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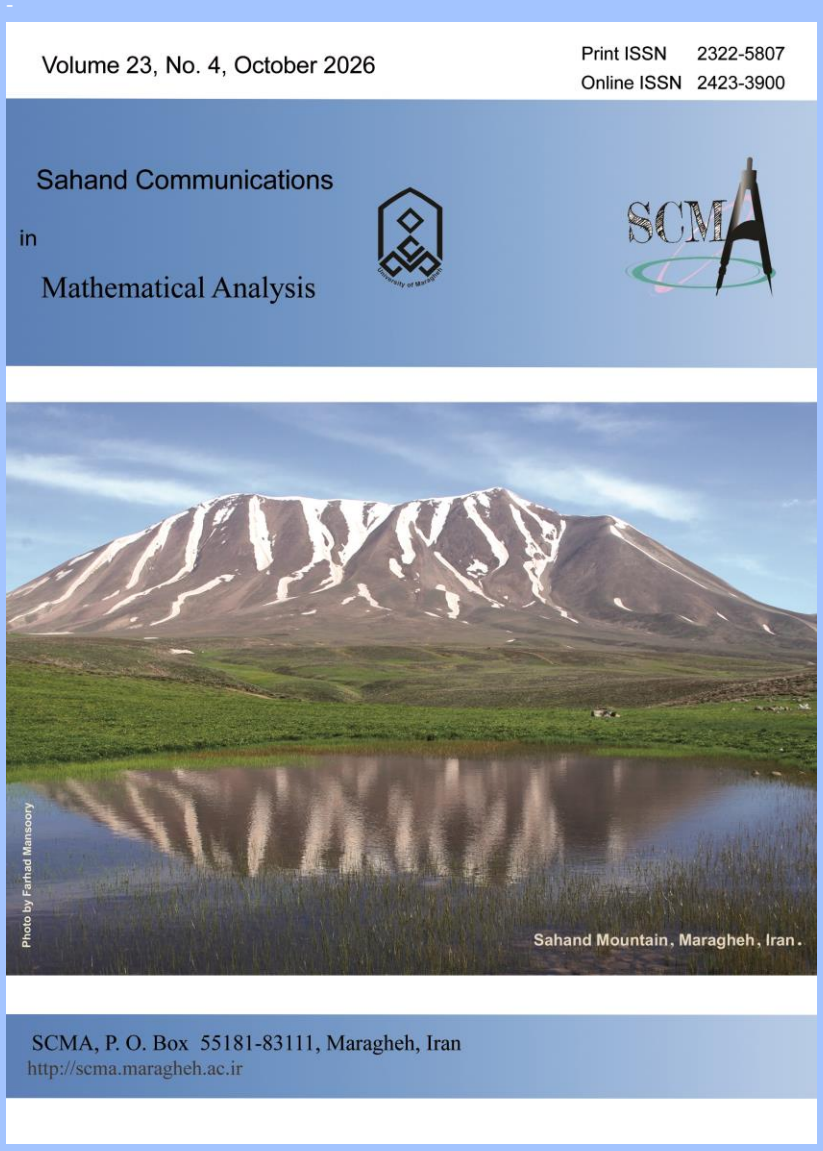

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Advanced Analysis of Hermite-Hadamard Mercer Type Inequalities for Twice Differentiable Functions with Application

Arslan Munir,[1*] Hasan Kara[2] and Fatih Hezenci[2]

Abstract. In recent years, fractional integral inequalities have emerged as powerful and widely applicable instruments in both pure and applied mathematics. Their versatility has motivated numerous researchers to propose broader and more general forms of classical inequalities, many of which rely on different types of fractional integral operators. In this work, we investigate Hermite-Jensen Mercer-type inequalities in the framework of s-convex functions by utilizing the Caputo-Fabrizio fractional integral operator, which is distinguished by its non-singular exponential kernel. By establishing a new fractional integral identity for twice-differentiable mappings, we derive a series of generalized inequalities that extend and unify several classical results. In addition, applications to the q-digamma function, and the modified-Bessel function are provided. Numerical computations and graphical illustrations are included to visualize the behaviour of the established bounds under various parameters. Many established results are recovered as special cases by taking appropriate parameter values.

1. Introduction

Fractional calculus has evolved into a significant area of modern mathematical research because of its wide-ranging applications in physics, engineering, image analysis, machine intelligence, and complex network modeling. Numerous studies have been conducted to develop identities and properties associated with fractional operators (see, for example,

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* Corresponding author.

[6, 7]). Unlike classical derivatives, which are often inadequate for describing memory-dependent or nonlocal phenomena, fractional derivatives provide a more flexible framework for modeling such processes. As a result, many real-world systems are more accurately represented by fractional differential equations. Several well-known fractional integral operators have been introduced in the literature, including the Riemann-Liouville operator, the Hadamard operator, and the Caputo-Fabrizio operator [15] among others. In this work, our attention is devoted exclusively to the Caputo-Fabrizio fractional integral operator. Within the ongoing development of fractional calculus, many researchers have proposed new operators designed to model a wide spectrum of phenomena arising in applied sciences. These operators often differ considerably in the behavior of their kernels, particularly with respect to singularity and locality. The Caputo-Fabrizio operator stands out among them because its kernel is non-singular, making it especially suitable for obtaining explicit or more stable solutions in several practical applications. Caputo and Fabrizio's presentation of the Caputo-Fabrizio fractional operator in [5] is an important addition to the rules and regulations. This operator is distinct from traditional fractional operators in that it doesn't depend on the Gamma function due to its nonsingular kernel. Its scientific and practical utility has been further improved by the novel fact that any real strength can be reduced to an integer order by applying the Laplace transform. This distinctive property makes it easier to solve several related problems, increasing its usefulness in a wide range of areas. Many different applications, including the fractional modeling for hearing loss due to Mumps virus [16] and the modeling of COVID-19 transmission dynamics [22], have significantly used the Caputo-Fabrizio fractional operator. These illustrations demonstrate the operator's ability to adjust and be efficient in using fractional calculus when dealing with challenges in the real world.

In recent years, the refinement and extension of integral inequalities through fractional operators has become an active area of investigation, leading to a substantial body of literature. Various classical inequalities such as those of Jensen-Mercer, Fejér, Opial, Ostrowski, and Simpson have been revisited within fractional frameworks to obtain sharper bounds and broader generalizations. Among these, the Hermite-Hadamard inequality holds a particularly prominent position because of its deep connection with the theory of convex functions. The enduring importance of this inequality is evident from the numerous research papers and monographs dedicated to its improvement, especially through

the incorporation of different fractional integral operators. The classical Hermite-Hadamard inequality for convex functions is given in its standard form by:

$$\Bbbk\left(\frac{\Theta+\Pi}{2}\right) \leq \frac{1}{\Pi-\Theta}\int_{\Theta}^{\Pi} \Bbbk\left(z\right) dz \leq \frac{\Bbbk\left(\Theta\right)+\Bbbk\left(\Pi\right)}{2}.$$

For concave functions, this inequality appears in its reverse form. The justification for this fact follows naturally from the Jensen inequality, a classical tool extensively studied in mathematical analysis. By employing Jensen's framework, Mehrez and Agarwal [12] demonstrated how one can derive refined forms of the Hermite-Hadamard inequality, presenting several new extensions within the setting of convex functions. In order to establish classical integral inequalities, fractional integral operators have been extensively studied in the literature. Agarwal et al. [1] established Hermite–Hadamard-type inequalities by employing generalized k-fractional integral operators. We suggest reading the articles referenced in [14, 23] for an extensive investigation of current research in fractional integral inequalities.

In [13] Mercer introduced a variant of Jensen's inequality, commonly referred to as the Jensen–Mercer inequality, which is stated as follows:

For a convex function $\Bbbk$ and any set of non-negative weights u_i such that $\sum\limits_{i=1}^{n} u_i = 1$, the inequality is given by:

$$\Bbbk\left(\sum_{i=1}^{n} u_i x_i\right) \leq \sum_{i=1}^{n} u_i \Bbbk\left(x_i\right),$$

This form generalizes the classic Jensen's inequality by considering a weighted average of points x_i.

Theorem 1.1. *For a convex mapping $\Bbbk : [\Theta, \Pi] \to \mathbb{R}$, the subsequent inequality is valid for all values of $x_i \in [\Theta, \Pi]\,(i = 1, 2, 3, \ldots, n)$:*

$$\Bbbk\left(\Theta+\Pi-\sum_{i=1}^{n} u_i x_i\right) \leq \Bbbk\left(\Theta\right)+\Bbbk\left(\Pi\right)-\sum_{i=1}^{n} u_i \Bbbk\left(x_i\right),$$

where $u_i \in [0,1]\,(i = 1, 2, 3, \ldots, n)$ and $\sum\limits_{i=1}^{n} u_i = 1$.

Moradi and Furuichi, in [17], introduced broadened and refined variations of inequalities of the Jensen-Mercer type. Subsequently, Adil et al. work in [10] demonstrated how these inequalities play an important role in problems arising from information theory. Their research also provided fresh quantitative bounds for Csiszár divergence and several related divergence measures. In addition, the Jensen-Mercer structure

enabled them to obtain improved estimates associated with the Zipf–Mandelbrot entropy. Later, employing a newly established refinement of Jensen's inequality, Kian et al. [11] formulated further versions of the Hermite-Hadamard inequality, which are presented below:

Theorem 1.2. *Let function* $\Bbbk : [\Theta, \Pi] \to \mathbb{R}$. *For any* $\Phi, \phi \in [\Theta, \Pi]$ *with* $\Phi < \phi$, *the following inequality is satisfied:*

$$\Bbbk\left(\Theta + \Pi - \frac{\Phi + \phi}{2}\right) \leq \Bbbk(\Theta) + \Bbbk(\Pi) - \frac{1}{\phi - \Phi}\int_{\phi}^{\Phi} \Bbbk(z)\, dz \leq \Bbbk(\Theta) + \Bbbk(\Pi) - \Bbbk\left(\frac{\Phi + \phi}{2}\right), \tag{1.1}$$

and

$$\begin{aligned}\Bbbk\left(\Theta + \Pi - \frac{\Phi + \phi}{2}\right) &\leq \frac{1}{\phi - \Phi}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\Phi} \Bbbk(z)\, dz \\ &\leq \frac{1}{2}\left[\Bbbk(\Theta + \Pi - \Phi) + \Bbbk(\Theta + \Pi - \phi)\right] \\ &\leq \Bbbk(\Theta) + \Bbbk(\Pi) - \frac{1}{2}\left[\Bbbk(\Phi) + \Bbbk(\phi)\right].\end{aligned}$$

Remark 1.3. If we put $\Phi = \Theta$ and $\phi = \Pi$ in inequality (1.1), one immediately recovers the standard classical Hermite-Hadamard inequality.

In recent years, considerable attention has been devoted to studying these inequalities, leading to progress through the development of various Hermite-Hadamard-Mercer-type extensions. To broaden this line of work, several studies such as [18] employed the Riemann-Liouville fractional integral to obtain new results. Likewise, Set et al. [21] introduced further Hermite-Hadamard Mercer variants by making use of generalized fractional integral operators. Extending these contributions, Butt et al. [4] derived additional bounds through the application of alternative harmonic techniques.

Definition 1.4 ([9])**.** The mapping of $\Bbbk : I \subseteq \mathbb{R} \to \mathbb{R}$ is defined as convex whenever the following condition holds:

$$\Bbbk(\gamma\Theta + (1-\gamma)\Pi) \leq \gamma\Bbbk(\Theta) + (1-\gamma)\Bbbk(\Pi),$$

for all $\Theta, \Pi \in I$ and $\gamma \in [0, 1]$.

Definition 1.5 ([8])**.** The mapping of $\Bbbk : I \subset [0, \infty) \to \mathbb{R}$ is defined as s-convex whenever the following condition holds:

$$\Bbbk(\gamma\Theta + (1-\gamma)\Pi) \leq \gamma^s\Bbbk(\Theta) + (1-\gamma)^s\Bbbk(\Pi),$$

for some fixed $s \in (0, 1]$, $\Theta, \Pi \in I$ and $\gamma \in [0, 1]$.

Definition 1.6 ([5])**.** Let $H^1(\Theta,\Pi)$ (denote the Sobolev space of order one),

$$H^1(\Theta,\Pi)=\left\{g\in L^2(\Theta,\Pi):g'\in L^2(\Theta,\Pi)\right\},$$

where

$$L^2(\Theta,\Pi)=\left\{g(z):\left(\int_\Theta^\Pi g^2(z)\,dz\right)^{\frac{1}{2}}<\infty\right\}.$$

Let $\Bbbk\in H^1(\Theta,\Pi)$, $\Theta<\Pi$, $\upsilon\in[0,1]$. The left-hand Caputo-Fabrizio derivative is defined as:

$$\left({}^{CFD}_{\Theta}D^{\upsilon}\Bbbk\right)(x)=\frac{\beta(\upsilon)}{1-\upsilon}\int_\Theta^x\Bbbk'(z)\,e^{\frac{-\rho(x-z)^{\upsilon}}{1-\upsilon}}dz,$$

for $x>\upsilon$ and the associated Caputo-Fabrizio integral operator is:

$$\left({}^{CF}_{\Theta}I^{\upsilon}\Bbbk\right)(x)=\frac{1-\upsilon}{\beta(\upsilon)}\Bbbk(x)+\frac{\upsilon}{\beta(\upsilon)}\int_\Theta^x\Bbbk(z)\,dz,$$

where $\beta(\upsilon)>0$ is a normalizer satisfying $\beta(0)=\beta(1)=1$. For $\rho=0$ and $\rho=1$. The left derivative is given by:

$$\left({}^{CFD}_{\Theta}D^{0}\Bbbk\right)(x)=\Bbbk'(x),\qquad\left({}^{CFD}_{\Theta}D^{1}\Bbbk\right)(x)=\Bbbk(x)-\Bbbk(\Theta).$$

For the right derivative, it is given by:

$$\left({}^{CFD}_{\Pi}D^{\upsilon}\Bbbk\right)(x)=\frac{-\beta(\upsilon)}{1-\upsilon}\int_x^\Pi\Bbbk'(z)\,e^{\frac{-\upsilon(z-x)^{\upsilon}}{1-\upsilon}}dz,$$

for $x<\Pi$ and the associated integral operator is:

$$\left({}^{CF}I^{\upsilon}_{\Pi}\Bbbk\right)(x)=\frac{1-\upsilon}{\beta(\upsilon)}\Bbbk(x)+\frac{\upsilon}{\beta(\upsilon)}\int_x^\Pi\Bbbk(z)\,dz.$$

Soubhagya et al. [19] established a set of Hermite-Hadamard-Mercer-type results involving the Caputo-Fabrizio fractional integral operator, which can be stated as follows:

$$\begin{aligned}\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)\leq\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\Bigg[&{}^{CF}_{(\Theta+\Pi-\frac{\Phi+\phi}{2})^-}I^{\upsilon}\Bbbk(\Theta+\Pi-\phi)\\&+{}^{CF}I^{\upsilon}_{(\Theta+\Pi-\frac{\Phi+\phi}{2})^+}\Bbbk(\Theta+\Pi-\Phi)\\&-\frac{1-\upsilon}{\beta(\upsilon)}\left[\Bbbk(\Theta+\Pi-\Phi)+\Bbbk(\Theta+\Pi-\phi)\right]\Bigg]\\\leq|\Bbbk(\Theta)|+|\Bbbk(\Pi)|-\frac{\Bbbk(\Phi)+\Bbbk(\phi)}{2}.\end{aligned}$$

The following is the outline of the article. It presents the fundamental concepts of Jensen-Mercer-type inequalities and fractional calculus. Using the Caputo-Fabrizio fractional operator for s-convex functions, new midpoint versions are introduced in Section 2. Additionally, for

twice-differentiable convex functions, a novel fractional integral identity of the Caputo-Fabrizio type is established in this section. Based on this identity, several error bounds are derived using the Hölder and power-mean inequalities. Furthermore, applications to the q-digamma function and the modified Bessel function are provided. Numerical computations and graphical illustrations are included to visualize the behavior of the established bounds under various parameter values.

2. An Essential Identity

In this section, we begin by proving an identity for twice-differentiable convex functions in the context of the Caputo-Fabrizio fractional integral operator and the Jensen-Mercer inequality. Using classical integral inequalities, such as the power-mean inequality and Hölder's inequality, we derive several new bounds.

Lemma 2.1. *Let $\Bbbk : [\Theta, \Pi] \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on (Θ, Π) and $\Bbbk'' \in L\ [\Theta, \Pi]$ with $\Theta \leq \Pi$ and $\Phi, \phi \in [\Theta, \Pi]$, then the following fractional identity holds:*

$$\begin{aligned}(2.1)\qquad &\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\\ &\quad-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\\ &=\frac{(\phi-\Phi)^2}{16}\left[\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\right.\\ &\quad\left.+\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\right].\end{aligned}$$

Proof. Let

$$\begin{aligned}J&=\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\\ &\quad+\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\\ &=J_1+J_2.\end{aligned}$$

Integration by parts, we have

(2.2)

$$\begin{aligned}J_1&=\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\\ &=\left.\frac{-2\gamma^2\Bbbk'\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)}{\phi-\Phi}\right|_0^1\end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{\phi-\Phi}\int_0^1 \Bbbk'\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\\
&=-\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\
&\quad+\frac{4}{\phi-\Phi}\left[\frac{-2\gamma}{\phi-\Phi}\Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)\Big|_0^1\right.\\
&\qquad\left.+\frac{2}{\phi-\Phi}\int_0^1 \Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\right]\\
&=-\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\
&\quad+\frac{8}{(\phi-\Phi)^2}\int_0^1 \Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\\
&=-\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\
&\quad+\frac{16}{(\phi-\Phi)^3}\int_{\Theta+\Pi-\frac{\Phi+\phi}{2}}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz.
\end{aligned}
$$

Similarly, we get

(2.3)

$$
\begin{aligned}
J_2&=\int_0^1 \gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\\
&=\left.\frac{2\gamma^2\Bbbk'\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)}{\phi-\Phi}\right|_0^1\\
&\quad-\frac{4}{\phi-\Phi}\int_0^1 \Bbbk'\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\\
&=\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\
&\quad-\frac{4}{\phi-\Phi}\left[\frac{2\gamma}{\phi-\Phi}\Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)\Big|_0^1\right.\\
&\qquad\left.-\frac{2}{\phi-\Phi}\int_0^1 \Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\right]\\
&=\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\
&\quad+\frac{8}{(\phi-\Phi)^2}\int_0^1 \Bbbk\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\\
&=\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)
\end{aligned}
$$

$$+\frac{16}{(\phi-\Phi)^3}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\frac{\Phi+\phi}{2}}\Bbbk(z)\,dz.$$

Adding the equalities (2.2) and (2.3), we have

(2.4)

$$\begin{aligned}J_1+J_2&=-\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\quad+\frac{16}{(\phi-\Phi)^3}\int_{\Theta+\Pi-\frac{\Phi+\phi}{2}}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz+\frac{2}{\phi-\Phi}\Bbbk'\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\quad-\frac{8}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)+\frac{16}{(\phi-\Phi)^3}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\frac{\Phi+\phi}{2}}\Bbbk(z)\,dz\\&=-\frac{16}{(\phi-\Phi)^2}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)+\frac{16}{(\phi-\Phi)^3}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz.\end{aligned}$$

Multiplying $\frac{\phi-\Phi}{16}$ on both sides of equality (2.4), we have

$$\begin{aligned}(J_1+J_2)\,\frac{\phi-\Phi}{16}&=-\frac{1}{\phi-\Phi}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\quad+\frac{1}{(\phi-\Phi)^2}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz.\end{aligned}\tag{2.5}$$

Multiplying on both sides by $\frac{\upsilon(\phi-\Phi)}{\beta(\upsilon)}$ and adding $\frac{2(1-\upsilon)}{\beta(\upsilon)(\phi-\Phi)}\Bbbk(x)$ to equality (2.5), we have

$$\begin{aligned}&(J_1+J_2)\,\frac{\upsilon(\phi-\Phi)^2}{16\beta(\upsilon)}+\frac{2(1-\upsilon)}{\beta(\upsilon)(\phi-\Phi)}\Bbbk(x)\\&\quad=-\frac{\upsilon}{\beta(\upsilon)}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\qquad+\left[\frac{\upsilon}{\beta(\upsilon)(\phi-\Phi)}\int_{\Theta+\Pi-\phi}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz+\frac{2(1-\upsilon)}{\beta(\upsilon)(\phi-\Phi)}\Bbbk(x)\right]\\&\quad=-\frac{\upsilon}{\beta(\upsilon)}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\qquad+\frac{1}{\phi-\Phi}\left[\frac{\upsilon}{\beta(\upsilon)}\int_{\Theta+\Pi-\phi}^{x}\Bbbk(z)\,dz+\frac{1-\upsilon}{\beta(\upsilon)}\Bbbk(x)\right.\\&\qquad\left.+\frac{\upsilon}{\beta(\upsilon)}\int_{x}^{\Theta+\Pi-\Phi}\Bbbk(z)\,dz+\frac{1-\upsilon}{\beta(\upsilon)}\Bbbk(x)\right]\\&\quad=-\frac{\upsilon}{\beta(\upsilon)}\Bbbk\left(\Theta+\Pi-\left(\frac{\phi+\Phi}{2}\right)\right)\\&\qquad+\frac{1}{\phi-\Phi}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right].\end{aligned}\tag{2.6}$$

By multiplying both sides of equality (2.6) by $\frac{\beta(\upsilon)}{\upsilon}$, we obtain

$$\begin{aligned}
&\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\\
&\quad-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\\
&=\frac{(\phi-\Phi)^2}{16}\left[\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)d\gamma\right.\\
&\quad\left.+\int_0^1\gamma^2\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)d\gamma\right].
\end{aligned}$$

The proof of Lemma 2.1 is now concluded. □

Corollary 2.2. *Substituting $\Phi=\Theta$ and $\phi=\Pi$ into Lemma 2.1, gives:*

$$\begin{aligned}
&\frac{\beta(\upsilon)}{\upsilon(\Pi-\Theta)}\left[\left({}^{CF}_{\Theta}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Pi}\Bbbk\right)(x)\right]\\
&\quad-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)-\frac{2(1-\upsilon)}{\upsilon(\Pi-\Theta)}\Bbbk(x)\\
&=\frac{(\Pi-\Theta)^2}{16}\left[\int_0^1\gamma^2\left\{\Bbbk''\left(\frac{\gamma}{2}\Pi+\frac{2-\gamma}{2}\Theta\right)+\Bbbk''\left(\frac{\gamma}{2}\Theta+\frac{2-\gamma}{2}\Pi\right)\right\}d\gamma\right].
\end{aligned}$$

Remark 2.3. Substituting $\upsilon=1$ and $\beta(0)=\beta(1)=1$ into Corollary 2.2, gives:

$$\begin{aligned}
&\frac{1}{\Pi-\Theta}\int_{\Theta}^{\Pi}\Bbbk(x)\,dx-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)\\
&=\frac{(\Pi-\Theta)^2}{16}\left[\int_0^1\gamma^2\left\{\Bbbk''\left(\frac{\gamma}{2}\Pi+\frac{2-\gamma}{2}\Theta\right)+\Bbbk''\left(\frac{\gamma}{2}\Theta+\frac{2-\gamma}{2}\Pi\right)\right\}d\gamma\right].
\end{aligned}$$

which was proved by Sarikaya et al. [20].

Theorem 2.4. *Let $\Bbbk:[\Theta,\Pi]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on (Θ,Π) and $|\Bbbk''|$ is s-convex on $[\Theta,\Pi]$ with $\Theta\leq\Pi$ and $\Phi,\phi\in[\Theta,\Pi]$, the subsequent fractional inequality is valid:*

$$\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\right.\\
&\quad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\right|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left[\frac{2}{3}\left(\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right)\right.\\
&\quad\left.-\left(\frac{2^{2-s}(2^{2+s}-s+3)}{(s^2+3s+2)(3+s)}\right)\left(\left|\Bbbk''(\Phi)\right|+\left|\Bbbk''(\phi)\right|\right)\right].
\end{aligned}$$

Proof. By applying the equality (2.1) in conjunction with the fractional Jensen–Mercer inequality, we derive:

$$\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\right.\\
&\qquad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\right|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left[\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)\right|d\gamma\right.\\
&\qquad\left.+\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)\right|d\gamma\right]\\
&\leq\frac{(\phi-\Phi)^2}{16}\left[\int_0^1\gamma^2\left(\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right.\right.\\
&\qquad\left.\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''(\phi)\right|+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''(\Phi)\right|\right)\right)d\gamma\right.\\
&\qquad\left.+\int_0^1\gamma^2\left(\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right.\right.\\
&\qquad\left.\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''(\Phi)\right|+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''(\phi)\right|\right)\right)d\gamma\right]\\
&=\frac{(\phi-\Phi)^2}{16}\left[\frac{2}{3}\left(\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right)\right.\\
&\qquad\left.-\left(\frac{2^{2-s}(2^{2+s}-s+3)}{(s^2+3s+2)(3+s)}\right)\left(\left|\Bbbk''(\Phi)\right|+\left|\Bbbk''(\phi)\right|\right)\right].
\end{aligned}$$

The proof of Theorem 2.4 is now finished. □

Corollary 2.5. *Substituting $s=1$ into Theorem 2.4, gives:*

$$\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\right.\\
&\qquad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\right|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left[\frac{2}{3}\left(\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right)\right.\\
&\qquad\left.-\frac{1}{3}\left(\left|\Bbbk''(\Phi)\right|+\left|\Bbbk''(\phi)\right|\right)\right].
\end{aligned}$$

Corollary 2.6. *Substituting $\Phi=\Theta$ and $\phi=\Pi$ into Corollary 2.5, gives:*

$$\left|\frac{\beta(\upsilon)}{\upsilon(\Pi-\Theta)}\left[\left({}^{CF}_{\Theta}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Pi}\Bbbk\right)(x)\right]\right.$$

$$-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)-\frac{2(1-\upsilon)}{\upsilon(\Pi-\Theta)}\Bbbk(x)\Big|$$
$$\leq\frac{(\Pi-\Theta)^2}{48}\left[\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right].$$

which was proved by Hong et al. [25, Corollary 2.5. ($\mu=0$)].

Corollary 2.7. *Substituting $\upsilon=1$ and $\beta(0)=\beta(1)=1$ into Corollary 2.6, gives:*

$$\left|\frac{1}{\Pi-\Theta}\int_{\Theta}^{\Pi}\Bbbk(x)\,dx-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)\right|\leq\frac{(\Pi-\Theta)^2}{48}\left[\left|\Bbbk''(\Theta)\right|+\left|\Bbbk''(\Pi)\right|\right].$$

Theorem 2.8. *Let $\Bbbk:[\Theta,\Pi]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on (Θ,Π) and $|\Bbbk''|^q$, $q>1$ is s-convex on $[\Theta,\Pi]$ with $\Theta\leq\Pi$ and $\Phi,\phi\in[\Theta,\Pi]$, the subsequent fractional inequality is valid:*

$$\begin{aligned}
&\Bigg|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\\
&\quad-\Bbbk\left(\Theta+\Pi-\frac{\Phi+\phi}{2}\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\Bigg|\\
&\leq\frac{(\phi-\Phi)^2}{16\times2^{p+1}}\Bigg[\Bigg(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\\
&\quad-\frac{1}{2^s(1+s)}\left|\Bbbk''(\Phi)\right|^q-\frac{2-2^{-s}}{1+s}\left|\Bbbk''(\phi)\right|^q\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\\
&\quad-\frac{2-2^{-s}}{1+s}\left|\Bbbk''(\Phi)\right|^q-\frac{1}{2^s(1+s)}\left|\Bbbk''(\phi)\right|^q\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Proof. By using the equality (2.1) and the well known Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\\
&\quad-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\Bigg|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left(\int_0^1\gamma^{2p}d\gamma\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\left(\int_0^1 \left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right]\\
&\leq \frac{(\phi-\Phi)^2}{16}\left(\int_0^1 \gamma^{2p} d\gamma\right)^{\frac{1}{p}}\left[\int_0^1 \left(\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)\right.\right.\\
&\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''(\phi)\right|^q+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''(\Phi)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\\
&+\int_0^1 \left(\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)\right.\\
&\left.\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''(\Phi)\right|^q+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''(\phi)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\right]\\
&\leq \frac{(\phi-\Phi)^2}{16\times 2^{p+1}}\left[\left(\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)\right.\right.\\
&\left.-\left(\frac{1}{2^s(1+s)}\left|\Bbbk''(\Phi)\right|^q+\left(\frac{2-2^{-s}}{(1+s)}\right)\left|\Bbbk''(\phi)\right|^q\right)\right)^{\frac{1}{q}}\\
&+\left(\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)\right.\\
&\left.\left.-\left(\left(\frac{2-2^{-s}}{(1+s)}\right)\left|\Bbbk''(\Phi)\right|^q+\frac{1}{2^s(1+s)}\left|\Bbbk''(\phi)\right|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The proof of Theorem 2.8 is completed. □

Corollary 2.9. *Substituting $s=1$ into Theorem 2.8, gives:*

$$
\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\right.\\
&\quad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\right|\\
&\leq \frac{(\phi-\Phi)^2}{16\times 2^{p+1}}\left[\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q-\left(\frac{\left|\Bbbk''(\Phi)\right|^q+3\left|\Bbbk''(\phi)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q-\left(\frac{3\left|\Bbbk''(\Phi)\right|^q+\left|\Bbbk''(\phi)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 2.10. *Substituting* $\Phi = \Theta$ *and* $\phi = \Pi$ *into Corollary 2.9, gives:*

$$\left| \frac{\beta(\upsilon)}{\upsilon(\Pi-\Theta)} \left[\left({}^{CF}_{\Theta} I^{\upsilon} \Bbbk \right)(x) + \left({}^{CF} I^{\upsilon}_{\Pi} \Bbbk \right)(x) \right] - \Bbbk\left(\frac{\Theta+\Pi}{2} \right) - \frac{2(1-\upsilon)}{\upsilon(\Pi-\Theta)} \Bbbk(x) \right| \leq \frac{(\Pi-\Theta)^2}{16 \times 2^{p+1}} \left[\left(\frac{|\Bbbk''(\Pi)|^q + 3|\Bbbk''(\Theta)|^q}{4} \right) + \left(\frac{3|\Bbbk''(\Pi)|^q + |\Bbbk''(\Theta)|^q}{4} \right) \right]^{\frac{1}{q}}.$$

Corollary 2.11. *Substituting* $\upsilon = 1$ *and* $\beta(0) = \beta(1) = 1$ *into Corollary 2.10, gives:*

$$\left| \frac{1}{\Pi-\Theta} \int_{\Theta}^{\Pi} \Bbbk(x)\, dx - \Bbbk\left(\frac{\Theta+\Pi}{2} \right) \right| \leq \frac{(\Pi-\Theta)^2}{16 \times 2^{p+1}} \left[\left(\frac{|\Bbbk''(\Pi)|^q + 3|\Bbbk''(\Theta)|^q}{4} \right) + \left(\frac{3|\Bbbk''(\Pi)|^q + |\Bbbk''(\Theta)|^q}{4} \right) \right]^{\frac{1}{q}}.$$

Theorem 2.12. *Let* $\Bbbk : [\Theta, \Pi] \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* (Θ, Π) *and* $|\Bbbk''|^q$, $q \geq 1$ *is s-convex on* $[\Theta, \Pi]$ *with* $\Theta \leq \Pi$ *and* $\Phi, \phi \in [\Theta, \Pi]$, *the subsequent fractional inequality is valid:*

$$\begin{aligned}
&\left| \frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)} \left[\left({}^{CF}_{\Theta+\Pi-\phi} I^{\upsilon} \Bbbk \right)(x) + \left({}^{CF} I^{\upsilon}_{\Theta+\Pi-\Phi} \Bbbk \right)(x) \right] - \Bbbk\left(\Theta + \Pi - \left(\frac{\Phi+\phi}{2} \right) \right) - \frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)} \Bbbk(x) \right| \\
&\leq \frac{(\phi-\Phi)^2}{16} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left[\left(\frac{1}{3} \left(|\Bbbk''(\Theta)|^q + |\Bbbk''(\Pi)|^q \right) - \left(\left(\frac{2^{-s}}{3+s} \right) |\Bbbk''(\Phi)|^q + \left(\frac{2^{-s}\left(2^{4+s} - 7s - s^2 - 14\right)}{(s^3 + 6s^2 + 11s + 6)} \right) |\Bbbk''(\phi)|^q \right) \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{1}{3} \left(|\Bbbk''(\Theta)|^q + |\Bbbk''(\Pi)|^q \right) - \left(\left(\frac{2^{-s}}{3+s} \right) |\Bbbk''(\phi)|^q + \left(\frac{2^{-s}\left(2^{4+s} - 7s - s^2 - 14\right)}{(s^3 + 6s^2 + 11s + 6)} \right) |\Bbbk''(\Phi)|^q \right) \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Proof. By using the equality (2.1) and the well known power-mean inequality, we have

$$\left| \frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)} \left[\left({}^{CF}_{\Theta+\Pi-\phi} I^{\upsilon} \Bbbk \right)(x) + \left({}^{CF} I^{\upsilon}_{\Theta+\Pi-\Phi} \Bbbk \right)(x) \right] \right.$$

$$
\begin{aligned}
&\left. -\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2\left(1-\upsilon\right)}{\upsilon\left(\phi-\Phi\right)}\Bbbk\left(x\right)\right| \\
&\leq \frac{\left(\phi-\Phi\right)^2}{16}\left[\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)\right|d\gamma\right. \\
&\quad\left.+\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)\right|d\gamma\right] \\
&\leq \frac{\left(\phi-\Phi\right)^2}{16}\int_0^1\left(\gamma^2d\gamma\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\phi+\frac{2-\gamma}{2}\Phi\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\gamma^2\left|\Bbbk''\left(\Theta+\Pi-\left(\frac{\gamma}{2}\Phi+\frac{2-\gamma}{2}\phi\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The Jensen–Mercer inequality allows us to represent this in the following form:

$$
\begin{aligned}
&\left|\frac{\beta\left(\upsilon\right)}{\upsilon\left(\phi-\Phi\right)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)\left(x\right)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)\left(x\right)\right]\right. \\
&\quad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2\left(1-\upsilon\right)}{\upsilon\left(\phi-\Phi\right)}\Bbbk\left(x\right)\right| \\
&\leq \frac{\left(\phi-\Phi\right)^2}{16}\left(\int_0^1\gamma^2d\gamma\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\gamma^2\left(\left(\left|\Bbbk''\left(\Theta\right)\right|^q+\left|\Bbbk''\left(\Pi\right)\right|^q\right)\right.\right.\right. \\
&\quad\left.\left.\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''\left(\phi\right)\right|^q+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''\left(\Phi\right)\right|^q\right)d\gamma\right)\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\gamma^2\left(\left(\left|\Bbbk''\left(\Theta\right)\right|^q+\left|\Bbbk''\left(\Pi\right)\right|^q\right)\right.\right.\right. \\
&\quad\left.\left.\left.-\left(\left(\frac{\gamma}{2}\right)^s\left|\Bbbk''\left(\Phi\right)\right|^q+\left(\frac{2-\gamma}{2}\right)^s\left|\Bbbk''\left(\phi\right)\right|^q\right)d\gamma\right)\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\left(\phi-\Phi\right)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left(\left|\Bbbk''\left(\Theta\right)\right|^q+\left|\Bbbk''\left(\Pi\right)\right|^q\right)\right.\right. \\
&\quad\left.\left.-\left(\left(\frac{2^{-s}}{3+s}\right)\left|\Bbbk''\left(\Phi\right)\right|^q+\left(\frac{2^{-s}\left(2^{4+s}-7s-s^2-14\right)}{\left(s^3+6s^2+11s+6\right)}\right)\left|\Bbbk''\left(\phi\right)\right|^q\right)\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{3}\left(\left|\Bbbk''\left(\Theta\right)\right|^q+\left|\Bbbk''\left(\Pi\right)\right|^q\right)\right.\right. \\
&\quad\left.\left.-\left(\left(\frac{2^{-s}}{3+s}\right)\left|\Bbbk''\left(\phi\right)\right|^q+\left(\frac{2^{-s}\left(2^{4+s}-7s-s^2-14\right)}{\left(s^3+6s^2+11s+6\right)}\right)\left|\Bbbk''\left(\Phi\right)\right|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The proof of Theorem 2.12 is completed. □

Corollary 2.13. *Substituting $s=1$ into Theorem 2.12, gives:*

$$\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\phi-\Phi)}\left[\left({}^{CF}_{\Theta+\Pi-\phi}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Theta+\Pi-\Phi}\Bbbk\right)(x)\right]\right.\\
&\qquad\left.-\Bbbk\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)-\frac{2(1-\upsilon)}{\upsilon(\phi-\Phi)}\Bbbk(x)\right|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\\
&\qquad\times\left[\left(\frac{1}{3}\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)-\left(\frac{1}{8}\left|\Bbbk''(\Phi)\right|^q+\frac{5}{24}\left|\Bbbk''(\phi)\right|^q\right)\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\frac{1}{3}\left(\left|\Bbbk''(\Theta)\right|^q+\left|\Bbbk''(\Pi)\right|^q\right)-\left(\frac{1}{8}\left|\Bbbk''(\phi)\right|^q+\frac{5}{24}\left|\Bbbk''(\Phi)\right|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.14. *Substituting $\Phi=\Theta$ and $\phi=\Pi$ into Corollary 2.13, gives:*

$$\begin{aligned}
&\left|\frac{\beta(\upsilon)}{\upsilon(\Pi-\Theta)}\left[\left({}^{CF}_{\Theta}I^{\upsilon}\Bbbk\right)(x)+\left({}^{CF}I^{\upsilon}_{\Pi}\Bbbk\right)(x)\right]\right.\\
&\qquad\left.-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)-\frac{2(1-\upsilon)}{\upsilon(\Pi-\Theta)}\Bbbk(x)\right|\\
&\leq\frac{(\Pi-\Theta)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\\
&\qquad\times\left[\left(\frac{3\left|\Bbbk''(\Pi)\right|^q+5\left|\Bbbk''(\Theta)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{5\left|\Bbbk''(\Pi)\right|^q+3\left|\Bbbk''(\Theta)\right|^q}{24}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.15. *Substituting $\upsilon=1$ and $\beta(0)=\beta(1)=1$ into Corollary 2.14, gives:*

$$\begin{aligned}
&\left|\frac{1}{\Pi-\Theta}\int_{\Theta}^{\Pi}\Bbbk(x)\,dx-\Bbbk\left(\frac{\Theta+\Pi}{2}\right)\right|\\
&\leq\frac{(\Pi-\Theta)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\\
&\qquad\times\left[\left(\frac{3\left|\Bbbk''(\Pi)\right|^q+5\left|\Bbbk''(\Theta)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{5\left|\Bbbk''(\Pi)\right|^q+3\left|\Bbbk''(\Theta)\right|^q}{24}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

3. Applications

3.1. **q-Digamma Function.** Before proceeding further, it is useful to introduce the q-digamma function and discuss several of its fundamental

features, including standard forms encountered in the literature. In what follows, we assume that the parameter q satisfies $0 < q < 1$. As established in (refer to [2, 3]), is denoted by

$$\begin{aligned}\Omega_q(z) &= -\ln(1-q) + \ln(q) \sum_{i=0}^{\infty} \frac{q^{i+z}}{1-q^{i+z}} \\ &= -\ln(1-q) + \ln(q) \sum_{i=0}^{\infty} \frac{q^{iz}}{1-q^{i}}.\end{aligned}$$

If $q > 1$ and $z > 0$, the q-digamma function Ω_q can be expressed as:

$$\begin{aligned}\Omega_q(z) &= -\ln(1-q) + \ln(q) \left[u - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-(i+z)}}{1-q^{-(i+z)}} \right] \\ &= -\ln(1-q) + \ln(q) \left[u - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-iz}}{1-q^{-i}} \right].\end{aligned}$$

Proposition 3.1. *Under the assumptions specified in Corollary 2.5 the following result is obtained:*

$$\begin{aligned}&\left| \frac{\Omega_q(\Theta + \Pi - \phi) + \Omega_q(\Theta + \Pi - \Phi)}{(\phi - \Phi)} - \Omega_q\left(\Theta + \Pi - \left(\frac{\Phi + \phi}{2}\right)\right) \right| \\ &\quad \leq \frac{(\phi - \Phi)^2}{16} \left[\frac{2}{3} \left(\left|\Omega_q''(\Theta)\right| + \left|\Omega_q''(\Pi)\right| \right) - \frac{1}{3} \left(\left|\Omega_q''(\Phi)\right| + \left|\Omega_q''(\phi)\right| \right) \right].\end{aligned}$$

Proof. If we set $\Bbbk(z) \to \Omega_q(z)$, the conclusion immediately follows, with $\beta(0) = \beta(1) = \alpha = 1$. □

Proposition 3.2. *Under the assumptions specified in Corollary 2.14 the following result is obtained:*

$$\begin{aligned}&\left| \frac{\Omega_q(\Theta) + \Omega_q(\Pi)}{(\phi - \Phi)} - \Omega_q\left(\frac{\Theta + \Pi}{2}\right) \right| \\ &\quad \leq \frac{(\Pi - \Theta)^2}{16} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \\ &\qquad \times \left[\left(\frac{3\left|\Omega_q''(\Pi)\right|^q + 5\left|\Omega_q''(\Theta)\right|^q}{24} \right)^{\frac{1}{q}} + \left(\frac{5\left|\Omega_q''(\Pi)\right|^q + 3\left|\Omega_q''(\Theta)\right|^q}{24} \right)^{\frac{1}{q}} \right].\end{aligned}$$

Proof. If we set $\Bbbk(z) \to \Omega_q(z)$, the conclusion immediately follows, with $\beta(0) = \beta(1) = \alpha = 1$. □

3.2. **Modified Bessel Function.** Consider the modified Bessel function of the first kind, denoted by I_σ, which admits the following series

expansion ([24, p.77])

$$I_{\sigma}(z)=\sum_{n\geq 0}\frac{\left(\frac{z}{2}\right)^{\sigma+2n}}{n!\Gamma(\sigma+n+1)},$$

where $z\in\mathbb{R}$ and $\sigma>-1$ second kind of Modified Bessel function $\mathbb{C}_{\sigma}$ [24, p.77] is defined as:

$$\mathbb{C}_{\sigma}(z)=\frac{\pi}{2}\frac{I_{-\sigma}(z)-I_{\sigma}(z)}{\sin\sigma\pi}.$$

Let the function $\Psi_{\sigma}(z):\mathbb{R}\to[1,\infty)$ defined by

$$\Psi_{\sigma}(z)=2^{\sigma}\Gamma(\sigma+1)z^{-\sigma}\mathbb{C}_{\sigma}(z),$$

The derivative of order one of the function $\Psi_{\sigma}(z)$ has been derived in [24, p.77], where it is expressed as:

$$\Psi'_{\sigma}(z)=\frac{z}{2(\sigma+1)}\Psi_{\sigma+1}(z), \tag{3.1}$$

moreover, the second derivative follows directly from equation (3.1) and is given by:

$$\Psi''_{\sigma}(z)=\frac{z^2}{4(\sigma+1)(\sigma+2)}\Psi_{\sigma+1}(z)+\frac{1}{2(\sigma+1)}\Psi_{\sigma+1}(z).$$

Proposition 3.3. *Let $[\Theta,\Pi]\in\mathbb{R}$ and $\sigma>-1$, then we have*

$$\begin{aligned}
&\left|\frac{\Psi_{\sigma}(\Theta+\Pi-\phi)+\Psi_{\sigma}(\Theta+\Pi-\Phi)}{(\phi-\Phi)}-\Psi_{\sigma}\left(\Theta+\Pi-\left(\frac{\Phi+\phi}{2}\right)\right)\right|\\
&\leq\frac{(\phi-\Phi)^2}{16}\left[\frac{2}{3}\left(\frac{\Theta^2}{4(\sigma+1)(\sigma+2)}\Psi_{\sigma+1}(\Theta)+\frac{1}{2(\sigma+1)}\Psi_{\sigma+1}(\Theta)\right.\right.\\
&\quad+\frac{\Pi^2}{4(\sigma+1)(\sigma+2)}\Psi_{\sigma+1}(\Pi)+\frac{1}{2(\sigma+1)}\Psi_{\sigma+1}(\Pi)\Big)\\
&\quad-\frac{1}{3}\left(\frac{\Phi^2}{4(\sigma+1)(\sigma+2)}\Psi_{\sigma+1}(\Phi)+\frac{1}{2(\sigma+1)}\Psi_{\sigma+1}(\Phi)\right.\\
&\quad\left.\left.+\frac{\phi^2}{4(\sigma+1)(\sigma+2)}\Psi_{\sigma+1}(\phi)+\frac{1}{2(\sigma+1)}\Psi_{\sigma+1}(\phi)\right)\right].
\end{aligned}$$

Proof. This conclusion can be derived immediately from Corollary 2.5 by setting $\Bbbk(z)=\Psi_{\sigma}(z)$ for $z>0$. □

4. Comparative Analysis

In this section, we provide a mathematical and graphical illustration which validate our findings.

Example 4.1. Let $\Bbbk(x) = x^4$, $x \in [0,2]$. Then $\Bbbk''(x) = 12x^2, |\Bbbk''(x)| = 12x^2$. Choosing $\Theta, \Pi \in [0,2]$ in Corollary 2.7, we obtain the following numerical verification.

x	Θ	Π	LHS	RHS
0.20	0.00	0.20	0.00022000	0.00040000
0.30	0.10	0.40	0.00291375	0.00382500
0.40	0.20	0.60	0.01312000	0.01600000
0.50	0.30	0.80	0.03859375	0.04562500
0.60	0.40	1.00	0.08982000	0.10440000
0.70	0.50	1.20	0.18001375	0.20702500
0.80	0.60	1.40	0.32512000	0.37120000
0.90	0.70	1.60	0.54381375	0.61762500
1.00	0.80	1.80	0.85750000	0.97000000
1.10	0.90	2.00	1.29031375	1.45502500

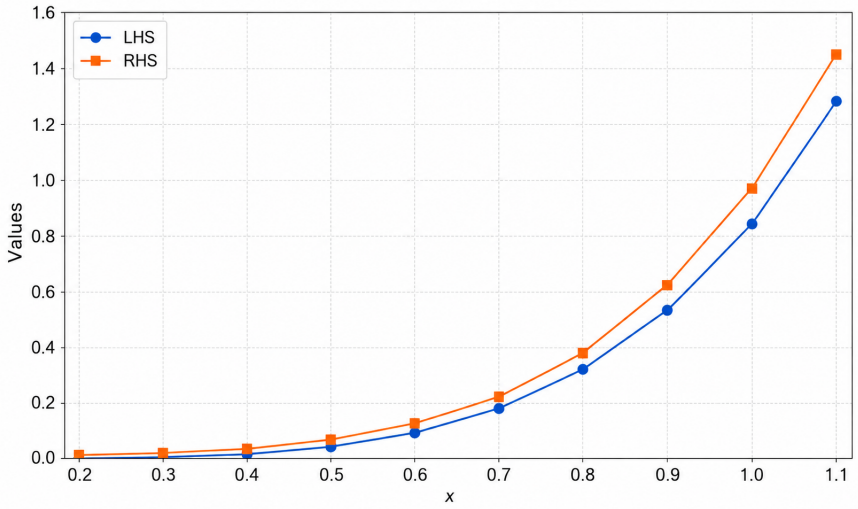

FIGURE 1. Graphical representation, which shows LHS is strictly below RHS.

Example 4.2. Let $\Bbbk(x) = 10^{-5}x^9$, $x \in [0,2]$. Then $\Bbbk''(x) = 72 \times 10^{-5}x^7, |\Bbbk''(x)| = 72 \times 10^{-5}x^7$. Since $\Bbbk''(x) \geq 0$ for all $x \in [0,2]$, the function $\Bbbk$ is convex and twice differentiable on $[0,2]$. Taking $p = q = 2$ in Corollary 2.11, and choosing $\Theta, \Pi \in [0,2]$, we obtain the following

numerical verification.

x	Θ	Π	LHS	RHS
0.20	0.00	0.20	5.02×10^{-13}	2.88×10^{-12}
0.30	0.10	0.40	3.11378×10^{-10}	8.29440×10^{-10}
0.40	0.20	0.60	1.2494848×10^{-8}	2.5194243×10^{-8}
0.50	0.30	0.80	$1.68682889 \times 10^{-7}$	$2.94912160 \times 10^{-7}$
0.60	0.40	1.00	$1.262955834 \times 10^{-6}$	$2.025002718 \times 10^{-6}$
0.70	0.50	1.20	$6.527773194 \times 10^{-6}$	$9.876165555 \times 10^{-6}$
0.80	0.60	1.40	$2.614927360 \times 10^{-5}$	$3.7948995251 \times 10^{-5}$
0.90	0.70	1.60	$8.6957809584 \times 10^{-5}$	$1.22306480225 \times 10^{-4}$
1.00	0.80	1.80	$2.50894354750 \times 10^{-4}$	$3.44375788429 \times 10^{-4}$
1.10	0.90	2.00	$6.47249415668 \times 10^{-4}$	$8.71206082212 \times 10^{-4}$

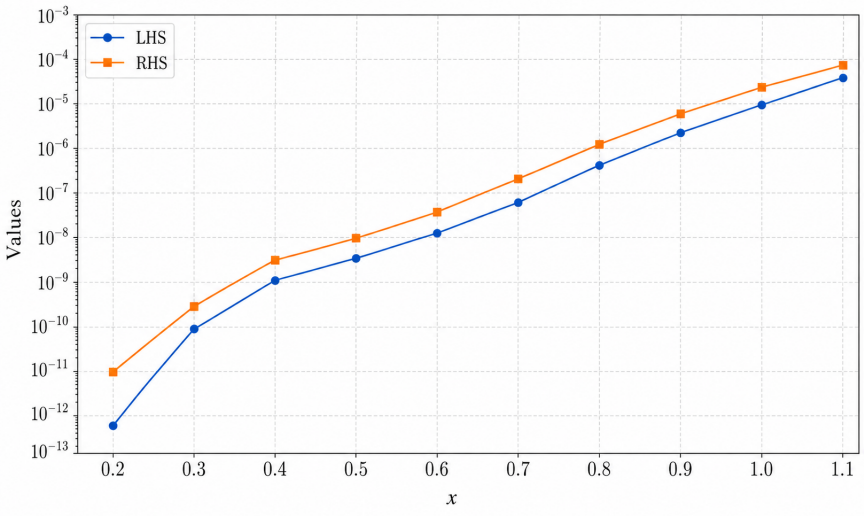

FIGURE 2. Graphical representation, which shows LHS is strictly below RHS.

Example 4.3. Let $\Bbbk(x) = x^4$, $x \in [0, 2]$. Then $\Bbbk''(x) = 12x^2, |\Bbbk''(x)| = 12x^2$. Taking $q = 2$ in Corollary 2.15, and choosing $\Theta, \Pi \in [0, 2]$, we

obtain the following numerical verification.

x	Θ	Π	LHS	RHS
0.20	0.00	0.20	0.00022000	0.00056118
0.30	0.10	0.40	0.00291375	0.00506109
0.40	0.20	0.60	0.01312000	0.02033471
0.50	0.30	0.80	0.03859375	0.05670512
0.60	0.40	1.00	0.08982000	0.12797260
0.70	0.50	1.20	0.18001375	0.25141429
0.80	0.60	1.40	0.32512000	0.44778421
0.90	0.70	1.60	0.54381375	0.74131335
1.00	0.80	1.80	0.85750000	1.15970962
1.10	0.90	2.00	1.29031375	1.73415789

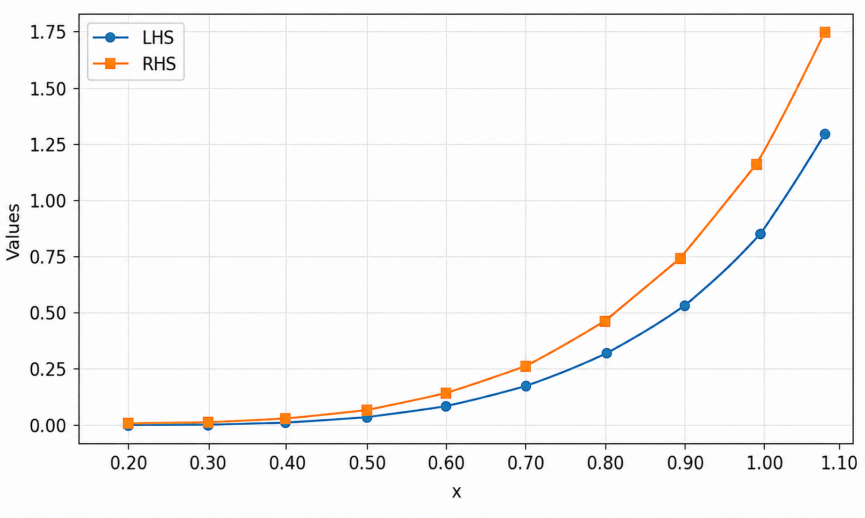

FIGURE 3. Graphical representation, which shows LHS is strictly below RHS.

5. Conclusions

Fractional calculus has gained considerable attention due to its wide range of applications in modeling real-world processes. As this field continues to evolve, there is an ongoing need to refine existing results and extend them to more general settings. In response, many researchers have developed new forms of fractional integral operators, employing various techniques and tools from fractional calculus to broaden the scope of the theory. In recent years, there has been significant interest

in using fractional calculus to derive various integral inequalities based on convex functions. The development of integral inequalities, mathematical modeling, and differential equations has been greatly facilitated by the Caputo-Fabrizio fractional operator.

In this work, we investigate Hermite-Jensen Mercer-type inequalities in the framework of s-convex functions by utilizing the Caputo-Fabrizio fractional integral operator. Additionally, for twice-differentiable convex functions, a novel fractional integral identity of the Caputo-Fabrizio fractional operator is established. Applications to the q-digamma function and the modified Bessel function are also provided. Numerical computations and graphical illustrations are included to visualize the behavior of the established bounds under various parameter values. Furthermore, many fractional results are recovered as special cases by choosing appropriate parameter values. It is anticipated that the findings presented here will stimulate further research in both fractional and quantum calculus. In particular, we expect that researchers will extend these ideas by employing various forms of generalized convexity and a range of fractional integral operators, including the k-fractional operator, the Riemann–Liouville integral, and the modified Caputo-Fabrizio integral.

Declarations

Availability of Data and Materia. No data were used to support this study.

Competing Interests. The authors declare that they have no competing interests.

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[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026 People's Republic of China.
Email address: munirarslan999@gmail.com

[2]Department of Mathematics, Düzce University, Düzce, Turkey.
Email address: hasan64kara@gmail.com

[3]Department of Mathematics, Düzce University, Düzce, Turkey.
Email address: fatihezenci@gmail.com