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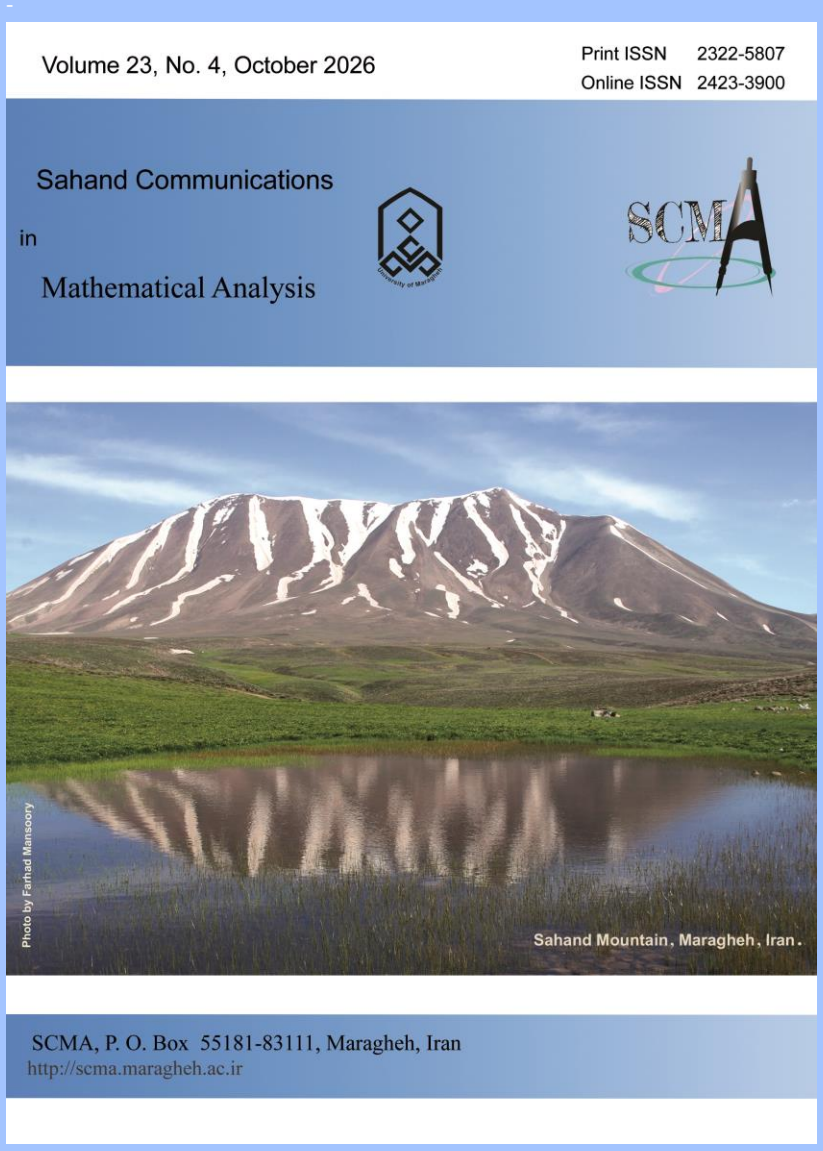

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Some New Results in Intuitionistic Fuzzy Topological Spaces

Behnam Talaee[1] and Rohollah Bakhshandeh-Chamazkoti[2*]

Abstract. This paper presents new results in the field of Intuitionistic Fuzzy Topological Spaces (IFTSs), a mathematical framework developed to handle uncertainty and vagueness in complex systems. Building on Atanassov's intuitionistic fuzzy set theory, this research expands the concept of topological spaces by incorporating intuitionistic fuzzy sets, leading to novel properties and theorems. The work addresses key aspects of IFTSs, including continuity, compactness, connectedness, separation axioms, and dimension theory. Additionally, the paper explores homeomorphisms in IFTSs, providing new results on the existence and uniqueness of such mappings. These findings contribute to the theoretical foundation of IFTSs and extend their applicability to real-world problems involving uncertainty, such as decision-making, pattern recognition, and artificial intelligence. The research also includes discussions on IF convergence and its interplay with continuity and compactness, further enriching the understanding of IFTSs.

The paper culminates with an exploration of Intuitionistic Fuzzy Topological Vector Spaces (IFTVS), examining the interaction between vector space operations and IF topology. Several results pertaining to IF continuous linear maps, their tangent properties, and scalar multiplication within the IFTVS framework are also explored. These findings open new avenues for applying fuzzy logic in topological vector spaces, with potential applications in functional analysis, optimization, and systems modeling.

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1. Introduction

In 1986, Atanassov proposed the concept of intuitionistic fuzzy set theory as an extension of Zadeh's fuzzy set theory. This innovative theory offers a flexible framework for addressing uncertainty and vagueness inherent in decision-making scenarios.

The theory of fuzzy topological spaces has evolved as a significant area of study within general topology and fuzzy set theory, contributing to a wide range of applications in mathematics and applied sciences. Over the past few decades, numerous research papers have been published, exploring various generalizations and extensions of classical topology within the framework of fuzzy sets. The intrinsic flexibility of fuzzy topological spaces in handling uncertainty and imprecision makes them ideal for modeling real-world phenomena where binary classifications fall short, [5–7, 12]

Intuitionistic Fuzzy Topological Spaces (IFTS), [8, 10] extend classical topology by integrating intuitionistic fuzzy sets, which consider both degrees of membership and non-membership, offering an effective tool for handling uncertainty and vagueness in applied mathematics and engineering. In applied mathematics, IFTS are employed in optimization problems, especially in multi-objective scenarios where incomplete or imprecise information needs to be modeled. They are also used in generalizing fixed point theorems and in game theory to manage strategic decisions under uncertainty, [3, 13, 16]. In engineering, IFTS find applications in control systems, [8], where they improve decision-making in systems with uncertain sensor inputs, and in signal processing, enhancing the ability to handle noisy or incomplete data. They are also valuable in robotics, aiding in environmental perception under ambiguous conditions, and in multi-criteria decision-making models that involve uncertainties in costs, risks, or system performance. The dual membership structure of IFTS provides a nuanced approach to problems in these fields, making it a powerful tool for modeling and decision-making under uncertainty.

The aim of this research paper is to present some new results in the field of intuitionistic fuzzy topological spaces. These results contribute to the theoretical foundation and practical applicability of IFTSs, paving the way for further advancements in this area. By exploring novel properties, theorems, and techniques, we aim to enhance the understanding and utility of IFTSs in various domains.

Intuitionistic fuzzy topological spaces (IFTS) were introduced as an extension of classical fuzzy topological spaces, incorporating the concept of intuitionistic fuzzy sets proposed by Atanassov in 1983. The development of intuitionistic fuzzy sets was motivated by the need to account

for both membership and non-membership degrees, providing a more flexible framework for dealing with uncertainty compared to traditional fuzzy set theory, which only involves membership functions.

The foundational concept of intuitionistic fuzzy sets (IFS) was introduced by Krassimir Atanassov in [2]. He extended Zadeh's fuzzy set theory by introducing two independent functions: a membership function $\mu(x)$ and a non-membership function $\nu(x)$, such that $0 \leq \mu(x) + \nu(x) \leq 1$. This theory provides a broader framework for uncertainty by allowing elements to have degrees of both membership and non-membership simultaneously.

Following the establishment of intuitionistic fuzzy sets, numerous authors extended these concepts to topology. The formal introduction of intuitionistic fuzzy topological spaces (IFTS) is credited to Çoker in [4], who laid the foundation for intuitionistic fuzzy topology by generalizing classical fuzzy topological structures to an intuitionistic fuzzy framework. In an IFTS, open sets are characterized by both membership and non-membership functions, resulting in new types of topological spaces based on intuitionistic fuzzy sets.

Building on the work of Atanassov and Çoker, several researchers have contributed to the development of IFTS. Notable works include studies on continuity, compactness, and separation axioms in intuitionistic fuzzy topological spaces, as discussed in [14, 15]. These studies explore different types of continuity within the intuitionistic fuzzy topological framework.

In [1], the authors investigated intuitionistic fuzzy compactness, introducing novel concepts of compactness and separation within the context of intuitionistic fuzzy sets.

More recently, there has been growing interest in applying intuitionistic fuzzy topology to practical problems. In [9, 11, 13], the authors examined the application of intuitionistic fuzzy topology to decision-making problems, where uncertainty plays a significant role. They extended traditional fuzzy decision-making models to intuitionistic fuzzy contexts, enabling more comprehensive representations of uncertainty.

The main sections of the paper are constructed as follows:

In Section 3, the formal definition of Intuitionistic Fuzzy Topological Spaces (IFTS) is provided, extending classical topology by incorporating both membership and non-membership functions for open sets. This section elaborates on the concept of IF topology, detailing the structure of open sets and introducing the notion of a proper IF topology, with several propositions and proofs to support these developments.

Section 4 focuses on continuity within the context of intuitionistic fuzzy spaces, exploring the properties of continuous and open mappings

and how these behave within the IFTS framework. It also discusses homeomorphisms between IFTS and their related properties.

Section 5 extends the results on IFTS to vector spaces, introducing the concept of Intuitionistic Fuzzy Topological Vector Spaces (IFTVS). It presents key results, such as the definition of IF continuous linear mappings, their scalar products, and the conditions under which vector space operations like addition and multiplication adhere to intuitionistic fuzzy topology principles. Moreover, the concept of mapping tangent to zero between Intuitionistic Fuzzy Topological Vector Spaces (IFTVS) is introduced to describe mappings that behave in a specific way around the zero point in such spaces.

2. Preliminary

An intuitionistic fuzzy topological space is a generalization of a classical topological space, where the notion of openness is extended to incorporate degrees of belief and disbelief. It allows for a more flexible representation of uncertainty and vagueness in the concept of open sets.

Definition 2.1. Suppose X be a nonempty fixed set. An intuitionistic fuzzy set (abbreviated as IFS) A on X can be represented by

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$$

where $\mu_A : X \longrightarrow [0,1]$ denotes the degree of membership $\mu_A(x)$ of an element x in X to the set A, and $\nu_A : X \longrightarrow [0,1]$ denotes the degree of non-membership $\nu_A(x)$ of the element x in X to the set A, subject to the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Definition 2.2. Suppose that

$$B = \{(x, \mu_B(x), \nu_B(x)) : x \in X\}$$

represents another IFS. Then, we can deduce the following relationships:

- $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
- $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) : x \in X\}$
- $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) : x \in X\}$
- Generally, if $\{A_i : i \in J\}$ is an arbitrary family of IFSs in X, then we have

$$\bigcap_{i \in J} A_i = \left\{ \left(x, \bigwedge_{i \in J} \mu_{A_i}(x), \bigvee_{i \in J} \nu_{A_i}(x) \right) : x \in X \right\}$$

$$\bigcup_{i \in J} A_i = \left\{ \left(x, \bigvee_{i \in J} \mu_{A_i}(x), \bigwedge_{i \in J} \nu_{A_i}(x) \right) : x \in X \right\}$$

Given that our primary goal is to establish the framework for building intuitionistic fuzzy topological spaces, it is essential to define the IFSs $(0,1)$ and $(1,0)$ in X as follows:

$$(0,1) = \{(x,0,1) : x \in X\}$$
$$(1,0) = \{(x,1,0) : x \in X\}$$

Definition 2.3. Let A be an IF set of the set X. For every $\alpha, \beta \in [0,1)$ satisfying $\alpha + \beta \leq 1$, we define

$$A_{\alpha,\beta} = \{x \in X \mid \mu_A(x) > \alpha \text{ or } \nu_A(x) < \beta\} \tag{2.1}$$

Definition 2.4. Suppose that X and Y are two nonempty sets and f is a function from X to Y. Let $B = \{(y, \mu_B(y), \nu_B(y)) : y \in Y\}$ is an IFS on Y, then the preimage of B under the function f, denoted by $f^{-1}(B)$, is the IFS in X which is given by

$$f^{-1}(B) = \Big\{(\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)}) \mid \mu_{f^{-1}(B)}(x) = \mu_B(f(x)),$$
$$\nu_{f^{-1}(B)}(x) = \nu_B(f(x))\Big\}$$

Suppose $A = \{(x, \lambda_A(x), \theta_A(x)) : x \in X\}$ represents an IFS on X. The image of A under the map f, denoted as $f(A)$, forms an IFS on Y given by

$$f(A) = \left\{(y, \lambda_{f(A)}(y), \theta_{f(A)}(y)) : y \in Y\right\},$$

where

$$\lambda_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$$\theta_{f(A)}(y)) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \theta_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 2.5. *Let X and Y be two sets, and let $f : X \to Y$ be a function. Suppose A_1, A_2 are IF sets in X, and B_1, B_2 are IF sets in Y. Then the following holds*

- *If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.*
- *If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.*

Remark 2.6. For an IF set $A = (\mu_A, \nu_A)$ on a set X, the expression $A = (\alpha, \beta)$ indicates that $\mu_A(x) = \alpha$ and $\nu_A(x) = \beta$.

Definition 2.7. Let X be a nonempty set and $x \in X$. For $\alpha, \beta \in [0,1]$, with $\alpha + \beta \leq 1$, define the IF set $x_{\alpha,\beta}$ of X as

$$x_{\alpha,\beta}(y) = \begin{cases} (\alpha, \beta) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x. \end{cases}$$

This set is referred to as an IF point of X.

3. Intuitionistic Fuzzy Topological Spaces (IFTS)

Definition 3.1. Let Λ be a family of Intuitionistic Fuzzy (IF) subsets of a set X that satisfies these conditions

i) The elements $(1,0)$ and $(0,1)$ are contained in Λ;
ii) The intersection of any two sets $A, B \in \Lambda$ is also in Λ;
iii) For any indexed family of sets $A_i \in \Lambda$, where $i \in I$, their union $\bigcup_{i\in I} A_i$ belongs to Λ.

Such a family Λ is called an *IF topology* on X. The pair (X, Λ) is referred to as an *IF topological space*. The elements of Λ are termed *IF open sets*, while their complements with respect to $(1,0)$ are called *IF closed sets* in the space (X, Λ). A subset $\mathcal{B} \subset \Lambda$ is an *open base for* Λ if every set in Λ can be expressed as a union of sets from $\mathcal{B}$.

Additionally, if the IF topology Λ satisfies the following condition, it is called a *proper IF topology*:

iv) For every $\alpha \in [0,1)$ and $\beta \in (0,1]$ such that $\alpha + \beta \leq 1$, the pair (α, β) belongs to Λ.

Proposition 3.2. *Let $\mathcal{B}$ be a family of IFS in X satisfying the following conditions:*

i) $\bigcup_{B_i \in \mathcal{B}} B_i = (1,0)$.
ii) *If B_1 and B_2 are both IFS in $\mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{B}$.*
iii) *For every $\alpha \in [0,1)$ and $\beta \in (0,1]$ with $\alpha + \beta \leq 1$ and $B \in \mathcal{B}$, then $(\alpha, \beta) \cap B \in \mathcal{B}$.*

The family of IFS in X, which can be expressed as a union of elements from $\mathcal{B}$, forms a proper fuzzy topology on X with $\mathcal{B}$ as its base.

Proof. Let $L(\mathcal{B})$ be the family of IFS that satisfies the three conditions mentioned above. Using condition i) $(1,0) \in L(\mathcal{B})$. Furthermore, if $A_i \in L(\mathcal{B})$ for $i \in I$, it is evident from condition i) that $\bigcup_{i\in I} A_i \in L(\mathcal{B})$. Now consider $A, C \in \mathcal{B}$. So $A = \cup_{i\in I} A_i$ and $C = \cup_{j\in J} C_j$ where $A_i, C_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. Now for every $x \in X$, we have

$$
\begin{aligned}
(A \cap C)(x) &= (\mu_A(x) \wedge \mu_C(x), \nu_A(x) \vee \nu_C(x)) \\
&= \left(\left(\bigvee_{i\in I} \mu_{A_i}(x) \right) \wedge \left(\bigvee_{j\in J} \mu_{C_j}(x) \right), \left(\bigwedge_{i\in I} \nu_{A_i}(x) \right) \vee \left(\bigwedge_{j\in J} \nu_{C_j}(x) \right) \right) \\
&= \left(\bigvee_{i,j} \left(\mu_{A_i}(x) \wedge \mu_{C_j}(x) \right), \bigwedge_{i,j} \left(\nu_{A_i}(x) \vee \nu_{C_j}(x) \right) \right)
\end{aligned}
$$

$$= \left(\bigcup_{i,j} (A_i \cap C_j) \right)(x)$$

This implies $A \cap C \in L(\mathcal{B})$. Finally, we will demonstrate that for every $\alpha \in [0,1)$ and $\beta \in (0,1]$ with $\alpha + \beta \leq 1$, we have $(\alpha, \beta) \in L(\mathcal{B})$. Using $i)$ for every $\alpha \in [0,1)$ and $\beta \in (0,1]$ with $\alpha + \beta \leq 1$, there exists $B \in \mathcal{B}$ such that $(\alpha, \beta) \subseteq B$, that leads to $\cup_{B \in \mathcal{B}} ((\alpha, \beta) \cap B) = (\alpha, \beta)$ and then by condition $iii)$, $(\alpha, \beta) \in L(\mathcal{B})$. So $L(\mathcal{B})$ is a proper IF topology. □

Theorem 3.3. *Let τ be a family of subsets of X. Then τ is a topology on X if and only if*

$$\tau^{\mathrm{IF}} = \left\{ \chi_A^{\mathrm{IF}} = (\chi_A, \chi_A^c) \mid A \in \tau \right\}$$

forms an IF topology on X.

Proof. Suppose that (X, τ) is a topological space. Since $\emptyset, X \in \tau$, it follows that $\chi_\emptyset^{\mathrm{IF}} = (0,1)$ and $\chi_X^{\mathrm{IF}} = (1,0)$ belong to τ^{IF}. Now assuming $\chi_{A_i}^{\mathrm{IF}} \in \tau^{\mathrm{IF}}$, $i = 1, 2, \ldots, n$, where $A_i \in \tau$ for all i. For every $x \in X$, we have

$$\begin{aligned} \left(\bigcap_{i=1}^{n} \chi_{A_i}^{\mathrm{IF}} \right)(x) &= \left(\bigwedge_{i=1}^{n} \chi_{A_i}(x), \bigvee_{i=1}^{n} \chi_{A_i}^c(x) \right), \\ &= \left(\chi_{\bigcap_{i=1}^{n} A_i}^{\mathrm{IF}} \right)(x). \end{aligned}$$

Since $\bigcap_{i=1}^{n} A_i \in \tau$, so it follows that $\left(\bigcap_{i=1}^{n} \chi_{A_i}^{\mathrm{IF}} \right) \left(= (\chi_{\bigcap_{i=1}^{n} A_i}^{\mathrm{IF}}) \right) \in \tau^{\mathrm{IF}}$.

Now suppose that $\{\chi_{A_i}^{\mathrm{IF}}\} \subseteq \tau^{\mathrm{IF}}$, where $A_i \in \tau$. Then for every $x \in X$, we have

$$\begin{aligned} \left(\bigcup_{i \in I} \chi_{A_i}^{\mathrm{IF}} \right)(x) &= \left(\bigvee_{i \in I} \chi_{A_i}(x), \bigwedge_{i \in I} \chi_{A_i}^c(x) \right), \\ &= \left(\chi_{\bigcup_{i \in I} A_i}^{\mathrm{IF}} \right)(x). \end{aligned}$$

Since $\bigcup_{i \in I} A_i \in \tau$, then we can conclude that

$$\left(\bigcup_{i \in I} \chi_{A_i}^{\mathrm{IF}} \right) = \left(\chi_{\bigcup_{i \in I} A_i}^{\mathrm{IF}} \right) \in \tau^{\mathrm{IF}}.$$

Thus (τ^{IF}, X) is an IFTS.

Now suppose that τ^{IF} is an IF topology on X. Since $\chi_\emptyset^{\mathrm{IF}} = (0,1)$ and $\chi_X^{\mathrm{IF}} = (1,0)$ belong to τ^{IF}, therefore $\emptyset, X \in \tau$. Let $A_i \in \tau$ for all $1 \leq i \leq n$. Therefore $\chi_{A_i}^{\mathrm{IF}} \in \tau^{\mathrm{IF}}$ and so $\bigcap_{i=1}^{n} \chi_{A_i}^{\mathrm{IF}}(x) = \chi_{\bigcap_{i=1}^{n} A_i} \in \tau^{\mathrm{IF}}$.

Hence $\bigcap_{i=1}^{n} A_i \in \tau$. Similarly, if $A_i \in \tau$ for all i in arbitrary index set I, we can immediately conclude $\bigcup_{i\in I} A_i \in \tau$. □

Theorem 3.4. *Assume X is a nonempty set and τ is an IF topology on X. Then, for every $\alpha \in [0,1)$ and $\beta \in (0,1]$ satisfying $\alpha+\beta \leq 1$, the family*

$$\tau_{\alpha,\beta} = \{A_{\alpha,\beta} \mid A \in \tau\}$$

constitutes a topology on X, where $A_{\alpha,\beta}$ is defined in (2.3).

Proof. For any $\alpha \in [0,1)$ and $\beta \in (0,1]$ such that $\alpha+\beta \leq 1$, we have

$$(1,0)_{\alpha,\beta} = \{x \in X \mid 1 > \alpha \ \vee \ 0 < \beta\} = X$$
$$(0,1)_{\alpha,\beta} = \{x \in X \mid 0 > \alpha \ \vee \ 1 < \beta\} = \emptyset$$

Since $(1,0),(0,1) \in \tau$, hence it follows that $X, \emptyset \in \tau_{\alpha,\beta}$. Now let $(A_i)_{\alpha,\beta} \in \tau_{\alpha,\beta}$, for some $A_i \in \tau$, $1 \leq i \leq n$. Then we have:

$$\begin{aligned}
\bigcap_{i=1}^{n}(A_i)_{\alpha,\beta} &= \{x \in X \mid \mu_{A_i}(x) > \alpha \ \vee \ \nu_{A_i}(x) < \beta, \ \forall i, i = 1,\ldots,n\}, \\
&= \left\{x \in X \;\middle|\; \bigwedge_{i=1}^{n} \mu_{A_i}(x) > \alpha \ \vee \ \bigvee_{i=1}^{n} \nu_{A_i}(x) < \beta\right\}, \\
&= \left\{x \in X \mid \mu_{\cap_{i=1}^{n} A_i}(x) > \alpha \ \vee \ \nu_{\cap_{i=1}^{n} A_i}(x) < \beta\right\}, \\
&= \left(\bigcap_{i=1}^{n} A_i\right)_{\alpha,\beta}.
\end{aligned}$$

Since $\bigcap_{i=1}^{n} A_i \in \tau$, it follows that $\bigcap_{i=1}^{n}(A_i)_{\alpha,\beta} \in \tau_{\alpha,\beta}$. Finally, let $\{(A_i)_{\alpha,\beta} \mid i \in I\} \subseteq \tau_{\alpha,\beta}$, where $A_i \in \tau$ for each $i \in I$. Then

$$\begin{aligned}
\bigcup_{i\in I}(A_i)_{\alpha,\beta} &= \left\{x \in X \mid \exists j \in I \text{ s.t. } \mu_{A_j}(x) > \alpha, \ \vee \ \nu_{A_j}(x) < \beta\right\} \\
&\subseteq \left\{x \in X \;\middle|\; \bigvee_{i\in I} \mu_{A_i}(x) > \alpha, \ \vee \ \bigwedge_{i\in I} \nu_{A_i}(x) < \beta\right\} \\
&= \left\{x \in X \;\middle|\; \mu_{\cup_{i\in I} A_i}(x) > \alpha, \ \vee \ \nu_{\cup_{i\in I} A_i}(x) < \beta\right\} \\
&= \left(\bigcup_{i\in I} A_i\right)_{\alpha,\beta}.
\end{aligned}$$

Conversely, assume $x \in \left(\bigcup_{i\in I} A_i\right)_{\alpha,\beta}$. Therefore, we have

$$\bigvee_{i\in I} \mu_{A_i}(x) > \alpha, \ \vee \ \bigwedge_{i\in I} \nu_{A_i}(x) < \beta,$$

and so

$$(\exists j \in I \ ; \ \mu_{A_j}(x) > \alpha), \ \vee \ (\exists k \in I \ ; \ \nu_{A_i}(x) < \beta).$$

Hence $x \in (A_j)_{\alpha,\beta}$ or $x \in (A_k)_{\alpha,\beta}$, implying that $x \in \bigcup_{i\in I}(A_i)_{\alpha,\beta}$. This leads us to conclude that $\bigcup_{i\in I}(A_i)_{\alpha,\beta} = \left(\bigcup_{i\in I} A_i\right)_{\alpha,\beta}$ which completes the proof. □

Theorem 3.5. *Suppose τ is a nonempty family of IF subsets of X such that $(1,0) \in \tau$. If the family $\tau_{0,1} = \{A_{0,1} \mid A \in \tau\}$ forms a topology on X, then it implies that τ is an IF topology on X.*

Proof. Since $\emptyset \in \tau_{0,1}$, there must be some $A \in \tau$ such that $A_{0,1} = \emptyset$. Recall that

$$A_{0,1} = \{x \in X \mid \mu_A(x) > 0 \text{ or } \nu_A(x) < 1\}.$$

Thus, we can deduce that $A = (0,1)$. By applying similar reasoning as demonstrated in the proof of Theorem 3.4, we can also conclude that

$$\bigcap_{i=1}^{n}(A_i)_{0,1} = \left(\bigcap_{i=1}^{n} A_i\right)_{0,1}$$

and

$$\bigcup_{i\in I}(A_i)_{0,1} = \left(\bigcup_{i\in I} A_i\right)_{0,1},$$

for every $A_i \in \tau$. This completes the proof. □

Definition 3.6. Let (X,τ) be an IF topology. A subfamily $\mathcal{S} \subseteq \tau$ is called subbase for τ if the family of all finite intersections of elements of $\mathcal{S}$ forms a base for τ.

Definition 3.7. Let $\{(X_i,\tau_i)\}_{i\in I}$ be a family of IF topology. Let $\mathcal{P} = \left\{p_i^{-1}(B) \mid B \in \tau_i\right\}_{i\in I}$, where p_i is the projection from $X = \sqcap_{i\in I}X_i$ onto X_i, for all $i \in I$. (Clearly, $p_i^{-1}(B)$ is an IF subset of X for all $i \in I$ and $B \in \tau_i$) Now let $\mathcal{B}$ be the family of all finite intersection of members of $\mathcal{P}$ and τ be the family of all unions of members of $\mathcal{B}$. Then τ is indeed an IF topology of X, with base $\mathcal{B}$ and subbase $\mathcal{P}$. This IF topology τ of X is called the product IF topology of $\{\tau_i\}_{i\in I}$ and denoted by $\tau = \sqcap_{i\in I}\tau_i$.

Definition 3.8. Let $\{X_i\}_{i=1}^n$ be a finite family of sets and for each $i \in I$, $A_i = (\mu_{A_i}, \nu_{A_i}) \overset{\text{IF}}{\subseteq} X_i$. The IF product $A = \sqcap_{i=1}^n A_i$ of the family $\{A_i\}_{i=1}^n$ is defined as the IF set in $X = \sqcap_{i=1}^n X_i$ with $A = (\mu_A, \nu_A)$ such that:

$$\mu_A(x) = \bigwedge_{i=1}^{n}\{\mu_{A_i}(p_i(x))\}, \qquad \nu_A(x) = \bigvee_{i=1}^{n}\{\nu_{A_i}(p_i(x))\}$$

for every $x = (x_1, \ldots, x_n) \in X$, where p_i is the projection map from X to X_i, for all $1 \leq i \leq n$.

Corollary 3.9. *Consider (X_i, τ_i), for $1 \leq i \leq n$, as intuitionistic fuzzy topological spaces. The product intuitionistic fuzzy topology (X, τ), where $X = \sqcap_{i=1}^{n} X_i$ and $\tau = \sqcap_{i=1}^{n} \tau_i$, has a basis formed by the collection of products of intuitionistic fuzzy sets $\sqcap_{i=1}^{n} A_i$, where $A_i \in \tau_i$ for each $i \in \{1, \ldots, n\}$.*

Proposition 3.10. *Let $\Lambda = \{(X_i, \tau_i)\}_{i \in I}$ be a family of IFTS. Suppose that (X, τ) is the IF product of the spaces in Λ, and let f be a mapping from an IF topological space (Y, ρ) to (X, τ). Then f is IF continuous if and only if $p_i \circ f$ is IF continuous for each $i \in I$, where $p_i : X = \prod_{i \in I} X_i \to X_i$ denotes the projection map.*

Proof. Suppose f is IF continuous. Let $A_i \in \tau_i$ is open. Then $p_i^{-1}(A_i)$ is obviously open in τ. Since f is IF continuous, it follows that $(p_i \circ f)^{-1}(A_i) = f^{-1}\left(p_i^{-1}(A_i)\right)$ is an open set in Y, as required. Conversely, assume that $p_i \circ f$ is IF continuous. Let $A_i \in \tau_i$ for arbitrary $i \in I$. Since $\left(f^{-1} \circ p_i^{-1}\right)(A_i)$ is ρ-open, therefore

$$\left\{\left(f^{-1} \circ p_i^{-1}\right)(A_i) \mid A_i \in \tau_i\right\}_{i \in I}$$

is a family of ρ-open IF set in Y. Recall that every member of τ is the union of finite intersections of members of family $\{p_i^{-1}(A_i) \mid A_i \in \tau_i\}_{i \in I}$. Also we know f^{-1} preserves unions and intersections. It follows that f^{-1} maps τ-open IF sets into the ρ-open IF sets in Y. This implies that f is continuous. □

Corollary 3.11. *Let $\Lambda = \{(X_i, \tau_i)\}_{i \in I}$ and $\Gamma = \{(Y_i, \rho_i)\}_{i \in I}$ be two families of IFTS and (X, τ), (Y, ρ) the IF product of members of Λ and Γ, respectively. For each $i \in I$, $f_i : X_i \longrightarrow Y_i$ be an IF continuous. Then for product mapping $f = \prod_{i \in I} f_i$ from (X, τ) into (Y, ρ), defined by $f((x_i)_{i \in I}) = (f_i(x_i))_{i \in I}$ is also IF continuous.*

Proof. For each $x = (x_i)_{i \in I} \in X$, we can express as $f(x) = (f_i(p_i(x)))_{i \in I}$. So by proposition 3.10, f is continuous. □

Proposition 3.12. *Consider the finite families of IFTS $\Lambda = \{(X_i, \tau_i)\}_{i=1}^{n}$ and $\Gamma = \{(Y_i, \rho_i)\}_{i=1}^{n}$, with (X, τ) and (Y, ρ) denoting the IFTS product of the members of Λ and Γ, respectively. For each $i \in \{1, \ldots, n\}$, assume $f_i : X_i \longrightarrow Y_i$ is an IF open mapping. Then, the product mapping $f = \prod_{i=1}^{n} f_i : X \longrightarrow Y$ defined by $(x_i)_{i-1}^{n} \mapsto (f_i(x_1))_{i=1}^{n}$, is IF open.*

Proof. Consider an open IF set A in τ. Then there exist open IF sets $A_{i_k} \in \tau_i$ for each $i \in \{1, \ldots, n\}$ and $k \in K$, where K is an index set,

such that $A = \cup_{k\in K} \prod_{i=1}^{n} A_{ik}$. The mapping $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ is then an IF set in (Y, ρ) such that

$$\begin{aligned}
\mu_{f(A)}(y) &= \mu_{\bigcup_{k\in K} f(\prod_{i=1}^{n} A_{ik})}(y) \\
&= \bigvee_{k\in K} \left(\bigvee_{z\in f^{-1}(y)} \mu_{\prod_{i=1}^{n} A_{ik}}(z) \right) \\
&= \bigvee_{k\in K} \bigvee_{z_1\in f_1^{-1}(y_1)} \cdots \bigvee_{z_n\in f_n^{-1}(y_n)} \bigwedge_{i=1}^{n} \mu_{A_{ik}}(z_i) \\
&= \bigvee_{k\in K} \left(\bigwedge_{i=1}^{n} \left\{ \bigvee_{z_i\in f_i^{-1}(y_i)} \mu_{A_{ik}}(z_i) \right\} \right) \\
&= \bigvee_{k\in K} \min \left\{ \mu_{f_1(A_{1k})}(y_1), \mu_{f_2(A_{2k})}(y_2), \ldots, \mu_{f_n(A_{nk})}(y_n) \right\} \\
&= \mu_{\bigcup_{k\in K} \prod_{i=1}^{n} (f_i(A_{ik}))}(y).
\end{aligned}$$

In a similar manner, $\nu_{f(A)}(y) = \nu_{\bigcup_{k\in K} \prod_{i=1}^{n} (f_i(A_{ik}))}(y)$. Therefore, $f(A) = \bigcup_{k\in K} \prod_{i=1}^{n} (f_i(A_{ik}))$. Since f_i is IF open for each $i \in \{1, \ldots, n\}$, it follows that $f(A)$ is open in ρ. □

4. IF Continuous Mapping

Definition 4.1. i) Let (X_1, τ_1) and (X_2, τ_2) be two Intuitionistic Fuzzy Topological Spaces (IFTSs). A mapping $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is called an IF continuous if for each open IF set $A \in \tau_2$, $f^{-1}(A)$ belongs to τ_1.

ii) A mapping f is called IF open if the image of every IF open set $B \subset X_1$ is an open set in X_2, i.e., $f(B) \in \tau_2$.

iii) A homeomorphism between two IF topological spaces X_1 and X_2 is a mapping f that is both IF continuous and IF open.

The mapping $f : (X_1, \tau_1) \longrightarrow (X_2, \tau_2)$ is said to be IF continuous at a point $x \in X_1$ if, for every open IF set A in τ_2 containing the IF point $y_{\alpha,\beta} = (f(x))_{\alpha,\beta}$, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, the pre-image of A under f, denoted by $f^{-1}(A)$, is an open IF set in τ_1 that contains $x_{\lambda,\gamma}$, with $0 < \lambda \leq \alpha$ and $\beta \leq \gamma \leq 1$.

Proposition 4.2. *Let (X, τ) and (Y, ρ) be IFTSs and $f : X \longrightarrow Y$ be a mapping. The following assertions hold equivalently:*

i) *f is IF continuous.*

ii) *For every IF set A in X, the pre-image of every neighborhood of $f(A)$, under f is a neighborhood of A.*

iii) *For every IF set A in X and every neighborhood N of $f(A)$, there exists a neighborhood U of A such that $f(U) \subset N$.*

Proof. *i)* $\rightarrow$ *ii)*: Let N be a neighborhood of $f(A)$. Then N contains an open neighborhood M of $f(A)$. Since $f(A) \subseteq M \subseteq N$, so we have

$$f^{-1}(f(A)) \subset f^{-1}(M) \subset f^{-1}(N).$$

Clearly, $A \subseteq f^{-1}(f(A))$. Also $f^{-1}(M)$ is open and hence $f^{-1}(N)$ is a neighborhood of A.

ii) $\rightarrow$ *iii)*: Since $f^{-1}(N)$ is a neighborhood of A, so $f\left(f^{-1}(N)\right) \subset N$. Taking $U = f^{-1}(N)$, yields the result. □

Remark 4.3. Let X_1, X_2 and X_3 are non-empty sets and A_1 and A_2 be IF subsets of X_1 and X_3 respectively. Let $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ be two maps. Then $(g \circ f)(A_1) = g(f(A_1))$ is an IF subset of X_3 and $(f \circ g)(A_3) = g(f(A_3))$ is an IF subset of X_1.

Corollary 4.4. *Suppose that $(X_1, \tau_1), (X_2, \tau_2)$ and (X_3, τ_3) are ITFSs. Let $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ and $(X_2, \tau_2) \xrightarrow{g} (X_3, \tau_3)$ be two maps. Then $g \circ f$ is IF continuous (open) if f and g are IF continuous (open).*

Example 4.5. Let $X_1 = X_2 = X = \{a, b\}$. Consider the intuitionistic fuzzy topology

$$\tau = \{(0, 1), (1, 0), U, V\},$$

where the intuitionistic fuzzy sets $U = (\mu_U, \nu_U)$ and $V = (\mu_V, \nu_V)$ are defined by

$$\mu_U(a) = 1, \qquad \nu_U(a) = 0, \qquad \mu_U(b) = 0, \qquad \nu_U(b) = 1,$$

and

$$\mu_V(a) = 0, \qquad \nu_V(a) = 1, \qquad \mu_V(b) = 1, \qquad \nu_V(b) = 0.$$

Clearly, τ contains the constant intuitionistic fuzzy sets $(0, 1)$ and $(1, 0)$ and is closed under arbitrary unions and finite intersections; hence τ is an intuitionistic fuzzy topology on X. Define the bijection

$$f : X \rightarrow X, \qquad f(a) = b, \qquad f(b) = a.$$

To verify the continuity of f, observe that the inverse image of every intuitionistic fuzzy open set in τ belongs to τ. Indeed,

$$f^{-1}((0, 1)) = (0, 1), \qquad f^{-1}((1, 0)) = (1, 0),$$

while

$$f^{-1}(U) = V, \qquad f^{-1}(V) = U.$$

Since $U, V \in \tau$, it follows that $f^{-1}(A) \in \tau$ for every $A \in \tau$. Therefore, f is intuitionistic fuzzy continuous. Furthermore,

$$f((0,1)) = (0,1), \qquad f((1,0)) = (1,0),$$

and

$$f(U) = V, \qquad f(V) = U.$$

Hence the image of every intuitionistic fuzzy open set is again intuitionistic fuzzy open, showing that f is an intuitionistic fuzzy open mapping.

Consequently, f is both intuitionistic fuzzy continuous and intuitionistic fuzzy open. Since f is bijective, it follows that f is an intuitionistic fuzzy homeomorphism.

5. IF Topological Vector Spaces

Definition 5.1. Let V be a vector space over the field K, where K can be either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. Consider a finite family $\{A_i\}_{i=1}^n$ of IF subsets of a vector space V. The sum of this family, denoted by $A = \sum_{i=1}^n A_i$, represents an IF subset of V with the following membership and non-membership functions:

$$\mu_A(x) = \bigvee_{\sum_{i=1}^n x_i = x} \left\{ \bigwedge_{i=1}^n \mu_{A_i}(x_i) \right\},$$

$$\nu_A(x) = \bigwedge_{\sum_{i=1}^n x_i = x} \left\{ \bigvee_{i=1}^n \nu_{A_i}(x_i) \right\},$$

for every $x \in V$.

Definition 5.2. Let B be an IF (Intuitionistic Fuzzy) subset of a vector space V over the field K. The scalar product aB, where $a \in K$, is an IF subset of V. For every $x \in V$, the corresponding membership and non-membership of aB are given by:

$$\mu_{aB}(x) = \mu_B\left(\frac{x}{a}\right), \qquad \nu_{aB}(x) = \nu_B\left(\frac{x}{a}\right), \quad \text{for } a \neq 0 \tag{5.1}$$

If $a = 0$, we define $0B(x) = 0_{(\alpha,\beta)}(x)$ as follows:

$$0B(x) = \begin{cases} (\alpha, \beta) & \text{if } x = 0, \\ (0,1) & \text{if } x \neq 0, \end{cases}$$

where $\alpha = \bigvee_{y \in V} \mu_B(y)$ and $\beta = \bigwedge_{y \in V} \nu_B(y)$.

Example 5.3. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Let $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $A_2 = (\mu_{A_2}, \nu_{A_2})$ be two IF subsets of $\mathbb{R}^2$ with membership and non-membership as below:

$$\mu_{A_1}(x,y) = \begin{cases} \frac{1}{3} & \text{if } x \neq 0, y = 0 \\ \frac{1}{2} & \text{if } x = 0, y \neq 0 \\ 1 & \text{if } x = y = 0, \\ 0 & \text{else} \end{cases} \qquad \nu_{A_1}(x,y) = \begin{cases} \frac{1}{4} & \text{if } x \neq 0, y = 0 \\ \frac{1}{3} & \text{if } x = 0, y \neq 0 \\ 0 & \text{if } x = y = 0, \\ \frac{1}{5} & \text{else} \end{cases}$$

and

$$\mu_{A_2}(x,y) = \begin{cases} \frac{2}{3} & \text{if } x = y \\ \frac{1}{3} & \text{if } x \neq y \end{cases} \qquad \nu_{A_2}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{1}{6} & \text{if } x = y \neq 0 \\ \frac{1}{2} & \text{if } x \neq y \end{cases}$$

Next, we will define the corresponding membership and non-membership functions for the sum of A_1 and A_2 that is denoted by $A = A_1 + A_2$ to represent it as an IF subset of $\mathbb{R}^2$.

Initially, we start with:

$$\begin{aligned} \mu_A(0,0) &= \bigvee \left\{ \mu_{A_1}(x_1,y_1) \wedge \mu_{A_2}(x_2,y_2) \;\middle|\; x_1 + x_2 = 0,\ y_1 + y_2 = 0 \right\} \\ &\geq \mu_{A_1}(0,0) \wedge \mu_{A_2}(0,0) \\ &= \frac{2}{3} \end{aligned}$$

This leads us to the conclusion that $\mu_A(0,0) = \frac{2}{3}$.

For any non-zero value of x, the following observations can be made:

$$\begin{aligned} \mu_A(x,0) &= \bigvee \left\{ \mu_{A_1}(x_1,y_1) \wedge \mu_{A_2}(x_2,y_2) \;\middle|\; x_1 + x_2 = x \neq 0,\ y_1 + y_2 = 0 \right\} \\ &\geq \mu_{A_1}(0,-x) \wedge \mu_{A_2}(x,x) \\ &= \frac{1}{2} \end{aligned}$$

In this case, if $x_1 = y_1 = 0$ and $(x_1, y_1) + (x_2, y_2) = (x, 0)$ where $x \neq 0$, then $y_2 = x_2 = x \neq 0$. Consequently, we find that $\mu_{A_1}(x_1, y_1) \wedge \mu_{A_2}(x_2, y_2) = \frac{1}{3}$. Thus, the statement $\mu_A(x, 0) = \frac{1}{2}$ is valid for all non-zero x.

For every non-zero y, the following inequality holds:

$$\mu_A(0,y) = \bigvee \left\{ \mu_{A_1}(x_1,y_1) \wedge \mu_{A_2}(x_2,y_2) \;\middle|\; x_1 + x_2 = 0,\ y_1 + y_2 = y \neq 0 \right\}$$

$$\geq \mu_{A_1}(0,y) \wedge \mu_{A_2}(0,0)$$

$$= \frac{1}{2}$$

In this context, if $x_1 = y_1 = 0$, then $\mu_{A_2}(x_2, y_2) = \frac{1}{3}$. Consequently, $\mu_A(0, y) = \frac{1}{2}$ holds true for all non-zero y.

When $x = y \neq 0$, it is evident that $\mu_A(x, x) = \frac{2}{3}$, as demonstrated by

$$\mu_A(x,y) = \bigvee \Big\{ \mu_{A_1}(x_1,y_1) \wedge \mu_{A_2}(x_2,y_2) \;\Big|\; x_1 + x_2 = x \neq 0, \; y_1 + y_2 = x \neq 0 \Big\}$$

$$\geq \mu_{A_1}(0,0) \wedge \mu_{A_2}(x,x)$$

$$= \frac{2}{3}$$

For $x \neq y$ with x and y being non-zero, we observe that

$$\mu_A(x,y) = \bigvee \Big\{ \mu_{A_1}(x_1,y_1) \wedge \mu_{A_2}(x_2,y_2) \;\Big|\; x_1 + x_2 = x \neq 0, \; y_1 + y_2 = x \neq 0, \; x \neq y \Big\}$$

$$\geq \mu_{A_1}(0,y-x) \wedge \mu_{A_2}(x,x)$$

$$= \frac{1}{2}$$

When $x_1 = y_1 = 0$, we have $\mu_{A_2}(x_2, y_2) = \frac{1}{3}$ in this case. Consequently, we can deduce that, $\mu_A(x, y) = \frac{1}{2}$ for $x \neq y$, $x \neq 0$, and $y \neq 0$.

For the non-membership function of A, we find that $\nu_A(0, 0) = 0$ because

$$\nu_A(0,0) = \bigwedge \Big\{ \mu_{A_1}(x_1,y_1) \vee \mu_{A_2}(x_2,y_2) \;\Big|\; x_1 + x_2 = 0, \; y_1 + y_2 = 0 \Big\}$$

$$\leq \nu_{A_1}(0,0) \wedge \nu_{A_2}(0,0)$$

$$= 0$$

For every nonzero x, we have $\nu_A(x, 0) = \frac{1}{5}$ since

$$\nu_A(x,0) = \bigwedge \Big\{ \mu_{A_1}(x_1,y_1) \vee \mu_{A_2}(x_2,y_2) \;\Big|\; x_1 + x_2 = x \neq 0, \; y_1 + y_2 = 0 \Big\}$$

$$\leq \nu_{A_1}(2x,x) \wedge \nu_{A_2}(-x,-x)$$

$$= \frac{1}{5}$$

Note that when $\nu_{A_2}(x_2, y_2) = \frac{1}{6}$, then $\nu_{A_1}(x_1, y_1) \geq \frac{1}{5}$ for all (x_1, y_1), which gives us the desired result.

For every non-zero y, we find that $\nu_A(0, y) = \frac{1}{5}$ due to the reasoning below:

$$\begin{aligned}\nu_A(x, 0) &= \bigwedge \left\{ \mu_{A_1}(x_1, y_1) \vee \mu_{A_2}(x_2, y_2) \;\middle|\; x_1 + x_2 = 0,\ y_1 + y_2 = y \neq 0 \right\} \\ &\leq \nu_{A_1}(y, 2y) \wedge \nu_{A_2}(-y, -y) \\ &= \frac{1}{5}\end{aligned}$$

It can be shown that $\nu_{A_1}(x_1, y_1) \vee \nu_{A_2}(x_2, y_2) \geq \frac{1}{5}$ for all pairs (x_1, y_1) and (x_2, y_2) in this case, which leads to the desired outcome.

In the situation where $x = y \neq 0$, we observe that

$$\begin{aligned}\nu_A(x, y) &= \bigwedge \left\{ \mu_{A_1}(x_1, y_1) \vee \mu_{A_2}(x_2, y_2) \;\middle|\; x_1 + x_2 = y_1 + y_2 = x \neq 0 \right\} \\ &\leq \nu_{A_1}(0, 0) \wedge \nu_{A_2}(x, x) \\ &= \frac{1}{6}\end{aligned}$$

Hence, it is evident that $\nu_A(x, x) = \frac{1}{6}$.

For non-zero x and y where $x \neq y$, it follows that $\nu_A(x, y) = \frac{1}{5}$, as demonstrated by the following calculation:

$$\begin{aligned}\nu_A(x, y) &= \bigwedge \left\{ \mu_{A_1}(x_1, y_1) \vee \mu_{A_2}(x_2, y_2) \;\middle|\; x_1 + x_2 = x \neq 0, \right. \\ &\qquad \left. y_1 + y_2 = y \neq 0 \right\} \\ &\leq \nu_{A_1}(x, y) \wedge \nu_{A_2}(0, 0) \\ &= \frac{1}{5}\end{aligned}$$

Moreover, it is evident that $\mu_{A_1}(x_1, y_1) \vee \mu_{A_2}(x_2, y_2) \geq \frac{1}{5}$ for all pairs (x_1, y_1) and (x_2, y_2) in this case, which establishes the desired outcome.

Consider $A = A_1 + A_2 = (\mu_A, \nu_A)$, with the functions defined by

$$\mu_A(x, y) = \begin{cases} \frac{2}{3} & \text{if } x = y \\ \frac{1}{2} & \text{if } x \neq y \end{cases} \qquad \nu_A(x, y) = \begin{cases} \frac{1}{5} & \text{if } x \neq y \\ 0 & \text{if } x = y = 0 \\ \frac{1}{6} & \text{if } x = y \neq 0 \end{cases}$$

This completes the example.

Example 5.4. Consider the IF subset A_1 of $\mathbb{R}^2$ in Example 5.3. It is easy to see that $aA_1 = A_1$ for every $0 \neq a \in \mathbb{R}$, and for $a = 0$,

$$0A_1(x,y) = \begin{cases} (0,1) & \text{if } (x,y) \neq (0,0) \\ (1,0) & \text{if } (x,y) = (0,0) \end{cases}$$

i.e,

$$\mu_{0A_1}(x,y) = \begin{cases} 0 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases} \qquad \nu_{0A_1}(x,y) = \begin{cases} 1 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Example 5.5. Let A be an intuitionistic fuzzy subset of $\mathbb{C}^2$ defined by the membership and non-membership functions

$$\mu_A(z_1,z_2) = \begin{cases} 1, & (z_1,z_2) = (0,0), \\ \frac{2}{3}, & \text{otherwise}, \end{cases}$$

and

$$\nu_A(z_1,z_2) = \begin{cases} 0, & (z_1,z_2) = (0,0), \\ \frac{1}{3}, & \text{otherwise}, \end{cases}$$

for all $(z_1,z_2) \in \mathbb{C}^2$. Let $a \in \mathbb{C}$ and define the scalar multiple $aA = (\mu_{aA}, \nu_{aA})$ as follows.

Case 1: $a \neq 0$**.** For every $(z_1,z_2) \in \mathbb{C}^2$, define

$$\mu_{aA}(z_1,z_2) = \mu_A\Big(\frac{z_1}{a}, \frac{z_2}{a}\Big), \qquad \nu_{aA}(z_1,z_2) = \nu_A\Big(\frac{z_1}{a}, \frac{z_2}{a}\Big).$$

Since $a \neq 0$, the transformation $(z_1,z_2) = (0,0)$ holds if and only if $\Big(\frac{z_1}{a}, \frac{z_2}{a}\Big) = (0,0)$. Hence, the membership and non-membership structure is preserved, and consequently, $aA = A$.

Case 2: $a = 0$**.** Define the scalar multiple $0A$ by

$$0A(z_1,z_2) = \begin{cases} (0,1), & (z_1,z_2) \neq (0,0), \\ (1,0), & (z_1,z_2) = (0,0), \end{cases}$$

that is,

$$\mu_{0A}(z_1,z_2) = \begin{cases} 0, & (z_1,z_2) \neq (0,0), \\ 1, & (z_1,z_2) = (0,0), \end{cases}$$

and

$$\nu_{0A}(z_1,z_2) = \begin{cases} 1, & (z_1,z_2) \neq (0,0), \\ 0, & (z_1,z_2) = (0,0). \end{cases}$$

Lemma 5.6. *Let V and W be vector spaces over the field K, where $K = \mathbb{R}$ or $\mathbb{C}$, and let $f : V \to W$ denote a linear map. Then, for any IF sets A and B in V, and for any $k \in K$, the following holds:*

$$f(A + B) = f(A) + f(B), \qquad f(kA) = kf(A).$$

Corollary 5.7. *Let V be a vector space over the field K, and A, B be two IF sets in V. For any scalar $k \in K$, the relation $k(A+B) = kA+kB$ holds.*

Definition 5.8. Let V be a vector space over the field $K = \mathbb{R}$ or $\mathbb{C}$. Consider τ as an IF topology on V and $\mathcal{X}$ as the standard topology on K . The vector space V is referred to as an IF topological vector space (IFTVS) over the field K if the following two mappings are IF continuous:

$$f : (V, \tau) \times (V, \tau) \longrightarrow (V, \tau)$$

$$(x, y) \longmapsto x + y$$

$$g : (K, \mathcal{X}) \times (V, \tau) \longrightarrow (V, \tau)$$

$$(r, x) \longmapsto rx$$

Definition 5.9. Let (X, τ) be an IFTS. An IF set A in X is called a neighborhood of an IF point $x_{\alpha,\beta}$ in X, if there exists an open IF set $B \in \tau$ such that $x_{\alpha,\beta} \subseteq B \subseteq A$.

Proposition 5.10. *Suppose that (X, τ) be an IFTS and let $\varphi(x_{\alpha,\beta})$ denote the set of all neighborhoods of an IF point $x_{\alpha,\beta}$ in X. Then we have*

- *Every IF set that contains an IF set belonging to $\varphi(x_{\alpha,\beta})$ also belongs to $\varphi(x_{\alpha,\beta})$.*
- *Any finite intersection of sets from $\varphi(x_{\alpha,\beta})$ is itself an element of $\varphi(x_{\alpha,\beta})$.*

Definition 5.11. Let V_1 and V_2 be two IF topological vector spaces, and let $\phi : V_1 \to V_2$ be a mapping. We say that ϕ is tangent to zero if, for every neighborhood N_2 of $0_{\alpha_2,\beta_2}$ in V_2, where $0 < \alpha_2 \leq 1$ and $\alpha_2 + \beta_2 \leq 1$, there exists a neighborhood N_1 of $0_{\alpha_1,\beta_1}$ in V_1, where $0 < \alpha_1 < \alpha_2$, $\beta_2 < \beta_1$, and $\alpha_1 + \beta_1 \leq 1$, such that

$$\phi[tN_1] \subset r(t)N_2$$

for some function $r(t)$ with $t \in \mathbb{R}$, where $\lim_{t\to 0} \dfrac{r(t)}{t} = 0$.

Example 5.12. If V be an IF topological vector space (IFTVS). The zero function on V, denotes by $\mathbf{0} : V \longrightarrow V$ is tangent to 0.

Lemma 5.13. *Let V, W, and Z be IF topological vector spaces over the field K. If the mapping $f : V \longrightarrow W$ is tangent to 0 and $g : W \longrightarrow Z$ is a linear map that is IF continuous at $0 \in W$, then $g \circ f$ is tangent to 0. Conversely, if f is linear and IF continuous at $0 \in V$, and g is tangent to 0, then $g \circ f$ remains tangent to 0.*

Proof. Since g is IF continuous at 0, for any neighborhood N of $0_{\alpha,\beta}$ in Z, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, there exists a neighborhood M of $0_{\lambda,\gamma}$ in W, where $0 < \lambda \leq \alpha$ and $\beta \leq \gamma \leq 1$, such that $g(M) \subseteq N$. Given any such M, since f is tangent to 0, there exists a neighborhood P of $0_{\delta,\sigma}$ in V, where $0 < \delta \leq \lambda$ and $\delta \leq \sigma \leq 1$, such that

$$f(tP) \subseteq r(t)M,$$

for some scaling function $r(t)$ with $\lim_{t\to 0} \dfrac{r(t)}{t} = 0$. Now, applying Lemmas 5.6 and 2.5, we obtain:

$$\begin{aligned} g(f(tP)) &\subseteq g(r(t)M) \\ &= r(t)g(M) \\ &\subseteq r(t)N. \end{aligned}$$

This shows that $g \circ f$ is tangent to 0. The second part of the lemma can be proven in a similar way. □

Corollary 5.14. *If f is a mapping tangent to 0, then the scalar product mapping kf is also tangent to 0 for any scalar $k \in K$.*

Definition 5.15. An IF topological space (X, τ) is called an T_1 IF space if each IF point is a closed IF subset of X.

Example 5.16. For any nonempty set X, the following IFTS

$$\tau = \{(1,0),(0,1)\} \cup \left\{x^c_{\alpha,\beta} : x \in X,\ \alpha,\beta \in [0,1],\ \alpha+\beta \leq 1\right\}$$

is an T_1 IF space, where

$$x^c_{\alpha,\beta} = \begin{cases} (\beta,\alpha) & \text{if } x = y, \\ (1,0) & \text{if } x \neq y. \end{cases} \tag{5.2}$$

Definition 5.17. Let V_1 and V_2 be IF topological vector spaces with T_1 IF topologies. An IF continuous mapping $f : V_1 \to V_2$ is said to be IF differentiable at a point $x \in V_1$ if there exists a linear IF continuous mapping $g : V_1 \to V_2$ such that for all $y \in V_1$, the following holds:

$$f(x+y) = f(x) + g(y) + \phi(y), \tag{5.3}$$

where ϕ is tangent to zero.

In this case, the mapping g is called the IF derivative of f at x, denoted $f'(x)$, and it is an IF continuous mapping from V_1 to V_2.

The mapping f is considered IF differentiable if it is IF differentiable at every point $x \in V_1$.

Proposition 5.18. *Let V and W be two IFTSs.*

i) *Any constant function from V to W is IF differentiable at every point of V, with its IF derivative being the zero map.*
ii) *Every linear IF continuous mapping $f : V \to W$ is differentiable at all points of V.*

Lemma 5.19. *Let N be a neighborhood of the IF point $0_{\alpha,\beta}$. Then there exists a neighborhood M of $0_{\alpha,\beta}$ such that $M + M \subseteq N$.*

Proof. Since the finite sum of functions is IF continuous, by Proposition 4.2, for the neighborhood N of $0_{\alpha,\beta}$, there exist neighborhoods M_1 and M_2 of $0_{\alpha,\beta}$ such that $M_1 + M_2 \subseteq N$. Let $M = M_1 \cap M_2$. Then $M + M \subseteq M_1 + M_2$, completing the proof. □

Lemma 5.20. *The sum of two mappings that are tangent to 0 is also tangent to 0.*

Proof. Suppose V and W are two IFVSs, and let f and g be mappings from V to W that are tangent to 0. For every neighborhood N of $0_{\alpha,\beta}$ in W, where $0 < \alpha, \beta \leq 1$, by Lemma 5.19, there exists a neighborhood M such that $M + M \subseteq N$. Now, by Corollary 5.7 and Lemma 2.5, we have $o(t)(M + M) \subseteq o(t)N$ in W, where $o(t)$ is a real-valued function with respect to t, and $\lim_{t\to 0} \frac{o(t)}{t} = 0$. According to Definition 5.11, there exist neighborhoods H_1 and H_2 of $0_{\lambda,\gamma}$ in V, with $0 < \lambda \leq \alpha$ and $\beta \leq \gamma \leq 1$, such that $f(tH_1) \subseteq o(t)M$ and $g(tH_2) \subseteq o(t)M$.

Let $H = H_1 \cap H_2$. Then we have $f(tH) \subseteq o(t)M$ and $g(tH) \subseteq o(t)M$. Finally, we obtain $f(tH) + g(tH) \subseteq o(t)N$, completing the proof. □

Proposition 5.21. *Let V_1, V_2, and V_3 be IFTVSs equipped with T_1 IF topologies. Suppose $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are IF continuous mappings. If $x \in V_1$ and $y = f(x)$, and if f is differentiable at x and g is differentiable at $f(x)$, then the composition $g \circ f$ is IF differentiable at x.*

Proof. Given that f is differentiable at x and g is differentiable at $f(x)$, by Definition 5.17, there exist mappings f' and ϕ from V_1 into V_2, as well as g' and ψ from V_2 into V_3, where f' and g' are linear and IF continuous, and ϕ and ψ are infinitesimal (tangent to zero), such that

$$\begin{aligned} f(x+u) - f(x) &= f'(u) + \phi(u), \\ g(y+v) - g(y) &= g'(v) + \psi(v), \end{aligned} \tag{5.4}$$

for every $u \in V_1$ and $v \in V_2$. Let $h = g \circ f$, and substituting into (5.4), we get

$$\begin{aligned} h(x+u) - h(x) &= g\left[f(x) + f'(u) + \phi(u)\right] - g(f(x)), \\ &= g(f(x)) + g'[f'(u) + \phi(u)] + \psi[f'(u) + \phi(u)] - g(f(x)), \\ &= (g' \circ f')(u) + (g' \circ \phi)(u) + \psi \circ (f' + \phi)(u), \end{aligned}$$

for every $u \in V_1$.

By Lemma 5.13, $g' \circ \phi$ is tangent to zero. Now, consider the mapping $\psi \circ (f' + \phi)$. For any neighborhood M of $0_{\alpha,\beta}$ in V_3 with $0 < \alpha, \beta \leq 1$, there exists a neighborhood N of $0_{\lambda,\gamma}$ in V_2, where $0 < \lambda \leq \alpha$ and $\beta \leq \gamma \leq 1$, such that $\psi(tN) \subseteq o(t)M$, where $o(t)$ is a real-valued function satisfying $\lim_{t\to 0} \frac{o(t)}{t} = 0$.

By Lemma 5.19, there exists a neighborhood N' of $0_{\lambda,\gamma}$ such that $N' + N' \subseteq N$. Since f' is IF continuous, by Proposition 4.2, there exists a neighborhood U of $0_{\delta,\sigma}$ in V_1, with $0 < \delta \leq \lambda$ and $\delta \leq \sigma \leq 1$, such that $f'(U) \subseteq N'$. This implies $tf'(U) \subseteq tN'$, or equivalently $f'(U) \subseteq tN'$. Given N', there exists a neighborhood R of $0_{\delta,\sigma}$ in V_1 such that $\phi(tR) \subseteq o(t)N'$, and for $\left|\frac{o(t)}{t}\right| \leq 1$, we have $o(t)N' \subseteq tN'$. Let $L = U \cap R$. By Lemma 2.5, we have $\phi(L) + f'(tL) \subseteq tN$, which implies

$$\begin{aligned} \psi(\phi(tL)) + f'(tL) &\subseteq \psi(tN), \\ &\subseteq o(t)M. \end{aligned}$$

Thus, the mapping $\psi \circ (f' + \phi) : V_1 \to V_3$ is tangent to zero. Finally, we conclude that

$$h(x+u) - h(x) = (g' \circ f')(u) + \xi(u)$$

for every $u \in V_1$, where $\xi = g' \circ \phi + \psi \circ (f' + \phi)$, with $g' \circ f'$ being linear and ξ being infinitesimal (tangent to zero), by Lemma 5.20. □

6. Conclusion

In conclusion, this paper presents significant advancements in the field of Intuitionistic Fuzzy Topological Spaces (IFTS) and their applications, particularly by extending foundational concepts such as continuity, compactness, and open mappings to the context of intuitionistic fuzzy topology. The exploration of homeomorphisms and their unique properties within IFTS adds depth to the theoretical framework. The research also extends these results to Intuitionistic Fuzzy Topological Vector Spaces (IFTVS), where key operations like scalar multiplication and vector addition are examined under the fuzzy topology framework. Additionally, the introduction and development of the concept of mappings tangent to zero between IFTVS offer a deeper understanding of

how these mappings behave near the origin, broadening the scope of applications in functional analysis and vector space theory. Overall, the findings contribute to both the theoretical enrichment of fuzzy topology and its practical relevance in addressing uncertainty and vagueness in various mathematical and real-world contexts, such as decision-making and optimization. Future research could explore further applications of these concepts, particularly in more complex systems and advanced mathematical structures.

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