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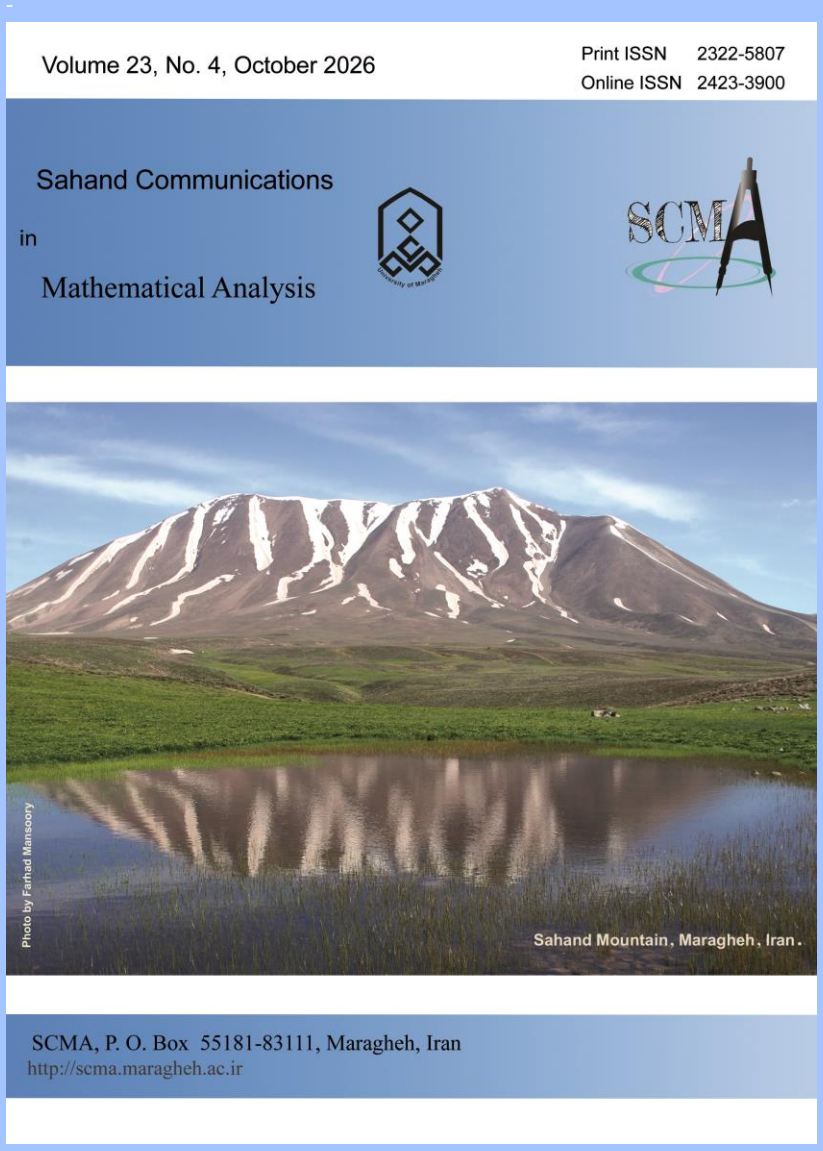

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New Approach for Certain Analytic Functions

Muhammet Kamali [1*] and Shigeyoshi Owa[2]

Abstract. Let A be the class of functions $f(z)$ in the open unit disc U with $f(0) = 0$. For this class A, we introduce the subclass $A_n(\alpha)$ of A consisting of

$$f(z) = \alpha z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}, \quad (n \in \mathbb{N}),$$

for $\alpha > 0$. For functions $f(z) \in A_n(\alpha)$, we discuss some coefficient problems, subordination problems and argument problems with some examples of the results.

1. Introduction

Let A be the class of analytic functions $f(z)$ in the open unit disc $U = \{z \in C : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Historically, we use the following functions $f(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

for $f(z) \in A$. Because, we are considering $f(z) \in A$ given by (1.1) using Taylor expansion. If we consider functions $f(z)$ given by

$$f(z) = \alpha z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}, \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1.2}$$

for $\alpha > 0$, then $f(z)$ is analytic in U with $f(0) = 0$ and $f'(0) = \alpha > 0$, where we take the principal value for $z^{\frac{k}{n}}$. If $\alpha = 1$ in (1.2), then $f(z)$ is analytic in U with $f(0) = 0$ and $f'(0) = 1$. Thus, for $\alpha = 1, f(z)$ given

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by (1.2) belongs to the class A. Therefore, $f(z)$ given by (1.2) is one of generalizations for (1.1). In this paper, we use $f(z) \in A_n(\alpha)$ for $f(z)$ given by (1.2).

If we consider a function $f(z) \in A$ given by

$$f(z) = \frac{z}{(1-z)^t} = z + tz^2 + \cdots \tag{1.3}$$

with

$$t = 2(1-\beta)\cos\gamma e^{-i\gamma}, \quad \left(|\gamma| < \frac{\pi}{2}, 0 \le \beta < 1\right)$$

then $f(z)$ has the form (1.1) and satisfies

$$\begin{aligned} \mathrm{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left\{e^{i\gamma} + 2(1-\beta)\cos\gamma\left(\frac{z}{1-z}\right)\right\} \\ &> \cos\gamma - (1-\beta)\cos\gamma \\ &= \beta\cos\gamma, \quad (z \in U). \end{aligned} \tag{1.4}$$

We call γ - spirallike function of order β for such $f(z)$ with the geometrical property. If $\gamma = 0$ in (1.3), $f(z)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad (z \in U),$$

for $0 \le \beta < 1$ and $f(z)$ is called as starlike of order β in U from the image domain of $f(z)$. If we consider a function $f(z)$ given by

$$f(z) = \frac{\alpha z}{(1-\sqrt[n]{z})^t} = \alpha z + \alpha t z^{1+\frac{1}{n}} + \cdots, \quad (\alpha > 0) \tag{1.5}$$

with

$$t = 2n(1-\beta)\cos\gamma e^{-i\gamma}, \quad \left(|\gamma| < \frac{\pi}{2}, 0 \le \beta < 1\right)$$

then $f(z)$ belongs to $A_n(\alpha)$ and satisfies

$$\begin{aligned} \mathrm{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left\{e^{i\gamma} + 2(1-\beta)\cos\gamma\frac{\sqrt[n]{z}}{1-\sqrt[n]{z}}\right\} \\ &> \cos\gamma - (1-\beta)\cos\gamma \\ &= \beta\cos\gamma, \quad (z \in U). \end{aligned} \tag{1.6}$$

Also, if $\gamma = 0$ in (1.5), then $f(z) \in A_n(\alpha)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad (z \in U). \tag{1.7}$$

For γ - spirallike functions of order β and starlike functions of order β, we see many research papers, but we don't see any research paper for $A_n(\alpha)$ satisfying (1.6) and (1.7).

With the above facts, we introduce the new class $A_n(\alpha)$ of $f(z)$ given by (1.2) and discuss some problems with

$$\operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > \beta\cos\gamma, \quad (z \in U), \tag{1.8}$$

with $|\gamma| < \frac{\pi}{2}$ and $\beta > 1$.

If we consider a function $f(z) \in A_n(\alpha)$ given by

$$f(z) = \alpha z(1 - \sqrt[n]{z})^t = \alpha z + \alpha t z^{1+\frac{1}{n}} + \cdots, \quad (\alpha > 0) \tag{1.9}$$

with

$$t = 2n(\beta - 1)\cos\gamma e^{-i\gamma}, \quad \left(|\gamma| < \frac{\pi}{2}, \beta > 1\right),$$

then $f(z)$ satisfies

$$\begin{aligned}\operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left\{e^{i\gamma} - 2(\beta-1)\cos\gamma\frac{\sqrt[n]{z}}{1-\sqrt[n]{z}}\right\} \\ &< \cos\gamma + (\beta-1)\cos\gamma \\ &= \beta\cos\gamma, \quad (z \in U).\end{aligned} \tag{1.10}$$

Therefore, $f(z)$ given by (1.9) satisfies (1.8) .

Also, if we consider a function $f(z) \in A_n(\alpha)$ given by

$$f(z) = (2\beta - \alpha)z - 2(\beta - \alpha)\cos\gamma e^{-i\gamma}\int_0^z \frac{dt}{1 - \sqrt[n]{t}}$$

with $|\gamma| < \frac{\pi}{2}$ and $0 < \alpha < \beta$, then $f(z)$ satisfies

$$\begin{aligned}\operatorname{Re}\left(e^{i\gamma}f'(z)\right) &= (2\beta-\alpha)\cos\gamma - 2(\beta-\alpha)\cos\gamma\operatorname{Re}\left(\frac{1}{1-\sqrt[n]{z}}\right) \\ &< (2\beta-\alpha)\cos\gamma - (\beta-\alpha)\cos\gamma \\ &= \beta\cos\gamma, \quad (z \in U).\end{aligned}$$

With above explanation, we would like to discuss some problems of $f(z) \in A_n(\alpha)$ concerning with (1.8) and (1.10). Actually, coefficient problems for $f(z) \in A_n(\alpha)$ and subordination problems using the result by Hallenbeck and Ruscheweyh [5] are considered. Further, applying the results due to Fejer and Riesz [2] or Nunokawa [9], we discuss some argument problems with examples for our results. We think it's very important to give some examples for the results. Finally, we would like to try to apply the result by Jack [6].

Studies in analytic function theory continue on various problems. In recent years, it has been possible to encounter some new problems in this theory. Some of these studies can be seen in [3, 7].

2. Coefficient Problems

We discuss some coefficient problems for $f(z) \in A_n(\alpha)$.

Theorem 2.1. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\sum_{k=1}^{\infty} \left(\frac{k}{n} + (\beta - 1)\cos\gamma \right) \left| a_{1+\frac{k}{n}} \right| \leq \alpha(\beta - 1)\cos\gamma, \tag{2.1}$$

for $\beta > 1$ and $|\gamma| < \frac{\pi}{2}$, then

$$\operatorname{Re}\left(e^{i\gamma} \frac{zf'(z)}{f(z)} \right) < \beta\cos\gamma, \quad (z \in U). \tag{2.2}$$

Proof. Consider a function $w(z)$ given by

$$\frac{zf'(z)}{f(z)} - 1 = (\beta - 1)\cos\gamma w(z), \quad (z \in U),$$

for $f(z) \in A_n(\alpha)$ satisfying (2.1). Then $w(z)$ is analytic in U with $w(0) = 0$. If $w(z)$ satisfies $|w(z)| < 1 (z \in U)$, then $f(z)$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < (\beta - 1)\cos\gamma, \quad (z \in U). \tag{2.3}$$

For such $f(z)$ and $w(z)$, $f(z)$ satisfies

$$\begin{aligned} \operatorname{Re}\left(e^{i\gamma} \frac{zf'(z)}{f(z)} \right) &= \operatorname{Re}\left(e^{i\gamma}(1 + (\beta - 1)\cos\gamma w(z)) \right) \\ &= \cos\gamma + (\beta - 1)\cos\gamma \operatorname{Re}\left(e^{i\gamma} w(z) \right) \\ &\leq \cos\gamma + (\beta - 1)\cos\gamma \left| e^{i\gamma} w(z) \right| \\ &< \cos\gamma + (\beta - 1)\cos\gamma \\ &= \beta\cos\gamma, \quad (z \in U). \end{aligned}$$

Therefore, if $f(z)$ satisfies (2.3), then $f(z)$ satisfies (2.2). It follows that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{\sum_{k=1}^{\infty} \left(\frac{k}{n}\right) a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}}{\alpha z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}} \right| \\ &< \frac{\sum_{k=1}^{\infty} \left(\frac{k}{n}\right) \left| a_{1+\frac{k}{n}} \right|}{\alpha - \sum_{k=1}^{\infty} \left| a_{1+\frac{k}{n}} \right|}. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we see that if $f(z)$ satisfies the coefficient inequality (2.1), then $f(z)$ satisfies (2.2).

Further, we consider the function $f(z) \in A_n(\alpha)$ given by

$$f(z) = \alpha z + \sum_{k=1}^{\infty} c_{1+\frac{k}{n}} z^{1+\frac{k}{n}},$$

with

$$c_{1+\frac{k}{n}} = \frac{\alpha(\beta-1)\cos\gamma e^{i\theta}}{\left(\frac{k}{n} + (\beta-1)\cos\gamma\right) k(k+1)}, \quad (0 \le \theta < 2\pi).$$

Then $f(z)$ satisfies

$$\begin{aligned}\sum_{k=1}^{\infty} \left(\frac{k}{n} + (\beta-1)\cos\gamma\right) \left|a_{1+\frac{k}{n}}\right| &\le \alpha(\beta-1)\cos\gamma \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\ &= \alpha(\beta-1)\cos\gamma. \qquad \square\end{aligned}$$

Letting $\gamma = 0$ in Theorem 2.1, we see

Corollary 2.2. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\sum_{k=1}^{\infty} \left(\frac{k}{n} + (\beta-1)\right) \left|a_{1+\frac{k}{n}}\right| \le \alpha(\beta-1),$$

for $\beta > 1$, then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad (z \in U).$$

When addressing the coefficient inequalities for $f(z) \in A_n(\alpha)$, we will use the following lemma, which is a generalization of the lemma due to Caratheodory [1].

Lemma 2.3. *If a function $p(z)$ given by*

$$p(z) = \alpha + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}}, \quad (n \in \mathbb{N}),$$

is analytic in U and $\operatorname{Re} p(z) > \beta (z \in U)$ for $0 \le \beta < \alpha$, then

$$\left|c_{\frac{k}{n}}\right| \le 2(\alpha-\beta), \quad (k \in \mathbb{N}). \tag{2.5}$$

The equality in (2.5) is given by

$$p(z) = \frac{\alpha + (\alpha - 2\beta)\sqrt[n]{z}}{1 - \sqrt[n]{z}} = \alpha + 2(\alpha-\beta)\sum_{k=1}^{\infty} z^{\frac{k}{n}}.$$

With Lemma 2.3, we derive

Theorem 2.4. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad (z \in U), \tag{2.6}$$

for some $\beta > 1$, then

$$\left|a_{1+\frac{1}{n}}\right| \leq 2\alpha n(\beta-1), \tag{2.7}$$

and

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2\alpha n(\beta-1)}{k} \prod_{l=1}^{k-1}\left(1+\frac{2n(\beta-1)}{l}\right), \tag{2.8}$$

where $0 < \gamma < \beta$. Equalities in (2.7) and (2.8) are given by

$$f(z) = \frac{\alpha z}{(1-\sqrt[n]{z})^{2n(1-\beta)}}.$$

Proof. We define $p(z)$ by

$$p(z) = \beta - \frac{zf'(z)}{f(z)} = (\beta-1) + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}}, \tag{2.9}$$

for $f(z) \in A_n(\alpha)$ which satisfies (2.6). Then $p(z)$ is analytic in $U, p(0) = \beta - 1$ and $\operatorname{Re} p(z) > 0 (z \in U)$. By Lemma 2.3, we know that

$$\left|c_{\frac{k}{n}}\right| \leq 2(\beta-1), \quad (k \in \mathbb{N}). \tag{2.10}$$

The inequality in (2.10) is given by

$$p(z) = \frac{(\beta-1)(1+\sqrt[n]{z})}{(1-\sqrt[n]{z})}.$$

Since (2.9) show us that

$$\begin{aligned} &\beta f(z) - zf'(z) = p(z)f(z) \\ &\Rightarrow \quad \beta f(z) - zf'(z) = \alpha(\beta-1)z + \sum_{k=1}^{\infty}\left(\beta - 1 - \frac{k}{n}\right) a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}, \end{aligned}$$

and

$$\begin{aligned} p(z)f(z) &= \left((\beta-1) + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}}\right)\left(\alpha z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}\right) \\ &= \alpha(\beta-1)z + \left\{\alpha c_{\frac{1}{n}} + (\beta-1)a_{1+\frac{1}{n}}\right\} z^{1+\frac{1}{n}} \\ &\quad + \left\{\alpha c_{\frac{2}{n}} + c_{\frac{1}{n}} a_{1+\frac{1}{n}} + (\beta-1)a_{1+\frac{2}{n}}\right\} z^{1+\frac{2}{n}} \\ &\quad + \left\{\alpha c_{\frac{3}{n}} + c_{\frac{2}{n}} a_{1+\frac{1}{n}} + c_{\frac{1}{n}} a_{1+\frac{2}{n}} + (\beta-1)a_{1+\frac{3}{n}}\right\} z^{1+\frac{3}{n}} + \cdots \\ &\quad + \left\{\begin{array}{c} \alpha c_{\frac{k}{n}} + c_{\frac{k-1}{n}} a_{1+\frac{1}{n}} + c_{\frac{k-2}{n}} a_{1+\frac{2}{n}} + \cdots \\ + c_{\frac{2}{n}} a_{1+\frac{k-2}{n}} + c_{\frac{1}{n}} a_{1+\frac{k-1}{n}} + (\beta-1)a_{1+\frac{k}{n}} \end{array}\right\} z^{1+\frac{k}{n}} + \cdots . \end{aligned}$$

We have for $k = 1$

$$\left(\beta - 1 - \frac{k}{n}\right) a_{1+\frac{1}{n}} = \left(\alpha c_{\frac{1}{n}} + (\beta - 1) a_{1+\frac{1}{n}}\right).$$

Therefore, with (2.10)

$$\left|a_{1+\frac{1}{n}}\right| \leq 2\alpha n(\beta - 1).$$

For $k = 2$, we have that

$$-\frac{2}{n} a_{1+\frac{2}{n}} = \alpha c_{\frac{2}{n}} + c_{\frac{1}{n}} a_{1+\frac{1}{n}},$$

and that

$$\begin{aligned}\left|a_{1+\frac{2}{n}}\right| &\leq \frac{2n(\beta-1)}{2}\left(\alpha + \left|a_{1+\frac{1}{n}}\right|\right) \\ &\leq \frac{2\alpha n(\beta-1)}{2}\left(1 + 2n(\beta-1)\right).\end{aligned}$$

Also, for $k = 3$, we see that

$$\begin{aligned}\left|a_{1+\frac{3}{n}}\right| &\leq \frac{2n(\beta-1)}{3}\left(\alpha + \left|a_{1+\frac{1}{n}}\right| + \left|a_{1+\frac{2}{n}}\right|\right) \\ &\leq \frac{2\alpha n(\beta-1)}{3}(1 + 2n(\beta-1))\left(1 + \frac{2n(\beta-1)}{2}\right).\end{aligned}$$

Using mathematical induction, we prove that

$$\begin{aligned}\left|a_{1+\frac{k}{n}}\right| &\leq \frac{2n(\beta-1)}{k}\left(\alpha + \left|a_{1+\frac{1}{n}}\right| + \left|a_{1+\frac{2}{n}}\right| + \cdots + \left|a_{1+\frac{k-1}{n}}\right|\right) \\ &\leq \frac{2\alpha n(\beta-1)}{k}\prod_{l=1}^{k-1}\left(1 + \frac{2n(\beta-1)}{l}\right),\end{aligned}$$

for $k = 2, 3, 4, \ldots$.

For equalities in (2.7) and (2.8), we consider

$$p(z) = \beta - \frac{zf'(z)}{f(z)} = \frac{(\beta-1)(1 + \sqrt[n]{z})}{(1 - \sqrt[n]{z})}.$$

This implies that

$$\frac{zf'(z)}{f(z)} = 1 + 2(1-\beta)\frac{\sqrt[n]{z}}{(1 - \sqrt[n]{z})},$$

and

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + 2(1-\beta)\frac{\sqrt[n]{z}}{z(1 - \sqrt[n]{z})}.$$

This means that

$$(\log f(z))' = (\log z)' - 2n(1-\beta)(\log(1 - \sqrt[n]{z}))'.$$

Thus we have

$$f(z) = \frac{\alpha z}{(1 - \sqrt[n]{z})^{2n(1-\beta)}}.$$

This completes the proof of the theorem. □

Next, we have

Theorem 2.5. *If $f(z) \in A_n(\alpha)$ with $0 < \alpha \leq 1$ satisfies*

$$\operatorname{Re} f'(z) < \beta, \quad (z \in U), \tag{2.11}$$

for $\beta > 1$, then

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2n(\beta - \alpha)}{k + n}, \quad (k \in \mathbb{N}). \tag{2.12}$$

The equality in (2.12) is attended for

$$f(z) = (2\beta - \alpha)z - 2(\beta - \alpha) \int_0^z \frac{dt}{1 - \sqrt[n]{t}}.$$

Proof. We consider a function $p(z)$ given by

$$p(z) = \beta - f'(z),$$

for $f(z)$ satisfying (2.11). Then $p(z)$ is analytic in $U, p(0) = \beta - \alpha > 0$ and $\operatorname{Re} p(z) > 0 (z \in U)$. By Lemma 2.3, if we write that

$$p(z) = \beta - \alpha + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}},$$

then $p(z)$ satisfies

$$\left|c_{\frac{k}{n}}\right| \leq 2(\beta - \alpha), \quad (k \in \mathbb{N}),$$

and the equality is given by

$$p(z) = \frac{(\beta - \alpha)(1 + \sqrt[n]{z})}{(1 - \sqrt[n]{z})}.$$

Noting that

$$(\beta - \alpha) + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}} = (\beta - \alpha) + \sum_{k=1}^{\infty} \left(1 + \frac{k}{n}\right) a_{1+\frac{k}{n}} z^{\frac{k}{n}},$$

we see that

$$\left|a_{1+\frac{k}{n}}\right| = \frac{n}{k + n} \left|c_{\frac{k}{n}}\right| \leq \frac{2n(\beta - \alpha)}{k + n}, \quad (k \in \mathbb{N}).$$

Considering $f(z) \in A_n(\alpha)$ given by

$$f'(z) = \beta - p(z) = \frac{\alpha + (\alpha - 2\beta)\sqrt[n]{z}}{(1 - \sqrt[n]{z})},$$

we have that

$$f(z) = (2\beta - \alpha)z - 2(\beta - \alpha)\int_0^z \frac{dt}{1 - \sqrt[n]{t}}. \qquad \square$$

Further, we derive

Theorem 2.6. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) < \beta\cos\gamma, \quad (z \in U),$$

for $|\gamma| < \frac{\pi}{2}$ and $\beta > 1$, then

$$\left|a_{1+\frac{1}{n}}\right| \leq 2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|, \tag{2.13}$$

and

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|}{k}\prod_{l=1}^{k-1}\left(1 + \frac{2n\left|\beta\cos\gamma - e^{i\gamma}\right|}{l}\right), \tag{2.14}$$

for $k = 2, 3, 4, \ldots$.

The equalities in (2.13) and (2.14) are given by

$$f(z) = \frac{\alpha z}{(1 - \sqrt[n]{z})t},$$

with

$$t = 2n\left(1 - \beta\cos\gamma e^{-i\gamma}\right). \tag{2.15}$$

Proof. We define a function $p(z)$ by

$$p(z) = \frac{\beta\cos\gamma - \left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)}{\beta\cos\gamma - e^{i\gamma}}. \tag{2.16}$$

Then $p(z)$ is analytic in $U, p(0) = 1, \operatorname{Re} p(z) > 0 (z \in U)$, and $p(z)$ is given by

$$p(z) = 1 + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}}.$$

Thus, using Lemma 2.3, we know that

$$\left|c_{\frac{k}{n}}\right| \leq 2, \quad (k \in \mathbb{N}), \tag{2.17}$$

and the equality is given by

$$p(z) = \frac{1 + \sqrt[n]{z}}{1 - \sqrt[n]{z}} = 1 + 2\sum_{k=1}^{\infty} z^{\frac{k}{n}}. \tag{2.18}$$

From (2.16), we have

$$\beta\cos\gamma f(z) - e^{i\gamma}zf'(z) = \left(\beta\cos\gamma - e^{i\gamma}\right)p(z)f(z)$$

(2.19)

$$\begin{aligned}\Rightarrow \quad & \beta\cos\gamma f(z) - e^{i\gamma} z f'(z) \\ & = \alpha\left(\beta\cos\gamma - e^{i\gamma}\right) z + \sum_{k=1}^{\infty}\left(\beta\cos\gamma - e^{i\gamma}\left(1+\frac{k}{n}\right)\right) a_{1+\frac{k}{n}} z^{1+\frac{k}{n}},\end{aligned}$$

and

(2.20)

$$\begin{aligned}&\left(\beta\cos\gamma - e^{i\gamma}\right) p(z) f(z) \\ &\quad = \alpha\left(\beta\cos\gamma - e^{i\gamma}\right) z \\ &\qquad + \left(\beta\cos\gamma - e^{i\gamma}\right)\left(\alpha c_{\frac{1}{n}} + a_{1+\frac{1}{n}}\right) z^{1+\frac{1}{n}} \\ &\qquad + \left(\beta\cos\gamma - e^{i\gamma}\right)\left(\alpha c_{\frac{2}{n}} + c_{\frac{1}{n}} a_{1+\frac{1}{n}} + a_{1+\frac{2}{n}}\right) z^{1+\frac{2}{n}} \\ &\qquad + \left(\beta\cos\gamma - e^{i\gamma}\right)\left(\alpha c_{\frac{3}{n}} + c_{\frac{2}{n}} a_{1+\frac{1}{n}} + c_{\frac{1}{n}} a_{1+\frac{2}{n}} + a_{1+\frac{3}{n}}\right) z^{1+\frac{3}{n}} \\ &\qquad \vdots \\ &\qquad + \left(\beta\cos\gamma - e^{i\gamma}\right) \\ &\qquad \times \left(\alpha c_{\frac{k}{n}} + c_{\frac{k-1}{n}} a_{1+\frac{1}{n}} + c_{\frac{k-2}{n}} a_{1+\frac{2}{n}} + \cdots + c_{\frac{1}{n}} a_{1+\frac{k-1}{n}} + a_{1+\frac{k}{n}}\right) z^{1+\frac{k}{n}} + \cdots\end{aligned}$$

Therefore, by (2.19) and (2.20), we see that

$$-\frac{e^{i\gamma}}{n} a_{1+\frac{1}{n}} = \alpha\left(\beta\cos\gamma - e^{i\gamma}\right) c_{\frac{1}{n}}. \tag{2.21}$$

Using (2.17),(2.21) shows

$$\left|a_{1+\frac{1}{n}}\right| = 2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|.$$

For $a_{1+\frac{2}{n}}$, we have

$$-\frac{2e^{i\gamma}}{n} a_{1+\frac{2}{n}} = \left(\beta\cos\gamma - e^{i\gamma}\right)\left(\alpha c_{\frac{2}{n}} + c_{\frac{1}{n}} a_{1+\frac{1}{n}}\right),$$

and

$$\begin{aligned}\left|a_{1+\frac{2}{n}}\right| &\le \frac{2n\left|\beta\cos\gamma - e^{i\gamma}\right|}{2}\left(\alpha + 2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|\right) \\ &= \frac{2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|}{2}\left(1 + 2n\left|\beta\cos\gamma - e^{i\gamma}\right|\right).\end{aligned}$$

Further, we see that

$$-\frac{3e^{i\gamma}}{n} a_{1+\frac{3}{n}} = \left(\beta\cos\gamma - e^{i\gamma}\right)\left(\alpha c_{\frac{3}{n}} + c_{\frac{2}{n}} a_{1+\frac{1}{n}} + c_{\frac{1}{n}} a_{1+\frac{2}{n}}\right),$$

and

$$\left|a_{1+\frac{3}{n}}\right| \leq \frac{2n\left|\beta\cos\gamma - e^{i\gamma}\right|}{3}\left(\alpha + \left|a_{1+\frac{1}{n}}\right| + \left|a_{1+\frac{2}{n}}\right|\right)$$
$$\leq \frac{2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|}{3}\left(1 + 2n\left|\beta\cos\gamma - e^{i\gamma}\right|\right)$$
$$\times\left(1 + \frac{2n\left|\beta\cos\gamma - e^{i\gamma}\right|}{2}\right).$$

Finally, applying mathematical induction, we obtain that

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2\alpha n\left|\beta\cos\gamma - e^{i\gamma}\right|}{k}\prod_{l=1}^{k-1}\left(1 + \frac{2n\left|\beta\cos\gamma - e^{i\gamma}\right|}{l}\right),$$

for $k = 2, 3, 4, \ldots$.

Let us consider $p(z)$ given by (2.18). Then $f(z)$ satisfies

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + 2\left(\beta\cos\gamma e^{-i\gamma} - 1\right)\frac{\sqrt[n]{z}}{z(1 - \sqrt[n]{z})},$$

and

$$\log f(z) = \log z + 2n\left(1 - \beta\cos\gamma e^{-i\gamma}\right)\log(1 - \sqrt[n]{z}).$$

This gives us that

$$f(z) = \frac{\alpha z}{(1 - \sqrt[n]{z})^t},$$

with t given by (2.15). □

Remark 2.7. If we take $\gamma = 0$ in Theorem 2.6, then we have Theorem 2.4 .

3. Subordination Problems

Let functions $f(z)$ and $g(z)$ be analytic in U. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z) \quad (z \in U)$, if there exists a function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ and such that $f(z) = g(w(z))$. If $g(z)$ is univalent in U, then $f(z) \prec g(z) \quad (z \in U)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (See Pommerenke [10] or Miller and Mocanu [8]). Hallenbeck and Ruscheweyh [5] have given the following result for subordinations.

Lemma 3.1. *Let $p(z)$ be analytic in U with $p(0) = 1$ and $h(z)$ be analytic and convex in U. If $p(z)$ satisfies*

$$p(z) + zp'(z) \prec h(z), \quad (z \in U),$$

then

$$p(z) \prec \frac{1}{z}\int_0^z h(t)dt, \quad (z \in U).$$

With the above lemma, we have

Theorem 3.2. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\frac{f'(z)+zf''(z)-\beta}{\alpha-\beta} \prec \frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \quad (z \in U), \tag{3.1}$$

for $\beta > 1$, then

$$\frac{f'(z)-\beta}{\alpha-\beta} \prec (2\beta-1) - \frac{2(\beta-1)}{z}\int_0^z \frac{dt}{1-\sqrt[n]{t}}, \quad (z \in U).$$

Proof. We define a function $p(z)$ by

$$p(z) = \frac{f'(z)-\beta}{\alpha-\beta},$$

and a function $h(z)$ by

$$h(z) = \frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \tag{3.2}$$

with $\beta > 1$. Then $p(z)$ be analytic in U and $p(0) = 1$, and $h(z)$ is convex in U with

$$\operatorname{Re} h(z) = 2\beta - 1 - 2(\beta-1)\operatorname{Re}\left(\frac{1}{1-\sqrt[n]{z}}\right) < \beta, \quad (z \in U).$$

It follows that

$$p(z) + zp'(z) = \frac{1}{\alpha-\beta}\left(f'(z)+zf''(z)-\beta\right).$$

Therefore, if $p(z)$ satisfies (3.1), then Lemma 3.1 implies that

$$\begin{aligned} p(z) &= \frac{f'(z)-\beta}{\alpha-\beta} \\ &\prec \frac{1}{z}\int_0^z \frac{1+(1-2\beta)\sqrt[n]{t}}{1-\sqrt[n]{t}}\,dt \\ &= \frac{1}{z}\int_0^z \left((2\beta-1) - 2(\beta-1)\frac{1}{1-\sqrt[n]{t}}\right)dt \\ &= (2\beta-1) - \frac{2(\beta-1)}{z}\int_0^z \frac{dt}{1-\sqrt[n]{t}}, \quad (z \in U). \end{aligned}$$

□

If we take $n = 1$ in Theorem 3.2, then we see

Corollary 3.3. *If $f(z) \in A_1(\alpha)$ satisfies*

$$\frac{f'(z)+zf''(z)-\beta}{\alpha-\beta} \prec \frac{1+(1-2\beta)z}{1-z}, \quad (z \in U),$$

for $\beta > 1$, *then*

$$\frac{f'(z)-\beta}{\alpha-\beta} \prec (2\beta-1)+2(\beta-1)\frac{\log(1-z)}{z}, \quad (z \in U).$$

Further, considering functions

$$p(z) = \frac{1}{\alpha-\beta}\left(\frac{f(z)}{z}-\beta\right),$$

for $f(z) \in A_n(\alpha)$ *and*

$$h(z) = \frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \quad (z \in U).$$

For $\beta > 1$, we obtain

Theorem 3.4. *If* $f(z) \in A_n(\alpha)$ *satisfies*

$$\frac{f'(z)-\beta}{\alpha-\beta} \prec \frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \quad (z \in U),$$

for $\beta > 1$, *then*

$$\frac{1}{\alpha-\beta}\left(\frac{f(z)}{z}-\beta\right) \prec (2\beta-1)-\frac{2(\beta-1)}{z}\int_0^z \frac{dt}{1-\sqrt[n]{t}}, \quad (z \in U).$$

Corollary 3.5. *If* $f(z) \in A_1(\alpha)$ *satisfies*

$$\frac{f'(z)-\beta}{\alpha-\beta} \prec \frac{1+(1-2\beta)z}{1-z}, \quad (z \in U),$$

for $\beta > 1$, *then*

$$\frac{1}{\alpha-\beta}\left(\frac{f(z)}{z}-\beta\right) \prec (2\beta-1)+2(\beta-1)\frac{\log(1-z)}{z}, \quad (z \in U).$$

Further, we see

Theorem 3.6. *If* $f(z) \in A_n(\alpha)$ *satisfies*

$$f'(z)+zf''(z) \prec \alpha\frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \quad (z \in U), \tag{3.3}$$

for $\beta > 1$, *then*

$$f'(z) \prec \alpha(2\beta-1)-\frac{2\alpha(\beta-1)}{z}\int_0^z \frac{dt}{1-\sqrt[n]{t}}, \quad (z \in U). \tag{3.4}$$

Proof. We consider functions $p(z)$ by $p(z) = \frac{f'(z)}{\alpha}$ and $h(z)$ by (3.2). Then Lemma 3.1 implies that if

$$p(z)+zp'(z) = \frac{1}{\alpha}\left(f'(z)+zf''(z)\right) \prec h(z), \quad (z \in U), \tag{3.5}$$

then

$$p(z) = \frac{f'(z)}{\alpha} \prec \frac{1}{z}\int_0^z h(t)dt, \quad (z \in U). \tag{3.6}$$

Thus, we see (3.5) is the same as (3.3) and (3.6) is the same as (3.4). □

4. Argument Problems

We consider a function $f(z) \in A_n(\alpha)$ given by

$$f(z) = \alpha \int_0^z \left(\frac{1+\sqrt[n]{t}}{1-\sqrt[n]{t}}\right)^{\gamma} dt, \quad (z \in U),$$

with $0 \leq \gamma < 1$. Then $f(z)$ satisfies

$$\left|\arg f'(z)\right| = \gamma \left|\arg\left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)\right| < \frac{\pi}{2}\gamma, \quad (z \in U). \tag{4.1}$$

We would like to discuss some conditions for $f(z) \in A_n(\alpha)$ which satisfies (4.1).

To do these problems, we need the following lemmas. The next lemma was given by Fejer and Riesz [2] (or by Tsuji [11]).

Lemma 4.1. *If a function $f(z)$ is analytic in $\bar{U}$, then*

$$\int_{-1}^{1} |f(z)|^{\rho}|dz| \leq \frac{1}{2}\int_{|z|=1} |f(z)|^{\rho}|dz|, \quad (\rho > 0), \tag{4.2}$$

where the integral on the left hand side is considered along the real axis.

Remark 4.2. Using the change of variables in Lemma 4.1, we know that (4.2) becomes

$$\int_{-r}^{r} \left|f\left(te^{i\theta}\right)\right|^{\rho} dt \leq \frac{r}{2}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\rho} d\theta. \tag{4.3}$$

Further, we need the following lemma due to Gwynme [4].

Lemma 4.3. *Let $f(z)$ be a complex valued harmonic function defined on a neighborhood of a closed disc of radius 1 and center 0 in the complex plane. Then*

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\rho}\right) \frac{1-r^2}{1+r^2-2r\cos(\theta-\rho)} d\rho,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\rho} d\rho = 1.$$

Now, we derive

Theorem 4.4. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\gamma}{2}\,\mathrm{Re}\left(\frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), \quad (z \in U),$$

for $0 < \gamma \leq 1$ and $\beta < 1$, then

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\gamma, \quad (z \in U).$$

Proof. We see that

$$\log f'(z) = \log\left|f'(z)\right| + i \arg f'(z)$$

and

$$\log f'(z) = \int_0^z \left(\log f'(t)\right) dt = \int_0^z \frac{f''(t)}{f'(t)} dt.$$

Thus, we have

$$\begin{aligned}
\left|\arg f'(z)\right| &= \left|Im\left(\log f'(z)\right)\right| \\
&= \left|Im \int_0^z \frac{f''(t)}{f'(t)}\, dt\right| \\
&\leq \left|Im \int_0^r \frac{f''\left(\rho e^{i\theta}\right)}{f'\left(\rho e^{i\theta}\right)} e^{i\theta}\, d\rho\right| \\
&= \int_0^r \left|Im\left(\frac{e^{i\theta} f''\left(\rho e^{i\theta}\right)}{f'\left(\rho e^{i\theta}\right)}\right)\right| d\rho \\
&\leq \int_{-r}^r \left|\frac{f''\left(\rho e^{i\theta}\right)}{f'\left(\rho e^{i\theta}\right)}\right| d\rho.
\end{aligned}$$

where $z = re^{i\theta}(0 \leq \theta \leq 2\pi), 0 \leq r < 1$ and $0 \leq \rho \leq r$. Thus, applying (4.3) with $\rho = 1$,

$$\begin{aligned}
&\left|\arg f'(z)\right| \\
&\leq \frac{r}{2} \int_0^{2\pi} \left|\frac{f''\left(re^{i\theta}\right)}{f'\left(re^{i\theta}\right)}\right| \\
&= \frac{1}{2} \int_0^{2\pi} \left|\frac{re^{i\theta} f''\left(re^{i\theta}\right)}{f'\left(re^{i\theta}\right)}\right| \\
&< \frac{\gamma^{2\pi}}{4} \int_0^{2\pi} Re\left(\frac{1+(1-2\beta)\sqrt[n]{r}e^{i\frac{\theta}{n}}}{1-\sqrt[n]{r}e^{i\frac{\theta}{n}}}\right) d\theta \\
&= \frac{\gamma^2}{4} \int_0^{2\pi} Re\left(\frac{1+(1-2\beta)\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)+i(1-2\beta)\sqrt[n]{r}\sin\left(\frac{\theta}{n}\right)}{1-\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)-i\sqrt[n]{r}\sin\left(\frac{\theta}{n}\right)}\right) d\theta
\end{aligned}$$

$$= \frac{\gamma}{4}\int_0^{2\pi} \frac{1-2\beta\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)-(1-2\beta)(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)}d\theta$$
$$= \frac{\gamma^{2\pi}}{4}\int_0^{2\pi}\left(\beta+(1-\beta)\frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)}\right)d\theta.$$

Using Lemma 4.3, we have

$$\int_0^{2\pi}\left(\frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)}\right)d\theta$$
$$= n\int_0^{\frac{2\pi}{n}}\left(\frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2-2\sqrt[n]{r}\cos\rho}\right)d\rho$$
$$\leq 2\pi.$$

Therefore, we prove that

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\gamma, \quad (z\in U). \qquad \square$$

Example 4.5. We consider a function $f(z)\in A_n(\alpha)$ given by

$$f'(z) = \alpha\left(\frac{2}{2-\sqrt[n]{z}}\right)^{3\gamma}, \quad (z\in U),$$

with $0<\gamma\leq 1$ and $n>6$. Then

$$\left|\arg f'(z)\right| = 3\gamma|\arg(2-\sqrt[n]{z})| < \frac{\pi}{2}\gamma, \quad (z\in U).$$

Further $f(z)$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| = \frac{3\gamma}{n}\left|\frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}\right|$$

and

$$w(z) = \frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}, \quad (z\in U),$$

implies

$$\left|w(z)-\frac{1}{3}\right| < \frac{2}{3}, \quad (z\in U).$$

This implies that

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3\gamma}{n}, \quad (z\in U).$$

Noting that

$$\mathrm{Re}\left(\frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) > \beta, \quad (z\in U).$$

If we take β such that $\frac{6}{n} \leq \beta < 1, f(z)$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\gamma}{2}\beta < \frac{\gamma}{2}\operatorname{Re}\left(\frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), \quad (z \in U).$$

Next, we prove

Theorem 4.6. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{\gamma}{2}\operatorname{Re}\left(\frac{1+(1-2\beta)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), \quad (z \in U),$$

for $0 < \gamma \leq 1$ and $\beta < 1$, then

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\gamma, \quad (z \in U).$$

Proof. We have that

$$\log\frac{f(z)}{z} = \log\left|\frac{f(z)}{z}\right| + i\arg\frac{f(z)}{z}$$

and

$$\begin{aligned}\log\frac{f(z)}{z} &= \int_0^z \left(\log\left(\frac{f(t)}{t}\right)\right)' dt \\ &= \int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right) dt.\end{aligned}$$

This gives us that

$$\begin{aligned}\left|\arg\frac{f(z)}{z}\right| &= \left|\operatorname{Im}\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right) dt\right| \\ &= \left|\operatorname{Im}\int_0^r \left(\frac{f'\left(\rho e^{i\theta}\right)}{f\left(\rho e^{i\theta}\right)} - \frac{1}{\rho e^{i\theta}}\right) e^{i\theta} d\rho\right| \\ &\leq \int_0^r \left|\operatorname{Im}\left(\frac{f'\left(\rho e^{i\theta}\right)}{f\left(\rho e^{i\theta}\right)} - \frac{1}{\rho e^{i\theta}}\right) e^{i\theta}\right| d\rho \\ &< \int_{-r}^r \left|Im\left(\frac{e^{i\theta}f'\left(\rho e^{i\theta}\right)}{f\left(\rho e^{i\theta}\right)} - \frac{1}{\rho}\right)\right| d\rho \\ &\leq \frac{1}{2}\int_0^{2\pi} \left|Im\left(\frac{re^{i\theta}f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} - 1\right)\right| dr,\end{aligned}$$

by (4.3). Thus, using the same method of the proof in Theorem 4.4, we see that

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\gamma}{4}\int_0^{2\pi} Re\left(\frac{1+(1-2\beta)\sqrt[n]{r}e^{i\frac{\theta}{n}}}{1-\sqrt[n]{r}e^{i\frac{\theta}{n}}}\right) d\theta$$

$$< \frac{\pi}{2}\gamma, \quad (z \in U). \qquad \square$$

Example 4.7. Consider a function $f(z) \in A_n(\alpha)$ given by

$$f(z) = \alpha z \left(\frac{2}{2 - \sqrt[n]{z}}\right)^{3\gamma}, \quad (z \in U),$$

with $0 < \gamma \leq 1$ and $n > 6$. Then

$$w(z) = \frac{\sqrt[n]{z}}{2 - \sqrt[n]{z}},$$

satisfies

$$\left|w(z) - \frac{4}{3}\right| < \frac{2}{3}, \quad (z \in U),$$

and

$$|\arg w(z)| = \left|\arg\left(\frac{2}{2 - \sqrt[n]{z}}\right)\right| < \frac{\pi}{6}, \quad (z \in U).$$

Thus, we see

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| = 3\gamma \left|\arg\left(\frac{2}{2 - \sqrt[n]{z}}\right)\right| < \frac{\pi}{2}\gamma, \quad (z \in U).$$

Further, $f(z)$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \frac{3\gamma}{n}\left|\frac{\sqrt[n]{z}}{2 - \sqrt[n]{z}}\right| < \frac{3\gamma}{n}, \quad (z \in U).$$

If we consider β such that $\frac{6}{n} \leq \beta < 1$, then $f(z)$ satisfies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{\gamma}{2}\beta < \frac{\gamma}{2}\operatorname{Re}\left(\frac{1 + (1 - 2\beta)\sqrt[n]{z}}{1 - \sqrt[n]{z}}\right), \quad (z \in U).$$

Here, we would like to introduce the following lemma by Nunokawa [9].

Lemma 4.8. *Let $g(z)$ be analytic in U with $g(0) = 1$ and $g(z) \neq 0 (z \in U)$. If there exists a point $z_0 \in U$ such that*

$$|\arg g(z)| < \frac{\pi}{2}\alpha, \quad (|z| < |z_0|)$$

and

$$|\arg g(z_0)| = \frac{\pi}{2}\alpha,$$

for some real $\alpha > 0$, then $g(z)$ satisfies

$$\frac{z_0 g'(z_0)}{g(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \left(for\ \arg(z_0) = \frac{\pi}{2}\alpha\right), \tag{4.4}$$

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right), \quad \left(for\ \arg(z_0) = -\frac{\pi}{2}\alpha\right), \tag{4.5}$$

and

$$g(z_0)^{\frac{1}{\alpha}} = \pm ia, \quad (a>0). \tag{4.6}$$

Now, we prove

Theorem 4.9. *If* $f(z) \in A_n(\alpha)$ *satisfies*

$$\left|\arg\left(\beta - f'(z) - zf''(z)\right)\right| < \frac{\pi}{2}\gamma + \tan^{-1}\gamma, \quad (z \in U), \tag{4.7}$$

for some $\beta > \alpha$ *and* $0 < \gamma < 1$, *then*

$$\left|\arg\left(\beta - f'(z)\right)\right| < \frac{\pi}{2}\gamma, \quad (z \in U). \tag{4.8}$$

Proof. We consider a function $g(z)$ given by

$$g(z) = \frac{\beta - f'(z)}{\beta - \alpha}, \tag{4.9}$$

for $f(z)$ which satisfies (4.7). Then $g(z)$ is analytic in U with $g(0) = 1$ and $g(z) \neq 0 (z \in U)$. Suppose that there exists a point $z_0 \in U$ such that

$$|\arg g(z)| = \left|\arg\left(\beta - f'(z)\right)\right| < \frac{\pi}{2}\gamma, \quad (|z| < |z_0|) \tag{4.10}$$

and

$$|\arg g(z_0)| = \left|\arg\left(\beta - f'(z_0)\right)\right| = \frac{\pi}{2}\gamma. \tag{4.11}$$

Then Lemma 4.8 implies that

$$\frac{z_0 g'(z_0)}{g(z_0)} = \frac{-z_0 f''(z_0)}{\beta - f'(z_0)} = ik\gamma, \tag{4.12}$$

where k is given by (4.4) and (4.5). If $\arg g(z_0) = \frac{\pi}{2}\gamma$, then

$$\begin{aligned}
\arg\left(g(z_0) + z_0 g'(z_0)\right) &= \arg\left(\beta - f'(z_0)\right)\left(1 - \frac{z_0 f''(z_0)}{\beta - f'(z_0)}\right) \\
&= \arg\left(\beta - f'(z_0) - z_0 f''(z_0)\right) \\
&= \frac{\pi}{2}\gamma + \arg(1 + ik\gamma) = \frac{\pi}{2}\gamma + \tan^{-1}(k\gamma) \\
&\geq \frac{\pi}{2}\gamma - \tan^{-1}\gamma.
\end{aligned}$$

This contradicts our condition (4.7).
If $\arg g(z_0) = -\frac{\pi}{2}\gamma$, then we also have

$$\begin{aligned}
\arg\left(g(z_0) + z_0 g'(z_0)\right) &= \arg\left(\beta - f(z_0) - z_0 f''(z_0)\right) \\
&= -\frac{\pi}{2}\gamma + \tan^{-1}(k\gamma)
\end{aligned}$$

$$\geq -\frac{\pi}{2}\gamma + \tan^{-1}\gamma.$$

Therefore, we say that there is no $z_0 \in U$ which satisfies (4.10) and (4.11). This shows us that $f(z)$ satisfies (4.8). □

Remark 4.10. If we take $\gamma = \frac{1}{\sqrt{3}}$ in Theorem 4.9, then we have that if $f(z) \in A_n(\alpha)$ satisfies

$$\left|\arg\left(\beta - f'(z) - zf''(z)\right)\right| < \frac{\sqrt{3}+1}{6}\pi, \quad (z \in U),$$

then

$$\left|\arg\left(\beta - f'(z)\right)\right| < \frac{\sqrt{3}}{6}\pi, \quad (z \in U).$$

Next we show

Theorem 4.11. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\left|\arg\left(\frac{\beta - f'(z)}{\beta - \alpha} - \frac{zf''(z)}{\beta - f'(z)}\right)\right| < \tan^{-1}\ \mathrm{h}(\gamma), \quad (z \in U), \tag{4.13}$$

for some $\beta > \alpha$ and $0 < \gamma < 1$, then

$$\left|\arg\left(\beta - f'(z)\right)\right| < \frac{\pi}{2}\gamma, \quad (z \in U), \tag{4.14}$$

where

(4.15)

$$h(\gamma) = \frac{\gamma}{2}\left(\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{2}} + \left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1+\gamma}{2}}\right)\sec\left(\frac{\pi}{2}\gamma\right) + \tan\left(\frac{\pi}{2}\gamma\right).$$

Proof. We define a function $g(z)$ by (4.9). If there exists a point $z_0 \in U$ such that (4.10) and (4.11), then $g(z)$ satisfies (4.12) as the proof of Theorem 4.9.

If $\arg g(z_0) = \frac{\pi}{2}\gamma$, then $g(z_0) = a^\gamma e^{i\frac{\pi}{2}\gamma}$ by (4.6). Thus, we have

$$\begin{aligned}
&\arg\left(g(z_0) + \frac{z_0 g'(z_0)}{g(z_0)}\right) \\
&= \arg\left(\frac{\beta - f'(z_0)}{\beta - \alpha} - \frac{z_0 f''(z_0)}{\beta - f'(z_0)}\right) \\
&= \arg\left(a^\gamma e^{i\frac{\pi}{2}\gamma} + ik\gamma\right) \\
&= \arg\left\{a^\gamma \cos\left(\frac{\pi}{2}\gamma\right) + i\left(k\gamma + a^\gamma \sin\left(\frac{\pi}{2}\gamma\right)\right)\right\} \\
&\geq \arg\left\{a^\gamma \cos\left(\frac{\pi}{2}\gamma\right) + i\left(\frac{\gamma}{2}\left(a + \frac{1}{a}\right) + a^\gamma \sin\left(\frac{\pi}{2}\gamma\right)\right)\right\} \\
&= \tan^{-1}\left\{\frac{\gamma}{2}\left(a^{1-\gamma} + a^{-1-\gamma}\right)\sec\left(\frac{\pi}{2}\gamma\right) + \tan\left(\frac{\pi}{2}\gamma\right)\right\}.
\end{aligned}$$

We write that $w(a) = a^{1-\gamma} + a^{-1-\gamma}$. Then $w(a)$ gives us

$$w'(a) = \frac{1}{a^{2+\gamma}}\left((1-\gamma)a^2 - (1+\gamma)\right).$$

This shows that $w(a)$ takes its minimum value

$$w\left(\sqrt{\frac{1+\gamma}{1-\gamma}}\right) = \left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{2}} + \left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1+\gamma}{2}},$$

for $a = \sqrt{\frac{1+\gamma}{1-\gamma}}$. Therefore, we have that

(4.16)

$$\begin{aligned}
&\arg\left(g(z_0) + \frac{z_0 g'(z_0)}{g(z_0)}\right) \\
&\quad \geq \tan^{-1}\left\{\frac{\gamma}{2}\left(\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{2}} + \left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1+\gamma}{2}}\right)\sec\left(\frac{\pi}{2}\gamma\right) + \tan\left(\frac{\pi}{2}\gamma\right)\right\} \\
&\quad = \tan^{-1} h(\gamma).
\end{aligned}$$

The inequality (4.16) contradicts our condition (4.13).
If $\arg g(z_0) = -\frac{\pi}{2}\gamma$, then $g(z_0) = a^{\gamma} e^{-i\frac{\pi}{2}\gamma}$ and

$$\begin{aligned}
&\arg\left(g(z_0) + \frac{z_0 g'(z_0)}{g(z_0)}\right) \\
&\quad = \arg\left(\frac{\beta - f'(z_0)}{\beta - \alpha} - \frac{z_0 f''(z_0)}{\beta - f'(z_0)}\right) \\
&\quad = \arg\left(a^{\gamma} e^{-i\frac{\pi}{2}\gamma} + ik\gamma\right) \\
&\quad \leq -\tan^{-1}\left\{\frac{\gamma}{2}\left(a^{1-\gamma} + a^{-1-\gamma}\right)\sec\left(\frac{\pi}{2}\gamma\right) + \tan\left(\frac{\pi}{2}\gamma\right)\right\} \\
&\quad \leq -\tan^{-1}\left\{\frac{\gamma}{2}\left(\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{2}} + \left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1+\gamma}{2}}\right)\sec\left(\frac{\pi}{2}\gamma\right) + \tan\left(\frac{\pi}{2}\gamma\right)\right\} \\
&\quad = -\tan^{-1} h(\gamma).
\end{aligned}$$

This also contradicts the condition (4.13). Therefore, we don't have any $z_0 \in U$ such that satisfies (4.10) and (4.11). This means that $f(z)$ satisfies the inequality (4.14). □

Remark 4.12. We consider $\gamma = \frac{1}{2}$ in (4.15). Then

$$h\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt[4]{3} + \frac{1}{\sqrt{3}}\right) + 1 = 1.6694\cdots$$

and

$$\frac{59}{180}\pi < \tan^{-1} h\left(\frac{1}{2}\right) < \frac{\pi}{3}.$$

For such $h\left(\frac{1}{2}\right)$, we see

$$\left|\arg\left(\beta - f'(z)\right)\right| < \frac{\pi}{4}, \quad (z \in U).$$

5. Applications of Jack's Lemma

We introduce Jack's lemma given by Jack [6] (or by Miller and Mocanu [8]).

Lemma 5.1. *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0), \quad (k \geq 1).$$

For the above lemma, we show

Theorem 5.2. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\left|zf'(z) - f(z)\right| < |f(z) - \alpha z|, \quad (z \in U), \tag{5.1}$$

then

$$\left|\frac{f(z)}{z} - \alpha\right| < \beta - \alpha, \quad (z \in U),$$

for $\alpha < \beta$.

Proof. We define a function $w(z)$ by

$$w(z) = \frac{\frac{f(z)}{z} - \alpha}{\beta - \alpha}, \quad (\beta > \alpha),$$

for $f(z)$ which satisfies (5.1). Then $w(z)$ is analytic in U with $w(0) = 0$. Suppose that $w(z)$ has its maximum value $|w(z_0)| = 1$ at a point z_0 on the circle $|z| = r < 1$, then Lemma 5.1 given us

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0 f'(z_0) - f(z_0)}{f(z_0) - \alpha z_0} = k \geq 1.$$

This contradicts the condition (5.1). Thus $w(z)$ should be $|w(z)| < 1$ for all $z \in U$ and

$$\left|\frac{f(z)}{z} - \alpha\right| < \beta - \alpha, \quad (z \in U). \qquad \square$$

Further, we derive

Theorem 5.3. *If $f(z) \in A_n(\alpha)$ satisfies*

$$\arg\left(zf''(z)\right) \neq \arg\left(f'(z) - \alpha\right), \quad (z \in U), \tag{5.2}$$

then

$$\left|f'(z) - \alpha\right| < 1, \quad (z \in U).$$

Proof. We consider a function $w(z)$ by

$$w(z) = f'(z) - \alpha, \quad (z \in U).$$

Then $w(z)$ is analytic in U and $w(0) = 0$. If $w(z)$ takes its maximum value

$$\max_{|z| \leq z_0|} |w(z)| = |w(z_0)| = 1,$$

then, by Lemma 5.1,we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0 f''(z_0)}{f'(z_0) - \alpha} = k \geq 1.$$

This means that

$$\arg\left(z_0 f''(z_0)\right) = \arg\left(f'(z_0) - \alpha\right),$$

and contradicts the condition (5.2). Therefore, $w(z)$ satisfies

$$|w(z)| = \left|f'(z) - \alpha\right| < 1, \quad (z \in U). \qquad \square$$

6. Conclusions

As we said that if $f(z) \in A$ satisfies the inequality (1.4), then $f(z)$ is said to be γ - spirallike of order β in U. If we take $\gamma = 0$ in (1.4), then $f(z) \in A$ is said to be starlike of order β in U. Historically, many mathematicians studied γ - spirallike functions of order β and starlike functions of order β and published a lot of research papers. But we didn't find any paper for $f(z) \in A_n(\alpha)$ satisfying the conditions

$$\operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > \beta\cos\gamma, \quad (z \in U)$$

for $|\gamma| < \frac{\pi}{2}$ and $\beta > 1$, or

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad (z \in U)$$

for $\beta > 1$. Our paper is the new analytically considerations for such functions. Thus we need to study the geometrically properties of such functions future.

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