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Subordination Criteria for Meromorphic m-Valent Functions with Balloon-Shaped Domains

Mallikarjun G. Shrigan[1*] and Suryakant H. Gond[2]

Abstract. In this paper, we introduce two new subclasses of meromorphic m-valent functions in the punctured unit disk $\mathbb{U}^*$, denoted by $\mathscr{MS}^*_{p,q}(bal)$ and $\mathscr{MC}_{p,q}(bal)$. These classes are defined via (p,q)-derivative operator and a subordination condition involving the balloon-shaped function

$$\mathcal{B}(\mathfrak{z}) = \frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}}.$$

Necessary and sufficient conditions for inclusion in these subclasses are established, extending several known results for meromorphic multivalent functions.

1. Introduction

Let $\mathscr{M}_p$ denote the class of meromorphic m-valent functions expressed as

$$\zeta(\mathfrak{z}) = \frac{1}{\mathfrak{z}^m} + \sum_{s=1}^{\infty} a_s \mathfrak{z}^{s-m}, \quad m \in \mathbb{N}, \tag{1.1}$$

which are analytic and m-valent in the punctured open unit disc $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$, where $\mathbb{U}= \{\mathfrak{z} \in \mathfrak{C} : |\mathfrak{z}| < 1\}$.

The function $\mathfrak{g}$ is said to be subordinate to ζ, if there exists a function ω analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(\mathfrak{z})| < 1$, such that $\mathfrak{g}(\mathfrak{z}) = \zeta(\omega(\mathfrak{z})), \mathfrak{z} \in \mathbb{U}$, denoted by $\mathfrak{g} \prec \zeta$. Especially, the above subordination is corresponding to $\mathfrak{g}(0) = \zeta(0)$ and $\zeta(\mathbb{U}) \subseteq \mathfrak{g}(\mathbb{U})$ if the function ζ is univalent in $\mathbb{U}$.

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If $\zeta \in \mathscr{M}_p$ satisfies

$$-\Re\left\{\frac{z\,\zeta'(\mathfrak{z})}{\zeta'(\mathfrak{z})}\right\} > \beta, \quad (\mathfrak{z} \in \mathbb{U},\, 0 \leq \beta < m),$$

then we say that ζ is meromorphic m-valent starlike of order β $(0 \leq \beta < m)$.

If $\zeta \in \mathscr{M}_p$ satisfies

$$-\Re\left\{1 + \frac{\mathfrak{z}\zeta''(\mathfrak{z})}{\zeta'(\mathfrak{z})}\right\} > \beta, \quad (\mathfrak{z} \in \mathbb{U},\, 0 \leq \beta < m),$$

then we say that ζ is meromorphic m-valent convex of order β.

For $\zeta \in \mathscr{M}_p$ given by (1.1) and $\varpi \in \mathscr{M}_p$ given by

$$\varpi(\mathfrak{z}) = \frac{1}{\mathfrak{z}^m} + \sum_{s=1}^{\infty} e_s \mathfrak{z}^{s-m}, \quad m \in \mathbb{N},$$

the Hadamard product [1] (or convolution) of ζ and ϖ is defined by

$$(\zeta * \varpi)(\mathfrak{z}) = \frac{1}{\mathfrak{z}^m} + \sum_{s=1}^{\infty} a_s e_s \mathfrak{z}^{s-m} = (\varpi * \zeta)(\mathfrak{z}), \quad m \in \mathbb{N}.$$

In recent years, researchers have actively explored the application of q-calculus across various domains of mathematics and physics. A generalization of standard calculus, known as (p, q)-calculus often referred to as post-quantum calculus extends the classical framework by eliminating the reliance on limit processes. The foundational concepts of the q-derivative and q-integral were introduced by Jackson [5, 12].

The theory of (p, q)-calculus operators, as developed in [3], has found applications in numerous scientific fields, including geometric function theory, particularly for the parameter range $0 < q \leq p \leq 1$. Within this framework, the (p, q)-derivative operator $\mathfrak{d}_{p,q} : \mathscr{M}_p \to \mathscr{M}_p$, defined for $\zeta \in \mathscr{M}_p$, is given by:

$$\mathfrak{d}_{p,q}\zeta(\mathfrak{z}) = \begin{cases} \dfrac{\zeta(p\mathfrak{z}) - \zeta(q\mathfrak{z})}{(p-q)\mathfrak{z}}, & \mathfrak{z} \neq 0,\ p \neq q, \\ \zeta^{'}(0), & \mathfrak{z} = 0, \end{cases} \tag{1.2}$$

where

$$\mathfrak{d}_{p,q}\zeta(\mathfrak{z}) = -\frac{[m]_{p,q}}{(pq)^m \mathfrak{z}^{m+1}} + \sum_{s=1}^{\infty} [s-m]_{p,q}\, a_s \mathfrak{z}^{s-m-1}$$

and

$$[s-m]_{p,q} = \frac{p^{s-m} - q^{s-m}}{p-q}.$$

The classes, which possess noteworthy geometric features, have been studied by various authors in Geometric Function Theory; see [6, 10, 13] for further details. Cho et al. [4] studied radius problems for a subclass of starlike functions associated with the sine function. They considered analytic functions in $\mathbb{U}$ satisfying

$$\Re\left(\frac{\mathfrak{z}\zeta'(\mathfrak{z})}{\sin(\zeta(\mathfrak{z}))}\right) > 0,$$

and obtained sharp radii for starlikeness, convexity, and close-to-convexity.

Kumar and Arora [7] introduced a class of starlike functions associated with the petal-shaped domain

$$\mathcal{P} = \left\{w \in \mathbb{C} : \Re\left(\frac{w}{1+w^2}\right) > 0\right\},$$

and derived inclusion results and coefficient bounds.

Mendiratta et al. [8] defined a subclass of strongly starlike functions linked with the exponential function by

$$\left|\arg\left(\frac{\mathfrak{z}\zeta'(\mathfrak{z})}{e^{\zeta(\mathfrak{z})}}\right)\right| < \frac{\pi\alpha}{2}, \quad (0 < \alpha \leq 1).$$

Mundula and Kumar [9] studied a subclass of starlike functions associated with the hyperbolic cosine function satisfying

$$\Re\left(\frac{\mathfrak{z}\zeta'(\mathfrak{z})}{\cosh(\zeta(\mathfrak{z}))}\right) > 0, \quad \mathfrak{z} \in \mathbb{U}.$$

Sharma et al. [11] introduced a class of starlike functions associated with the cardioid domain

$$\mathscr{S}^* = \left\{w \in \mathbb{C} : \left|w - \tfrac{1}{2}\right| < \tfrac{1}{2}\right\}.$$

They analyzed the coefficient estimates and convolution properties for this class.

Inspired by the foundational works of various authors as mentioned above, this study introduces and defines specific subclasses of meromorphic m-valent functions associated with balloon-shaped domains within the framework of (p,q)-calculus.

Definition 1.1. Let $0 < q \leq p \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. A function $\zeta \in \mathscr{M}_p$ is said to belong to the class $\mathscr{M}\mathscr{S}^*_{p,q}(bal)$ if it satisfies the subordination condition

$$1 - \frac{1}{b}\left[\frac{\mathfrak{z}\,\mathfrak{d}_{p,q}\zeta(\mathfrak{z})}{\zeta(\mathfrak{z})} + \frac{[m]_{p,q}}{(pq)^m}\right] \prec \mathcal{B}(\mathfrak{z}). \tag{1.3}$$

Definition 1.2. Let $0 < q \leq p \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. A function $\zeta \in \mathscr{M}_p$ is said to belong to the class $\mathscr{MC}_{p,q}(bal)$ if it satisfies the subordination condition

$$1 - \frac{1}{b}\left[\frac{\mathfrak{d}_{p,q}\left(\mathfrak{z}\,\mathfrak{d}_{p,q}\zeta(\mathfrak{z})\right)}{\mathfrak{d}_{p,q}\zeta(\mathfrak{z})} + \frac{[m]_{p,q}}{(pq)^m}\right] \prec \mathcal{B}(\mathfrak{z}), \tag{1.4}$$

where

$$\mathcal{B}(\mathfrak{z}) = \frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}},$$

as illustrated in Figure 1.

The simple computation shows

$$\mathfrak{z} \in \mathscr{MC}_{p,q}(bal) \quad \Leftrightarrow \quad -\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\mathfrak{d}_{p,q}\zeta(\mathfrak{z}) \in \mathscr{MS}^{\star}_{p,q}(bal). \tag{1.5}$$

The generating function [2]

$$\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}},$$

plays a central role in describing the associated image domain, as illustrated in Figure 1. It maps the unit disk onto a bounded region with symmetric geometric properties.

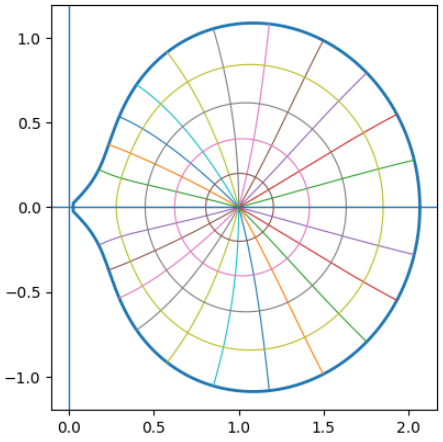
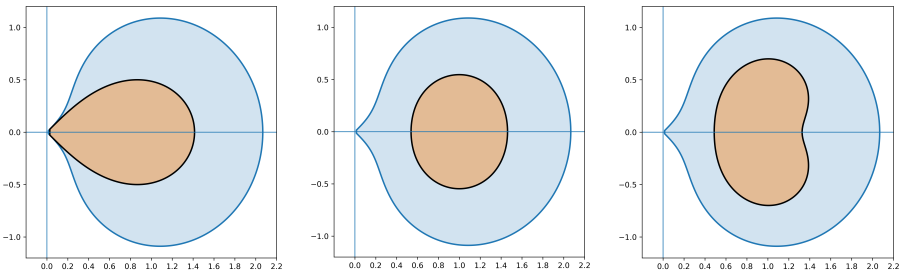


FIGURE 1. Graph of the function $\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}}$.

In support of the theoretical formulation, the functions

$$f_1(\mathfrak{z}) = \sqrt{1+\mathfrak{z}}, \qquad f_2(\mathfrak{z}) = \frac{2}{1+e^{-\mathfrak{z}}}, \qquad f_3(\mathfrak{z}) = \sqrt{1+\tanh\mathfrak{z}}$$

serve as concrete examples belonging to the defined class, their corresponding image domains, presented in Figure 2, demonstrate boundedness and symmetry consistent with the prescribed subordination condition, thereby confirming their admissibility that the class $\mathscr{M}\mathscr{S}^*_{p,q}(bal)$ is nonempty.



(A) $\sqrt{1+\mathfrak{z}}$ (B) $\frac{2}{1+e^{-\mathfrak{z}}}$ (C) $\sqrt{1+\tanh\mathfrak{z}}$

FIGURE 2. Members of the class $\mathscr{M}\mathscr{S}^*_{p,q}(bal)$.

The classes $\mathscr{M}\mathscr{S}^*_{p,q}(bal)$ and $\mathscr{M}\mathscr{C}_{p,q}(bal)$ are introduced to establish a cohesive analytical setting that unifies symmetric starlike and symmetric convex function behaviors. The transition between these geometric characteristics is regulated by an auxiliary parameter, which provides a continuous shift in structure: the starlike tendency is emphasized when the parameter is near zero, whereas convexity becomes increasingly dominant as it approaches unity. The following functions belong to the class $\mathscr{M}\mathscr{S}^*_{p,q}(bal)$:

(i) $\zeta_1(\mathfrak{z}) = \dfrac{1}{\mathfrak{z}^m}(1+\mathfrak{z})^\alpha$, $0 < \alpha \leq 1$,

(ii) $\zeta_2(\mathfrak{z}) = \dfrac{1}{\mathfrak{z}^m}e^{\lambda\mathfrak{z}}$, $|\lambda| \leq 1$.

Now consider the particular case $p = 1$, $q = \frac{1}{2}$, $m = 1$, and $b = 2$, then

$$\zeta_1(\mathfrak{z}) = \frac{1}{\mathfrak{z}}\sqrt{1+\mathfrak{z}}, \qquad \zeta_2(\mathfrak{z}) = \frac{1}{\mathfrak{z}}e^{\mathfrak{z}}.$$

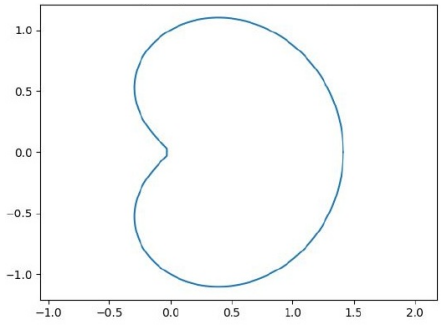


FIGURE 3. Image of the unit circle $|\mathfrak{z}| = 1$ under $\zeta_1(\mathfrak{z}) = \frac{1}{\mathfrak{z}}\sqrt{1+\mathfrak{z}}$.

These geometric observations confirm that both functions satisfy the subordination condition with respect to the dominant function $\mathcal{B}(\mathfrak{z})$. We give here a few examples of the functions which belong to the class $\mathscr{MC}_{p,q}(bal)$. For this purpose, we set

$$\zeta(\mathfrak{z}) = \frac{1}{\mathfrak{z}^2} + \varepsilon\left(\mathfrak{z} + \frac{\mathfrak{z}^2}{2!} + \frac{\mathfrak{z}^3}{3!} + \frac{\mathfrak{z}^4}{4!}\right), \quad 0 < \varepsilon \leq 1,$$

Choosing the parameters $p = 1$, $q = \frac{1}{2}$, $m = 2$, and $b = 6$, we compute

$$[m]_{p,q} = \frac{1-q^m}{1-q} = \frac{1-(1/2)^2}{1-1/2} = \frac{3}{2}, \qquad (pq)^m = \left(\frac{1}{2}\right)^2 = \frac{1}{4},$$

and hence

$$\frac{[m]_{p,q}}{(pq)^m} = 6.$$

Further, using the (p,q)-derivative, the dominant term yields

$$\frac{\mathfrak{d}_{p,q}\left(\mathfrak{z}\,\mathfrak{d}_{p,q}\zeta(\mathfrak{z})\right)}{\mathfrak{d}_{p,q}\zeta(\mathfrak{z})} \approx -2, \quad \text{as } \mathfrak{z} \to 0.$$

Therefore

$$1 - \frac{1}{b}\left[\frac{\mathfrak{d}_{p,q}\left(\mathfrak{z}\,\mathfrak{d}_{p,q}\zeta(\mathfrak{z})\right)}{\mathfrak{d}_{p,q}\zeta(\mathfrak{z})} + \frac{[m]_{p,q}}{(pq)^m}\right] = 1 - \frac{1}{6}(-2+6) = \frac{1}{3}.$$

Since $\frac{1}{3}$ lies in the range of the function

$$\mathcal{B}(\mathfrak{z}) = \frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}},$$

it follows the Equation (1.4).

In this study, we examine various properties of meromorphic m-valent functions associated with a balloon-shaped domain in the context of (p,q)-calculus. Initially, we derive the necessary and sufficient conditions for functions that belong to these subclasses $\mathscr{MS}^*_{p,q}(bal)$ and $\mathscr{MC}_{p,q}(bal)$, providing insights into the coefficient bounds and geometric characteristics pertinent to this domain.

2. Main Results

Throughout this section, unless otherwise specified, we assume that $0 < q \leq p \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$.

Theorem 2.1. *If $\zeta \in \mathscr{M}_p$ then $\zeta \in \mathscr{MS}^*_{p,q}$ if and only if*

$$\mathfrak{z}^m \left[\zeta(\mathfrak{z}) * \left(\frac{1+(\mathfrak{B}(bal)-\mathfrak{z})}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})}\right)\right] \neq 0, \tag{2.1}$$

where

$$\mathfrak{B}(bal) = b(pq)^m \mathfrak{z}\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right). \tag{2.2}$$

Proof. It is readily verified that the above condition holds for any function $\zeta \in \mathscr{M}_p$. Hence, we have

$$\zeta(\mathfrak{z}) = \zeta(\mathfrak{z}) * \frac{1}{\mathfrak{z}^m(1-\mathfrak{z})} \tag{2.3}$$

and

$$-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\mathfrak{d}_{p,q}\zeta(\mathfrak{z}) = \zeta(\mathfrak{z}) * \frac{1-(pq+\frac{1}{[m]_{p,q}})\mathfrak{z}}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})}. \tag{2.4}$$

To begin with, suppose that $\zeta \in \mathscr{MS}^*_{p,q}(bal)$. To establish the validity of (2.1), we reformulate (1.3) using the definition of subordination, namely,

$$-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\frac{\mathfrak{d}_{p,q}\zeta(\mathfrak{z})}{\zeta(\mathfrak{z})} = 1 - \frac{b(pq)^m}{[m]_{p,q}}(1 - \frac{2\sqrt{1+w(\mathfrak{z})}}{(1+e^{-w(\mathfrak{z})})},$$

where w is a Schwarz function, hence

$$\mathfrak{z}^m \left[-(pq)^m \mathfrak{z}\, \mathfrak{d}_{p,q}\zeta(\mathfrak{z}) - \left([m]_{p,q} + b(pq)^m \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right)\right)\zeta(\mathfrak{z})\right] \neq 0. \tag{2.5}$$

Substituting (2.3) and (2.4) into (2.5), we obtain

$$\mathfrak{z}^m \left[\left(\zeta(\mathfrak{z}) * \frac{\left(1-\left(pq+\frac{1}{[m]_{p,q}}\right)\mathfrak{z}\right)[m]_{p,q}}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})}\right) - \left([m]_{p,q} + b(pq)^m\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right)\right)\left(\zeta(\mathfrak{z}) * \frac{1}{\mathfrak{z}^m(1-\mathfrak{z})}\right)\right] \neq 0, \quad \mathfrak{z} \in \mathbb{U}^*,$$

which is equivalent to

$$\mathfrak{z}^m \left[\zeta(\mathfrak{z}) * \frac{1 + \left(-pq + \frac{1}{b(pq)^m \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1 \right)} \right) \mathfrak{z}}{\mathfrak{z}^m (1-\mathfrak{z})(1-pq\mathfrak{z})} \left(-b(pq)^m \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} \right) \right) \right] \neq 0$$

or

$$\mathfrak{z}^m \left[\zeta(\mathfrak{z}) * \frac{1 + \left(b(pq)^{m+1} \mathfrak{z} \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1 \right) \right) - \mathfrak{z}}{\mathfrak{z}^m (1-\mathfrak{z})(1-pq\mathfrak{z})} \right] \neq 0$$

or

$$\mathfrak{z}^m \left[\zeta(\mathfrak{z}) * \frac{1 + \mathfrak{B}(\text{bal}) - \mathfrak{z}}{\mathfrak{z}^m (1-\mathfrak{z})(1-pq\mathfrak{z})} \right] \neq 0,$$

where $\mathfrak{B}(\text{bal}) = b(pq)^m \mathfrak{z} \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1 \right)$ which completes the proof of the necessity in Theorem 2.1.

Conversely, assume that $\zeta \in \mathscr{M}_p$ satisfies condition (2.1). Since the first part of the proof established the equivalence between (2.1) and (2.4), it follows that

$$-\frac{(pq)^m}{[m]_{p,q}} \mathfrak{z} \cdot \frac{\mathfrak{d}_{p,q} \zeta(\mathfrak{z})}{\zeta(\mathfrak{z})} \neq 1 - \frac{b(pq)^m}{[m]_{p,q}} \left(1 - \frac{2\sqrt{1+\mathfrak{z}}}{(1+e^{-\mathfrak{z}})} \right), \quad \mathfrak{z} \in \mathbb{U}^* \tag{2.6}$$

and let

$$\mho(\mathfrak{z}) = -\frac{(pq)^m}{[m]_{p,q}} \frac{\mathfrak{z} \mathfrak{d}_{p,q} \zeta(\mathfrak{z})}{\zeta(\mathfrak{z})}, \qquad \Im(\mathfrak{z}) = 1 - \frac{b(pq)^m}{[m]_{p,q}} \left(1 - \frac{2\sqrt{1+\mathfrak{z}}}{(1+e^{-\mathfrak{z}})} \right).$$

The relation (2.6) implies that

$$\Im(\mathfrak{z}) \cap \mho(\mathfrak{z}) = \emptyset.$$

Hence, the simply connected domain lies entirely within a connected component of $\mathbb{C} \setminus \Im(\mathfrak{d}\mathbb{U}^*)$. Since $\mho(0) = \Im(0)$ and $\Im$ is univalent, it follows from the principle of subordination that

$$\mho(\mathfrak{z}) \prec \Im(\mathfrak{z}),$$

which consequently implies that $\zeta \in \mathscr{M}\mathscr{S}^*_{p,q}(bal)$.

Hence, the proof of Theorem 2.1 is concluded. □

Theorem 2.2. *A function ζ given by equation (1.1), belongs to the class $\mathscr{M}\mathscr{S}^*_{p,\,q}(bal)$ if and only if it satisfies the following condition is that*

$$1 + \sum_{s=1}^{\infty} \left(1 + b(pq)^{m+1} \right) \mathfrak{z} \left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1 \right) [s]_{p,q} a_s \mathfrak{z}^s \neq 0, \quad \mathfrak{z} \in \mathbb{U}^*. \tag{2.7}$$

Proof. In view of Theorem 2.1, we consider a function ζ belongs to $\mathscr{MS}^*_{p,q}(bal)$ if and only if (2.1) is satisfied. Since

$$\frac{1}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})} = \frac{1}{\mathfrak{z}^m} + (1+pq)\mathfrak{z}^{1-m} + \left(1+pq+(pq)^2\right)\mathfrak{z}^{2-m} + \left(1+pq+(pq)^2+(pq)^3\right)\mathfrak{z}^{3-m} + \cdots, \quad (\mathfrak{z}\in\mathbb{U}^*).$$

Hence

$$\begin{aligned} &1 + \frac{[\mathfrak{B}(\text{bal}) - \mathfrak{z}]}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})} \\ &= \frac{1}{\mathfrak{z}^m} + \sum_{s=1}^{\infty}\left(1+b(pq)^{m+1}\right)\mathfrak{z}\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right)[s]_{p,q}\mathfrak{z}^{s-m} \\ &= \frac{1}{z^m} + \sum_{s=1}^{\infty}[1+\mathfrak{B}(\text{bal})][s]_{p,q}\mathfrak{z}^{s-m}. \end{aligned}$$

A straightforward computation reveals that (2.1) is equivalent to (2.7), thereby completing the proof of Theorem 2.2. □

Theorem 2.3. *If $\zeta \in \mathscr{M}_p$ satisfies the inequality*

$$\sum_{s=1}^{\infty}\left[(1+e) + |b|(pq)^{m+1}\left(2\sqrt{2} - (1+e)\right)\right][s]_{p,q}|a_s| \le (1+e) \tag{2.8}$$

then $\zeta \in \mathscr{MS}^*_{p,q}(bal)$.

Proof. Since

$$\begin{aligned} &\left|1 + \sum_{s=1}^{\infty}\left(1+b(pq)^{m+1}\right)\mathfrak{z}\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right)[s]_{p,q}a_s\mathfrak{z}^m\right| \\ &\ge 1 - \left|\sum_{s=1}^{\infty}\left[1 + b(pq)^m\mathfrak{z}\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}} - 1\right)\right][s]_{p,q}a_s z^m\right| \\ &\ge 1 - \sum_{s=1}^{\infty}\left[1 + b(pq)^m|\mathfrak{z}|\left(\frac{2|\sqrt{1+\mathfrak{z}}|}{|1+e^{-\mathfrak{z}}|} - 1\right)\right][s]_{p,q}a_s|\mathfrak{z}^m|. \end{aligned}$$

This implies that

$$\sum_{s=1}^{\infty}(1+e) + |b|(pq)^{m+1}\left(2\sqrt{2} - (1+e)\right)[s]_{p,q}|a_s| \le (1+e).$$

Hence, the proof of Theorem 2.3 is concluded. □

Theorem 2.4. *If $\zeta \in \mathscr{M}_p$, then $\zeta \in \mathscr{MC}_{p,q}(bal))$ if and only if*

$$\mathfrak{z}^m\left\{\zeta(\mathfrak{z}) * -\frac{1}{[m]_{p,q}(p-q)\mathfrak{z}^m}\left[\frac{q^{m-1}\mathfrak{B}_p(\text{bal}) - pq}{(1-p\mathfrak{z})(1-p^2q\,\mathfrak{z})} - \frac{p^{m-1}\mathfrak{B}_p(\text{bal}) - pq}{(1-q\mathfrak{z})(1-q^2p\,\mathfrak{z})}\right]\right\} \tag{2.9}$$

$\neq 0,$

where

$$\mathfrak{B}_p(bal) = b(pq)^{m+1}\left(\frac{2\sqrt{1+p\mathfrak{z}}}{1+e^{-p\mathfrak{z}}}-1\right),$$
$$\mathfrak{B}_q(bal) = b(pq)^{m+1}\left(\frac{2\sqrt{1+q\mathfrak{z}}}{1+e^{-q\mathfrak{z}}}-1\right).$$

Proof. By definition (1.1), it follows that $\zeta \in \mathscr{MC}_{p,q}(bal))$ if and only if

$$-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\mathfrak{D}_{p,q}\zeta(\mathfrak{z}) \in \mathscr{MS}^*_{p,q}(bal).$$

Subsequently, applying Theorem 2.3 yields, $-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\mathfrak{D}_{p,q}\zeta \in \mathscr{MS}^*_{pq}(bal)$ if and only if

$$\mathfrak{z}^m\left[-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{z}\mathfrak{D}_{p,q}\zeta * \mathfrak{g}(\mathfrak{z})\right] \neq 0, \quad \mathfrak{z} \in \mathbb{U}^*, \tag{2.10}$$

where

$$\mathfrak{g}(\mathfrak{z}) = 1 + \frac{[b(pq)^m\mathfrak{z}\left(\frac{2\sqrt{1+\mathfrak{z}}}{1+e^{-\mathfrak{z}}}-1\right)-\mathfrak{z}]}{\mathfrak{z}^m(1-\mathfrak{z})(1-pq\mathfrak{z})}.$$

Upon performing elementary computation, we observe that,

$$\mathfrak{D}_{p,q}g(\mathfrak{z}) = \frac{1}{(p-q)\mathfrak{z}}\left[\frac{b(pq)^m(p\mathfrak{z})\left(\frac{2\sqrt{1+p\mathfrak{z}}}{1+e^{-p\mathfrak{z}}}-1\right)-p\mathfrak{z}}{(p\mathfrak{z})^m(1-p\mathfrak{z})(1-pq\,p\mathfrak{z})}\right.$$
$$\left.-\frac{b(pq)^m(q\mathfrak{z})\left(\frac{2\sqrt{1+q\mathfrak{z}}}{1+e^{-q\mathfrak{z}}}-1\right)-q\mathfrak{z}}{(q\mathfrak{z})^m(1-q\mathfrak{z})(1-pq\,q\mathfrak{z})}\right],$$

and

$$-\frac{(pq)^m}{[m]_{p,q}}\mathfrak{D}_{p,q}g(\mathfrak{z})$$
$$= -\frac{(pq)^m}{[m]_{p,q}}\frac{1}{(p-q)\mathfrak{z}}\left[\frac{b(pq)^m(p\mathfrak{z})\left(\frac{2\sqrt{1+p\mathfrak{z}}}{1+e^{-p\mathfrak{z}}}-1\right)-p\mathfrak{z}}{(p\mathfrak{z})^m(1-p\mathfrak{z})(1-pq\,p\mathfrak{z})}\right.$$
$$\left.-\frac{b(pq)^m(q\mathfrak{z})\left(\frac{2\sqrt{1+q\mathfrak{z}}}{1+e^{-q\mathfrak{z}}}-1\right)-q\mathfrak{z}}{(q\mathfrak{z})^m(1-q\mathfrak{z})(1-pq\,q\mathfrak{z})}\right]$$

$$= -\frac{(pq)^m}{[m]_{p,q}} \cdot \frac{1}{(p-q)\mathfrak{z}^{m+1}} \left[\frac{1}{p^m} \frac{b(pq)^m p\mathfrak{z} \left(\frac{2\sqrt{1+p\mathfrak{z}}}{1+e^{-p\mathfrak{z}}} - 1 \right) - p\mathfrak{z}}{(1-p\mathfrak{z})(1-p^2q\,\mathfrak{z})} \right.$$

$$\left. - \frac{1}{q^m} \frac{b(pq)^m q\mathfrak{z} \left(\frac{2\sqrt{1+q\mathfrak{z}}}{1+e^{-q\mathfrak{z}}} - 1 \right) - q\mathfrak{z}}{(1-q\mathfrak{z})(1-pq^2\,\mathfrak{z})} \right]$$

$$= -\frac{(pq)^m}{[m]_{p,q}} \cdot \frac{1}{(p-q)\mathfrak{z}^m} \left[\frac{1}{p^m} \frac{\frac{\mathfrak{B}_p(\text{bal})}{q} - p}{(1-p\mathfrak{z})(1-p^2q\,\mathfrak{z})} - \frac{1}{q^m} \frac{\frac{\mathfrak{B}_q(\text{bal})}{p} - q}{(1-q\mathfrak{z})(1-pq^2\,\mathfrak{z})} \right]$$

$$= -\frac{1}{[m]_{p,q}(p-q)\mathfrak{z}^m} \left[\frac{q^{m-1}\mathfrak{B}_p(\text{bal}) - pq}{(1-p\mathfrak{z})(1-p^2q\,\mathfrak{z})} - \frac{p^{m-1}\mathfrak{B}_p(\text{bal}) - pq}{(1-q\mathfrak{z})(1-q^2p\,\mathfrak{z})} \right].$$

Consequently, equations (2.10) and (2.9) are equivalent, thereby concluding the proof of Theorem 2.4. □

Theorem 2.5. *A function ζ, defined by equation* (1.1)*, belongs to the class $\mathscr{MC}_{p,q}(bal)$ if and only if it satisfies the following condition*

$$\frac{1}{[m]_{p,q}} \left[\sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{2k-j} q^{k+m-j} + \sum_{j=0}^{k} p^{2k+1-j} q^{k+m-j} |\mathfrak{B}|(bal) \right) \right.$$

$$\left. - \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{2k-j} p^{k+m-j} + \sum_{j=0}^{k} q^{2k+1-j} p^{k+m-j} |\mathfrak{B}(bal)| \right) \right]$$

$$\neq 0.$$

Proof. According to Theorem 2.4, a function $\zeta \in \mathscr{MC}_{p,q}(bal))$ if and only if condition (2.9) is satisfied. Consider

$$\frac{1}{\mathfrak{z}^m(1-p\mathfrak{z})(1-p^2\mathfrak{z})} = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{j+2(k-j)} q^{k-j} \right) \mathfrak{z}^{k-m} + \cdots$$

and

$$\frac{1}{\mathfrak{z}^m(1-q\mathfrak{z})(1-q^2\mathfrak{z})} = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{j+2(k-j)} p^{k-j} \right) \mathfrak{z}^{k-m} + \cdots,$$

so that

$$\frac{q^m \left(1 + p\mathfrak{z}\,\mathfrak{B}(bal)\right)}{[m]_{p,q}\,\mathfrak{z}^m(1-p\mathfrak{z})(1-p^2q\mathfrak{z})}$$

$$= \frac{q^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{2k-j} q^{k-j} + \sum_{j=0}^{k} p^{2k+1-j} q^{k-j} \mathfrak{z}\mathfrak{B}(bal) \right) \mathfrak{z}^{k-m}$$

and

$$\frac{p^m \left(1 + q\mathfrak{z}\, \mathfrak{B}(bal)\right)}{[m]_{p,q}\, \mathfrak{z}^m (1 - q\mathfrak{z})(1 - q^2 p\mathfrak{z})}$$
$$= \frac{p^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{2k-j} p^{k-j} + \sum_{j=0}^{k} q^{2k+1-j} p^{k-j} \mathfrak{z}\mathfrak{B}(bal) \right) \mathfrak{z}^{k-m}.$$

Now

$$\left| \frac{q^m \left(1 + p\mathfrak{z}\, \mathfrak{B}(bal)\right)}{[m]_{p,q}\, \mathfrak{z}^m (1 - p\mathfrak{z})(1 - p^2 q\mathfrak{z})} - \frac{p^m \left(1 + q\mathfrak{z}\, \mathfrak{B}(bal)\right)}{[m]_{p,q}\, \mathfrak{z}^m (1 - q\mathfrak{z})(1 - q^2 p\mathfrak{z})} \right| \neq 0,$$

therefore

$$\left| \frac{q^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{2k-j} q^{k-j} + \sum_{j=0}^{k} p^{2k+1-j} q^{k-j}\, \mathfrak{z}\mathfrak{B}(\mathrm{bal}) \right) \mathfrak{z}^{k-m} \right.$$
$$\left. - \frac{p^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{2k-j} p^{k-j} + \sum_{j=0}^{k} q^{2k+1-j} p^{k-j}\, \mathfrak{z}\mathfrak{B}(\mathrm{bal}) \right) \mathfrak{z}^{k-m} \right|$$
$$\neq 0,$$

implies that

$$\frac{q^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{2k-j} q^{k-j} + \sum_{j=0}^{k} p^{2k+1-j} q^{k-j}\, |\mathfrak{B}|(\mathrm{bal}) \right)$$
$$- \frac{p^m}{[m]_{p,q}} \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{2k-j} p^{k-j} + \sum_{j=0}^{k} q^{2k+1-j} p^{k-j}\, |\mathfrak{B}(\mathrm{bal})| \right) \mathfrak{z}^{k-m}$$
$$\neq 0$$

and

$$\frac{1}{[m]_{p,q}} \left[\sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} p^{2k-j} q^{k+m-j} + \sum_{j=0}^{k} p^{2k+1-j} q^{k+m-j}\, |\mathfrak{B}|(\mathrm{bal}) \right) \right.$$
$$\left. - \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q^{2k-j} p^{k+m-j} + \sum_{j=0}^{k} q^{2k+1-j} p^{k+m-j}\, |\mathfrak{B}(\mathrm{bal})| \right) \right]$$
$$\neq 0.$$

Hence, the proof of Theorem 2.5 is concluded. □

Theorem 2.6. *If $\zeta \in \mathscr{M}_p$, then it satisfies the inequality*

$$\left[\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k} p^{2k-j}q^{k+m-j} + \sum_{j=0}^{k} p^{2k+1-j}q^{k+m-j}\,|\mathfrak{B}|(bal)\right) - \sum_{k=1}^{\infty}\left(\sum_{j=0}^{k} q^{2k-j}p^{k+m-j} + \sum_{j=0}^{k} q^{2k+1-j}p^{k+m-j}\,|\mathfrak{B}(bal)|\right)\right] < [m]_{p,q}$$

then $\zeta \in \mathscr{MC}_{p,q}(bal)$.

Proof. The proof is analogous to that of the previous theorems and hence is omitted. □

3. Conclusion

In this study, we examine various properties of meromorphic m-valent functions associated with a balloon-shaped domain in the context of (p, q)-calculus. Initially, we derive the necessary and sufficient conditions for functions belong to the subclasses $\mathscr{MS}^*_{p,q}(bal)$ and $\mathscr{MC}_{p,q}(bal)$.

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