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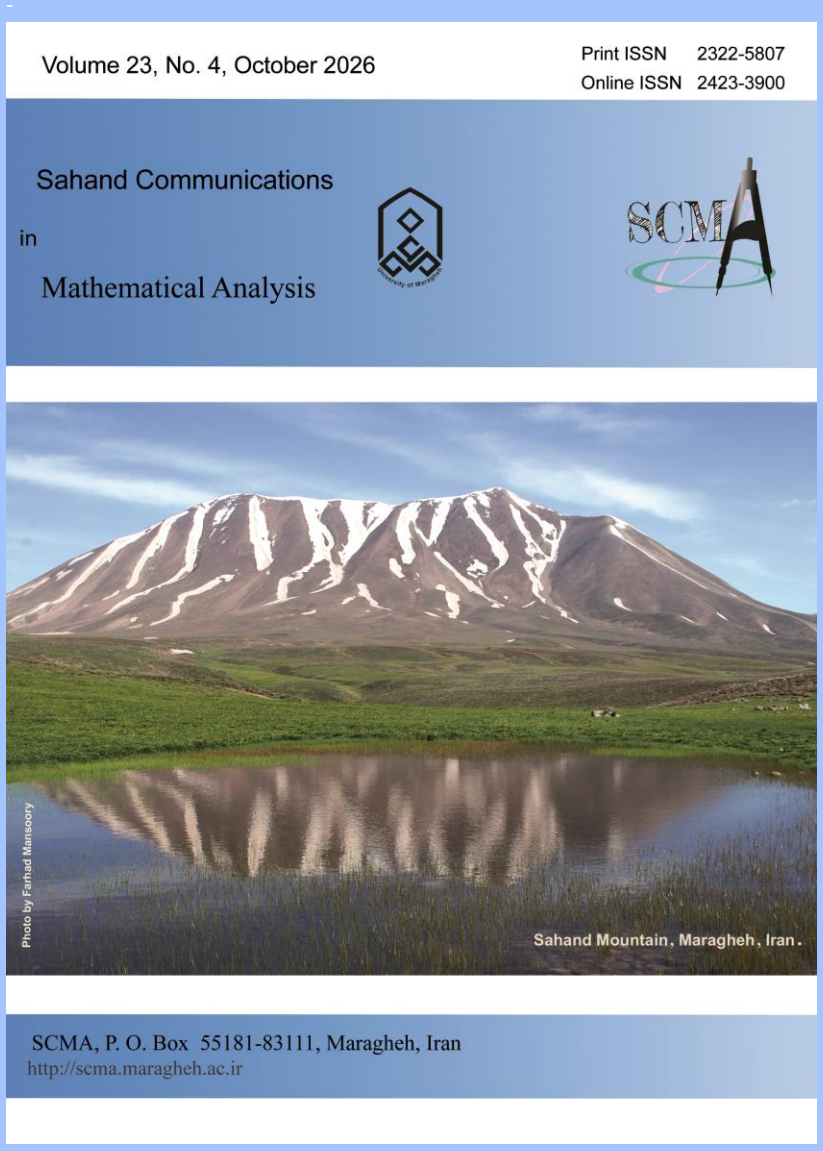

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Toeplitz and Hankel Determinants for Univalent Functions Subordinate to $\cosh z$

Safa Salehian[1*] and Ahmad Motamednezhad[2]

Abstract. This paper establishes upper bounds for the fourth-order Hankel and Toeplitz determinants associated with reciprocal bounded-turning functions that are subordinate to $\cosh z$, along with their corresponding inverses. Additionally, the Zalcman conjecture is examined for specific values of n within these function classes, and precise upper bounds for the Fekete-Szegö inequalities are derived.

1. Introduction

Consider the class $\mathcal{A}$, which consists of analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Each function $f(z)$ in $\mathcal{A}$ can be expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

The class of univalent functions that map the unit disk $\mathbb{D}$ onto a univalent domain in $\mathbb{C}$ is denoted by $\mathcal{S}$.

The Koebe One-Quarter Theorem [5] guarantees that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which satisfies: $f^{-1}(f(z)) = z, (z \in \mathbb{D})$ and $f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$, where

$$\begin{aligned} f^{-1}(w) = w - a_3 w^3 - a_4 w^4 + \left(3a_3^2 - a_5\right) w^5 + (7a_3 a_4 - a_6) w^6 \\ + \left(8a_3 a_5 + 4a_4^2 - 12a_3^3 - a_7\right) w^7 + \cdots . \end{aligned} \tag{1.2}$$

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In this paper, we investigate the following class:

$$\mathcal{H}_{\cosh z} := \left\{ f \in \mathcal{A} : f'(z) \prec \cosh z \right\},$$

where the symbol "$\prec$" denotes the usual subordination. For a comprehensive theory of univalent functions and their significance in coefficient problems, we refer to the book by Duren [5].

Let Ω be the class of Schwarz functions, that is, analytic functions $\omega : \mathbb{D} \longrightarrow \mathbb{D}$ with $\omega(0) = 0$. Functions $\omega \in \Omega$ can be expressed as a power series:

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{1.3}$$

Pommerenke [16] introduced the q-th Hankel determinant for the sequence $a_n, a_{n+1}, a_{n+2}, \cdots$, where $n,\ q \geq 1$

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & a_{n+q+1} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

For instance, for $n = 1$ and $q = 3, 4$, we have

$$H_3(1) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5, \tag{1.4}$$

and

$$\begin{aligned} H_4(1) = {} & a_4^4 - 3a_3a_4^2a_5 + a_3^2a_5^2 + 2a_2a_4a_5^2 - a_5^3 + 2a_3^2a_4a_6 \\ & - 2a_2a_4^2a_6 - 2a_2a_3a_5a_6 + 2a_4a_5a_6 + a_2^2a_6^2 - a_3a_6^2 \\ & - a_3^3a_7 + 2a_2a_3a_4a_7 - a_4^2a_7 - a_2^2a_5a_7 + a_3a_5a_7. \end{aligned} \tag{1.5}$$

Ali et al. [1] studied the symmetric Toeplitz determinant $T_q(n)$ in 2018. It is given by

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}.$$

For example, for $n = 1$ and $q = 3, 4$, we have

$$\begin{aligned} T_3(1) = {} & \left(1 - a_2^2\right)^2 - \left(a_2a_3 - a_4\right)^2 + \left(a_2^3 - a_2a_4\right)^2 - \left(a_2 - a_2a_3\right)^2 \\ & + 2\left(a_2^2 - a_3\right)\left(a_3 - a_2a_4\right) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} T_4(1) = {} & \left(a_2^2 - a_3^2\right)^2 - \left(a_3a_4 - a_2a_5\right)^2 + \left(a_4^2 - a_3a_5\right)^2 \\ & - \left(a_2a_3 - a_3a_4\right)^2 + 2\left(a_3^2 - a_2a_4\right)\left(a_2a_4 - a_3a_5\right). \end{aligned} \tag{1.7}$$

The study of third Hankel determinants for certain subclasses of class $\mathcal{S}$, pioneered by Babalola [3], Bansal [4], Zaprawa [20], Orhan and Zaprawa [14], Kowalczyk et al. [9], and Kwon et al. [10], directly motivated Arif et al. [2] to investigate the fourth Hankel determinant for bounded turning functions and analytic functions associated with the Bernoulli Lemniscate, as documented in [2]. For further insight, the studies in references [6, 13, 15] are also recommended.

Inspired by Libera and Zlotkiewicz's work [11] on the early coefficients of the inverse of regular convex functions and coefficient bounds for functions with derivatives in $\mathcal{P}$, subsequent research by various authors has explored the inverses of different subclasses of class $\mathcal{S}$ (see [7, 8, 17–19]).

The non-univalent analytic function $\phi(z) = \cosh z$ maps the unit disk onto a circular region in the right half-plane. This region exhibits symmetry about the real axis and satisfies the normalization conditions $\phi(0) = 1$ and $\phi'(0) = 0$.

Building on these previous studies, the present work introduces two novel subclasses, denoted as $\mathcal{H}_{\cosh z}$ and ${\mathcal{H}^*}_{\cosh z}$. We then investigate the fourth-order Hankel and Toeplitz determinants for functions belonging to these subclasses.

Definition 1.1. Let f be a function such that $f \in \mathcal{A}$. Then f belongs to the class $\mathcal{H}_{\cosh z}$, if and only if

$$f'(z) \prec \cosh z, \quad (z \in \mathbb{D}). \tag{1.8}$$

Definition 1.2. A function g belongs to the class ${\mathcal{H}^*}_{\cosh z}$, if it is the inverse of a function $f \in \mathcal{H}_{\cosh z}$ and can be represented by the series

$$g(w) = f^{-1}(w) = w + \sum_{n\geq 2} d_n w^n, \quad \left(|w| \leq r_0(f), r_0(f) \geq \frac{1}{4}\right). \tag{1.9}$$

2. Lemmas

Let $\mathcal{P}$ be the family of holomorphic functions p in the unit disk $\mathbb{D}$ that satisfy the following conditions:

1. p(0)=1
2. Real $(p(z)) > 0$ for all $z \in \mathbb{D}$ (the real part of $p(z)$ is positive in the unit disk).

These functions have the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots . \tag{2.1}$$

Functions belonging to this family $\mathcal{P}$ are referred to as functions with a positive real part.

The following lemmas, which pertain to functions with a positive real part of the form given in equation (2.1), will be useful in the subsequent discussion.

Lemma 2.1 ([16]). *Let $p \in \mathcal{P}$, then*

$$|p_n| \leq 2, \quad (n \geq 2).$$

Lemma 2.2 ([2]). *Let p belong to the class $\mathcal{P}$. Then, for all real constants J, K, L, the following sharp inequality holds*

$$\left|Jp_1^3 - Kp_1p_2 + Lp_3\right| \leq 2\left(|J| + |K - 2J| + |J - K + L|\right).$$

For $J = 1$, $K = 2$ and $L = 1$, we have $\left|p_1^3 - 2p_1p_2 + p_3\right| \leq 2$.

Lemma 2.3 ([12]). *For any function p in the class $\mathcal{P}$, the following inequality holds*

$$\left|p_2 - \mu p_1^2\right| \leq 2\max\{1, |2\mu - 1|\}.$$

Lemma 2.4 ([20]). *Let p be an element of the class $\mathcal{P}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leq 2, \quad \forall k, n = 1, 2, 3, \ldots, \quad n > k, \quad \mu \in [0, 1].$$

In the case when $\mu = 1$, we have $|p_n - p_k p_{n-k}| \leq 2$.

Remark 2.5. If $\omega(z) \in \Omega$ and $p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then

$$\begin{aligned}
w(z) &= \frac{p(z) - 1}{p(z) + 1} \\
&= \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)z^3 \\
&\quad + \left(-\frac{c_1^4}{16} + \frac{3c_1^2c_2}{8} - \frac{c_1c_3}{2} - \frac{c_2^2}{4} + \frac{c_4}{2}\right)z^4 \\
&\quad + \left(\frac{c_1^5}{32} - \frac{c_1^3c_2}{4} + \frac{3c_1^2c_3}{8} + \frac{3c_1c_2^2}{8} - \frac{c_1c_4}{2} - \frac{c_2c_3}{2} + \frac{c_5}{2}\right)z^5 \\
&\quad + \left(-\frac{c_1^6}{64} + \frac{5c_1^4c_2}{32} - \frac{c_1^3c_3}{4} - \frac{3c_1^2c_2^2}{8} + \frac{3c_1^2c_4}{8} + \frac{3c_1c_2c_3}{4}\right. \\
&\quad \left. - \frac{c_1c_5}{2} + \frac{c_2^3}{8} - \frac{c_3^2}{4} - \frac{c_2c_4}{2} + \frac{c_6}{2}\right)z^6 + \cdots.
\end{aligned}$$

Also

$$\begin{aligned}
&\cosh\left(w(z)\right) \\
&\quad = 1 + \frac{c_1^2}{8}z^2 + \left(\frac{c_1c_2}{4} - \frac{c_1^3}{8}\right)z^3 + \left(\frac{37c_1^4}{384} - \frac{3c_1^2c_2}{8} + \frac{c_1c_3}{4} + \frac{c_2^2}{8}\right)z^4
\end{aligned}$$

$$+\left(-\frac{13c_1^5}{192}+\frac{37c_1^3c_2}{96}-\frac{3c_1^2c_3}{8}-\frac{3c_1c_2^2}{8}+\frac{c_1c_4}{4}+\frac{c_2c_3}{4}\right)z^5$$
$$+\left(\frac{2101c_1^6}{46080}-\frac{65c_1^4c_2}{192}+\frac{37c_1^3c_3}{96}+\frac{37c_1^2c_2^2}{64}\right.$$
$$\left.-\frac{3c_1^2c_4}{8}-\frac{3c_1c_2c_3}{4}+\frac{c_3^2}{8}-\frac{c_2^3}{8}+\frac{c_2c_4}{4}+\frac{c_1c_5}{4}\right)z^6+\cdots.$$

3. Main Results

Theorem 3.1. *If $f\in\mathcal{H}_{\cosh z}$, then*

$$|a_2|=0,\quad |a_3|\le\frac{1}{6},\quad |a_4|\le\frac{1}{4},\quad |a_5|\le\frac{83}{240},\quad |a_6|\le\frac{5}{6},\quad |a_7|\le\frac{23}{7}.$$

This result is sharp for the follwing function

$$f_1(z)=\int_0^z\cosh t\,dt=z+\frac{1}{6}z^3+\cdots.$$

Proof. Let $f\in\mathcal{H}_{\cosh z}$. By the definition of subordination, there exists a function $\omega\in\Omega$, such that

$$f'(z)=\cosh(\omega(z)). \tag{3.1}$$

Let

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}\quad\Rightarrow\quad\omega(z)=\frac{p(z)-1}{p(z)+1}.$$

From Remark 2.5 and equation (3.1), we have

(3.2)

$$\begin{aligned}&\cosh\left(w(z)\right)\\&=1+\sum_{n=2}^{\infty}na_nz^{n-1}\\&=1+\frac{c_1^2}{8}z^2+\left(\frac{c_1c_2}{4}-\frac{c_1^3}{8}\right)z^3+\left(\frac{37c_1^4}{384}-\frac{3c_1^2c_2}{8}+\frac{c_1c_3}{4}+\frac{c_2^2}{8}\right)z^4\\&\quad+\left(-\frac{13c_1^5}{192}+\frac{37c_1^3c_2}{96}-\frac{3c_1^2c_3}{8}-\frac{3c_1c_2^2}{8}+\frac{c_1c_4}{4}+\frac{c_2c_3}{4}\right)z^5\\&\quad+\left(\frac{2101c_1^6}{46080}-\frac{65c_1^4c_2}{192}+\frac{37c_1^3c_3}{96}+\frac{37c_1^2c_2^2}{64}-\frac{3c_1^2c_4}{8}-\frac{3c_1c_2c_3}{4}\right.\\&\quad\left.+\frac{c_3^2}{8}-\frac{c_2^3}{8}+\frac{c_2c_4}{4}+\frac{c_1c_5}{4}\right)z^6+\cdots.\end{aligned}$$

Equating the coefficients on both sides of the equation (3.2), we get

$$2a_2=0, \tag{3.3}$$

$$3a_3 = \frac{c_1^2}{8}, \tag{3.4}$$

$$4a_4 = \left(\frac{c_1c_2}{4} - \frac{c_1^3}{8}\right), \tag{3.5}$$

$$5a_5 = \left(\frac{37c_1^4}{384} - \frac{3c_1^2c_2}{8} + \frac{c_1c_3}{4} + \frac{c_2^2}{8}\right), \tag{3.6}$$

$$6a_6 = \left(-\frac{13c_1^5}{192} + \frac{37c_1^3c_2}{96} - \frac{3c_1^2c_3}{8} - \frac{3c_1c_2^2}{8} + \frac{c_1c_4}{4} + \frac{c_2c_3}{4}\right), \tag{3.7}$$

and

$$\begin{aligned} 7a_7 = \Big(&\frac{2101c_1^6}{46080} - \frac{65c_1^4c_2}{192} + \frac{37c_1^3c_3}{96} + \frac{37c_1^2c_2^2}{64} - \frac{3c_1^2c_4}{8} - \frac{3c_1c_2c_3}{4} \\ &+\frac{c_3^2}{8} - \frac{c_2^3}{8} + \frac{c_2c_4}{4} + \frac{c_1c_5}{4}\Big). \end{aligned} \tag{3.8}$$

Applying Lemmas 2.1, 2.2, 2.3, and 2.4 to the relations (3.4) to (3.8) yields

$$a_2 = 0, \tag{3.9}$$

$$3|a_3| = \frac{|c_1|^2}{8} \le \frac{1}{2}, \tag{3.10}$$

$$4|a_4| = \left|\frac{c_1c_2}{4} - \frac{c_1^3}{8}\right| = \frac{|c_1|}{4}\left|c_2 - \frac{c_1^2}{2}\right| \le 1, \tag{3.11}$$

$$\begin{aligned} 5|a_5| &= \left|\frac{37c_1^4}{384} - \frac{3c_1^2c_2}{8} + \frac{c_1c_3}{4} + \frac{c_2^2}{8}\right| \\ &\le \frac{|c_1|}{384}\left|37c_1^3 - 144c_1c_2 + 96c_3\right| + \frac{|c_2|^2}{8} \\ &\le \frac{59}{48} + \frac{1}{2} = \frac{83}{48}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} 6|a_6| &= \left|-\frac{13c_1^5}{192} + \frac{37c_1^3c_2}{96} - \frac{3c_1^2c_3}{8} - \frac{3c_1c_2^2}{8} + \frac{c_1c_4}{4} + \frac{c_2c_3}{4}\right| \\ &\le \frac{|c_1|^2}{8}\left|-\frac{13c_1^3}{24} + \frac{37c_1c_2}{12} - 3c_3\right| + \left|\frac{c_1c_4}{4} - \frac{2c_1c_2^2}{8}\right| \\ &\quad + \left|\frac{c_2c_3}{4} - \frac{c_1c_2^2}{8}\right| \\ &\le \frac{|c_1|^2}{192}\left|13c_1^3 - 74c_1c_2 + 72c_3\right| + \frac{|c_1|}{4}\left|c_4 - c_2^2\right| \\ &\quad + \frac{|c_2|}{4}\left|c_3 - \frac{c_1c_2}{2}\right| \end{aligned} \tag{3.13}$$

$$\leq \frac{1}{24}(13+48+11)+2=5$$

and

$$\begin{aligned}7|a_7| &= \left|\frac{2101c_1^6}{46080}-\frac{65c_1^4c_2}{192}+\frac{37c_1^3c_3}{96}+\frac{37c_1^2c_2^2}{64}-\frac{3c_1^2c_4}{8}-\frac{3c_1c_2c_3}{4}+\frac{c_3^2}{8}\right.\\ &\quad \left.-\frac{c_2^3}{8}+\frac{c_2c_4}{4}+\frac{c_1c_5}{4}\right|\\ &\leq \frac{65|c_1|^4}{192}\left|c_2-\frac{2101c_1^2}{15600}\right|+\frac{3}{4}|c_1c_2|\left|c_3-\frac{37}{48}c_1c_2\right|+\frac{|c_2|}{4}\left|c_4-\frac{c_2^2}{2}\right|\\ &\quad +\frac{3}{8}|c_1|^2|c_4-c_1c_3|+\frac{|c_1c_5|}{4}+\frac{|c_3|^2}{8}+\frac{|c_1|^3|c_3|}{96},\end{aligned}$$

so

$$|a_7| \leq \frac{23}{7}. \tag{3.14}$$ □

Theorem 3.2. *Let $f \in \mathcal{H}_{\cosh z}$. Then*

$$|a_3-\mu a_2^2| \leq \frac{1}{6}$$

where μ is any complex number. This result is sharp for the function $f_1(z)$.

Proof. Given the relations (3.3) and (3.4) and the condition $|c_1| \leq 2$, we have:

$$|a_3-\mu a_2^2| = \frac{|c_1|^2}{24} = \frac{1}{6}.$$ □

Theorem 3.3. *Let $f \in \mathcal{H}_{\cosh z}$. Then*

$$|H_3(1)| \leq \frac{539}{4320}.$$

Proof. Utilizing relation (1.4), we obtain

$$|H_3(1)| = \left|2a_2a_3a_4-a_3^3-a_4^2+a_3a_5-a_2^2a_5\right|.$$

From relations (3.9) to (3.12), we obtain:

$$|H_3(1)| = \left|a_3a_5-a_3^3-a_4^2\right| \leq |a_3||a_5|+|a_3|^3+|a_4|^2 \leq \frac{539}{4320}.$$ □

Theorem 3.4. *If $f \in \mathcal{H}_{\cosh z}$, then*

$$|H_4(1)| \leq 0.74076343.$$

Proof. From relation (1.5), we have

$$\begin{aligned}|H_4(1)| = \big|&a_4^4-3a_3a_4^2a_5+a_3^2a_5^2+2a_2a_4a_5^2-a_5^3+2a_3^2a_4a_6-2a_2a_4^2a_6\\ &-2a_2a_3a_5a_6+2a_4a_5a_6+a_2^2a_6^2-a_3a_6^2-a_3^3a_7+2a_2a_3a_4a_7\end{aligned}$$

$$
\begin{aligned}
&-a_4^2a_7 - a_2^2a_5a_7 + a_3a_5a_7\big| \\
&\le |a_4|^4 + 3|a_3||a_4|^2|a_5| + |a_5|^2\left|2a_2a_4 + a_3^2\right| + |a_5|^3 \\
&\quad + 2|a_4||a_6|\left|a_3^2 - a_2a_4\right| + 2|a_5||a_6|\left|a_4 - a_2a_3\right| + |a_6|^2\left|a_2^2 - a_3\right| \\
&\quad + |a_7|\left|2a_2a_3a_4 - a_3^3 - a_4^2 - a_2^2a_5 + a_3a_5\right|.
\end{aligned}
$$

The application of relations (3.3) and (3.10) to (3.14) leads to

$$
\begin{aligned}
|H_4(1)| &\le |a_4|^4 + 3|a_3||a_4|^2|a_5| + |a_3|^2|a_5|^2 + |a_5|^3 + 2|a_3|^2|a_4||a_6| \\
&\quad + 2|a_4||a_5||a_6| + |a_3||a_6|^2 + |a_7|\,|H_3(1)| \\
&\le \frac{1}{256} + \left(\frac{1}{32}\times\frac{83}{240}\right) + \left(\frac{1}{36}\times\frac{(83)^2}{(240)^2}\right) + \left(\frac{83}{240}\right)^3 \\
&\quad + \left(\frac{5}{36\times 12}\right) + \left(\frac{83}{240}\times\frac{5}{12}\right) + \left(\frac{25}{216}\right) + \left(\frac{23}{7}\times\frac{539}{4320}\right) \\
&\simeq 0.74076343.
\end{aligned}
$$

□

Theorem 3.5. *Let $f \in \mathcal{H}_{\cosh z}$ be defined as in* (1.1). *Then*

$$
\left|a_2^2 - a_3\right| \le \frac{1}{6}, \qquad \left|a_3^2 - a_5\right| \le \frac{259}{720}, \qquad \left|a_4^2 - a_7\right| \le \frac{189}{56}.
$$

Proof. First part: $\left|a_2^2 - a_3\right| \le \frac{1}{6}$ follows from Theorem 3.2 with $\mu = 1$.

Second part: From relations (3.4) and (3.6), we get

$$
\begin{aligned}
a_3^2 - a_5 &= \frac{c_1^4}{576} - \frac{1}{5}\left(\frac{37c_1^4}{384} - \frac{3c_1^2c_2}{8} + \frac{c_1c_3}{4} + \frac{c_2^2}{8}\right) \\
&= c_1\left(\frac{-101c_1^3}{5760} + \frac{3c_1c_2}{40} - \frac{c_3}{20}\right) - \frac{c_2^2}{40}
\end{aligned}
$$

By using Lemmas 2.1 and 2.2, we obtain

$$
\begin{aligned}
\left|a_3^2 - a_5\right| &\le \frac{|c_1|}{5760}\left|101c_1^3 - 432c_1c_2 + 288c_3\right| + \frac{|c_2|^2}{40} \\
&\le \frac{1}{1440}\left(101 + |432 - 202| + |101 - 432 + 288|\right) + \frac{1}{10} = \frac{259}{720}.
\end{aligned}
$$

Third part: By applying relations (3.5) and (3.8), we have

$$
\begin{aligned}
a_4^2 - a_7 = -\frac{1}{7}\Bigg(&\frac{2101c_1^6}{46080} - \frac{65c_1^4c_2}{192} + \frac{37c_1^3c_3}{96} + \frac{37c_1^2c_2^2}{64} - \frac{3c_1^2c_4}{8} - \frac{3c_1c_2c_3}{4} \\
&+ \frac{c_3^2}{8} - \frac{c_2^3}{8} + \frac{c_2c_4}{4} + \frac{c_1c_5}{4} - \frac{7c_1^6}{1024} + \frac{7c_1^4c_2}{256} - \frac{7c_1^2c_2^2}{256}\Bigg).
\end{aligned}
$$

By applying Lemmas 2.1, 2.3 and 2.4, we obtain

$$
\left|a_4^2 - a_7\right| = \frac{1}{7}\left|\frac{893c_1^6}{23040} - \frac{239c_1^4c_2}{768} + \frac{37c_1^3c_3}{96} + \frac{141c_1^2c_2^2}{256} - \frac{3c_1^2c_4}{8}\right.
$$

$$
\begin{aligned}
&\left. -\frac{3c_1c_2c_3}{4}+\frac{c_3^2}{8}-\frac{c_2^3}{8}+\frac{c_2c_4}{4}+\frac{c_1c_5}{4}\right| \\
&\leq \frac{1}{7}\left[\frac{239|c_1|^4}{768}\left|c_2-\frac{893}{7170}c_1^2\right|+\frac{3|c_1||c_2|}{4}\left|c_3-\frac{47}{64}c_1c_2\right|\right. \\
&\quad +\frac{|c_3|^2}{8}+\frac{|c_2|}{4}\left|c_4-\frac{c_2^2}{2}\right|+\frac{|c_1|}{4}|c_5-c_1c_4| \\
&\quad \left.+\frac{|c_1|^2}{8}|c_1c_3-c_4|+\frac{25}{96}|c_1|^3|c_3|\right] \\
&\leq \frac{1}{7}\left[\frac{239}{24}+6+1+\frac{1}{2}+1+1+\frac{25}{6}\right] \\
&= \frac{189}{56}. \qquad \square
\end{aligned}
$$

Remark 3.6. A conjecture proposed by Zalcman asserts that the coefficients of functions in the class $\mathcal{S}$ satisfy a specific inequality $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for $n \geq 2$.

We can confidently state that this conjecture holds true for the initial values of n (2, 3 and 4) when dealing with functions from the class $\mathcal{H}_{\cosh z}$

Theorem 3.7. *If $f \in \mathcal{H}_{\cosh z}$ is defined as in* (1.1)*, then*

$$|T_4(1)| \leq \frac{161}{144}.$$

Proof. From relation (1.6), we have

$$
\begin{aligned}
|T_4(1)| = \Big|&\left(1-a_2^2\right)^2-(a_2a_3-a_4)^2+\left(a_2^3-a_2a_4\right)^2-(a_2-a_2a_3)^2 \\
&+2\left(a_2^2-a_3\right)(a_3-a_2a_4)\Big|.
\end{aligned}
$$

Using relations (3.9) to (3.11), we obtain

$$|T_4(1)| \leq 1+|a_4|^2+2|a_3|^2 \leq 1+\frac{1}{16}+\frac{1}{18}=\frac{161}{144}. \qquad \square$$

Theorem 3.8. *Let $f \in \mathcal{H}_{\cosh z}$. Then*

$$|T_4(2)| \leq \frac{6461}{518400}.$$

Proof. According to relation (1.7), we obtain

$$
\begin{aligned}
|T_4(2)| = \Big|&\left(a_2^2-a_3^2\right)^2-(a_3a_4-a_2a_5)^2+\left(a_4^2-a_3a_5\right)^2-(a_2a_3-a_3a_4)^2 \\
&+2\left(a_3^2-a_2a_4\right)(a_2a_4-a_3a_5)\Big|.
\end{aligned}
$$

Since $a_2 = 0$, we get

$$|T_4(2)| \leq |a_3|^4+2|a_3|^2|a_4|^2+\left|a_4^2-a_3a_5\right|^2+2|a_3|^3|a_5|. \tag{3.15}$$

Now, observe that

$$
\begin{aligned}
(3.16)\quad &\left|a_4^2 - a_3a_5\right| \\
&= \left|\left(\frac{c_1c_2}{16} - \frac{c_1^3}{32}\right)^2 - \frac{1}{120}\left(\frac{37c_1^6}{384} - \frac{3c_1^4c_2}{8} + \frac{c_1^3c_3}{4} + \frac{c_1^2c_2^2}{8}\right)\right| \\
&= \frac{|c_1|^2}{120}\left|-\frac{c_1^4}{48} + \frac{3c_1^2c_2}{32} + \frac{c_1c_3}{4} - \frac{11c_2^2}{32}\right| \\
&\leq \frac{|c_1|^2}{120}\left[\frac{3|c_1|^2}{32}\left|c_2 - \frac{2c_1^2}{9}\right| + \frac{11}{32}\left|c_4 - \frac{8}{11}c_1c_3\right| + \frac{11}{32}\left|c_4 - c_2^2\right|\right] \\
&\leq \frac{17}{240}.
\end{aligned}
$$

From relations (3.10) to (3.12), (3.15) and (3.16), we conclude

$$
\begin{aligned}
|T_4(2)| &\leq |a_3|^4 + 2|a_3|^2|a_4|^2 + \left|a_4^2 - a_3a_5\right|^2 + 2|a_3|^3|a_5| \\
&\leq \frac{1}{1296} + \frac{1}{288} + \frac{289}{57600} + \frac{83}{25920} \\
&= \frac{6461}{518400}.
\end{aligned}
$$

□

4. The Relationship Between Fourth Hankel and Toeplitz Determinants and Inverses of Analytic Functions of $\cosh z$

Let $g \in \mathcal{H}^*_{\cosh z}$. By definition, there exists a function $f \in \mathcal{H}_{\cosh z}$ such that $w = f(g(w))$. Substituting the series expansions of g from equation (1.9) and f from equation (1.1) into the identity $w = f(g(w)) = f\left(f^{-1}(w)\right)$, and using the coefficient values a_i derived from relations (3.3) to (3.8), we obtain the following

$$d_2 = 0, \tag{4.1}$$

$$d_3 = -\frac{c_1^2}{24}, \tag{4.2}$$

$$d_4 = \frac{c_1^3}{32} - \frac{c_1c_2}{16}, \tag{4.3}$$

$$d_5 = \frac{1}{20}\left(-\frac{9c_1^4}{32} - \frac{3c_1^2c_2}{2} + c_1c_3 + \frac{c_2^2}{2}\right) \tag{4.4}$$

$$d_6 = \frac{1}{6}\left(\frac{5c_1^5}{384} - \frac{53c_1^3c_2}{192} + \frac{3c_1^2c_3}{8} + \frac{3c_1c_2^2}{8} - \frac{c_1c_4}{4} - \frac{c_2c_3}{4}\right), \tag{4.5}$$

and

$$7d_7 = \left(\frac{317c_1^6}{15360} + \frac{13c_1^4c_2}{240} - \frac{43c_1^3c_3}{160} - \frac{197c_1^2c_2^2}{480} + \frac{3c_1^2c_4}{8} + \frac{3c_1c_2c_3}{4}\right. \tag{4.6}$$

$$-\frac{c_3^2}{8}+\frac{c_2^3}{8}-\frac{c_2c_4}{4}-\frac{c_1c_5}{4}\Bigg).$$

Theorem 4.1. *Let* $g(w)=w+\sum_{n=2}^{\infty}d_nw^n\in\mathcal{H}^*_{\cosh z}$. *Then, we have the following coefficient bounds:*

$$|d_2|=0,\quad |d_3|\le\frac{1}{6},\quad |d_4|\le\frac{1}{4},\quad |d_5|\le\frac{29}{40},\quad |d_6|\le\frac{5}{6},\quad |d_7|\le\frac{397}{210}.$$

Proof. From relations (4.1) to (4.6), and by applying Lemmas 2.1, 2.2, 2.3, and 2.4, we obtain the desired bounds for the coefficients:

$$\begin{aligned}
|d_2|&=0,\\
|d_3|&=\frac{|c_1|^2}{24}\le\frac{1}{6},\\
|d_4|&=\frac{|c_1|}{16}\left|\frac{c_1^2}{2}-c_2\right|\le\frac{1}{4},\\
|d_5|&=\frac{1}{20}\left(|c_1|\frac{\left|9c_1^3+48c_1c_2-32c_3\right|}{32}+\frac{|c_2|^2}{2}\right)\le\frac{29}{40},
\end{aligned}$$

and

$$\begin{aligned}
6|d_6|&=\left|\frac{5c_1^5}{384}-\frac{53c_1^3c_2}{192}+\frac{3c_1^2c_3}{8}+\frac{3c_1c_2^2}{8}-\frac{c_1c_4}{4}-\frac{c_2c_3}{4}\right|\\
&\le|c_1|^2\frac{\left|5c_1^3-106c_1c_2+144c_3\right|}{384}+\frac{|c_1|}{4}\left|c_2^2-c_4\right|\\
&\quad+\frac{|c_2|}{4}\left|c_3-\frac{c_1c_2}{2}\right|\\
&\le 5
\end{aligned}$$

and

$$\begin{aligned}
7|d_7|&=\left|\frac{317c_1^6}{15360}+\frac{13c_1^4c_2}{240}-\frac{43c_1^3c_3}{160}-\frac{197c_1^2c_2^2}{480}+\frac{3c_1^2c_4}{8}\right.\\
&\quad\left.+\frac{3c_1c_2c_3}{4}-\frac{c_3^2}{8}+\frac{c_2^3}{8}-\frac{c_2c_4}{4}-\frac{c_1c_5}{4}\right|\\
&\le\frac{13|c_1|^4}{240}\left|c_2+\frac{317c_1^2}{832}\right|+\frac{3|c_1||c_2|}{4}\left|c_3-\frac{197c_1c_2}{360}\right|\\
&\quad+\frac{3|c_1|^2}{8}\left|c_4-\frac{43}{60}c_1c_3\right|+\frac{|c_2|}{4}\left|c_4-\frac{c_2^2}{2}\right|+\frac{|c_3|^2}{8}+\frac{|c_1||c_5|}{4},
\end{aligned}$$

so

$$7|d_7|\le\frac{397}{30}.$$

□

Theorem 4.2. *If $g \in \mathcal{H}^*{}_{\cosh z}$ is of the form given in equation* (1.9)*, then*

$$|d_3 - \mu d_2^2| \leq \frac{1}{6}.$$

Proof. Since $g \in \mathcal{H}^*{}_{\cosh z}$, from relations (4.1) and (4.2), we obtain the desired bound:

$$|d_3 - \mu d_2^2| = |d_3| \leq \frac{1}{6}. \qquad \square$$

Theorem 4.3. *If $g \in \mathcal{H}^*{}_{\cosh z}$ is expressed as in equation* (1.9)*, then*

$$|H_3(1)| \leq \frac{203}{1080}.$$

Proof. Using relations (1.4), (4.1) to (4.4), and Lemmas 2.1, 2.2, 2.3, and 2.4, we obtain

$$|H_3(1)| = \left|2d_2d_3d_4 - d_3^3 - d_4^2 + d_3d_5 - d_2^2d_5\right|.$$

From Theorem 4.1, we get

$$|H_3(1)| = \left|d_3^3 + d_4^2 - d_3d_5\right| \leq |d_3|^3 + |d_4|^2 + |d_3||d_5| \leq \frac{203}{1080}. \qquad \square$$

Theorem 4.4. *Let $g \in \mathcal{H}^*{}_{\cosh z}$ be defined as in equation* (1.9)*. Then*

$$|H_4(1)| \simeq 1.23618036.$$

Proof. From relation (1.5), we have

$$\begin{aligned}
|H_4(1)| = {} & \left|d_4^4 - 3d_3d_4^2d_5 + d_3^2d_5^2 + 2d_2d_4d_5^2 - d_5^3 + 2d_3^2d_4d_6\right. \\
& -2d_2d_4^2d_6 - 2d_2d_3d_5d_6 + 2d_4d_5d_6 + d_2^2d_6^2 - d_3d_6^2 \\
& \left.-d_3^3d_7 + 2d_2d_3d_4d_7 - d_4^2d_7 - d_2^2d_5d_7 + d_3d_5d_7\right| \\
\leq {} & |d_4|^4 + 3|d_3||d_4|^2|d_5| + |d_5|^2\left|d_2d_4 - d_3^2\right| + |d_5|^2\left|d_2d_4 + 2d_3^2\right| \\
& + |d_5|^3 + 2|d_4||d_6|\left|d_3^2 - d_2d_4\right| + 2|d_5||d_6|\left|d_4 - d_2d_3\right| \\
& + |d_6|^2\left|d_2^2 - d_3\right| + |d_7|\left|2d_2d_3d_4 - d_3^3 - d_4^2 - d_2^2d_5 + d_3da_5\right|.
\end{aligned}$$

From Theorems 4.1 and 4.3, we get

$$\begin{aligned}
|H_4(1)| \leq {} & |d_4|^4 + 3|d_3||d_4|^2|d_5| + 3|d_3|^2|d_5|^2 + |d_5|^3 + 2|d_3|^2|d_4||d_6| \\
& + 2|d_4||d_5||d_6| + |d_3||d_6|^2 + |d_7|\left|H_3(1)\right|.
\end{aligned}$$

So

$$\begin{aligned}
|H_4(1)| &\leq \frac{1}{256} + \frac{29}{1280} + \frac{841}{19200} + \frac{24389}{64000} + \frac{5}{432} + \frac{29}{96} + \frac{25}{216} + \frac{80591}{226800} \\
&\simeq 1.23618036. \qquad \square
\end{aligned}$$

Theorem 4.5. *Suppose $g \in \mathcal{H}^*_{\cosh z}$ has the form given by equation (1.9). Then*

$$\left|d_2^2 - d_3\right| \leq \frac{1}{6}, \qquad \left|d_3^2 - d_5\right| \leq \frac{271}{360}, \qquad \left|d_4^2 - d_7\right| \leq \frac{1729}{840}.$$

Proof. The first part, $\left|d_2^2 - d_3\right| \leq \frac{1}{6}$, follows directly from Theorem 4.2 with $\mu = 1$.

For the second part, using relations (4.2) and (4.4), we have

$$\begin{aligned} d_3^2 - d_5 &= \frac{c_1^4}{576} - \frac{1}{20}\left(-\frac{9c_1^4}{32} - \frac{3c_1^2c_2}{2} + c_1c_3 + \frac{c_2^2}{2}\right) \\ &= \frac{1}{20}\left(\frac{91c_1^4}{288} + \frac{3c_1^2c_2}{2} - c_1c_3 - \frac{c_2^2}{2}\right) \end{aligned}$$

Using Lemmas 2.1 and 2.2, we get

$$\begin{aligned} \left|d_3^2 - d_5\right| &\leq \frac{|c_1|}{20}\frac{\left|91c_1^3 + 432c_1c_2 - 288c_3\right|}{288} + \frac{|c_2|^2}{40} \\ &\leq \frac{47}{72} + \frac{1}{10} = \frac{271}{360}. \end{aligned}$$

For the third part, from relations (4.3) and (4.6), we get

$$\begin{aligned} d_4^2 - d_7 = {} & \frac{1}{7}\left(\frac{7c_1^6}{1024} - \frac{7c_1^4c_2}{256} + \frac{7c_1^2c_2^2}{256}\right) - \frac{1}{7}\left(\frac{951c_1^6}{46080} + \frac{13c_1^4c_2}{240} - \frac{197c_1^2c_2^2}{480}\right. \\ & \left. - \frac{43c_1^3c_3}{160} + \frac{3c_1^2c_4}{8} + \frac{3c_1c_2c_3}{4} - \frac{c_3^2}{8} + \frac{c_2^3}{8} - \frac{c_2c_4}{4} - \frac{c_1c_5}{4}\right) \\ = {} & \frac{1}{7}\left(-\frac{53c_1^6}{3840} - \frac{313c_1^4c_2}{3840} + \frac{1681c_1^2c_2^2}{3840} + \frac{43c_1^3c_3}{160} - \frac{3c_1^2c_4}{8}\right. \\ & \left. - \frac{3c_1c_2c_3}{4} + \frac{c_3^2}{8} - \frac{c_2^3}{8} + \frac{c_2c_4}{4} + \frac{c_1c_5}{4}\right) \end{aligned}$$

By using Lemmas 2.1, 2.3 and 2.4, we get

$$\begin{aligned} \left|d_4^2 - d_7\right| \leq {} & \frac{1}{7}\left(\frac{313|c_1|^4}{3840}\left|\frac{53}{313}c_1^2 + c_2\right| + \frac{3|c_1||c_2|}{4}\left|\frac{1681c_1c_2}{2880} - c_3\right|\right. \\ & + \frac{|c_2|}{4}\left|c_4 - \frac{c_2^2}{2}\right| + \frac{|c_1|}{4}\left|c_5 - c_1c_4\right| + \frac{|c_1|^2}{8}\left|-c_4 + c_1c_3\right| \\ & \left. + \frac{23|c_1|^3|c_3|}{160} + \frac{|c_3|^2}{8}\right) \\ \leq {} & \frac{1729}{840}. \end{aligned}$$

□

Remark 4.6. Remark 4.6 points out that Zalcman's conjecture holds true for $n = 2, 3$ and 4 within the class $\mathcal{H}^*_{\cosh z}$. Zalcman's conjecture suggests that the coefficients of functions in a certain subclass of analytic functions satisfy the inequality:

$$|a_n - a_{n-1}| \leq (n-1)^2, \quad \text{for } n \geq 2.$$

This remark indicates that you have verified Zalcman's conjecture for these initial values of n in the context of the functions in the class $\mathcal{H}^*_{\cosh z}$.

Theorem 4.7. *Consider $g \in \mathcal{H}^*_{\cosh z}$ with the representation given by* (1.9)*. Then*

$$|T_4(1)| \leq \frac{161}{144}.$$

Proof. From relation (1.6), we have

$$\begin{aligned} |T_4(1)| = \Big| &\left(1 - d_2^2\right)^2 - (d_2 d_3 - d_4)^2 + \left(d_2^3 - d_2 d_4\right)^2 - (d_2 - d_2 d_3)^2 \\ &+2\left(d_2^2 - d_3\right)(d_3 - d_2 d_4)\Big| \end{aligned}$$

By using Theorem 4.1, we have

$$|T_4(1)| \leq 1 + |d_4|^2 + 2|d_3|^2 \leq \frac{161}{144}. \qquad \square$$

Theorem 4.8. *If the function $g \in \mathcal{H}^*_{\cosh z}$ is defined by* (1.9)*, then*

$$|T_4(2)| \leq \frac{10069}{129600} \simeq 0.0776929012.$$

Proof. Considering relation (1.7) and the condition $d_2 = 0$, we get

$$\begin{aligned} |T_4(2)| = \Big| &\left(d_2^2 - d_3^2\right)^2 - (d_3 d_4 - d_2 d_5)^2 + \left(d_4^2 - d_3 d_5\right)^2 - (d_2 d_3 - d_3 d_4)^2 \\ &+2\left(d_3^2 - d_2 d_4\right)(d_2 d_4 - d_3 d_5)\Big| \\ \leq\ &|d_3|^4 + 2|d_3|^2|d_4|^2 + \left|d_4^2 - d_3 d_5\right|^2 + 2|d_3|^3|d_5|. \end{aligned}$$

From relations (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} d_4^2 - d_3 d_5 &= \left(\frac{c_1^6}{1024} + \frac{c_1^2 c_2^2}{256} - \frac{c_1^4 c_2}{256}\right) + \frac{c_1^2}{480}\left(-\frac{9c_1^4}{32} - \frac{3c_1^2 c_2}{2} + c_1 c_3 + \frac{c_2^2}{2}\right) \\ &= \frac{c_1^3}{7680}\left[3c_1^3 - 54 c_1 c_2 + 16 c_3\right] + \frac{19 c_1^2 c_2^2}{3840}. \end{aligned}$$

By applying Lemmas 2.1 and 2.2, we get

$$\begin{aligned} |d_4^2 - d_3 d_5| &\leq \frac{|c_1|^3}{7680}\left|3c_1^3 - 54 c_1 c_2 + 16 c_3\right| + \frac{19|c_1|^2|c_2|^2}{3840} \\ &\leq \frac{16}{7680}\left[3 + |54 - 6| + |3 - 54 + 16|\right] + \frac{256}{3840} \end{aligned} \tag{4.7}$$

$$\leq \frac{31}{120}.$$

At the end, from (4.7) and Theorem 4.1, we have

$$|T_4(2)| \leq \frac{10069}{129600} \simeq 0.0776929012. \qquad \square$$

5. Conclusions

In this paper, we introduce two novel subclasses of analytic functions $\mathcal{H}_{\cosh z}$ and $\mathcal{H}^*_{\cosh z}$ which are defined in terms of subordination to the hyperbolic cosine function, $\cosh z$ as a fundamental framework. We then explore sharp bounds on the coefficients of functions that belong to these subclasses. Additionally, we derive the fourth-order Hankel and Toeplitz determinants for several well-known subclasses as important consequences of our results. Furthermore, we address the Zalcman conjecture for selected values of n within these classes of functions, and obtain precise upper bounds for the Fekete-Szegö inequalities.

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