpha)$ |
|---|---|---|---|
| 0.2 | −0.4147 | 0.5484 | 1.0850 |
| 0.4 | −0.7239 | 1.0105 | 2.1748 |
| 0.6 | −0.8981 | 1.2863 | 3.0048 |
| 0.8 | −0.9639 | 1.3767 | 3.4779 |

The Figure 3 and computational numerical values given in Table 3 illustrate the accuracy of Theorem 3.9 and bounds of mean-value integrals.

**Example 4.5.** For $\mathcal{Z}(x) = \frac{x^3}{3}$, $\Theta(\mu_2, \mu_1) = \mu_2 - \mu_1$ and $[\mu_1, \mu_2] = [0, 2]$, then Theorem 3.10 can be written as:

- $$m_1 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} - \frac{2\left(\frac{\frac{1}{2^\alpha}-1}{\alpha+1}+\frac{2}{3}\right)}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)},$$
- $$m_2 = \frac{\alpha}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}\left(\frac{2}{\alpha(\alpha+1)(\alpha+2)}+\frac{1}{3(\alpha+3)} + \frac{2}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{3\alpha}+\frac{1}{\alpha(\alpha+1)}\right),$$
- $$m_3 = \frac{4}{3\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)} + \frac{2\left(\frac{\frac{1}{2^\alpha}-1}{\alpha+1}+\frac{2}{3}\right)}{\left(\frac{1-\alpha}{\alpha+1}+\alpha\right)\Gamma(\alpha)}.$$

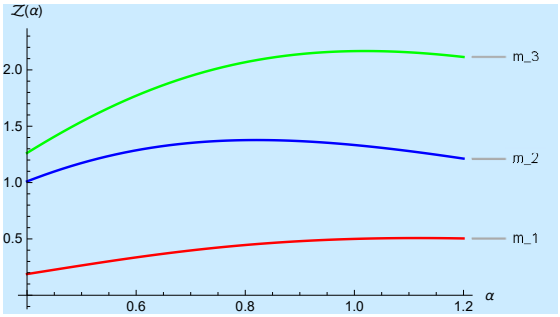



FIGURE 4. Graphical validations of Theorem 3.10

TABLE 4. Numerical comparison of Theorem 3.10

| $\alpha$ | $m_1(\alpha)$ | $m_2(\alpha)$ | $m_3(\alpha)$ |
|---|---|---|---|
| 0.2 | 0.0542 | 0.5484 | 0.6160 |
| 0.4 | 0.1882 | 1.0105 | 1.2627 |
| 0.6 | 0.3360 | 1.2863 | 1.7707 |
| 0.8 | 0.4459 | 1.3767 | 2.0681 |

The Figure 4 and computational numerical values given in Table 4 illustrate the accuracy of Theorem 3.10 and bounds of mean-value integrals.

## 5. Application

5.1. **Special Means.** We will now recapture special means for binary numbers and explain some fascinating relations between means.

$$A(\mu_1,\mu_2)=\frac{\mu_1+\mu_2}{2}.$$

$$_n(\mu_1,\mu_2)=\left[\frac{{\mu_2}^{n+1}-{\mu_1}^{n+1}}{(\mu_2-\mu_1)(n+1)}\right]^{\frac{1}{n}}.$$

**Proposition 5.1.** *Assume the assumption of theorem 3.4 are satisfied, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{1}{2^{n-1}}A^n(4\mu_1,\Theta(\mu_2,\mu_1))-A^n(2\mu_1,\Theta(\mu_2,\mu_1))\right.\right.\\&\qquad\left.+\frac{1}{2^{n-1}}A^n(4\mu_1,3\Theta(\mu_2,\mu_1))\right]\\&\qquad\left.-\left(\frac{\Theta(\mu_2,\mu_1)}{2}\right)^2L_n^n(\mu_1+\Theta(\mu_2,\mu_1),\mu_1)\right|\\&\leq\frac{5\Theta(\mu_2,\mu_1)}{24}A(na^{n-\mu_1},nb^{n-1}).\end{aligned}$$

*Proof.* By setting $\alpha=1,k=1$ and $\mathcal{Z}(x)=x^n$ in theorem 3.4, the proof becomes quite straightforward. □

**Proposition 5.2.** *Suppose that the assumption of theorem 3.7 are satisfied, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{1}{2^{n-1}}A^n(4\mu_1,\Theta(\mu_2,\mu_1))-A^n(2\mu_1,\Theta(\mu_2,\mu_1))\right.\right.\\&\qquad\left.\left.+\frac{1}{2^{n-1}}A^n(4\mu_1,3\Theta(\mu_2,\mu_1))\right]-\left(\frac{\Theta(\mu_2,\mu_1)}{2}\right)^2L_n^n(\mu_1+\Theta(\mu_2,\mu_1),\mu_1)\right|\\&\leq\frac{\Theta(\mu_2,\mu_1)}{4}\left\{\left(\frac{1}{p+1}\left(\frac{1}{2}\right)^{p+1}\right)^{\frac{1}{p}}\right.\\&\qquad\times\left[\left(\frac{1}{8}A(7|na^{n-1}|^q,|nb^{n-1}|^q)\right)^{\frac{1}{q}}+\left(\frac{1}{8}A(|na^{n-1}|^q,7|nb^{n-1}|^q)\right)^{\frac{1}{q}}\right]\\&\qquad+\left(\frac{1}{p+1}\left(\frac{5}{6}\right)^{p+1}-\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{8}A(5|na^{n-1}|^q,3|nb^{n-1}|^q)\right)^{\frac{1}{q}}\right.\\&\qquad\left.\left.+\left(\frac{1}{8}A(3|na^{n-1}|^q,5|nb^{n-1}|^q)\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

*Proof.* The proof follows directly by taking $\alpha = 1, k = 1$ and $\mathcal{Z}(x) = x^n$ in theorem 3.7. □

5.2. **Error bounds.** Now we deduce some error bounds of the Milne quadrature rule from our main findings. Let $\hat{S} : \mu_1 = x_0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots x_n = \mu_2$ be a partition of $[\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)]$ and $[x_i, x_{i+1}]$ be any arbitrary subset of $[\mu_1, \mu_1 + \Theta(\mu_2, \mu_1)]$. And we set $h = x_{i+1} - x_i$.

$$T(\hat{S}, \mathcal{Z}) = \frac{\Theta(\mu_2, \mu_1)}{3}\left[2\mathcal{Z}\left(\frac{4\mu_1 + \Theta(\mu_2, \mu_1)}{4}\right) - \mathcal{Z}\left(\frac{2\mu_1 + \Theta(\mu_2, \mu_1)}{2}\right) + 2\mathcal{Z}\left(\frac{4\mu_1 + 3\Theta(\mu_2, \mu_1)}{4}\right)\right]$$

and

$$\int_{\mu_1}^{\mu_2} \mathcal{Z}(x)\mathrm{d}x = T(\hat{S}, \mathcal{Z}) + \bar{R}(\hat{S}, \mathcal{Z}).$$

**Proposition 5.3.** *All assumption of Theorem 3.10 are satisfied then*

$$|\bar{R}(\hat{(}S), \mathcal{Z})| \leq \sum_{i=0}^{n} \frac{(\Theta(x_{i+1}, x_i))^2}{4}\left\{\frac{\kappa}{\alpha + k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}}} - 1\right) + \frac{2}{3}\right\}(M - m).$$

*Proof.* Applying Theorem 3.10 over $[x_i, x_{i+1}]$ and taking sum from 0 to $n$, we obtain our required result. □

Next, we explain the impacts of our created outcomes in matrices. In this setting, $\mathbb{C}^\delta$ indicate the collection of $\delta \times \delta$ complex matrices, $\mathbb{M}^\delta$ symbolises the algebra of all $\delta \times \delta$ complex matrices and $M^+\delta$ refers to the subset of $\mathbb{M}\delta$ comprised of strictly positive matrices. A matrix $\mathcal{A} \in \mathbb{M}^+\delta$ satisfies $\langle Aw, w\rangle > 0$ for every non-zero vector $\varpi \in \mathbb{C}^\delta$. The theory of matrices and convexity is consolidated by Sababheh in [28].

$$\mathcal{S}(\varpi) = \left\|\mathcal{A}^\varpi \mathcal{X}\mathcal{V}^{1-\varpi} + \mathcal{A}^{1-\varpi}\mathcal{X}\mathcal{V}^\varpi\right\|,$$

where $\mathcal{A}, B \in \mathbb{M}_\delta^+, \mathcal{X} \in \mathbb{M}_\delta$ is convex for all $\varpi \in [0, 1]$.

**Proposition 5.4.** *By using Theorem 3.10, we have*

$$\begin{aligned}
&\left|\frac{1}{3}\Bigg(2\left\|\mathcal{A}^{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}}\mathcal{X}\mathcal{V}^{\frac{4(1-\mu_1)-\Theta(\mu_2,\mu_1)}{4}} + \mathcal{A}^{\frac{4(1-\mu_1)+\Theta(\mu_2,\mu_1)}{4}}\mathcal{X}\mathcal{V}^{\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}}\right\|\right.\\
&\quad - \left\|\mathcal{A}^{\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}}\mathcal{X}\mathcal{V}^{\frac{2(1-\mu_1)-\Theta(\mu_2,\mu_1)}{2}} + \mathcal{A}^{\frac{2(1-\mu_1)-\Theta(\mu_2,\mu_1)}{2}}\mathcal{X}\mathcal{V}^{\frac{2(1-\mu_1)-\Theta(\mu_2,\mu_1)}{2}}\right\|\\
&\quad \left.+2\left\|\mathcal{A}^{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}}\mathcal{X}\mathcal{V}^{\frac{4(1-\mu_1)-3\Theta(\mu_2,\mu_1)}{4}} + \mathcal{A}^{\frac{4(1-\mu_1)+3\Theta(\mu_2,\mu_1)}{4}}\mathcal{X}\mathcal{V}^{\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}}\right\|\Bigg)\right.\\
&\quad + \frac{2^{\frac{\alpha}{\kappa}-1}\Gamma_k(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\Bigg(\left\|\mathcal{A}^{\mu_1}\mathcal{X}\mathcal{V}^{1-\mu_1} + \mathcal{A}^{1-\mu_1}\mathcal{X}\mathcal{V}^{\mu_1}\right\|\\
&\quad + \left\|\mathcal{A}^{\mu_1+\Theta(\mu_2,\mu_1)}\mathcal{X}\mathcal{V}^{1-(\mu_1+\Theta(\mu_2,\mu_1))} + \mathcal{A}^{1-(\mu_1+\Theta(\mu_2,\mu_1))}\mathcal{X}\mathcal{V}^{\mu_1+\Theta(\mu_2,\mu_1)}\right\|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2^{\frac{\alpha}{\kappa}}\Gamma_k(\alpha)(1-\alpha)}{(\Theta(\mu_2,\mu_1))^{\frac{\alpha}{\kappa}}}\Big({}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^-}\left\|\mathcal{A}^{\mu_1}\mathcal{X}\mathcal{V}^{1-\mu_1}+\mathcal{A}^{1-\mu_1}\mathcal{X}\mathcal{V}^{\mu_1}\right\|\\
&+{}^{AB}_{\kappa}I^{\alpha}_{\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)^+}\\
&\times\left\|\mathcal{A}^{\mu_1+\Theta(\mu_2,\mu_1)}\mathcal{X}\mathcal{V}^{1-(\mu_1+\Theta(\mu_2,\mu_1))}+\mathcal{A}^{1-(\mu_1+\Theta(\mu_2,\mu_1))}\mathcal{X}\mathcal{V}^{\mu_1+\Theta(\mu_2,\mu_1)}\right\|\Big)\Big|\\
&\leq\frac{\Theta(\mu_2,\mu_1)}{4}\left\{\frac{\kappa}{\alpha+k}\left(\frac{1}{2^{\frac{\alpha}{\kappa}}}-1\right)+\frac{2}{3}\right\}(M-m).
\end{aligned}
$$

5.3. **$\mathfrak{q}$-digamma Function.** First, we revisit the notion of the $\mathfrak{q}$-digamma function and its mathematical representations: Assume that $0<\mathfrak{q}<1$. The $\mathfrak{q}$-digamma function $\Upsilon_{\mathfrak{q}}(\mathfrak{u})$ (for further information, refer to [4]) can be expressed as

$$
\begin{aligned}
\Upsilon_{\mathfrak{q}}(\mathfrak{u})&=-\ln(1-\mathfrak{q})+\ln(\mathfrak{q})\sum_{i=0}^{\infty}\frac{\mathfrak{q}^{i+\mathfrak{u}}}{1-\mathfrak{q}^{i+\mathfrak{u}}}.\\
&=-\ln(1-\mathfrak{q})+\ln(\mathfrak{q})\sum_{i=0}^{\infty}\frac{\mathfrak{q}^{iu}}{1-\mathfrak{q}^{iu}}.
\end{aligned}
$$

If $\mathfrak{q}>1$ and $\mathfrak{u}>0$, the $\mathfrak{q}$-digamma function $\Upsilon_{\mathfrak{q}}$ can be represented as:

$$
\begin{aligned}
\Upsilon_{\mathfrak{q}}(\mathfrak{u})&=-\ln(\mathfrak{q}-1)+\ln(\mathfrak{q})\left[\mathfrak{u}-\frac{1}{2}-\sum_{i=0}^{\infty}\frac{\mathfrak{q}^{-(i+\mathfrak{u})}}{1-\mathfrak{q}^{-(i+\mathfrak{u})}}\right]\\
&=-\ln(\mathfrak{q}-1)+\ln(\mathfrak{q})\left[\mathfrak{u}-\frac{1}{2}-\sum_{i=0}^{\infty}\frac{\mathfrak{q}^{-iu}}{1-\mathfrak{q}^{-iu}}\right].
\end{aligned}
$$

The notion briefed above shows that for $\mathfrak{q}>0$, the function $\Upsilon'_{\mathfrak{q}}(\mathfrak{u})$ is completely monotonic on the interval $(0,\infty)$, which implies that it is a convex mapping. From these facts, we can formulate the following important findings concerning the $\mathfrak{q}$-digamma function.

**Proposition 5.5.** *Considering Theorem 3.4 with $\alpha=k$, and $k=1$, we obtain,*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\Upsilon'_{\mathfrak{q}}\left(\frac{4\mu_1+\Theta(\mu_2,\mu_1)}{4}\right)-\Upsilon'_{\mathfrak{q}}\left(\frac{2\mu_1+\Theta(\mu_2,\mu_1)}{2}\right)\right.\right.\\
&\qquad\left.\left.+2\Upsilon'_{\mathfrak{q}}\left(\frac{4\mu_1+3\Theta(\mu_2,\mu_1)}{4}\right)\right]-\frac{\Upsilon'_{\mathfrak{q}}(\mu_2)-\Upsilon'_{\mathfrak{q}}(\mu_1)}{\mu_2-\mu_1}\right|\\
&\leq\frac{\Theta(\mu_2,\mu_1)}{12}[|\Upsilon''_{\mathfrak{q}}(\mu_1)|+|\Upsilon''_{\mathfrak{q}}(\mu_2)|].
\end{aligned}
$$

*Proof.* By applying $\mathcal{Z}(\varkappa_1)\mapsto\Upsilon'_q(\varkappa_1)$, then desired result follows directly. □

## 6. Conclusion

In this manuscript, we have derived the novel versions of Milne's kinds of inequalities by leveraging the modified Atangana-Baleanu fractional operators and preinvex functions. The novelty of the proposed results is that under certain conditions, they coincide with classical counterparts, and from these results, one can calculate the bounds for non-convex functions. In the future, we will try to expand this idea to compute tight bounds involving convex functions and their generalizations. Also, we will establish fuzzy and quantum forms of Milne's inequality. We anticipate that these findings will play an important role in optimization theory and will create new insights for researchers. An interesting problem is how tight bounds can be developed.

**Acknowledgment.** The author wishes to thank the editor and reviewers for their valuable comments and suggestions for the improvement of this paper.

## References

1. M. Alomari, M. Darus and S.S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA RRC, 10 (1) (2007).
2. M. Alomari, *New error estimations for the Milne's quadrature formula in terms of at most first derivatives*, Konuralp J. Math., 1 (1) (2013), pp. 17–23.
3. M.W. Alomari and S.S. Dragomir, *Various error estimations for several Newton–Cotes quadrature formulae in terms of at most first derivative and applications in numerical integration*, Jordan J. Math. Stat., 7 (2) (2014), pp. 89–108.
4. R. Askey, *The q-gamma and q-beta functions*, Appl. Anal., 8 (2) (1978/79), pp. 125–141.
5. S.I. Butt, I. Javed, P. Agarwal and J.J. Nieto, *Newton–Simpson-type inequalities via majorization*, J. Inequal. Appl., 2023 (1) (2023), pp. 16.
6. B. Bin-Mohsin, M.Z. Javed, M.U. Awan, A.G. Khan, T. Cesarano and M.A. Noor, *Exploration of quantum Milne–Mercer-type inequalities with applications*, Symmetry, 15 (5) (2023), pp. 1096.
7. H. Budak, P. Kösem and H. Kara, *On new Milne-type inequalities for fractional integrals*, J. Inequal. Appl., 2023 (1) (2023), pp. 1–15.
8. Y.M. Chu, M.U. Awan, M.Z. Javed and A.G. Khan, *Bounds for the remainder in Simpson's inequality via n-polynomial convex functions of higher order using Katugampola fractional integrals*, J. Math., 2020 (2020), pp. 4189036.

9. S.S. Dragomir, R.P. Agarwal and P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl., 5 (2000), pp. 533–579.
10. S.S. Dragomir and R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., 11 (5) (1998), pp. 91–95.
11. F. Hezenci and H. Budak, *Maclaurin-type inequalities for Riemann–Liouville fractional integrals*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A, 76 (2) (2023), pp. 15–32.
12. W. Haider, H. Budak, A. Shehzadi, F. Hezenci and H. Chen, *Analysing Milne-type inequalities using tempered fractional integrals*, Anal. Math. Phys., 14 (5) (2024), pp. 101.
13. M. Iqbal, S. Qaisar and S. Hussain, *On Simpson's type inequalities utilizing fractional integrals*, J. Comput. Anal. Appl., 23 (6) (2017), pp. 1137–1145.
14. M.Z. Javed, M.U. Awan, B. Bin-Mohsin, H. Budak and S.S. Dragomir, *Some classical inequalities associated with generic identity and applications*, Axioms, 13 (8) (2024), pp. 533.
15. U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., 147 (1) (2004), pp. 137–146.
16. S. Kermausuor and E.R. Nwaeze, *New fractional integral inequalities via k-Atangana–Baleanu fractional integral operators*, Fractal Fract., 7 (10) (2023), pp. 740.
17. A. Kashuri and M.Z. Sarikaya, *Parameterized inequalities of different types for preinvex functions with respect to another function via generalized fractional integral operators and their applications*, Ukr. Math. J., 73 (9) (2022), pp. 1371–1397.
18. A. Lakhdari, H. Budak, M.U. Awan and B. Meftah, *Extension of Milne-type inequalities to Katugampola fractional integrals*, Bound. Value Probl., 2024 (1) (2024), pp. 1–16.
19. M.A. Latif, *Estimates of Hermite–Hadamard inequality for twice differentiable harmonically convex functions with applications*, J. Math., Punjab Univ., 50 (1) (2020).
20. B. Meftah, A. Souahi and M. Merad, *Some local fractional Maclaurin-type inequalities for generalized convex functions and their applications*, Chaos Solitons Fractals, 162 (2022), pp. 112504.
21. S. Mubeen and S. Iqbal, *Grüss-type integral inequalities for generalized Riemann–Liouville k-fractional integrals*, J. Inequal. Appl., 2016 (1) (2016), pp. 109.
22. S. Mubeen, S. Habib and M.N. Naeem, *The Minkowski inequality involving generalized k-fractional conformable integrals*, J. Inequal. Appl., 2019 (2019), pp. 1–18.

23. M.A. Noor, K.I. Noor, M.U. Awan and J. Li, *On Hermite–Hadamard inequalities for h-preinvex functions*, Filomat, 28 (7) (2014), pp. 1463–1474.
24. Y. Peng and T. Du, *Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions*, Filomat, 37 (28) (2023), pp. 9497–9509.
25. M. Rostamian Delavar, A. Kashuri and M. De La Sen, *On weighted Simpson's $\frac{3}{8}$ rule*, Symmetry, 13 (10) (2021), pp. 1933.
26. R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1997.
27. M.Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, 57 (9–10) (2013), pp. 2403–2407.
28. M. Sababheh, *Convex functions and means of matrices*, Math. Inequal. Appl., 20 (1) (2017), pp. 29–47.
29. F. Safdar and M. Attique, *Extended weighted Simpson-like type inequalities for preinvex functions and their use in physical systems*, J. Math., Punjab Univ., 54 (10) (2023).
30. T. Sitthiwirattham, K. Nonlaopon, M.A. Ali and H. Budak, *Riemann–Liouville fractional Newton-type inequalities for differentiable convex functions*, Fractal Fract., 6 (3) (2022), pp. 175.
31. Y. Shuang, Y. Wang and F. Qi, *Integral inequalities of Simpson's type for $(\alpha, m)$-convex functions*, J. Nonlinear Sci. Appl., 9 (12) (2016), pp. 6364–6370.
32. W. Sun, *Hermite–Hadamard type local fractional integral inequalities for generalized s-preinvex functions and their generalization*, Fractals, 29 (4) (2021), pp. 2150098.
33. M. Tariq, S.K. Ntouyas and A.A. Shaikh, *A comprehensive review of the Hermite–Hadamard inequality pertaining to fractional integral operators*, Mathematics, 11 (8) (2023), pp. 1953.
34. T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl., 136 (1) (1988), pp. 29–38.
35. T. Weir and V. Jeyakumar, *A class of nonconvex functions and mathematical programming*, Bull. Aust. Math. Soc., 38 (2) (1988), pp. 177–189.

---

[1] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan.
*Email address*: mianusamaasif11@gmail.com

[2] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan.
*Email address*: muawan@gcuf.edu.pk