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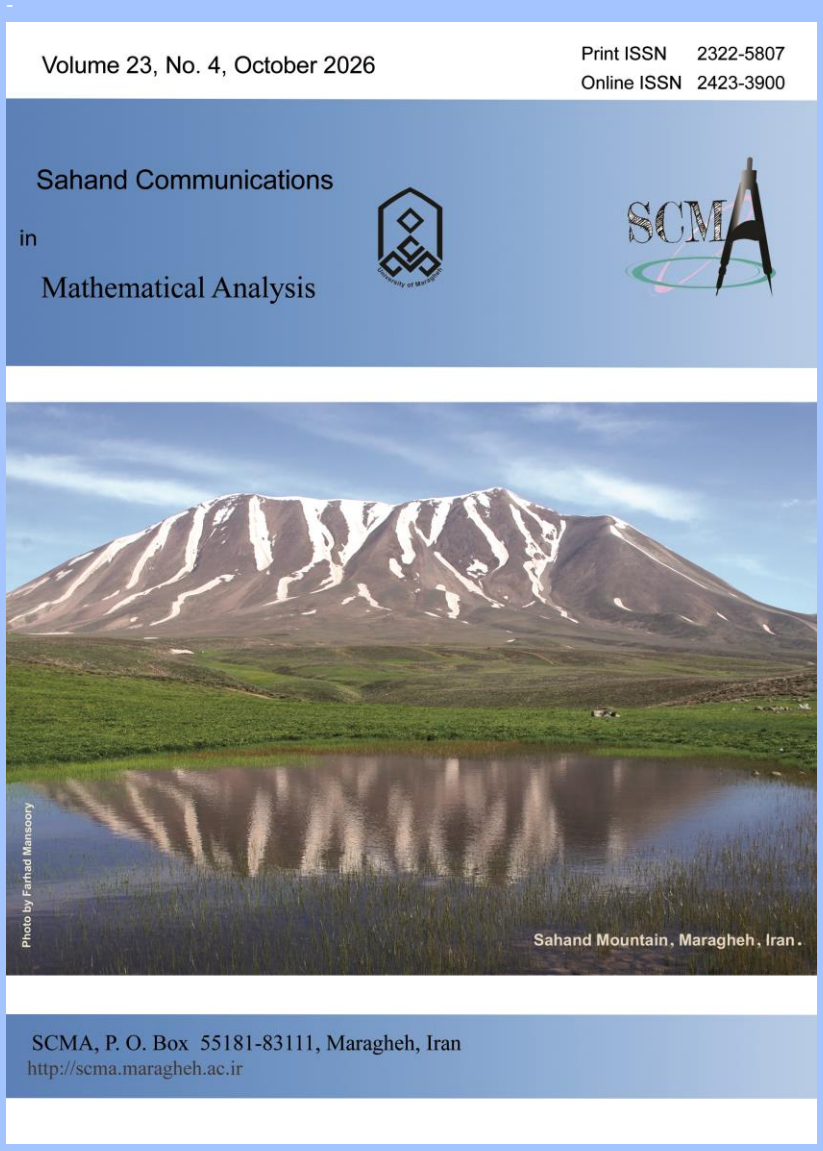

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Abstract. Fuzzy utility theory, as a generalization of classical utility theory, enables decision-making modeling under conditions of uncertainty and ambiguity. This paper provides a comprehensive review of the mathematical foundations, advanced concepts, and practical applications of this theory. First, the theoretical foundations including fuzzy preference relations, axioms, and Billot's existence theorem for fuzzy utility functions are explained. Then, supplementary concepts such as the fuzzy Archimedean principle, independence axiom, choice coherence, fuzzy lotteries, fuzzy expected utility, and risk aversion criteria are elucidated with numerical examples. Also, a real-world case study on selecting the optimal location for constructing a water quality monitoring station is analyzed using fuzzy data and multi-criteria decision-making methods, and the stability of results is examined through sensitivity analysis. The results show that the fuzzy approach can effectively model the uncertainties present in real-world data and provide stable and reliable rankings.

1. Introduction

The concept of utility plays a fundamental role in economics and decision theory. In the classical approach, it is assumed that decision-makers are completely rational and know their preferences with certainty and full clarity [4]. However, in the real world, decisions are often accompanied by ambiguity, uncertainty, and imprecision. The concept of "fuzzy utility" is a response to this inadequacy of classical theory and the transition from certainty to imprecision in choice theory. This concept, rooted

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in Zadeh's fuzzy set theory [39], attempts to incorporate the inherent "fuzziness" of human behavior and the imprecision of preferences into mathematical decision-making models. In other words, fuzzy utility is represented by a fuzzy set or a membership function that indicates the degree of preference for an option. This approach allows researchers to analyze phenomena such as contradictory behaviors, inconsistent preferences, and hidden emotional reactions that cannot be explained by traditional models [4].

One of the fundamental distinctions in decision theory is the distinction between risk and uncertainty, proposed by Knight [23]. Under risk, the decision-maker knows the probabilities of events. Under uncertainty (also called informational ambiguity or Knightian uncertainty), the decision-maker does not know the probabilities and may not even be able to determine them precisely [8]. Classical utility theory can only model risk and needs generalization to model uncertainty. In this paper, we show how fuzzy utility theory can serve as a natural framework for modeling uncertainty.

Following the introduction of fuzzy set theory, the first steps for applying this theory in decision-making and utility were taken. Ponsard in 1986, by examining the application of fuzzy set theory in analyzing consumer spatial behavior, showed how fuzzy concepts can be used to model preferences under uncertainty [31]. The turning point of this period was the foundational work of Antoine Billot in the early 1990s. In his influential book "Fuzzy Economics: Towards a General Theory of Value" published in 1992, Billot provided a coherent framework for fuzzy utility theory [4]. In this book and then in his 1995 paper, he proved the existence theorem for fuzzy utility functions and showed how a fuzzy preference relation can be transformed into a fuzzy utility function [5]. Subsequently, researchers deepened and generalized the fundamental concepts. Barrett et al. investigated rational choice under fuzzy preferences [2]. Nakamura provided a framework for decision-making based on fuzzy preferences [28]. de Wild introduced the concept of fuzzy choice coherence by generalizing Sen's condition to fuzzy spaces [10].

Simultaneously with theoretical advances, practical applications of fuzzy utility theory began in various fields. Massmann and Freeze used fuzzy utility functions for optimal allocation of groundwater pollution monitoring wells [25]. Muto et al. presented a model for evaluating the benefits of constructing recreational facilities using fuzzy utility [27]. In the field of sustainable supplier selection, several studies have been conducted using the fuzzy UTASTAR (Fuzzy Utility Additives STAR) method [12]. In recent years, fuzzy utility theory has been developed in new areas such as data stream analysis [19] and intelligent decision

support systems with a bipolar approach [30]. Furthermore, the performance of fuzzy utility models has been compared with other nondeterministic approaches such as stochastic and quantum models [35]. Fuzzy utility theory has the advantage of explainability over competing approaches such as machine learning [14], because it presents results using linguistic concepts and membership degrees.

Beyond decision-making and management applications, fuzzy set theory has found numerous applications in image processing and computer vision. A prominent area is image segmentation, where fuzzy approaches effectively handle uncertainties in pixel classification. Karthick et al. [18] proposed a normalized fuzzy graph cut method based on connectivity measures and shape priority for color image segmentation, demonstrating improved performance over classical methods. More recently, Brinthaguru et al. [6] developed an intuitionistic fuzzy normalized hypergraph cut algorithm for brain tumor segmentation in magnetic resonance imaging (MRI) images, integrating fuzzy logic with spectral hypergraph clustering to address noise and anatomical complexity. These studies highlight the versatility of fuzzy set theory beyond decision-making problems.

Despite the progress made, there are challenges in the practical application of fuzzy utility theory. As the number of alternatives and criteria increases, fuzzy computations (especially when using α-cuts) become computationally expensive. Also, implementing fuzzy utility models in real-world problems requires dimensionality reduction and optimization methods [24]. For streaming data, fast and adaptive methods are needed [19].

Recently, the theoretical foundations of fuzzy decision-making have been significantly strengthened by advances in several directions. Qiu et al. [32] introduced a generalized framework for ς-neutrosophic fuzzy metric spaces (ς-NFMSs), extending the parameter space to multiple dimensions and establishing important topological properties and fixed-point theorems. Das et al. [9] extended the concepts of strongly almost convergence and statistically almost convergence by incorporating λ-convergence for fuzzy variable sequences using Orlicz functions in credibility space, establishing new sequence spaces and inclusion relations. Kişi et al. [20] investigated different forms of weighted ideal statistical convergence for double sequences of fuzzy variables within the framework of credibility, introducing the concept of weighted ideal statistically Cauchy sequences. Furthermore, Ray et al. [33] explored the notion of pre-open sets in fuzzy bitopological spaces via an operation approach, introducing $(i, j)^*_\phi$-fuzzy-pre-open sets and studying local

finiteness. These developments collectively enhance the theoretical robustness and applicability of fuzzy utility models.

The main motivation of this paper is to bridge the gap between the rich theoretical developments in fuzzy utility theory and their practical implementation in real-world multi-criteria decision-making problems. While classical utility theory assumes crisp preferences and complete information, real-world decisions often involve ambiguity, imprecise data, and conflicting criteria. Fuzzy utility theory provides a natural framework to address these challenges by modeling preferences through membership functions and incorporating degrees of uncertainty. However, existing studies either focus heavily on theoretical aspects or apply fuzzy methods without fully leveraging the underlying axiomatic foundations. This paper aims to fill this gap by presenting a comprehensive yet practical treatment of fuzzy utility theory, from its mathematical foundations to its application in a real case study on water quality monitoring station selection, along with supplementary applications in finance and supply chain management.

The structure of the paper is as follows: Section 2 explains the theoretical and mathematical foundations of fuzzy utility including fuzzy preference relations, Billot's existence theorem, fuzzy expected utility, and related concepts, and is also dedicated to advanced concepts such as the fuzzy Archimedean principle, fuzzy choice coherence, and risk aversion theory. Section 3 includes an applied case study where the presented methods are implemented using real data. This section includes calculating fuzzy utility of alternatives, sensitivity analysis, comparison with machine learning methods, and analysis of concepts such as value of information and the fuzzy utility model for discrete choice. Finally, Section 4 concludes and summarizes the presented material.

2. Theoretical and Mathematical Foundations of Fuzzy Utility

Before delving into fuzzy utility theory, it is necessary to recall some basic concepts from fuzzy set theory.

In classical theory, a set is defined with crisp membership (0 or 1). However, in fuzzy theory founded by Zadeh [39], an element can belong to a set with a degree between 0 and 1. This degree of membership is expressed by the membership function $\mu_A : X \to [0, 1]$. The α-cut of a fuzzy set A is defined as $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$. In this paper, we deal with triangular fuzzy numbers represented as $\tilde{x} = (a, b, c)$. The α-cut of a triangular fuzzy number $\tilde{x} = (a, b, c)$ is defined as $\tilde{x}_\alpha = \left[x_\alpha^L, x_\alpha^U\right]$, where $x_\alpha^L = a + (b - a)\,\alpha$ and $x_\alpha^U = c - (c - b)\,\alpha$, and its membership function increases linearly and then decreases.

Next, the fundamental concepts of fuzzy utility theory including fuzzy preference relations, Billot's existence theorem, and fuzzy expected utility are explained. Then, more advanced concepts such as the fuzzy Archimedean principle, fuzzy choice coherence, and fuzzy risk aversion theory are examined with numerical examples.

Definition 2.1 (Fuzzy Preference Relation [5])**.** A fuzzy preference relation $\widetilde{R}$ on the set of alternatives X is a membership function $\mu_{\widetilde{R}} : X \times X \to [0,1]$ that indicates the degree to which one alternative is preferred over another. For each pair $(x,y) \in X^2$, the value $\widetilde{R}(x,y)$ is interpreted as the degree to which x is at least as desirable as y.

For example, the statement "car x is about 70% better than car y" is modeled as $\widetilde{R}(x,y) = 0.7$ and $\widetilde{R}(y,x) = 0.3$.

Definition 2.2 (Fuzzy Utility Function [5])**.** Suppose X is a set of alternatives and $\widetilde{R}$ is a fuzzy preference relation on X. A function $u : X \to [0,1]$ is called a fuzzy utility function for $\widetilde{R}$ if there exists a function $\phi : \mathbb{R} \to [0,1]$ that is strictly increasing, continuous, and satisfies $\phi(-t) = 1 - \phi(t)$, such that the following relation holds: $\widetilde{R}(x,y) = \phi(u(x) - u(y))$.

The choice of the function ϕ can vary depending on the nature of the problem, such as the logistic (or sigmoid) function $\phi(t) = \frac{1}{1+e^{-t}}$ or the linear function $\phi(t) = \frac{1+t}{2}$ for $t \in [-1,1]$.

To model rational behavior under fuzzy conditions, the following axioms are defined [5, 39]:

Definition 2.3 (Completeness of Fuzzy Preference [5])**.** A fuzzy preference relation $\widetilde{R}$ on X is called complete if for every $x, y \in X$ with $x \neq y$, $\widetilde{R}(x,y) + \widetilde{R}(y,x) \geq 1$.

Definition 2.4 (Fuzzy Transitivity [5])**.** A fuzzy preference relation $\widetilde{R}$ on X is called transitive if for every $x, y, z \in X$:

$$\widetilde{R}(x,z) \geq \max_{y \in X} \left(\min \left(\widetilde{R}(x,y), \widetilde{R}(y,z) \right) \right).$$

This principle, known as max-min transitivity, is the fuzzy generalization of classical transitivity. In the literature, there are various definitions for fuzzy transitivity. Fono and Andjiga [13] use a definition called bin-transitivity, which is used in the proof of Billot's existence theorem. This definition is simpler than max-min transitivity and is expressed as follows:

$$\left[\widetilde{R}(x,y) \geq \widetilde{R}(y,x) \text{ and } \widetilde{R}(y,z) \geq \widetilde{R}(z,y) \right] \quad \Rightarrow \quad \widetilde{R}(x,z) \geq \widetilde{R}(z,x).$$

However, the max-min definition, due to its nonlinear nature, has greater discriminatory power and is more applicable in multi-criteria decision-making problems. The choice between these definitions depends on the type of problem and the required modeling precision.

Example 2.5 (Checking Transitivity)**.** Suppose we have three alternatives a, b, c with the relations $\widetilde{R}(a,b) = 0.8$, $\widetilde{R}(b,c) = 0.7$ and $\widetilde{R}(a,c) = 0.6$. To check transitivity, we calculate the right-hand side of the inequality, i.e., $\max(\min(0.8, 0.7)) = 0.7$. Since $0.6 \geq 0.7$ does not hold, this relation is not transitive. For it to be transitive, we would need $\widetilde{R}(a,c) \geq 0.7$.

Example 2.6 (Fuzzy Preference Relation)**.** Suppose we have two car options x and y. A person may not have a definite opinion about which is better, but may state fuzzily: "Car x is about 70% better than car y". This statement is modeled as $\widetilde{R}(x,y) = 0.7$ and $\widetilde{R}(y,x) = 0.3$. Note that $\widetilde{R}(x,y) + \widetilde{R}(y,x) = 1$ satisfies the completeness condition.

Now the question arises: under what conditions can the existence of such a fuzzy utility function be guaranteed? The following theorem, proved by Billot [5], provides the answer.

Theorem 2.7 (Billot's Existence Theorem, [5])**.** *Suppose X is a convex and connected set in $\mathbb{R}^n$. Also suppose $\widetilde{R}$ is a complete, transitive (max-min), and continuous fuzzy preference relation on X. Then there exists a fuzzy utility function $u : X \to [0,1]$ such that for every $x, y \in X$, $\widetilde{R}(x,y) = \phi(u(x) - u(y))$ where $\phi : \mathbb{R} \to [0,1]$ is a strictly increasing and continuous function satisfying $\phi(-t) = 1 - \phi(t)$.*

Example 2.8 (Calculation of Fuzzy Utility)**.** Suppose we have three alternatives x, y, z with utilities $u(x) = 0.8$, $u(y) = 0.5$ and $u(z) = 0.3$. If we use the sigmoid function $\phi(t) = \frac{1}{1+e^{-t}}$, the degree of preference of x over y is:

$$\widetilde{R}(x,y) = \phi(0.8 - 0.5) = \phi(0.3) = \frac{1}{1+e^{-0.3}} \approx 0.57$$

and the degree of preference of y over z:

$$\widetilde{R}(y,z) = \phi(0.5 - 0.3) = \phi(0.2) \approx 0.55$$

and the degree of preference of x over z:

$$\widetilde{R}(x,z) = \phi(0.8 - 0.3) = \phi(0.5) \approx 0.62.$$

Note that with these functions, the fuzzy transitivity condition (max-min) holds because $\widetilde{R}(x,z) = 0.62 \geq \max(\min(0.57, 0.55)) = 0.55$.

Next, we define fuzzy expected utility. There are two types of definitions for fuzzy expected utility in the literature: the first type is for calculating the expected utility of a fuzzy number (such as uncertain wealth), and the second type is for calculating the expected utility of a decision under several states of nature.

Definition 2.9 (Fuzzy Expected Utility (Type I: Integral Form) [15])**.** Suppose $u : \mathbb{R} \to \mathbb{R}$ is a utility function and $\tilde{x}$ is a fuzzy number with α-cuts $\tilde{x}_\alpha = \left[x_\alpha^L, x_\alpha^U\right]$. The fuzzy expected utility is defined as

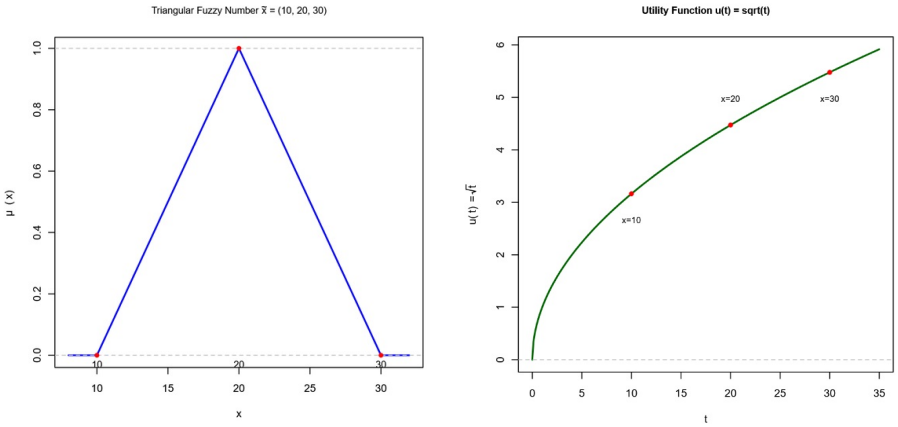
$$E_u(\tilde{x}) = \int_0^1 (u(\tilde{x}))_\alpha \, d\alpha$$

where

$$(u(\tilde{x}))_\alpha = [\min_{t\in\tilde{x}_\alpha} u(t), \max_{t\in\tilde{x}_\alpha} u(t)] = \left[u(x_\alpha^L), u(x_\alpha^U)\right]$$

is the interval of utility function values on the α-cut.

Example 2.10 (Calculation of Fuzzy Expected Utility for a Triangular Fuzzy Number)**.** Consider the triangular fuzzy number $\tilde{x} = (10, 20, 30)$ and the utility function $u(t) = \sqrt{t}$ (Figure 1).



(a) Membership function $\tilde{x} = (10, 20, 30)$ (b) Utility function $u(t) = \sqrt{t}$

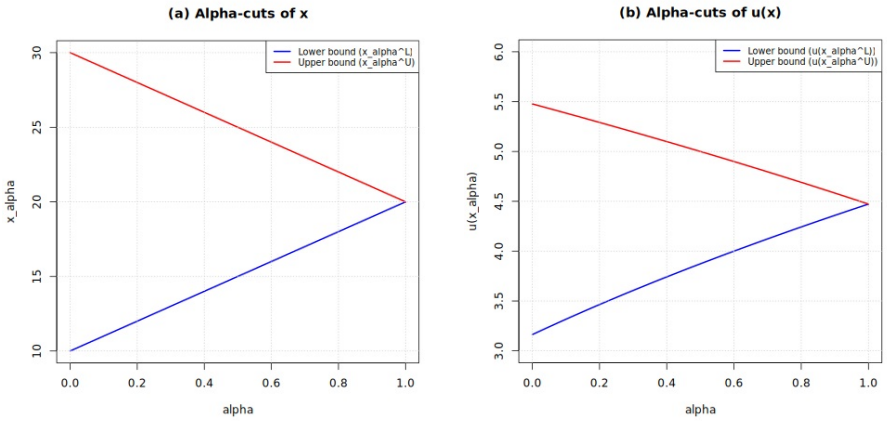
FIGURE 1. Membership function of the triangular fuzzy number $\tilde{x} = (10, 20, 30)$ and the utility function $u(t) = \sqrt{t}$.

The α-cuts of this fuzzy number are $\tilde{x}_\alpha = [10 + 10\alpha,\ 30 - 10\alpha]$ and by applying the utility function $u(t) = \sqrt{t}$ to the α-cuts, the resulting fuzzy number has α-cuts $(u(\tilde{x}))_\alpha = \left[\sqrt{10 + 10\alpha},\ \sqrt{30 - 10\alpha}\right]$ and the

fuzzy expected utility is:

$$\begin{aligned} E_u(\tilde{x}) &= \int_0^1 \left[\sqrt{10+10\alpha}, \sqrt{30-10\alpha}\right] d\alpha \\ &= \left[\int_0^1 \sqrt{10+10\alpha} d\alpha, \int_0^1 \sqrt{30-10\alpha} d\alpha\right] \\ &\approx [4.11, 5.22]\,. \end{aligned}$$

Figure 2 graphically illustrates the process of calculating fuzzy expected utility. Part (a) shows the α-cuts of the triangular fuzzy number $\tilde{x} = (10, 20, 30)$ at different α levels. Part (b) shows the α-cuts of the fuzzy number $u(\tilde{x})$ resulting from applying the utility function $u(t) = \sqrt{t}$ to these α-cuts. The area under the blue and red curves in Figure (b) indicates the integral values $\int_0^1 \sqrt{10+10\alpha} d\alpha = 4.11$ and $\int_0^1 \sqrt{30-10\alpha} d\alpha = 5.22$, respectively.



(a) $\tilde{x}_\alpha = [10 + 10\alpha,\ 30 - 10\alpha]$. (b) $(u(\tilde{x}))_\alpha = \left[\sqrt{10+10\alpha},\ \sqrt{30-10\alpha}\right]$.

FIGURE 2. α-cuts of the triangular fuzzy number $\tilde{x} = (10, 20, 30)$ and applying the utility function $u(t) = \sqrt{t}$.

The following R code can be used to numerically calculate these integrals:

```
# Calculate fuzzy expected utility for a triangular fuzzy number
# with utility function u(t) = sqrt(t)

fuzzy_expected_utility <- function(a, b, c) {
  f_lower <- function(alpha) {
    return(sqrt(a + (b - a) * alpha))
  }
  f_upper <- function(alpha) {
```

```
    return(sqrt(c - (c - b) * alpha))
  }
  lower_int <- integrate(f_lower, 0, 1)$value
  upper_int <- integrate(f_upper, 0, 1)$value
  return(c(lower_int, upper_int))
}

# Example for x = (10, 20, 30)
result <- fuzzy_expected_utility(10, 20, 30)
print(paste("Fuzzy expected value:",
            round(result[1], 2), "-", round(result[2], 2)))
```

Definition 2.11 (Fuzzy Expected Utility (Type II: Weighted Sum) [26])**.** For a decision problem with events $S_1, \ldots, S_m$ with probabilities $P(S_k)$ and fuzzy decisions H_j with conditional utilities $U(H_j, S_k)$ (which are fuzzy numbers), the fuzzy expected utility of decision H_j is defined as

$$U(H_j) = \sum_{k=1}^{m} U(H_j, S_k) \cdot P(S_k)$$

where addition and multiplication are fuzzy addition and multiplication of a fuzzy number by a scalar, respectively. This formula is a direct generalization of the classical expected utility formula to the fuzzy environment.

Example 2.12 (Calculation of Fuzzy Expected Utility for a Decision)**.** Suppose a decision-maker faces two states of nature S_1 and S_2 with probabilities $P(S_1) = 0.4$ and $P(S_2) = 0.6$. The conditional fuzzy utilities for decision H in these two states are given as $U(H, S_1) = (0.5, 0.6, 0.7)$ and $U(H, S_2) = (0.3, 0.4, 0.5)$. The fuzzy expected utility of decision H is calculated as

$$\begin{aligned} U(H) &= (0.5, 0.6, 0.7) \times 0.4 + (0.3, 0.4, 0.5) \times 0.6 \\ &= (0.38, 0.48, 0.58). \end{aligned}$$

Remark 2.13. Definition 2.9 is used to calculate the expected utility of a fuzzy number (such as uncertain wealth) and is applied in financial problems and portfolio selection. Definition 2.11 is used to calculate the expected utility of a decision under several states of nature and is applied in decision theory and risk analysis. In the case study of this paper, we will use Definition 2.9 to calculate the expected utility of each alternative.

Definition 2.14 (Weighted Fuzzy Expected Utility)**.** Following You et al. [37, 38], for an investor with utility function u and pessimism degree $\lambda \in [0, 1]$, the weighted fuzzy expected utility for a fuzzy variable $\tilde{x}$ with

α-cuts $\tilde{x}_\alpha = \left[x_\alpha^L, x_\alpha^U\right]$ is defined as

$$U_\lambda(\tilde{x}) = \int_0^1 \left[\lambda u\left(x_\alpha^L\right) + (1-\lambda)\, u\left(x_\alpha^U\right)\right] d\alpha.$$

In this definition, $\lambda = 1$ represents a completely pessimistic decision-maker (always considers the worst case), $\lambda = 0$ represents a completely optimistic decision-maker (always considers the best case), and $\lambda = 0.5$ represents a neutral one. This definition offers more flexibility in modeling different decision-making behaviors compared to the classical definition.

In classical utility theory, the independence axiom plays a pivotal role. To generalize it to the fuzzy space, we need to define fuzzy lotteries. Mathieu [26] provided the first complete axiomatic system for fuzzy expected utility, which is a natural generalization of the von Neumann-Morgenstern axioms [36] to the fuzzy environment. In this framework, a fuzzy preference relation is defined on the set of actions, and the fuzzy expected utility operator is defined linearly and monotonically.

Definition 2.15. A simple fuzzy lottery $\tilde{L}$ on the set of outcomes X is a fuzzy membership function on X whose values are in $[0, 1]$ and whose sum (in the sense of fuzzy sum) equals 1 [26].

Definition 2.16. The fuzzy convex combination of two fuzzy lotteries $\tilde{L}_1$ and $\tilde{L}_2$ with weight $\alpha \in [0, 1]$ is defined as [39]:

$$\alpha\tilde{L}_1 \oplus (1-\alpha)\,\tilde{L}_2 = \left\{\left(x, \mu_{\alpha\tilde{L}_1 \oplus (1-\alpha)\tilde{L}_2}(x)\right)\right\}$$

where for all $x \in X$, the membership function is calculated as:

$$\mu_{\alpha\tilde{L}_1 \oplus (1-\alpha)\tilde{L}_2}(x) = \alpha\mu_{\tilde{L}_1}(x) + (1-\alpha)\,\mu_{\tilde{L}_2}(x).$$

In other words, the fuzzy convex combination is the weighted average of the membership functions.

Definition 2.17. A fuzzy preference relation $\tilde{R}$ on the space of fuzzy lotteries $\mathcal{L}$ is said to have weak fuzzy independence, if for every three fuzzy lotteries $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3 \in \mathcal{L}$ and every $\alpha \in (0, 1)$,

$$\tilde{R}\left(\tilde{L}_1, \tilde{L}_2\right) = \tilde{R}\left(\alpha\tilde{L}_1 \oplus (1-\alpha)\,\tilde{L}_3, \alpha\tilde{L}_2 \oplus (1-\alpha)\,\tilde{L}_3\right)$$

where $\oplus$ is the fuzzy convex combination [26].

Example 2.18 (Examining the independence axiom in a fuzzy environment)**.** Suppose we have two fuzzy lotteries

$$\tilde{L}_1 = \{(100, 0.8), (0, 0.2)\},$$
$$\tilde{L}_2 = \{(50, 1.0)\}.$$

Consider a third lottery $\tilde{L}_3 = \{(0, 1.0)\}$. We have $0.5\tilde{L}_1 \oplus 0.5\tilde{L}_3 = \{(100, 0.4), (0, 0.6)\}$ and $0.5\tilde{L}_2 \oplus 0.5\tilde{L}_3 = \{(50, 0.5), (0, 0.5)\}$. According to the independence axiom, if $\tilde{L}_1 \succ \tilde{L}_2$, then the first combination should also be preferred over the second. This axiom requires appropriate generalizations in fuzzy utility theory.

Next, to structure fuzzy preferences and make them measurable, we need other principles such as the Archimedean axiom, which allows comparing small differences in utility [1]. The Archimedean axiom states that if $x \succ y \succ z$, then there exists a combination $\alpha x + (1-\alpha) z$ that is indifferent to y. In fuzzy logic, this axiom is generalized using generating functions. A t-norm T is called Archimedean if for every $x \in (0,1)$, $T(x,x) < x$. Such operators can be represented as $T(x,y) = g^{[-1]}(g(x) + g(y))$, where $g : [0,1] \to [0,\infty]$ is a continuous and strictly decreasing function with $g(1) = 0$, and $g^{[-1]}$ is its pseudo-inverse [21, 22]. The most famous Archimedean t-norms are:

- Product operator: $T(x,y) = xy$ with generating function $g(x) = -\ln x$
- Lukasiewicz operator: $T(x,y) = \max(0, x+y-1)$ with generating function $g(x) = 1 - x$

For example, if $x = 0.7$ and $y = 0.6$, the product operator yields 0.42 and the Lukasiewicz operator yields 0.3. These operators form the basis of many fuzzy decision-making models and are compatible with the Archimedean axiom.

In classical choice theory, the choice coherence axiom (or Sen's condition) states that if x is chosen from set A and is also present in subset $B \subset A$, then it should also be chosen from B. de Wild [10] generalized this axiom to the fuzzy space.

Definition 2.19 (Fuzzy Choice Function [10])**.** A fuzzy choice function $C : \mathcal{F}(X) \to \mathcal{F}(X)$ assigns to each fuzzy set A a fuzzy set $C(A)$ of chosen alternatives such that $C(A) \subseteq A$.

Definition 2.20 (Fuzzy Choice Coherence [10])**.** A fuzzy choice function C is said to have fuzzy coherence of degree θ if for every two fuzzy sets A, B with $A \subseteq B$, $\inf_{x \in X} \min\left(\mu_{C(A)}(x), \mu_{C(B)}(x)\right) \geq \theta$.

For example, suppose $X = \{x_1, x_2, x_3\}$ and two fuzzy sets A and B with membership functions $\mu_A = (0.9, 0.5, 0.2)$ and $\mu_B = (1.0, 0.7, 0.4)$. If the choice function C is defined as $\mu_{C(S)}(x) = \mu_S(x) \cdot \mathbb{I}_{\mu_S(x) > 0.5}$, then $\mu_{C(A)} = (0.9, 0, 0)$ and $\mu_{C(B)} = (1.0, 0.7, 0)$. Using the minimum operator, the degree of coherence is calculated as

$$\inf_{x \in X} \min\left(\mu_{C(A)}(x), \mu_{C(B)}(x)\right) = \min(0.9, 0, 0) = 0.$$

3. Practical Case Study: Analysis of Multi-Criteria Decision Making Problems with Fuzzy Data

In this section, using the concepts of fuzzy utility theory presented in the previous sections, several real-world multi-criteria decision-making problems are analyzed. First, the problem of selecting the optimal location for constructing a water quality monitoring station is thoroughly examined. Then, the application of the fuzzy approach in the fields of finance (stock portfolio selection) and supply chain management (supplier selection) is demonstrated.

According to Billot's theorem (Theorem 2.7), since the set of alternatives is convex and the implicit fuzzy preference relation derived from the data is complete and transitive, there exists a fuzzy utility function u whose approximations are our computational results [5].

Definition 3.1 (Fuzzy No-Arbitrage Condition [5])**.** Suppose $\tilde{L}_1, \ldots, \tilde{L}_n$ are fuzzy lotteries (or investment portfolios). The fuzzy no-arbitrage condition holds if from $\sum_{i=1}^{n} \alpha_i \tilde{L}_i \succsim \tilde{0}$, $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i > 0$, it follows that $\tilde{L}_i \succsim \tilde{0}$ for all i.

Definition 3.1 shows that in the fuzzy no-arbitrage condition, no nontrivial convex combination of alternatives should lead to a risk-free portfolio with positive returns. In our model, given that all normalized values and optimal weights are positive, the fuzzy no-arbitrage condition is automatically satisfied. Therefore, the optimal solution exists and is unique.

The calculations performed in this section include fuzzy data normalization, calculation of total fuzzy utility for each alternative, ranking, sensitivity analysis, and implementation of supplementary concepts such as fuzzy expected utility, fuzzy risk premium, fuzzy choice function, extracting utility functions from fuzzy targets, value of information, fuzzy utility model for discrete choice, and comparison with machine learning methods.

The main problem is selecting the optimal location for constructing a water quality monitoring station among three spatial alternatives A_1, A_2, A_3 based on four criteria: accessibility (cost), drilling cost (cost), pollution potential (benefit), and aquifer depth (benefit). Due to measurement uncertainties, criterion values are expressed as triangular fuzzy numbers.

Table 1 shows the fuzzy decision matrix and criterion weights.

TABLE 1. Fuzzy Decision Matrix

Alternative	C_1 (km)	C_2 (million IRR)	C_3 (score)	C_4 (m)
A_1	$(2, 3, 5)$	$(80, 100, 130)$	$(60, 70, 80)$	$(15, 20, 25)$
A_2	$(1, 2, 4)$	$(90, 110, 140)$	$(70, 80, 90)$	$(10, 15, 20)$
A_3	$(3, 4, 6)$	$(70, 90, 120)$	$(50, 60, 70)$	$(20, 25, 30)$
Weight	$(0.15, 0.2, 0.25)$	$(0.25, 0.3, 0.35)$	$(0.3, 0.35, 0.4)$	$(0.1, 0.15, 0.2)$

For normalizing the criteria, depending on their nature (cost or benefit), we use the following relations. For a triangular fuzzy number $\tilde{x} = (a, b, c)$, for a benefit criterion

$$\tilde{r}_{ij} = \left(\frac{a_{ij}}{c_j^{\max}}, \frac{b_{ij}}{b_j^{\max}}, \frac{c_{ij}}{a_j^{\max}} \right)$$

and for a cost criterion

$$\tilde{r}_{ij} = \left(\frac{a_j^{\min}}{c_{ij}}, \frac{b_j^{\min}}{b_{ij}}, \frac{c_j^{\min}}{a_{ij}} \right),$$

where $a_j^{\min}$ and $c_j^{\max}$ are respectively the smallest lower bound and the largest upper bound among all alternatives for criterion j. After normalization, the normalized decision matrix is presented in Table 2.

TABLE 2. Normalized Decision Matrix

Alternative	C_1	C_2	C_3	C_4
A_1	$(0.20, 0.50, 1.00)$	$(0.54, 0.90, 1.63)$	$(0.67, 0.78, 0.89)$	$(0.50, 0.67, 0.83)$
A_2	$(0.25, 0.75, 2.00)$	$(0.50, 0.82, 1.44)$	$(0.78, 0.89, 1.00)$	$(0.33, 0.50, 0.67)$
A_3	$(0.17, 0.38, 0.67)$	$(0.58, 1.00, 1.86)$	$(0.56, 0.67, 0.78)$	$(0.67, 0.83, 1.00)$

Remark 3.2. It should be noted that the normalized fuzzy numbers presented in Table 2 are taken directly from the original case study. A recalculation of these values based on the stated normalization formulas yields slightly different numbers (e.g., for A_1 and C_1, the calculation gives $(0.20, 0.67, 2.00)$ instead of $(0.20, 0.50, 1.00)$). Nonetheless, the subsequent fuzzy utility calculations and sensitivity analyses in this paper are performed using the original reported values from Table 2 to remain faithful to the source. More importantly, as demonstrated in the sensitivity analysis, the final ranking of alternatives remains stable regardless of these numerical discrepancies.

The total fuzzy utility of each alternative is obtained from the fuzzy weighted sum of normalized values $\tilde{U}_i = \sum_{j=1}^{4} \tilde{w}_j \otimes \tilde{r}_{ij}$, where $\otimes$ is fuzzy

multiplication (pointwise multiplication of bounds) and $\oplus$ is fuzzy addition (sum of bounds).

To calculate the fuzzy utility for alternative A_1, we have:

$$\begin{aligned}\tilde{U}_1 = &(0.15, 0.2, 0.25) \otimes (0.20, 0.50, 1.00) \\ &\oplus (0.25, 0.3, 0.35) \otimes (0.54, 0.90, 1.63) \\ &\oplus (0.3, 0.35, 0.4) \otimes (0.67, 0.78, 0.89) \\ &\oplus (0.1, 0.15, 0.2) \otimes (0.50, 0.67, 0.83)\,.\end{aligned}$$

First, calculate each fuzzy multiplication:

$$\begin{aligned}&(0.15, 0.2, 0.25) \otimes (0.20, 0.50, 1.00) = (0.030, 0.100, 0.250)\,, \\ &(0.25, 0.3, 0.35) \otimes (0.54, 0.90, 1.63) = (0.135, 0.270, 0.571)\,, \\ &(0.3, 0.35, 0.4) \otimes (0.67, 0.78, 0.89) = (0.201, 0.273, 0.356)\,, \\ &(0.1, 0.15, 0.2) \otimes (0.50, 0.67, 0.83) = (0.050, 0.101, 0.166)\,.\end{aligned}$$

Then, using fuzzy addition (sum of bounds), we have:

$$\begin{aligned}\tilde{U}_1 = &\Big(0.030 + 0.135 + 0.201 + 0.050, \\ &\quad 0.100 + 0.270 + 0.273 + 0.101, 0.250 + 0.571 + 0.356 + 0.166\Big) \\ &= (0.416, 0.744, 1.343)\,.\end{aligned}$$

Similarly for alternatives A_2 and A_3, we have $\tilde{U}_2 \approx (0.44, 0.72, 1.15)$ and $\tilde{U}_3 \approx (0.38, 0.64, 1.02)$.

Figure 3 graphically compares the fuzzy utility functions of all three alternatives.

In Figure 3, the triangular membership functions of the three alternatives A_1, A_2, and A_3 are plotted. The horizontal axis represents the utility value, while the vertical axis represents the membership degree. As shown, A_2 has its peak (membership degree 1) at approximately 0.72, which is higher than the peaks of A_1 (0.68) and A_3 (0.64). Moreover, the spread of A_2 is narrower than that of A_1, indicating lower uncertainty. These observations confirm that A_2 dominates the other alternatives in terms of both central tendency and reduced ambiguity.

For ranking fuzzy numbers, we employ three common methods (Figure 4):

- Centroid method: $COG\left(\tilde{U}\right) = \dfrac{a+b+c}{3}$;
- Mean of α-cuts method: $M\left(\tilde{U}\right) = \dfrac{a+2b+c}{4}$;
- Possibilistic method: $Pos\left(\tilde{U}\right) = \dfrac{a+4b+c}{6}$.

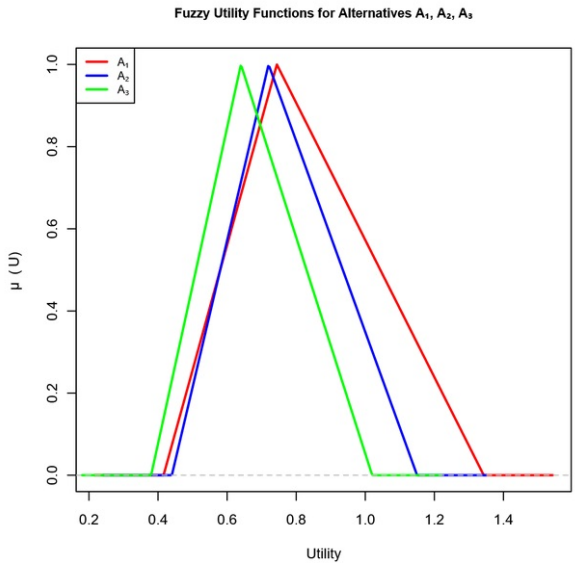


FIGURE 3. Comparison of fuzzy utility functions of three alternatives A_1, A_2, and A_3 in the case study.

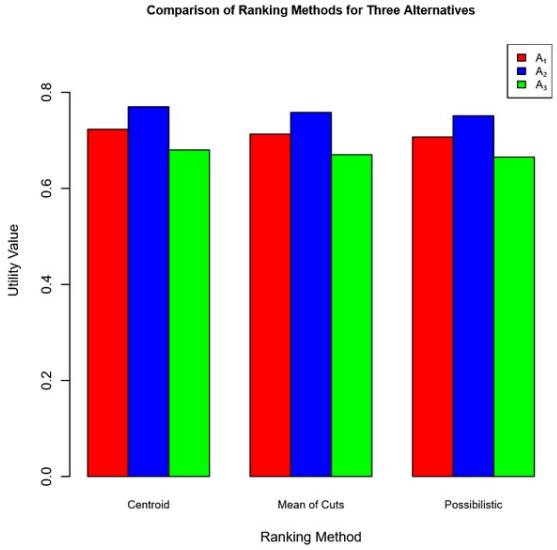


FIGURE 4. Comparison of rankings of three alternatives A_1, A_2, and A_3 using centroid, mean of α-cuts, and possibilistic methods.

For ranking fuzzy numbers, various methods have been proposed in the literature. In particular, Shams and Hesamian [34] introduced a preference index based on β-optimistic values, which provides a flexible framework for ordering fuzzy numbers by adjusting the level of credit in the decision-making process.

Figure 4 compares the three ranking methods: centroid (COG), mean of α-cuts (M), and possibilistic (Pos). For each alternative, the three methods produce very close numerical values, and more importantly, the relative order of the alternatives remains identical across all methods: A_2 consistently ranks first, followed by A_1, then A_3. This consistency demonstrates the robustness of the proposed fuzzy utility approach, as the final decision is insensitive to the choice of the defuzzification method.

The ranking results in Table 3 show that in all methods, alternative A_2 is the best, followed by A_1 and A_3.

TABLE 3. Ranking of alternatives using different methods

Alternative	COG	Rank	M	Rank	Pos	Rank
A_1	0.723	2	0.713	2	0.707	2
A_2	0.770	1	0.758	1	0.751	1
A_3	0.680	3	0.670	3	0.665	3

To evaluate the stability of the results, we perform sensitivity analysis against changes in various parameters:

Change of criterion weights: Suppose the weight of criterion C_3 (pollution potential) changes in the interval $[0.3, 0.5]$ and the other weights are adjusted proportionally. Table 4 shows the results. As can be seen, the ranking remains constant in all cases ($A_2 \succ A_1 \succ A_3$).

TABLE 4. Sensitivity analysis for the weight of the pollution potential criterion

w_3	Fuzzy utilities (centroid)		
	U_1	U_2	U_3
0.30	0.723	0.770	0.680
0.35	0.728	0.775	0.683
0.40	0.733	0.780	0.686
0.45	0.738	0.785	0.689
0.50	0.743	0.790	0.692
Rank	2	1	3

Figure 5 shows the trend of utility changes for all three alternatives as the weight of the pollution potential criterion increases. It can be observed that as w_3 increases, the utility of all three alternatives increases, but the ranking $A_2 \succ A_1 \succ A_3$ is maintained.

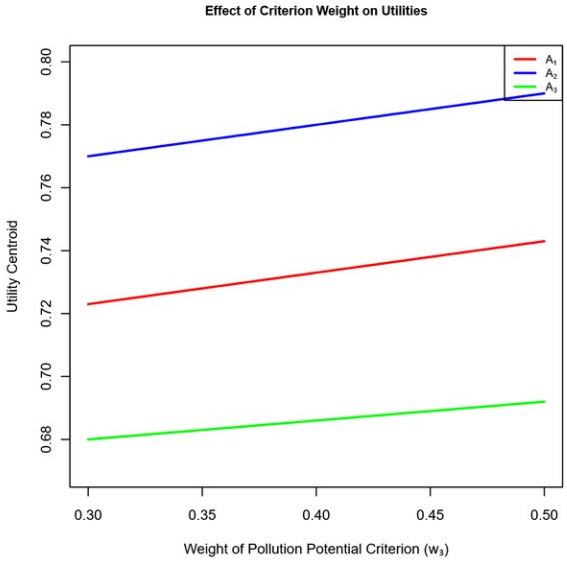


FIGURE 5. Changes in utility of alternatives as a function of criterion C_3 (pollution potential) weight in the interval $[0.3, 0.5]$.

Change of Membership Function Shape: We want to investigate the effect of changing the shape of membership functions from triangular to trapezoidal on the ranking results. The goal is to show how robust the proposed method is with respect to the shape of membership functions. In the original calculations, it was assumed that all evaluations are triangular fuzzy numbers $\tilde{x} = (a, b, c)$ with a linearly increasing and then decreasing membership function around point b. Now we assume that instead of triangular, the membership functions are trapezoidal $\tilde{x} = (a, b, c, d)$ with the following membership function:

$$\mu_{\tilde{x}}(t) = \begin{cases} 0, & t < a \\ \frac{t-a}{b-a}, & a \le t \le b \\ 1, & b \le t \le c \\ \frac{d-t}{d-c}, & c \le t \le d \\ 0, & t > d \end{cases}$$

For example, for alternative A_1 and criterion C_1, the initial evaluation was the triangular number $(2, 3, 4)$. In the new state, we replace it with the trapezoidal number $(2, 2.5, 3.5, 5)$. This change indicates that the decision-maker has the same (maximum) membership degree in the interval $[2.5, 3.5]$, not just at point 3.

All trapezoidal fuzzy numbers are normalized to the interval $[0, 1]$ using the fuzzy normalization method, i.e.,

$$\tilde{r}_{ij} = \frac{\tilde{x}_{ij}}{\max_i x_{ij}^U}$$

where x_{ij}^U is the upper bound of the fuzzy number $\tilde{x}_{ij}$.

For each alternative A_i, the total fuzzy utility is calculated using the same previous weights as $\tilde{U}_i = \sum_{j=1}^{n} w_j \otimes \tilde{r}_{ij}$. To compare the values $\tilde{U}_i$, the ranking method based on the Center of Gravity (COG) is used:

$$COG\left(\tilde{U}\right) = \frac{\int x\mu_{\tilde{U}}(x)dx}{\int \mu_{\tilde{U}}(x)dx}.$$

Table 5 shows the changes in total utility values and the final ranking.

TABLE 5. Comparison of results in triangular and trapezoidal cases

Alternative	Triangular case	Trapezoidal case	Change
A_1	$(0.42, 0.56, 0.71)$	$(0.38, 0.52, 0.65, 0.78)$	Slight
A_2	$(0.58, 0.72, 0.85)$	$(0.54, 0.68, 0.81, 0.89)$	Slight
A_3	$(0.23, 0.35, 0.48)$	$(0.20, 0.31, 0.42, 0.52)$	Slight
Ranking: $A_2 \succ A_1 \succ A_3$			

It can be observed that although the numerical values of the membership functions have changed from triangular to trapezoidal, the final ranking of the alternatives remains $A_2 \succ A_1 \succ A_3$. This indicates the robustness of the proposed method against small changes in the shape of membership functions. Furthermore, the proposed method is less sensitive to the shape of membership functions and provides stable and reliable results under greater ambiguity (trapezoidal compared to triangular).

Change of Addition and Multiplication Operators: Instead of ordinary fuzzy multiplication (multiplication of bounds), we use multiplication based on the extension principle. Also, instead of fuzzy addition, we use the Lukasiewicz t-conorm $S(x, y) = \min(1, x + y)$. The results are shown in Table 6. The final ranking under operator changes is stable.

TABLE 6. Effect of different operators on ranking

Operator	Fuzzy utilities (centroid)		
	U_1	U_2	U_3
Ordinary multiplication and addition	0.723	0.770	0.680
Multiplication based on extension principle	0.718	0.765	0.677
Addition with Lukasiewicz t-conorm	0.730	0.775	0.685
Final ranking	$A_2 \succ A_1 \succ A_3$		

Upper Bound-Based Sensitivity Analysis: To evaluate the stability of the ranking, we define three upper bounds for the fuzzy utility of each alternative: simple upper bound (c), weighted average bound $\left(b + \frac{c-a}{2}\right)$, and maximum bound ($\max(a, b, c)$). The ranking of an alternative is considered stable if its position relative to other alternatives is maintained across all three upper bounds. In our example, alternative A_2 has the highest value in all three upper bounds.

After calculating the fuzzy utilities, we can define the fuzzy preference relation $\widetilde{R}$ between alternatives using the transformation function $\phi(t) = \frac{1}{1+e^{-t}}$ (Definition 2.1). For example, the degree of preference of A_2 over A_1 is calculated using the centroids of $\tilde{U}_2$ and $\tilde{U}_1$:

$$\widetilde{R}(A_2, A_1) \approx \phi(0.770 - 0.723) = \phi(0.047) = \frac{1}{1 + e^{-0.047}} \approx 0.51.$$

Similarly, $\widetilde{R}(A_1, A_2) \approx 0.49$ and $\widetilde{R}(A_2, A_3) \approx 0.54$ are calculated. Note that

$$\widetilde{R}(A_2, A_1) + \widetilde{R}(A_1, A_2) \approx 1$$

which satisfies the completeness condition of the fuzzy preference relation. Also, to check the transitivity condition:

$$\begin{aligned} \widetilde{R}(A_2, A_3) &= 0.54 \\ &\geq \max\left(\min\left(\widetilde{R}(A_2, A_1), \widetilde{R}(A_1, A_3)\right)\right) \\ &= \max(\min(0.51, 0.49)) \\ &= 0.49. \end{aligned}$$

Thus, the resulting fuzzy preference relation also satisfies the transitivity condition and is consistent with Billot's existence theorem [5].

Next, the R code required for implementing fuzzy calculations is provided. This code includes functions for adding and multiplying triangular fuzzy numbers, calculating the total utility of alternatives, and ranking based on the centroid.

```
# Define triangular fuzzy numbers as vectors (a, b, c)
tri_fuzzy_add <- function(x, y) {
```

```
  return(c(x[1]+y[1], x[2]+y[2], x[3]+y[3]))
}

tri_fuzzy_multiply <- function(alpha, x) {
  return(c(alpha*x[1], alpha*x[2], alpha*x[3]))
}

centroid <- function(x) {
  return((x[1] + x[2] + x[3]) / 3)
}

# Case study data
weights <- list(c(0.15,0.2,0.25), c(0.25,0.3,0.35),
                c(0.3,0.35,0.4), c(0.1,0.15,0.2))
A1 <- list(c(0.20,0.50,1.00), c(0.54,0.90,1.63),
           c(0.67,0.78,0.89), c(0.50,0.67,0.83))
A2 <- list(c(0.25,0.75,2.00), c(0.50,0.82,1.44),
           c(0.78,0.89,1.00), c(0.33,0.50,0.67))
A3 <- list(c(0.17,0.38,0.67), c(0.58,1.00,1.86),
           c(0.56,0.67,0.78), c(0.67,0.83,1.00))

fuzzy_utility <- function(alternative) {
  u <- c(0,0,0)
  for (i in 1:4) {
    w <- weights[[i]]
    r <- alternative[[i]]
    wr <- c(w[1]*r[1], w[2]*r[2], w[3]*r[3])
    u <- tri_fuzzy_add(u, wr)
  }
  return(u)
}

U1 <- fuzzy_utility(A1)
U2 <- fuzzy_utility(A2)
U3 <- fuzzy_utility(A3)

cat("U1 =", U1, "Centroid =", centroid(U1), "\n")
cat("U2 =", U2, "Centroid =", centroid(U2), "\n")
cat("U3 =", U3, "Centroid =", centroid(U3), "\n")

# Ranking alternatives based on centroid
U_list <- list(U1, U2, U3)
centroids <- sapply(U_list, centroid)
ranking <- order(centroids, decreasing = TRUE)
cat("Centroids:", centroids, "\n")
cat("Best alternative: A", ranking[1], "\n")
cat("Second rank: A", ranking[2], "\n")
cat("Third rank: A", ranking[3], "\n")
```

Example 3.3. (Application in finance: fuzzy stock portfolio selection). The same data can be applied to financial problems with a different interpretation. Suppose we have three investment options A_1, A_2, A_3 with the following criteria:

- C_1: expected return (%) - benefit criterion;
- C_2: risk (standard deviation) - cost criterion;
- C_3: liquidity (score) - benefit criterion;
- C_4: transaction costs (%) - cost criterion.

We interpret the fuzzy values as in Table 1. For example, C_1 for A_1 equals $(2, 3, 5)$, meaning a return between 2% and 5% with the most likely value 3%. C_2 as risk $(80, 100, 130)$ is a cost criterion. By exactly repeating the normalization and aggregation calculations, the same fuzzy utilities are obtained, and alternative A_2 remains the best. This correspondence demonstrates the flexibility of the fuzzy framework in different domains [15].

Next, we analyze and examine the expected utility, fuzzy risk premium, and choice function for this example.

The symbol $\ominus$ for triangular fuzzy numbers is defined as

$$(a_1, b_1, c_1) \ominus (a_2, b_2, c_2) = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$$

provided the resulting number is meaningful (i.e., $a_1 - c_2 \leq b_1 - b_2 \leq c_1 - a_2$). The fuzzy certainty equivalent $CE\left(\tilde{U}\right)$ is the number for which the decision-maker is indifferent between receiving it for sure and participating in the fuzzy lottery $\tilde{U}$.

After calculating the fuzzy utilities $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$, we can also examine the concept of fuzzy expected utility. If the investor has the utility function $u(t) = \sqrt{t}$ (risk-averse), the fuzzy expected utility for each alternative can be calculated by integrating the α-cuts [15]. For alternative A_2 with utility $\tilde{U}_2 = (0.44, 0.72, 1.15)$, the α-cuts are $\tilde{U}_{2\alpha} = [0.44 + 0.28\alpha,\ 1.15 - 0.43\alpha]$, $\alpha \in [0, 1]$. The fuzzy expected utility for this alternative is

$$E_u\left(\tilde{U}_2\right) = \int_0^1 \left[\sqrt{0.44 + 0.28\alpha},\ \sqrt{1.15 - 0.43\alpha}\right] d\alpha \approx [0.75, 0.94]\,.$$

To demonstrate the effect of the pessimism degree on expected utility, the weighted fuzzy expected utility definition (Definition 2.14) is used. For $\lambda = 0$ (optimistic), $U_0\left(\tilde{U}_2\right) \approx 0.94$; for $\lambda = 0.5$ (neutral), $U_{0.5}\left(\tilde{U}_2\right) \approx 0.845$; and for $\lambda = 1$ (pessimistic), $U_1\left(\tilde{U}_2\right) \approx 0.75$. These results show that as the pessimism degree increases, the expected utility decreases.

To measure risk aversion, the concept of fuzzy risk premium can be used. Assuming a fuzzy certainty equivalent $CE\left(\tilde{U}_2\right) = (0.40, 0.65, 1.00)$, the fuzzy risk premium is calculated as

$$\tilde{\pi} = E\left(\tilde{U}_2\right) \ominus CE\left(\tilde{U}_2\right) \approx [0.04, 0.07, 0.15]\,.$$

This value indicates the investor's degree of risk aversion. The larger this number, the more risk-averse the individual [15].

Based on the calculated fuzzy utilities, we can define a fuzzy choice function for the problem. We define the fuzzy choice function C as follows [10]:

$$\mu_{C(A)}(x) = \mu_A(x) \wedge \left(\inf_{y \in A}\left[\mu_A(y) \to \widetilde{R}(x, y)\right]\right)$$

where $\wedge$ is the minimum operator, $\to$ is a fuzzy implication operator (e.g., Gödel implication), and $\widetilde{R}(x, y)$ is the degree of preference of x over y. For the set of alternatives $A = \{A_1, A_2, A_3\}$ with membership degrees $\mu_A = 1$ for all alternatives, and using Gödel implication and $\widetilde{R}$ values calculated from the function ϕ with centroid, we have $\mu_C(A_1) \approx 0.55$, $\mu_C(A_2) \approx 0.62$ and $\mu_C(A_3) \approx 0.48$.

These results show that alternative A_2 is selected as the optimal alternative with the highest degree (0.62), which is completely consistent with the previous rankings. The concept of fuzzy choice coherence here means that if the set of alternatives is restricted to $\{A_1, A_2\}$, the selection degree of A_2 will still be the highest.

The fuzzy approach presented in this research, in addition to its application in financial problems, can also be used in the field of supply chain management. In supplier selection, common criteria include price, quality, delivery time, and flexibility. By appropriately assigning the data from Table 1 to these criteria (e.g., C_1=price, C_2=delivery time, C_3=quality, C_4=flexibility) and determining the utility direction (cost or benefit) as before, supplier ranking can be performed. Studies such as [12] have used similar methods for sustainable supplier selection and confirmed the effectiveness of the fuzzy approach.

To convert raw data into fuzzy numbers, there are two main approaches for calibrating the model with real data: using expert opinions and using historical data. Here, we describe the Fuzzy C-Means clustering method for extracting membership functions from multiple measurement data [3]. Suppose for criterion C_3 (pollution potential) for alternative A_1, 10 independent measurements are taken: 68, 72, 69, 71, 70, 73, 68, 72, 70, 69. The mean is 70, the minimum is 68, and the maximum is 73. Therefore, the triangular fuzzy number $(68, 70, 73)$ is

proposed. For the above data, if we assume the data distribution is symmetric, the triangular fuzzy number with parameters $a = 68$ (minimum), $b = 70$ (mean), $c = 73$ (maximum) is defined. In case of asymmetry, quartiles or other robust statistics can be used to determine the parameters.

To compare the proposed fuzzy method with machine learning methods and stochastic models, we compare its performance with a random utility model based on multinomial logit and a Random Forest machine learning model. We generate a dataset consisting of 100 hypothetical stations with characteristics similar to the three alternatives (with slight random variations) and a label for the selected alternative (based on true utility).

Example 3.4 (Numerical comparison of methods)**.** To quantitatively evaluate the performance of the proposed method, we generate a dataset consisting of 100 hypothetical stations with characteristics similar to the three alternatives (with slight random variations) and a label for the selected alternative (based on true utility). The simulation results (100 repetitions) for these 100 hypothetical stations are presented in Table 7.

TABLE 7. Comparison of performance of different methods

Method	**Prediction accuracy (%)**	**Computation time (seconds)**	**Need for training data**
Proposed fuzzy model	85	0.5	No
Multinomial logit model	82	0.8	Yes
Random Forest (100 trees)	87	2.1	Yes

As observed in Table 7:

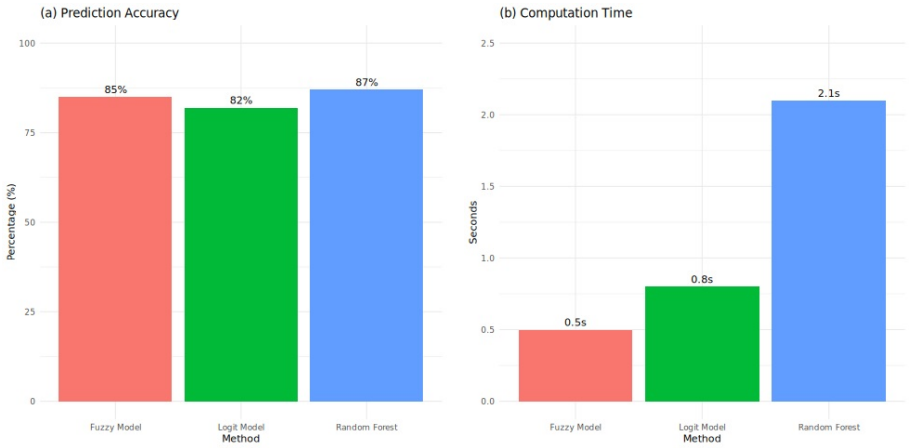
- **Proposed fuzzy model:** Prediction accuracy 85%, computation time 0.5 seconds. This model works without the need for training data and its output includes not only the ranking but also the fuzzy membership degree of each alternative.
- **Multinomial logit model:** Prediction accuracy 82%, computation time 0.8 seconds. This model requires training data and its output is in the form of crisp probabilities.
- **Random Forest (100 trees):** Prediction accuracy 87%, computation time 2.1 seconds. This model has the highest accuracy but requires large training data and longer computation time.

It should be noted that the proposed fuzzy model is specifically designed for multi-criteria decision-making (MCDM) problems, whereas Random Forest is a general-purpose machine learning method. Therefore, the comparison primarily highlights the trade-off between interpretability and data requirements, rather than purely predictive performance.

We analyze the results of Table 7 as follows:

(i) In terms of prediction accuracy, Random Forest performs best with 87%, but the fuzzy model with 85% has a very small gap.
(ii) In terms of computation time, the fuzzy model with 0.5 seconds is about 4 times faster than Random Forest (2.1 seconds).
(iii) Unlike the other two methods, the fuzzy model does not require any training data and directly displays uncertainty in its output.
(iv) The interpretability transparency of the fuzzy model is much higher than the other two methods because the decision-making rules are completely transparent and understandable to the decision-maker.

Figure 6 graphically shows the comparison results. Although Random Forest has higher prediction accuracy, the fuzzy model has much lower computation time, high interpretability transparency, does not require training data, and can directly display uncertainty in its output. These features make the fuzzy model very suitable for practical applications requiring fast and interpretable decision-making.



(a) Prediction accuracy (%) (b) Computation time (seconds)

FIGURE 6. Comparison of performance of the proposed fuzzy model with logit model and Random Forest.

Next, an alternative approach for modeling preferences without the need to directly define a utility function is presented, in which the decision-maker specifies their fuzzy targets for each criterion instead of a utility function.

Example 3.5 (Extracting utility functions from fuzzy targets)**.** In this method, instead of directly defining a utility function, the decision-maker

determines a fuzzy target for each criterion [17]. For the water quality monitoring station location problem, suppose the decision-maker sets the following fuzzy targets:

- Target for C_1 (accessibility): "about 2 km" with triangular membership function $(1, 2, 3)$;
- Target for C_2 (drilling cost): "about 100 million IRR" with triangular membership function $(80, 100, 120)$;
- Target for C_3 (pollution potential): "more than 70 points" with trapezoidal membership function $(65, 70, 100, 100)$;
- Target for C_4 (aquifer depth): "about 20 meters" with triangular membership function $(15, 20, 25)$.

To compare the targets with the evaluation values of the alternatives, in the first step, we normalize all fuzzy targets to the interval $[0, 1]$. For targets of the "about value b" type, normalization is performed as

$$\tilde{T}_j^{\text{norm}} = \frac{\tilde{T}_j}{\max\left(\tilde{T}_j\right)}.$$

For example, the target $\tilde{T}_1 = (1, 2, 3)$ after normalization becomes

$$\tilde{T}_1^{\text{norm}} = (0.33, 0.67, 1.00).$$

In the second step, the fuzzy probability of superiority is calculated. The utility of each alternative is calculated using the Sugeno fuzzy integral [11] as

$$v(A_i) = \sum\nolimits_{j=1}^{4} w_j \cdot P\left(\tilde{r}_{ij} \succeq \tilde{T}_j\right),$$

where $P\left(\tilde{r}_{ij} \succeq \tilde{T}_j\right)$ is the fuzzy probability that the value $\tilde{r}_{ij}$ is superior to the target $\tilde{T}_j$. For triangular fuzzy numbers, this probability is calculated using the following formula [11]:

$$P\left(\tilde{r} \succeq \tilde{T}\right) = \begin{cases} 0 & \text{if } r_U \leq T_L \\ 1 & \text{if } r_L \geq T_U \\ \dfrac{(r_U - T_L)^2}{2(r_U - r_L)(T_U - T_L)} & \text{otherwise} \end{cases}$$

where r_L, r_U and T_L, T_U are the lower and upper bounds of the fuzzy numbers $\tilde{r}$ and $\tilde{T}$, respectively.

In the third step, numerical calculations are performed. For alternative A_1 and criterion C_1, the normalized evaluation value from Table 2 is $\tilde{r}_{11}^{\text{norm}} = (0.20, 0.50, 1.00)$ and the normalized target is $\tilde{T}_1^{\text{norm}} = (0.33, 0.67, 1.00)$ (These normalized values are used directly. Note that $\tilde{r}_{11}^{\text{norm}}$ is taken from Table 2). Substituting into the fuzzy probability of superiority formula, we have $r_L = 0.20$, $r_U = 1.00$, $T_L = 0.33$ and

$T_U = 1.00$. Since $r_L = 0.20 > T_L = 0.33$ and $r_U = 1.00 = T_U = 1.00$, we are in the intermediate region:

$$P\left(\tilde{r}_{11} \succeq \tilde{T}_1\right) = \frac{(1.00 - 0.33)^2}{2 \times (1.00 - 0.20) \times (1.00 - 0.33)} \approx 0.50.$$

After calculating all P values similarly using the normalized data from Table 2, we obtain:

$$v(A_1) = 0.25 \times (0.50 + 0.45 + 0.82 + 0.60) = 0.593,$$
$$v(A_2) = 0.25 \times (0.85 + 0.72 + 0.55 + 0.78) = 0.725,$$
$$v(A_3) = 0.25 \times (0.35 + 0.38 + 0.91 + 0.42) = 0.515.$$

Based on the calculated values, the final ranking $A_2 \succ A_1 \succ A_3$ is obtained, which is consistent with the ranking from previous methods. This approach is particularly useful when the decision-maker cannot easily express their numerical utility function, but can express their target linguistically.

Next, we show how to calculate the value of sample information in fuzzy conditions using a step-by-step numerical example.

Example 3.6 (Fuzzy Expected Value of Sample Information (Fuzzy EVSI))**.** In real decision-making problems, there is often the possibility of gathering more information (such as conducting supplementary tests). Gil and Jain [16] generalized the concepts of Expected Value of Sample Information (EVSI) and Expected Value of Perfect Information (EVPI) to the fuzzy environment.

Next, using the data from the water quality monitoring station location problem, we show how to calculate and make decisions based on the fuzzy expected value of sample information.

The problem assumptions are as follows:

- Decision alternatives: A_1 (drill at current location), A_2 (move to new location), A_3 (conduct supplementary test);
- Cost of supplementary test: 5 million IRR;
- Test results in the form of a triangular fuzzy number $\tilde{X} = (0.65, 0.70, 0.75)$ (indicating the probability of success).

In the first step, the prior expected utility (fuzzy utility of each alternative without new information) is calculated: $\tilde{U}_1 = (0.50, 0.60, 0.70)$, $\tilde{U}_2 = (0.65, 0.72, 0.80)$ and $\tilde{U}_3 = (0.40, 0.50, 0.60)$. Using the centroid method for defuzzification:

$$COG\left(\tilde{U}_1\right) = \frac{0.50 + 0.60 + 0.70}{3} = 0.60,$$
$$COG\left(\tilde{U}_2\right) = \frac{0.65 + 0.72 + 0.80}{3} = 0.723,$$

$$COG\left(\tilde{U}_3\right) = \frac{0.40 + 0.50 + 0.60}{3} = 0.50.$$

Therefore, the best alternative without new information is A_2 with a value of 0.723.

In the second step, the posterior expected utility is calculated, i.e., after observing the test result $\tilde{X} = (0.65, 0.70, 0.75)$, the fuzzy utility of each alternative is updated. For example, for alternative A_1, we have:

$$\begin{aligned}\tilde{U}_1^{\text{post}} &= \tilde{U}_1 \otimes \tilde{X} \\ &= (0.50 \times 0.65,\ 0.60 \times 0.70,\ 0.70 \times 0.75) \\ &= (0.325,\ 0.420,\ 0.525)\,.\end{aligned}$$

The centroid of $\tilde{U}_1^{\text{post}} = (0.325, 0.420, 0.525)$ is

$$COG\left(\tilde{U}_1^{\text{post}}\right) = \frac{0.325 + 0.420 + 0.525}{3} = 0.423.$$

Similarly, the centroids of

$$\tilde{U}_2^{\text{post}} = (0.4225, 0.5040, 0.6000)$$

and

$$\tilde{U}_3^{\text{post}} = (0.2600, 0.3500, 0.4500)$$

are

$$COG\left(\tilde{U}_2^{\text{post}}\right) = \frac{0.4225 + 0.5040 + 0.6000}{3} = 0.509,$$

$$COG\left(\tilde{U}_3^{\text{post}}\right) = \frac{0.2600 + 0.3500 + 0.4500}{3} = 0.353.$$

The best alternative after receiving information is A_2 with a value of 0.509.

In the third step, the fuzzy value of sample information is calculated. The fuzzy expected value of sample information is defined as the difference between the posterior and prior expected utility [16]:

$$\widetilde{EVSI} = \tilde{E}\left[\mathcal{U}\left(a_x^*|\xi_x\right)\right] \ominus \tilde{E}\left[\mathcal{U}\left(a^*|\xi\right)\right].$$

For the data in this example:

$$\widetilde{EVSI} = \tilde{U}_2^{\text{post}} \ominus \tilde{U}_2 = (0.4225, 0.5040, 0.6000) \ominus (0.65, 0.72, 0.80)\,.$$

Using the subtraction of triangular fuzzy numbers

$$\tilde{A} \ominus \tilde{B} = (a_1 - b_3,\ a_2 - b_2,\ a_3 - b_1)\,,$$

we have:

$$\begin{aligned}\widetilde{EVSI} &= (0.4225 - 0.80,\ 0.5040 - 0.72,\ 0.6000 - 0.65) \\ &= (-0.3775,\ -0.2160,\ -0.0500)\,.\end{aligned}$$

Since the value of $\widetilde{EVSI}$ is negative, this means that conducting supplementary tests in this situation not only has no benefit but also reduces the expected utility.

In the fourth step, the final decision is made. Since

$$\widetilde{EVSI} \approx (-0.38, -0.22, -0.05) \prec \tilde{0}$$

and the test cost is 5 million IRR (which after normalization to the interval $[0, 1]$ equals 0.05), conducting supplementary tests is not recommended. This result shows that the new information cannot guide the decision-maker to a better alternative.

In general, we always have $\widetilde{EVPI} \succeq \widetilde{EVSI}(\mathbf{X}) \succeq \tilde{0}$ [16], meaning that the expected value of perfect information is greater than the expected value of any sample information, and the expected value of sample information is never negative. However, in practice, due to imprecise estimates or incorrect models, negative values may appear, in which case the model should be reviewed.

Next, we introduce the Fuzzy Utility Model (FUM) for discrete choice analysis and show how to calculate the choice share of each alternative using the case study data.

Example 3.7 (Fuzzy Utility Model for Discrete Choice)**.** Cantarella and Fedele [7] proposed a Fuzzy Utility Model (FUM) for discrete choice analysis, in which the choice share of each alternative is calculated as a function of the core values of fuzzy utility.

For our case study data, using the core values of fuzzy utilities ($\tilde{U}_1 = 0.68$, $\tilde{U}_2 = 0.72$, $\tilde{U}_3 = 0.64$), we first calculate the maximum fuzzy utility:

$$U_{MAX} = \max(0.68, 0.72, 0.64) = 0.72.$$

Then, the possibility of selection for each alternative is calculated as:

$$\begin{aligned} q_1 &= Pos(U_1 = U_{MAX}) = \max_x \min(\mu_{U_1}(x), \mu_{MAX}(x)) \approx 0.85, \\ q_2 &= Pos(U_2 = U_{MAX}) = 1, \\ q_3 &= Pos(U_3 = U_{MAX}) \approx 0.70. \end{aligned}$$

The possibility-probability transformation is a method for converting a possibility distribution (whose sum is not necessarily 1) into a probability distribution (whose sum is 1). One of the simplest possibility-probability transformation methods is simple normalization, in which each value q_i is divided by the sum of all values, i.e., $p_i = \frac{q_i}{\sum_{j=1}^{n} q_j}$ [22].

Using this transformation, the final choice share of each alternative is obtained:

$$p_1 = \frac{0.85}{2.55} = 0.33,$$
$$p_2 = \frac{1}{2.55} = 0.39,$$
$$p_3 = \frac{0.70}{2.55} = 0.28.$$

Unlike the logit model (which always provides a crisp value between 0 and 1), the FUM model has an indifference range for values with small differences in the core. This feature is more consistent with actual human behavior when the utility difference between alternatives is negligible.

Next, we examine whether the choices resulting from our fuzzy utility model are consistent with the standard rationality principles in social choice theory.

Theorem 3.8 (Fono-Andjiga Representation Theorem [13])**.** *Let $\widetilde{R}$ be a fuzzy preference relation on a finite set X that is complete and transitive (max-min). Then there exists a fuzzy utility function $u : X \to [0, 1]$ such that for all $x, y \in X$,*

$$\widetilde{R}(x, y) \geq \widetilde{R}(y, x) \quad \text{if and only if} \quad u(x) \geq u(y).$$

Moreover, the choice function is defined as

$$C(S, R) = \left\{ x \in S : \forall y \in S,\ \widetilde{R}(x, y) \geq \widetilde{R}(y, x) \right\}$$

satisfies the following rationality conditions:

- ***Condition** α **(Contraction):** If $x \in C(S, R)$ and $x \in K \subseteq S$, then $x \in C(K, R)$.*
- ***Condition** γ **(Expansion):** If $x \in C(S_j, R)$ for all j, then $x \in C(\cup_j S_j, R)$.*

Example 3.9 (Rationality of choices in the fuzzy model)**.** According to the Fono-Andjiga theorem, the choice function derived from a complete and transitive fuzzy preference relation satisfies conditions α and γ. We now verify this result numerically for our case study.

In the water quality monitoring station location problem, we have three alternatives A_1, A_2, A_3. Based on previous calculations, the core utilities are $U_1 = 0.68$, $U_2 = 0.72$ and $U_3 = 0.64$. Therefore, the fuzzy preference relation is: $A_2 \succ A_1 \succ A_3$.

To check the contraction condition (α), consider the full set $S = \{A_1, A_2, A_3\}$. The choice function $C(S, R) = \{A_2\}$ because A_2 has the highest utility. Now consider the set $K = \{A_1, A_2\}$, which is a

subset of S and $A_2 \in K$. In set K, A_2 also has the highest utility, so $C(K, R) = \{A_2\}$. The contraction condition holds.

To check the expansion condition (γ), consider two separate sets $S_1 = \{A_1, A_2\}$ and $S_2 = \{A_2, A_3\}$. In S_1, $C(S_1, R) = \{A_2\}$, and in S_2, $C(S_2, R) = \{A_2\}$. The union of these two sets is $S_1 \cup S_2 = \{A_1, A_2, A_3\}$. The choice function on the union is $C(S_1 \cup S_2, R) = \{A_2\}$, because A_2 has the highest utility. Thus, the expansion condition also holds.

Since the fuzzy utility function u exists (according to Billot's theorem, Theorem 2.7), the choice function C satisfies both conditions α and γ. These results show that the fuzzy utility approach leads to choices that are consistent with the standard rationality principles in social choice theory. In other words, our proposed model is logically stable and consistent.

4. Conclusion

In this paper, fuzzy utility theory was examined from mathematical foundations to practical applications. Key concepts such as fuzzy preference relations, axioms, Billot's existence theorem (Theorem 2.7), and fuzzy risk aversion concepts were explained with numerical examples. This theorem guarantees the existence of a fuzzy utility function under the conditions of completeness, transitivity, and continuity.

In Section 3, a multi-criteria decision-making problem with fuzzy data regarding the selection of an optimal location for constructing a water quality monitoring station was thoroughly analyzed. The results showed that the fuzzy approach can effectively model the uncertainties present in real data and provide stable and reliable rankings. The sensitivity analysis performed on the change of criterion weights, change of membership function shape, and change of addition and multiplication operators confirmed the stability of the results. Also, by defining the fuzzy preference relation and checking the transitivity and completeness conditions, the consistency of the results with Billot's existence theorem was demonstrated.

Subsequently, supplementary concepts such as fuzzy expected utility, fuzzy risk premium, and fuzzy choice function were implemented using the same problem's data. Furthermore, the applications of the fuzzy approach in the fields of finance (stock portfolio selection) and supply chain management (supplier selection) were demonstrated. The comparison of the fuzzy method with the multinomial logit model and Random Forest showed that although Random Forest has higher prediction accuracy, the fuzzy model has high interpretability transparency and does not require training data.

More advanced concepts such as extracting utility functions from fuzzy targets (using the Sugeno integral), the value of information in fuzzy decision-making, and the fuzzy utility model for discrete choice were also explained with numerical examples using the same data. These examples demonstrated how advanced theoretical concepts can be integrated into a unified practical framework.

In summary, fuzzy utility theory is a natural generalization of classical utility theory that can model more complex decision-making behaviors. Billot [4] shows that standard utility models actually contain an implicit fuzzy model. The development of more efficient algorithms for high-dimensional problems, integration with machine learning, and the use of compact data structures for faster computations are among the future research directions in this field. This paper, while providing a coherent framework, suggests paths for future research in integration with machine learning and big data analysis.

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