Sahand Communications in Mathematical Analysis (SCMA) Vol. 24 No. 1 (2027), 1-25
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2064232.2238

Fixed Points on Symmetric Spaces via Boyd and Wong Idea

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Abstract. We overcome the obstacle of the missing triangle inequality in symmetric spaces by introducing two new concepts: the bounded shift property (BSP) and the uniform bounded shift property with respect to a class $\mathcal{D}$ of AR sequences (UBSP$_{\mathcal{D}}$). These notions are inspired by the classical Boyd–Wong argument and are designed to capture the behavior of non-Cauchy asymptotically regular sequences in symmetric spaces. We prove that several important classes of symmetric spaces, including Φ-symmetric, γ-symmetric and JMS-spaces, satisfy these properties. As a consequence, we establish new fixed point theorems for ψ-contractions and ζ-contractions under relaxed assumptions, extending and unifying earlier results in the literature. In particular, our approach allows a natural generalization of the standard contraction principle by weakening the hypothesis $\psi \in \Psi$ to $\psi \in \Psi(\varepsilon)$, which is of independent interest. The results are applied to b-metric spaces, interpolative metric spaces and bounded symmetric spaces. Several examples and counterexamples are provided to illustrate the sharpness of the results and the novelty of the concepts introduced.

1. Introduction

The fixed point theory of various classes of mappings in metric spaces and other generalized metric structures has been extensively investigated by numerous researchers. In this context, the development and analysis of conditions ensuring the existence and uniqueness of fixed points for mappings have led to significant advancements, particularly through the introduction of concepts such as Picard operators (PO) and weakly Picard operators (WPO); see [13] for their definitions. These operators

2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Generalized simulation function, Fixed points, Symmetric spaces, b-metric spaces, Boyd and Wong's property, Bounded shift property.
Received: 2025-06-26, Accepted: 2026-04-29.

play a pivotal role in fixed-point theory and its wide-ranging applications (see, for example, [13, 14, 21–26] and references therein), which serve as fundamental tools in tackling complex problems arising in numerical analysis, game theory, mathematical economics, differential and integral equations, etc.

By omitting the third axiom of metric spaces (the triangle inequality), we arrive at the concept of a symmetric space. Unlike metric spaces, symmetric spaces dispense with the triangle inequality, providing a more flexible mathematical framework that encompasses a variety of distance-like structures including b-metric, Φ-symmetric, γ-symmetric and JMS-spaces. A symmetric space is a pair (X, s), where X is a nonempty set and $s : X \times X \to [0, +\infty)$ is a symmetric function that is zero only when both arguments are equal.

A family of closed sets can be defined as follows to induce a topology τ_s on X: The definition and characteristics of a closed subset $A \subseteq X$ are found in [4, 8, 10]. If every ball centered at x is a neighborhood of x in the topology τ_s [4, 8, 10], then the symmetric space is referred to as semi-metric. This is generally not the case [11]; see also [8, Remark 2.2].

Definition 1.1 ([5, 19, 20])**.** Assume that the space (X, s) is symmetric.

(i) A triangle function for s is a function $\Phi : [0, +\infty)^2 \to [0, +\infty)$ such that
   (a) Φ is non-decreasing in each of its variables,
   (b) Φ is continuous at (0,0),and
   (c) for all $x, y, z \in X$, $s(x, y) \leq \Phi(s(x, z), s(z, y))$.

   If Φ exists for symmetry s, then (X, s) is called a Φ-symmetric space. Assume $b \geq 1$. b-ultrametric space and b-metric space are symmetric spaces with the triangle functions $\Phi(u, v) = b \max\{u, v\}$ and $\Phi(u, v) = b(u + v)$, where $u, v \geq 0$, respectively. However, in [5], Φ was presumed symmetric, but we do not include it in our definition, therefore the functions $\Phi(u, v) = bu + b'v, u, v \geq 0$ or $\Phi(u, v) = u + bv, u, v \geq 0$ are also triangles, where $b, b' \geq 1$; hence, the following is a special case of this one.

(ii) If a triangle function Φ fulfills a stronger requirement than (ic), as follows:

$$|s(x, y) - s(z, w)| \leq \Phi\left(s(x, z), s(y, w)\right),$$

   for all $x, y, z, w \in X$, then (X, s) is regarded as regular Φ-symmetric space.

(iii) A γ-triangle function for s is defined as a function $\gamma : (0, +\infty) \to (0, +\infty)$ satisfying

$$\text{if } s(x, y), s(y, z) < \gamma(\epsilon) \text{ then } s(x, z) < \epsilon, \tag{1.1}$$

for all $\epsilon > 0, x, y, z \in X$. If γ exists for the symmetry (X, s), we call it γ-symmetric space. Every b-metric space with b-triangular coefficient b has γ-triangle function, as $\gamma(t) = \frac{t}{2b}, t > 0$ fulfills.

Some preliminaries can be found in [10]. These foundational concepts are essential for understanding the subsequent analyses and discussions presented in this paper.

Definition 1.2 ([10])**.** Let x_n be a sequence in a symmetric space (X, s) and let x be an element of X

(i) We say the sequence x_n converges to $x \in X$ ($\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} s(x_n, x) = 0$.
(ii) A self-map T on X is continuous if and only if for every sequence x_n, the limit $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} Tx_n = Tx$.
(iii) We say the sequence x_n is Cauchy if $\lim_{n,m\to\infty} s(x_n, x_m) = 0$. (X, s) is called complete if every Cauchy sequence converges.
(iv) A sequence y_i is termed $\sum$-Cauchy if $\sum_{i=1}^{\infty} s(y_i, y_{i+1}) < \infty$. The space (X, s) is referred to as $\sum$-complete if all $\sum$-Cauchy sequences converge.

Remark 1.3 ([10])**.** x_n is not a Cauchy sequence provided $\epsilon > 0$ and increasing sequences r_k and s_k of $\mathbb{N}$ exist such that $\epsilon \leq s(x_{r_k}, x_{s_k})$ for all $k \in \mathbb{N}$.

The rest of the paper is structured as follows: Section 2 introduces the Boyd–Wong property (BWP) on symmetric spaces and proves that regular Φ-symmetric spaces satisfy this property. Section 3 focuses on fixed point results in symmetric spaces using simulation functions. It defines ζ-contractions, weak ζ-contractions and explores their relationship with Boyd–Wong-type properties. Several theorems establish conditions for the existence and uniqueness of fixed points under these generalized contractions. Section 4 introduces two new structural properties: the bounded shift property (BSP) and the uniform bounded shift property with respect to a class $\mathcal{D}$ of AR sequences (UBSP$_{\mathcal{D}}$). It presents further fixed point theorems for ψ-contractions under the relaxed assumption $\psi \in \Psi(\varepsilon)$ and demonstrates their validity in a broad class of symmetric spaces, including Φ-symmetric, γ-symmetric, JMS-spaces and bounded symmetric spaces. Section 5 concludes the paper by summarizing the main contributions and outlining directions for future research.

2. Boyd and Wong Idea

A well-known result that dates back to Boyd and Wong's idea is used in articles to prove fixed point results in metric spaces (see [7], as well

as [28, Lemma 2.1] and [2, 6, 9, 15, 25, 27–29]): In a metric space (X, d) for any non-Cauchy sequence x_n such that $d(x_{n+1}, x_n) \to 0$ there exists $\varepsilon > 0$ and two subsequences x_{r_k}, x_{s_k} of x_n such that $\lim_{k\to+\infty} d(x_{r_k}, x_{s_k}) = \lim_{k\to+\infty} d(x_{r_k+1}, x_{s_k+1}) = \varepsilon^+$. In light of these works, we introduce certain new concepts related to properties of symmetric spaces in this section (and Sec. 4), along with a weaker definition applicable to a broader range of symmetric spaces. Throughout this paper, $x_{r_k} \ll x_n$ indicates that x_{r_k} is a subsequence of x_n.

Definition 2.1. Assume that (X, s) is a symmetric space.

(i) Let $\mathcal{A}$ denote the class of all AR sequences x_n in X, i.e., sequences satisfying
$$\lim_{n\to\infty} s\left(x_{n+1}, x_n\right) = 0.$$

(ii) We say that (X, s) has the Boyd–Wong property (briefly, BWP) if for every $x_n \in \mathcal{A}$ that is not a Cauchy sequence, there exist $\varepsilon > 0$ and subsequences $x_{r_k}, x_{s_k} \ll x_n$ such that $s(x_{r_k+1}, x_{s_k+1})$ and $s(x_{r_k}, x_{s_k}) \geq \varepsilon$ are non-increasing sequences and
$$\lim_{k\to+\infty} s\left(x_{r_k}, x_{s_k}\right) = \lim_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) = \varepsilon^+.$$

According to [25] and Theorem 2.2, metric spaces have BWP. Continuing the articles [2, 6, 7, 9, 15, 25, 27–29] on metric spaces, the following theorem demonstrates with a novel and straightforward proof that any regular Φ-symmetric space has BWP. Thus, every strong b-metric has also BWP.

Theorem 2.2. *Let (X, s) be a regular Φ-symmetric space and $x_n \in \mathcal{A}$ then*

(I) for every $x_{r_k}, x_{s_k} \ll x_n$,
$$\lim_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) = \lim_{k\to+\infty} s\left(x_{r_k}, x_{s_k}\right). \tag{2.1}$$

(II) (X, s) has BWP.

Proof. (I) From [30, Lemma 5] there exists a nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$ such that ψ is continuous at 0, $\psi(0) = 0$ and the inequality
$$s(x, y) \leq s\left(x, z\right) + \psi\left(s\left(z, y\right)\right), \tag{2.2}$$
holds, for all $x, y, z \in X$. From inequality (2.2), we have

(2.3)
$$\begin{aligned} s\left(x_{r_k}, x_{s_k}\right) &\leq s\left(x_{s_k}, x_{r_k+1}\right) + \psi\left(s\left(x_{r_k+1}, x_{r_k}\right)\right) \\ &\leq s\left(x_{r_k+1}, x_{s_k+1}\right) + \psi\left(s\left(x_{s_k+1}, x_{s_k}\right)\right) + \psi\left(s\left(x_{r_k+1}, x_{r_k}\right)\right), \end{aligned}$$

and

$$\begin{aligned} s\left(x_{r_k+1}, x_{s_k+1}\right) &\leq s\left(x_{r_k}, x_{s_k+1}\right) + \psi\left(s\left(x_{r_k+1}, x_{r_k}\right)\right) \\ &\leq s\left(x_{r_k}, x_{s_k}\right) + \psi\left(s\left(x_{s_k+1}, x_{s_k}\right)\right) + \psi\left(s\left(x_{r_k+1}, x_{r_k}\right)\right), \end{aligned} \tag{2.4}$$

for all $k \in \mathbb{N}$. Since $x_n \in \mathcal{A}$ then the inequalities (2.3), (2.4) and the continuity of ψ at 0 show that (2.1) holds while $k \to +\infty$.

(II) Suppose that x_n is not a Cauchy sequence, then there exist $\varepsilon' > 0$ and increasing sequences r_k and s_k such that $t_k := s\left(x_{r_k}, x_{s_k}\right) \geq \varepsilon'$. Thus, there exist a non-increasing subsequence $\varepsilon \leq t_{k_l}$ of t_k such that $\lim\limits_{l\to+\infty} t_{k_l} = \varepsilon := \liminf_k t_k \geq \varepsilon'$. Let us reuse t_k to showing this subsequence once again for convenience. From (2.1) we get $t'_k := \lim\limits_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) = \varepsilon$. Then, there exists non-increasing subsequence t'_{k_l} of $t'_k := s\left(x_{r_k+1}, x_{s_k+1}\right)$ such that $\lim\limits_{l\to+\infty} t'_{k_l} = \varepsilon$, so

$$\lim_{k\to+\infty} s\left(x_{r_{k_l}+1}, x_{r_{k_l}+1}\right) = \lim_{k\to+\infty} s\left(x_{r_{k_l}}, x_{r_{k_l}}\right) = \varepsilon.$$

Thus, (ii) holds.

□

It is easy to see that the symmetric space (X, s) has BWP iff the symmetric space $(X, s^\alpha), \alpha > 0$ has BWP, thus, typical examples of b-metric spaces (e.g., [10, Sec 2: Examples 1, 2, 3]) have BWP. Furthermore, the following example shows that BWP does not hold in b-metric spaces generally, while the following theorem indicates that b-metric spaces have weaker BWP.

Example 2.3. Let $X = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ and

$$D(x,y) = \begin{cases} 0, & x = y, \\ 2, & (x,y) = \left(\frac{1}{2k-1}, \frac{1}{2k'-1}\right), \quad k \neq k', \\ 4, & (x,y) = \left(\frac{1}{2k}, \frac{1}{2k'}\right), \quad k \neq k', \\ 1 - 2^{-\max\{x,y\}}, & \text{otherwise.} \end{cases} \tag{2.5}$$

(X, D) is not a metric space, since

$$\begin{aligned} 4 = D\left(\frac{1}{2}, \frac{1}{4}\right) &= 4\left[D\left(1, \frac{1}{2}\right) + D\left(1, \frac{1}{4}\right)\right] \\ &= 4\left(1 - 2^{-1} + 1 - 2^{-1}\right). \end{aligned}$$

We will show that (X, D) is a b-metric on X with b-triangular coefficient $b = 4$. We have $\text{Image}(D) = \{0, 2, 4\} \cup \left\{1 - 2^{-\frac{1}{n}}, n \in \mathbb{N}\right\}$ and (X, D) is a symmetric space.

(i) If $D(x, y) = 4 \in \text{Image}(D)$ then from (2.5) we have $x = \frac{1}{2k}, y = \frac{1}{2k'}, k \neq k'$. Two cases happens:
  (a) If $z = \frac{1}{2k''-1}, k'' \in \mathbb{N}$ then

$$\begin{aligned} &4\left[D\left(x, z\right) + D\left(z, y\right)\right] \\ &\geq 4\left[1 - 2^{-\max\left\{\frac{1}{2k}, \frac{1}{2k''-1}\right\}} + 1 - 2^{-\max\left\{\frac{1}{2k'}, \frac{1}{2k''-1}\right\}}\right] \\ &\geq 4\left(1 - 2^{-1} + 1 - 2^{-1}\right) = D\left(x, y\right). \end{aligned}$$

  (b) If $z = \frac{1}{2k''}, k'' \in \mathbb{N}$ then the b-triangular inequality is obvious.

(ii) If $D(x, y) = 2 \in \text{Image}(D)$ then from (2.5) we have $x = \frac{1}{2k-1}, y = \frac{1}{2k'-1}, k \neq k'$. Two cases happens:
  (a) If $z = \frac{1}{2k''}, k'' \in \mathbb{N}$ then

$$4\left[D\left(x, z\right) + D\left(z, y\right)\right] \geq 4\left(1 - 2^{-1} + 1 - 2^{-1}\right) > 2 = D\left(x, y\right)$$

  (b) If $z = \frac{1}{2k''-1}, k'' \in \mathbb{N}$ then the b-triangular inequality is obvious.

(iii) If $D(x, y) = 1 - 2^{-\frac{1}{n}} \in \text{Image}(D)$ then from (2.5) one of the x or y is odd and the other is even. Thus, for any $z \in X$ one of the $D(x, z)$ or $D(y, z)$ is greater than 2. Therefore the b-triangular inequality holds.

Let $x_n = \frac{1}{n}$, then $D\left(x_n, x_{n+1}\right) = 1 - 2^{-\frac{1}{n}} \to 0$, that is $x_n \in \mathcal{A}$. x_n is not a Cauchy sequence since $D\left(x_{2n}, x_{2n+2}\right) = 4$, for all $n \in \mathbb{N}$. Suppose that x_{r_k}, y_{r_k} are two subsequence of x_n such that $t_k := D\left(x_{r_k}, y_{r_k}\right) \to \varepsilon^+ > 0$. Since the only accumulation point of $\text{Image}\left(D\right)$ is 0, thus there exits $K_0 \in \mathbb{N}$ such that all t_k, $k > K_0$ equal to one of the numbers 2, 4, or $1 - 2^{-\frac{1}{r_0}}, r_0 \in \mathbb{N}$. In case, $t_k = 2$, for all $k > K_0$ then from (2.5), x_{r_k}, y_{r_k} are both odd subsequence. Thus, x_{r_k+1}, y_{r_k+1} are both even subsequence. From (2.5), equality (ii) does not hold. The case, $t_k = 4$, for all $k > K_0$ is similar to previous one. In case, $t_k = 1 - 2^{-\frac{1}{r_0}}$, for all $k > K_0$, where $r_0 \in \mathbb{N}$: From (2.5) we get $D\left(x_{r_k+1}, y_{r_k+1}\right) = 1 - 2^{-\frac{1}{r_0+1}}$, for all $k > K_0$, which shows equality (ii) does not hold.

Despite the preceding example, next theorem shows that a symmetric space (X, s) with triangle function Φ which satisfies

$$\begin{cases} \Phi(0,u) \leq M_1 u, & \text{for all } u \in [0,+\infty), \\ \Phi(u,0) \leq M_2 u, & \text{for all } u \in [0,+\infty). \end{cases} \tag{2.6}$$

where $M_1, M_2 \geq 1$ are constants, has weaker property. Notice that by taking $z = x$ in triangular inequality (ic) we conclude that for any triangle function we get $s(x,y) \leq \Phi(0, s(x,y)) \leq M_1 s(x,y)$ and $s(x,y) \leq \Phi(s(x,y), 0) \leq M_2 s(x,y)$, for $x, y \in X$, thus, we must have $M_1, M_2 \geq 1$.

Theorem 2.4. *Suppose that (X, s) is a symmetric space with a triangle function Φ that is continuous. Given $M_1, M_2 \geq 1$, such that (2.6) holds. If $x_n \in \mathcal{A}$ that is not Cauchy, then $\varepsilon > 0$ and $x_{r_k}, x_{s_k} \ll x_n$ exist such that $s(x_{r_k}, x_{s_k})$ and $s(x_{r_k+1}, x_{s_k+1}) \geq \varepsilon$ are non-increasing sequences and i and ii hold separately. In this case, separately implies that the ε and the subsequences obtained in parts i and ii may differ.*

(i) $\lim_k s(x_{r_k}, x_{s_k}) = \varepsilon^+$ *and*

$$\frac{1}{M_1 M_2}\varepsilon \leq \lim_{k\to+\infty} s(x_{r_k+1}, x_{s_k+1}) \leq M_1 M_2 \varepsilon. \tag{2.7}$$

(ii) $\lim_k s(x_{r_k+1}, x_{s_k+1}) = \varepsilon^+$ *and*

$$\frac{1}{M_1 M_2}\varepsilon \leq \lim_{k\to+\infty} s(x_{r_k}, x_{s_k}) \leq M_1 M_2 \varepsilon. \tag{2.8}$$

Proof. To prove part i, suppose that x_n is not a Cauchy sequence, then there exist $\varepsilon_1 > 0$ and $x_{r_k}, x_{s_k} \ll x_n$ such that $\varepsilon_1 \leq t_k := s(x_{r_k}, x_{s_k}) < +\infty$. Thus, there exist a non-increasing subsequence $\varepsilon_1 \leq t_{k_l}$ of t_k such that $\lim_{l\to+\infty} t_{k_l} = \varepsilon^+ := \liminf_k t_k \geq \varepsilon_1$. For convenience let us show this subsequence again with t_k. We have

$$\begin{aligned} s(x_{r_k+1}, x_{s_k+1}) &\leq \Phi\left(s(x_{r_k+1}, x_{r_k}), s(x_{r_k}, x_{s_k+1})\right) \\ &\leq \Phi\left(s(x_{r_k+1}, x_{r_k}), \Phi\left(s(x_{r_k}, x_{s_k}), s(x_{s_k+1}, x_{s_k})\right)\right), \end{aligned} \tag{2.9}$$

for all $k \in \mathbb{N}$. Take the limit as $k \to +\infty$ from (2.9) and non-increasing condition on t_k and Φ (see (ia) in Definition 1.1) we get

$$\begin{aligned} \limsup_{k\to+\infty} s(x_{r_k+1}, x_{s_k+1}) &\leq \Phi\left(0, \Phi\left(\lim_{k\to+\infty} s(x_{r_k}, x_{s_k}), 0\right)\right) \\ &\leq \Phi(0, \Phi(\varepsilon, 0)) \leq M_1 M_2 \varepsilon. \end{aligned} \tag{2.10}$$

There exists non-increasing subsequence t'_{k_l} of $t'_k := s(x_{r_k+1}, x_{s_k+1})$ such that $\lim_{l\to+\infty} t'_{k_l} = L \leq M_1 M_2 \varepsilon$ exists. Again for convenience let us show

the subsequences t_{k_l} and t'_{k_l} (of t_k and t'_k) with t_k and t'_k, respectively. Thus, we have $\lim_k s\left(x_{r_k}, x_{s_k}\right) = \varepsilon^+$ and

(2.11)
$$\begin{aligned} s\left(x_{r_k}, x_{s_k}\right) &\leq \Phi\left(s\left(x_{r_k+1}, x_{r_k}\right), s\left(x_{s_k}, x_{r_k+1}\right)\right) \\ &\leq \Phi\left(s\left(x_{r_k+1}, x_{r_k}\right), \Phi\left(s\left(x_{r_k+1}, x_{s_k+1}\right), s\left(x_{s_k+1}, x_{s_k}\right)\right)\right), \end{aligned}$$

for all $k \in \mathbb{N}$. Take the limit as $k \to +\infty$ from (2.11) and non-increasing condition on t'_k and Φ (see (ia) in Definition 1.1) we get

$$\begin{aligned} \varepsilon = \lim_{k\to+\infty} s\left(x_{r_k}, x_{s_k}\right) &\leq \Phi\left(0, \Phi\left(\lim_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right), 0\right)\right) \\ &\leq M_1 M_2 \lim_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right). \end{aligned} \tag{2.12}$$

From (2.10) and (2.12) inequalities in (2.7) obtain.

To prove part ii similar proof as part i may be used: Since x_n is not a Cauchy sequence, then there exist $\varepsilon_1 > 0$ and $x_{r_k+1}, x_{s_k+1} \ll x_n$ such that $\varepsilon_1 \leq t'_k := s(x_{r_k+1}, x_{s_k+1}) < +\infty$. Again by passing to subsequences and from (2.9) and (2.11), inequalities in (2.8) hold. □

Corollary 2.5 (see [1, 2])**.** *Suppose that (X, s) is a b-metric space. If $x_n \in \mathcal{A}$ that is not a Cauchy sequence, then $\varepsilon > 0$ and $x_{r_k}, x_{s_k} \ll x_n$ exist such that $s\left(x_{r_k}, x_{s_k}\right)$ and $s\left(x_{r_k+1}, x_{s_k+1}\right) \geq \varepsilon$ are non-increasing sequences and i and ii hold separately. In this case, separately implies that the ε and the subsequences obtained in parts i and ii may differ.*

(i) $\lim_k s\left(x_{r_k}, x_{s_k}\right) = \varepsilon^+$ *and*

$$\frac{1}{b^2}\varepsilon \leq \lim_{k\to+\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) \leq b^2\varepsilon.$$

(ii) $\lim_k s\left(x_{r_k+1}, x_{s_k+1}\right) = \varepsilon^+$ *and*

$$\frac{1}{b^2}\varepsilon \leq \lim_{k\to+\infty} s\left(x_{r_k}, x_{s_k}\right) \leq b^2\varepsilon,$$

Proof. It is a conclusion of Theorem 2.4 for $M_1 = M_2 = b$. □

3. Applications to Fixed Points on Symmetric Spaces via Simulation Functions

The generalized simulation function presented in [25] is examined in this section. Consider the functions $\zeta : [0, +\infty)^2 \to \mathbb{R}$ and $\phi : [0, +\infty) \to \mathbb{R}$. If (3.1) and (3.2) are met, ζ is called a simulation function (of type I).

$$\zeta(t, s) \geq 0 \quad \Rightarrow \quad \zeta(t, s) \leq \phi(s) - \phi(t), \quad \forall s, t > 0, \tag{3.1}$$

and

$$\lim_{n\to\infty} \zeta\left(t_n, s_n\right) \neq 0, \tag{3.2}$$

where t_n, s_n are non-increasing sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$.

Let T be a self-map on X and (X,s) be a symmetric space. $\zeta : [0,+\infty)^2 \to \mathbb{R}$ is a function. A self-map T is referred to as a ζ-contraction [18] if

$$0 \leq \zeta\left(s(Tu,Tv), s(u,v)\right), \quad \forall u,v \in X,$$

T is called a weak ζ-contraction [25] if

$$0 \leq \zeta\left(s\left(Tu,T^2u\right), s\left(u,Tu\right)\right), \quad \forall u \in X.$$

Additionally, if $s(Tu,T^2u) \leq s(u,Tu), \forall u \in X$, then T is called a weak nonexpansive map [25].

The following are a few examples of simulation functions that may be found in [25, Examples 2.2 (i)-(x)].

Example 3.1. (i) The general form of a simulation function can be considered as:

$$\zeta_\chi(t,s) = \phi(s) - \chi(t,s) - \phi(t), \quad \forall s,t \in [0,+\infty).$$

where $\chi : [0,+\infty)^2 \to [0,+\infty), \phi : [0,+\infty) \to \mathbb{R}$ are functions. Obviously, ζ_χ satisfy (3.1). If $\chi(t,s) = \eta(s)$, where $\eta : [0,+\infty) \to [0,+\infty)$ is a function such that $\eta^{-1}(0) = \{0\}$, then ζ_χ change to [27, Example 2.2-(iv)]:

$$\bar{\zeta}_{\text{Weak}}(t,s) = \phi(s) - \eta(s) - \phi(t), \quad \forall s,t \in [0,+\infty). \tag{3.3}$$

Some conditions may be impose on χ and ϕ such that (3.2) holds too: If $\eta(t) \to 0 \Leftrightarrow t \to 0$ and ϕ satisfies

$$\text{if } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \quad \text{then} \quad \lim_{n\to\infty} \phi(t_n) = \lim_{n\to\infty} \phi(s_n), \tag{3.4}$$

for all non-increasing sequences $t_n, s_n \in (0,+\infty)$. Then $\bar{\zeta}_{\text{Weak}}$ satisfies (3.2) since

$$\begin{aligned}\lim_{n\to\infty} \bar{\zeta}_{\text{Weak}}\left(s_n,t_n\right) &= \lim_{n\to\infty} \left(\phi\left(s_n\right) - \eta\left(s_n\right) - \phi\left(t_n\right)\right) \\ &= \lim_{n\to\infty} \left(-\eta(s_n)\right) \neq 0.\end{aligned}$$

Condition (3.4) holds for a continues or non-decreasing function ϕ [27, Remark 2.5].

(ii) $\bar{\zeta}_{\text{Weak}}$ yields $\zeta_{\text{G-BW}}$ and ζ_{BW} when $\eta(t) = \phi(s) - \psi(s)$ and $\eta(t) = s - \psi(s)$, respectively, see [25, Example 2.2-(ix) and -(i)].

In some circumstances on ζ and ϕ, when T is a ζ-contraction, [25, Theorem 2.4-(i)] proves that fixed points are unique. When T is a weak ζ-contraction, [25, Theorem 2.4-(ii)] demonstrates that Picard sequences on metric spaces are in class $\mathcal{A}$. It is easy to observe that these parts in the same way apply similarly to any symmetric space. In [25, Theorem

2.4-(iii)], the fixed point theorem on metric spaces was derived via a simulation function using BWP on metric spaces. By making a little modification, the proof of it may be reconstructed in the next theorem (i) on symmetric spaces with BWP. Also, in a similar manner as [25, Example 2.2-(iv)], part (ii) in the next theorem holds on $\sum$-complete symmetric spaces.

Definition 3.2. We call an SM space (X, s) having the property W3, see [10, Subsec 1.3], if for all sequences x_n and x, y in X the following holds

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} x_n = y \quad \Rightarrow \quad x = y.$$

Lemma 3.3. *Assume that the symmetric space (X, s) has the property W3 and T is a continuous self-map on X then every convergent Picard sequence in X is a fixed point of T.*

Proof. Let $x_0 \in X$ and $\lim_n T^n(x_0) = u$. Since T is continues we get $\lim_n T^{n+1}x_0 = Tu$. Thus, property W3 implies $Tu = u$. □

Theorem 3.4. *Assume that the symmetric space (X, s) has the property W3.*

(i) If (X, s) is complete and has BWP, then T is a WPO if ϕ meets (3.4)*, T is a continuous, weak nonexpansive and ζ-contraction map, where ζ is a simulation function. Furthermore, T is a PO if [25, Theorem 2.4-(i)] is true.*

(ii) If T is a weak nonexpansive self-map on X and weakly $\bar{\zeta}_{Weak}$-contraction, where $\bar{\zeta}_{Weak}$ is given as Equation (3.3) *and $\phi : [0, +\infty) \to \mathbb{R}$ is a lower bounded function such that*

$$\frac{\eta(t)}{t} > 0 \text{ be a non-increasing function on } (0, c),$$

where $0 < c \leq +\infty$. Then every Picard sequence is a $\sum$-Cauchy sequence in X. Also, if (X, s) is $\sum$-complete and T is a continuous, then T is a WPO.

Proof. (i) The proof is exactly similar as [25, Theorem 2.4-(iii)].

(ii) Let $t_i = s(x_{i+1}, x_i) = s\left(T^{i+1}x_0, T^i x_0\right) \in s(\text{graph}(T)) = \{s(x, Tx); x \in X\}$and proceed as the lines of the proof in [25, Theorem 2.4-(iv)]. □

Sec 2 indicates plenty of symmetric space have BWP and Sec 3 indicates some results in the theory of simulation function and its consequence (see [25]) can be obtained for these spaces. For example, we present a few of them as a consequence of Theorem 3.4, which can also be found in the other literature as follows:

Corollary 3.5 (see [25, Corollary 3.1])**.** *Assume (X, s) is a symmetric space, T is a weak nonexpansive self-map on X and $\psi < \phi : [0, +\infty) \to \mathbb{R}$ are functions with lower-bounded ϕ. Assume*

$$\phi\left(s\left(Tu, T^2u\right)\right) \leq \psi\left(s\left(u, Tu\right)\right), \quad \forall u \in X.$$

Then each Picard sequence is a $\sum$-Cauchy. In addition, if (X, s) is $\sum$-complete and T is continuous, then T is a WPO.

Proof. Follows from Theorem 3.4-(ii). □

The following generalizes the main theorem of [28, Theorem 3.6] to strong b-metric space.

Corollary 3.6. *Suppose T is a continuous self-map on X and (X, s) is a complete b-metric space. Suppose*

$$\phi\left(s\left(Tu, Tv\right)\right) \leq \psi\left(s\left(u, v\right)\right), \quad \forall u, v \in X,$$

where ϕ and ψ hold under the circumstance of [25, Example 2.2-(ix)], that is, $\psi < \phi$ and

$$\limsup_{t \to \varepsilon^+} \psi\left(t\right) < \liminf_{t \to \varepsilon^+} \phi\left(t\right), \quad \forall \varepsilon > 0.$$

Then T is a PO.

Proof. Follows from Theorem 3.4-(i) and Corollary 2.5. □

3.1. **Applications to Fixed Point of Interpolative Metric Spaces.** Since the fixed point results in the previous section rely essentially on the BWP, it is natural to investigate whether other important generalized metric structures also enjoy this property. In this subsection, we show that the class of (α, c)-interpolative metric spaces indeed has the BWP; consequently, all the conclusions of Section 3 apply to these spaces.

Definition 3.7 ([5, 17])**.** Let (X, s) be a symmetric space. We say that (X, s) is an (α, c)-interpolative metric space if there exist constants $\alpha \in (0, 1)$ and $c \geq 0$ such that

$$s\left(x, y\right) \leq s\left(x, z\right) + s\left(z, y\right) + c\, s\left(x, z\right)^{\alpha}\, s\left(z, y\right)^{1-\alpha}$$

for all $x, y, z \in X$.

Theorem 3.8. *Every (α, c)-interpolative metric space (X, s) has BWP.*

Proof. Let $x_n \in \mathcal{A}$ that is not Cauchy. By the selection argument (as in Theorem 2.2), there exist subsequences x_{r_k}, x_{s_k} and a constant $\varepsilon > 0$ such that

$$a_k := s\left(x_{r_k}, x_{s_k}\right) \geq \varepsilon, \quad a_k \text{ is non-increasing}, \quad \lim_{k \to \infty} a_k = \varepsilon^+.$$

Put $b_k := s(x_{r_k+1}, x_{s_k+1})$, $u_k := s(x_{r_k}, x_{r_k+1})$, $v_k := s(x_{s_k}, x_{s_k+1})$. Since x_n is AR, $u_k, v_k \to 0$.

We shall prove that $\lim_{k\to\infty} b_k = \varepsilon$.

To estimate a_k in terms of b_k, we apply the interpolative triangle inequality twice: first to the three points $x_{r_k}, x_{r_k+1}, x_{s_k}$ to relate $s(x_{r_k}, x_{s_k})$ to $s(x_{r_k+1}, x_{s_k})$ and then to $x_{r_k+1}, x_{s_k+1}, x_{s_k}$ to relate $s(x_{r_k+1}, x_{s_k})$ to b_k. This yields

$$a_k \leq b_k + \delta_k, \tag{3.5}$$

where

$$\delta_k := u_k + v_k + c\, b_k^{\alpha} v_k^{1-\alpha} + c\, u_k^{\alpha} \left(b_k + v_k + c\, b_k^{\alpha} v_k^{1-\alpha}\right)^{1-\alpha}.$$

Similarly, applying the inequality to the pairs of consecutive points $x_{r_k+1}, x_{r_k}, x_{s_k+1}$ and then $x_{r_k}, x_{s_k}, x_{s_k+1}$, we get the reverse estimate

$$b_k \leq a_k + \gamma_k, \tag{3.6}$$

where

$$\gamma_k := u_k + v_k + c\, a_k^{\alpha} v_k^{1-\alpha} + c\, u_k^{\alpha} \left(a_k + v_k + c\, a_k^{\alpha} v_k^{1-\alpha}\right)^{1-\alpha}.$$

Since a_k is bounded (it converges to ε), the terms a_k and b_k are uniformly bounded. Therefore, all terms involving u_k or v_k in δ_k and γ_k tend to 0 (they are products of bounded factors with factors that tend to 0). Hence,

$$\lim_{k\to\infty} \delta_k = 0, \qquad \lim_{k\to\infty} \gamma_k = 0.$$

Taking the lim sup in (3.5) and (3.6) gives

$$\limsup_{k\to\infty} a_k \leq \limsup_{k\to\infty} b_k, \qquad \limsup_{k\to\infty} b_k \leq \limsup_{k\to\infty} a_k.$$

Thus $\limsup a_k = \limsup b_k$. But $a_k \to \varepsilon$, so b_k also converges to ε. Consequently, (ii) holds. This proves that (X, s) satisfies the BWP. □

Corollary 3.9. *All the conclusions of Section 3 hold for a (α, c)-interpolative metric space.*

Remark 3.10.

(i) The key observation of the preceding theorem is that the interpolative inequality is strong enough to ensure that the error terms vanish as $k \to +\infty$, exactly as the metric triangle inequality does.

(ii) It is important to note that not every b-metric space is regular. Example 2.3 provides a concrete b-metric space (with $b = 4$) that fails to have BWP and hence it cannot be regular. Therefore, when applying Theorem 3.4-(i) (which requires BWP), one must either verify BWP directly or rely on a sufficient condition such as regularity. Strong b-metric spaces (where the b-triangle

inequality holds with the same constant and additional compatibility conditions) are regular and thus possess BWP.

(iii) In Theorem 3.8, we prove that every (α, c)-interpolative metric space satisfies BWP. Since interpolative metric spaces form a proper subclass of b-metric spaces and are not necessarily regular, this shows that BWP can hold without the assumption of regularity. Thus, regularity is sufficient but not necessary for BWP.

(iv) The author has no idea that how properties i or ii in Theorem 2.4 (weak-BMP) can be used for fixed point results on symmetric spaces (like as Definitions 2.1 or 4.1), but in Section 4 we will see that with appropriate modifications of these properties, important theorems in the fixed point theory of symmetric spaces can be derived.

4. A New Approach to the Boyd–Wong Idea

The simulation function introduced in Example 3.1-(ii) generates the Boyd–Wong contraction. In this setting, one may consider concepts weaker from that given in Definition 2.1, which consequently leads to weaker results in the literature (see Subsection 4.4).

4.1. **Bounded Shift Property.**

Definition 4.1. Let (X, s) be a symmetric space. We say that (X, s) has the bounded shift property (briefly, BSP) if, for every $x_n \in \mathcal{A}$ that is not Cauchy, there exist constants $K \in \mathbb{N}$, $\epsilon > 0$, $\varepsilon > 0$ (depending only on the sequence x_n) and subsequences $x_{r_k}, x_{s_k} \ll x_n$ such that, for all $k \geq K$ and for some $i_k \geq k$,

$$\epsilon \leq s\left(x_{r_k+i_k}, x_{s_k+i_k}\right) \tag{4.1}$$

and

$$s\left(x_{r_k}, x_{s_k}\right) \leq \varepsilon. \tag{4.2}$$

Any bounded symmetric space has BSP, where $\varepsilon := \operatorname{diam}(X) = \sup\{s(x, y) : x, y \in X\} < +\infty$. Every b-metric space has BSP (by Theorem 4.3, since it is a Φ-symmetric space). The following example shows that there is a semi-metric space that does not have BSP.

Example 4.2 (see [12, Example 2] and [4, Corollary 2])**.** Let $X = \mathbb{N}$ and $s\left(n, m\right) = \frac{|n-m|}{2^{\min\{n,m\}}}$, for all $m, n \in X$. Then (X, s) is a semi-metric space. The sequence $x_n = n$ is AR since $s\left(n, n+1\right) = \frac{1}{2^n} \to 0$. Also, x_n is not Cauchy since $\lim_m s\left(n, m\right) = +\infty$ for any fixed $n \in \mathbb{N}$. For every subsequences x_{r_k}, x_{s_k} of x_n, we have $r_k \to \infty$ and $s_k \to \infty$. Thus, for

all $M > 2$, there exists $K > 0$ such that $r_k, s_k \geq M$ for all $k \geq K$. Since $\frac{x}{2^x}$ is decreasing on $(2, +\infty)$, we get

$$s\left(x_{r_k}, x_{s_k}\right) = \frac{|r_k - s_k|}{\min\left\{2^{s_k}, 2^{r_k}\right\}} \leq \max\left\{\frac{r_k}{2^{r_k}}, \frac{s_k}{2^{s_k}}\right\} \leq \frac{M}{2^M}.$$

This proves $\lim_{k\to\infty} s\left(x_{r_k}, x_{s_k}\right) = 0$. Thus (X, s) does not have BSP. Also, this example does not have the JMS property defined below; see [4, Corollary 2].

Theorem 4.3. *Every Φ-symmetric space with a triangle function Φ continuous at $(0, 0)$ has BSP.*

Proof. Let $x_n \in \mathcal{A}$ be a sequence that is not Cauchy. Then there exists $\varepsilon_1 > 0$ such that for every $N \in \mathbb{N}$ there exist indices $m \geq n \geq N$ satisfying

$$s\left(x_m, x_n\right) \geq \varepsilon_1.$$

Since x_n is asymptotically regular, for each $\varepsilon_2 > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$s(x_{n+1}, x_n) < \varepsilon_2, \quad \text{for all } n > N_0. \tag{4.3}$$

Step 1: For each $n > N_0$, define

$$S_n\left(N_0\right) := \left\{m \geq n : s\left(x_{n+1}, x_{m+1}\right) \geq \varepsilon_1\right\}.$$

We claim that $S_n(N_0) \neq \emptyset$ for every $n > N_0$. Indeed, if $S_n(N_0) = \emptyset$ for some $n > N_0$, then

$$s\left(x_{n+1}, x_{m+1}\right) < \varepsilon_1, \quad \text{for all } m \geq n.$$

This contradicts the assumption that x_n is not Cauchy, since there exist arbitrarily large indices m such that $s\left(x_{m+1}, x_{n+1}\right) \geq \varepsilon_1$. Hence, $S_n(N_0) \neq \emptyset$ for every $n > N_0$.

Now define two sequences r_k and s_k by

$$r_k := k, \quad s_k := \min S_k\left(N_0\right), \quad k > N_0.$$

Clearly, $s_k \geq k$ and $s\left(x_{r_k+1}, x_{s_k+1}\right) \geq \varepsilon_1$, for all $k > N_0$. Moreover, by the minimality of s_k, we have

$$s_k - 1 \notin S_{r_k}\left(N_0\right) \quad \Rightarrow \quad s\left(x_{r_k+1}, x_{s_k}\right) < \varepsilon_1, \tag{4.4}$$

for all $k > N_0$.

Step 2: We now estimate an upper bound for $s(x_{r_k}, x_{s_k})$. Fix $\varepsilon_2 > 0$. By (4.3), (4.4), the Φ-triangle inequality (ic) and the monotonicity of Φ, we obtain

$$s\left(x_{r_k}, x_{s_k}\right) \leq \Phi\left(s\left(x_{r_k}, x_{r_k+1}\right), s\left(x_{r_k+1}, x_{s_k}\right)\right) \leq \varepsilon,$$

for every $k > N_0$, where $\varepsilon := \Phi\left(\varepsilon_2, \varepsilon_1\right)$ in the Definition 4.1.

Step 3: We now estimate a lower bound for a subsequence of

$s\left(x_{r_k+k}, x_{s_k+k}\right)$. Suppose, to the contrary, that $\lim_{k\to\infty} s\left(x_{r_k+k}, x_{s_k+k}\right) = 0$. Then, by the Φ-triangle inequality, we have

$$\begin{aligned} s\left(x_{r_k+k-1}, x_{s_k+k-1}\right) &\leq \Phi\left(s\left(x_{r_k+k}, x_{r_k+k-1}\right), s\left(x_{s_k+k-1}, x_{r_k+k}\right)\right) \\ &\leq \Phi\left(s\left(x_{r_k+k}, x_{r_k+k-1}\right),\right. \\ &\quad \left.\Phi\left(s\left(x_{s_k+k-1}, x_{s_k+k}\right), s\left(x_{s_k+k}, x_{r_k+k}\right)\right)\right). \end{aligned}$$

Using the asymptotic regularity of x_n and continuity of Φ at $(0,0)$, it follows that $\lim_{k\to\infty} s\left(x_{r_k+k-1}, x_{s_k+k-1}\right) = 0$. Repeating this argument yields $\lim_{k\to\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) = 0$, which contradicts the construction of the sequences r_k and s_k.

Therefore, $\limsup_{k\to\infty} s\left(x_{r_k+k}, x_{s_k+k}\right) > 0$. Hence, there exist $0 < \epsilon$, $K \in \mathbb{N}$ and a subsequence $s\left(x_{r_{k_i}+k_i}, x_{s_{k_i}+k_i}\right)$ such that

$$s\left(x_{r_{k_i}+k_i}, x_{s_{k_i}+k_i}\right) \geq \epsilon, \quad \text{for all } i \geq K.$$

where $k_i \geq i$. Thus, the sequences $s\left(x_{r_{k_i}}, x_{s_{k_i}}\right)$ and $s\left(x_{r_{k_i}+k_i}, x_{s_{k_i}+k_i}\right)$ satisfy (4.1) and (4.2). Therefore, (X, s) has the BSP. □

Theorem 4.4. *Every γ-symmetric space has BSP.*

Proof. Proceed as in the proof of Theorem 4.3.

Step 2. Without loss of generality, we may replace ε_1 by $\min\left\{\varepsilon_1, \gamma\left(\varepsilon_1\right)\right\}$. Set $\varepsilon_2 := \gamma\left(\varepsilon_1\right)$. Then, by (4.3), (4.4) and the γ-triangle inequality (1.1), we obtain

$$s\left(x_{r_k}, x_{r_k+1}\right) \leq \varepsilon_1 \leq \gamma\left(\varepsilon_1\right), \qquad s\left(x_{r_k+1}, x_{s_k}\right) \leq \gamma\left(\varepsilon_1\right),$$

which implies that $s\left(x_{r_k}, x_{s_k}\right) \leq \varepsilon_1$. Hence, $s\left(x_{r_k}, x_{s_k}\right) \leq \varepsilon$, for every $k > N_0$, where $\varepsilon := \varepsilon_1$ in the Definition 4.1.

Step 3. Suppose, to the contrary, that $\lim_{k\to\infty} s(x_{r_k+k}, x_{s_k+k}) = 0$.

Let $\varepsilon > 0$ be arbitrary and put $l := \gamma(\varepsilon)$. For all sufficiently large k, asymptotic regularity and the above limit imply that

$$s\left(x_{r_k+k}, x_{s_k+k}\right) < \gamma(l), \tag{4.5}$$

$$s\left(x_{r_k+k}, x_{r_k+k-1}\right) < \gamma(l), \tag{4.6}$$

$$s\left(x_{s_k+k-1}, x_{s_k+k}\right) < l. \tag{4.7}$$

Applying the γ-triangle inequality (1.1) to (4.5) and (4.6), we obtain $s\left(x_{r_k+k-1}, x_{s_k+k}\right) < l$. Combining this inequality with (4.7) and applying (1.1) once more yields $s\left(x_{r_k+k-1}, x_{s_k+k-1}\right) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{k\to\infty} s\left(x_{r_k+k-1}, x_{s_k+k-1}\right) = 0$. Repeating the above argument inductively, we obtain $\lim_{k\to\infty} s\left(x_{r_k+m}, x_{s_k+m}\right) = 0$ for every fixed integer $m \geq 0$. In particular, $\lim_{k\to\infty} s\left(x_{r_k+1}, x_{s_k+1}\right) = 0$,

which contradicts the construction of the sequences r_k and s_k. The proof can then be continued as in Theorem 4.3. □

4.2. **Uniform Bounded Shift Property.** The theorem by Boyd and Wong in semi-metric spaces was developed by Jachymski et al. [12] based on the definition below and later, Alshehri et al. [3] and Arandelović and Kečkić [4] extended this theorem to symmetric spaces, as discussed in Subsec 4.4 and [10, Subsec 1.5].

Definition 4.5 ([12])**.** Assume (X, s) is a symmetric space. We say that (X, s) is a JMS-space with parameter (δ, η) if there exist $\delta, \eta > 0$ such that

$$s(x, z) + s(z, y) < \delta \quad \Rightarrow \quad s(x, y) < \eta, \tag{4.8}$$

for all $x, y, z \in X$.

Remark 4.6. Every b-metric has the JMS property with parameter $\delta = 1, \eta = b$ since if $s(x, z) + s(z, y) < 1$, then $s(x, y) < b$ for all $x, y, z \in X$. Since $\delta = \frac{1}{2}$ and $\eta = \frac{b}{2}$ also fulfill (4.8), the parameter η is not unique.

Definition 4.7. Let $\mathcal{D}$ be the class of all AR sequences x_n such that there exist constants $\delta, \eta > 0$ satisfying the following shift conditions:

$$\text{if } s\left(x_{m+n}, x_n\right) \leq \eta, \quad \text{then } s\left(x_{m+n+1}, x_{n+1}\right) \leq \delta, \tag{4.9}$$

$$\text{if } s\left(x_m, x_n\right) \leq \eta, \quad \text{then } s\left(x_{m+1}, x_{n+1}\right) \leq \eta, \tag{4.10}$$

for all $m, n \in \mathbb{N}$.

We say that (X, s) has the uniform bounded shift property with respect to $\mathcal{D}$ (briefly, $\text{UBSP}_{\mathcal{D}}$) with parameters (δ, η) if, for every sequence $x_n \in \mathcal{D}$ that is not Cauchy, there exist $K \in \mathbb{N}$ and subsequences $x_{r_k}, x_{s_k} \ll x_n$ such that for all $k \geq K$ and some $i_k \geq k$, the relations (4.1) and (4.2) hold, where $\epsilon > 0$ may depend on the sequence, while the constants δ and η (which define the class $\mathcal{D}$) are fixed independently of the sequence.

Any bounded symmetric space has $\text{UBSP}_{\mathcal{D}}$ with the parameters $\delta = \eta = \text{diam}(X) = \sup\{s(x, y), x, y \in X\} < +\infty$, since the shift conditions (4.9) and (4.10) are automatically satisfied and the existence of the required subsequences follows directly from the definition of non-Cauchyness. Also, every b-metric space has BSP according to Corollary 2.5; moreover, every b-metric space has $\text{UBSP}_{\mathcal{D}}$ according to [4, Proposition 5] and Theorem 4.8.

Theorem 4.8. *Every JMS-space (X, s) with parameters (δ, η) has the $\text{UBSP}_{\mathcal{D}}$ with parameter $(\delta/2, \eta)$.*

Proof. Let $x_n \in \mathcal{D}$ be a sequence that is not Cauchy.

Step 1 : Since x_n is not Cauchy, there exists $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there exist $m > n \geq N$ satisfying $s(x_n, x_m) \geq \varepsilon_0$.

Choose $N_0 \geq N$ such that

$$s\left(x_n, x_{n+1}\right) < \varepsilon := \min\left\{\varepsilon_0, \frac{\delta}{2}, \eta\right\}, \quad n \geq N_0. \tag{4.11}$$

For every $k > N_0$, let $i_k := k$, and define j_k to be the smallest integer greater than i_k satisfying $s\left(x_{i_k}, x_{j_k}\right) > \varepsilon$. Such an index exists because x_n is not Cauchy.

By the minimality of j_k, $s\left(x_{i_k}, x_{j_k-1}\right) \leq \varepsilon$. From (4.11) we get $s\left(x_{j_k-1}, x_{j_k}\right) < \varepsilon$.

Hence

$$s\left(x_{i_k}, x_{j_k-1}\right) + s\left(x_{j_k-1}, x_{j_k}\right) < 2\varepsilon \leq \delta.$$

Applying the JMS inequality we obtain

$$s\left(x_{i_k}, x_{j_k}\right) \leq \eta, \quad k > N_0. \tag{4.12}$$

Step 2 : Next we prove that

$$s\left(x_{i_k}, x_{i_k+n}\right) \leq \eta, \quad k \geq N_0, n \in \mathbb{N}. \tag{4.13}$$

Fix $k \geq K_0$. From (4.11) the assertion follows directly from the choice of k that (4.13) holds for $n = 1$.

Assume that $s\left(x_{i_k}, x_{i_k+n}\right) \leq \eta$, for some $n \geq 1$. From (4.9) we obtain $s\left(x_{i_k+1}, x_{i_k+n+1}\right) \leq \frac{\delta}{2}$. From (4.11) we have $s\left(x_{i_k}, x_{i_k+1}\right) < \frac{\delta}{2}$, thus, $s\left(x_{i_k}, x_{i_k+1}\right) + s\left(x_{i_k+1}, x_{i_k+n+1}\right) < \delta$. The JMS inequality therefore yields $s\left(x_{i_k}, x_{i_k+n+1}\right) < \eta$. Hence (4.13) follows by induction. Moreover, by (4.10), $s(x_{i_k}, x_{i_k+n}) \leq \eta$ yields $s(x_{i_k+1}, x_{i_k+n+1}) \leq \eta$. Repeating this argument inductively, we obtain

$$s(x_{i_k+m}, x_{i_k+n+m}) \leq \eta, \quad \text{for every } m, n \geq N_0.$$

Consequently, $s(x_{i_k+i}, x_{i_k+j}) \leq \eta$ for all $i, j \in \mathbb{N}$ and $k > N_0$. Since x_n is not Cauchy, for each $k > N_0$ there exist $\varepsilon_1 > 0$ and indices $r_k, s_k \geq k$ such that $s\left(x_{i_k+r_k}, x_{i_k+s_k}\right) \geq \varepsilon_1$. Thus, from this together with (4.12), (X, s) satisfies the $\mathrm{UBSP}_{\mathcal{D}}$ property with parameters $\delta/2$ and η. □

The shifted (η,δ)-conditions (4.9)-(4.10) are precisely what allows the shift-monotonicity chain in the induction step. This condition replaces the need for a triangle inequality and is the key to proving the $\mathrm{UBSP}_{\mathcal{D}}$ for JMS spaces.

Remark 4.9. Definitions 4.5 and 4.7 serve different purposes in our development. Definition 4.5 introduces the JMS property as a condition on the space itself, requiring that whenever the sum of two distances is

below δ, the direct distance is bounded by η, uniformly for all points in X.

Definition 4.7, on the other hand, introduces a class $\mathcal{D}$ of sequences satisfying two shift conditions (equations (4.9) and (4.10)) and then defines the property $\mathrm{UBSP}_{\mathcal{D}}$ with respect to this class. This definition is tailored to capture the shift-behaviour of sequences (such as Picard sequences, see Theorem 4.13) within a JMS-space. The class $\mathcal{D}$ serves as a bridge between the structural JMS condition and the sequence-based bounded shift property needed for fixed point arguments.

4.3. **Fixed Point Consequences.** Two distinct fixed-point theorems are obtained from the preceding theorems, which need the following definition.

Definition 4.10. Suppose $\varepsilon > 0$. The set of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi^{-1}(0) = \{0\}$ and

$$\lim_{n\to\infty} \psi^n(t) = 0, \quad \forall t \leq \varepsilon, \tag{4.14}$$

is denoted by $\Psi(\varepsilon)$; also put $\Psi = \bigcap_{\varepsilon>0} \Psi(\varepsilon)$. If $\psi \in \Psi(\varepsilon)$ for some $\varepsilon > 0$ and T is a self-map on a symmetric space (X, s) satisfying

$$s(Tu, Tv) \leq \psi(s(u, v)), \quad \forall u, v \in X, \tag{4.15}$$

with restriction $s(u, v) \leq \varepsilon$, then T is called a ψ-ε-contraction. If $\psi \in \Psi$ and (4.15) holds without restriction, T is called a ψ-contraction.

The following lemma indicates that every ψ-ε-contraction and ψ-contraction is continuous.

Lemma 4.11 (see [4, Lemma 1], [23])**.** *(i) If $\psi \in \Psi(\varepsilon)$ for some $\varepsilon > 0$, then $t > \psi(t)$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $0 < t \leq \varepsilon$.*

(ii) Let (X, s) be a symmetric space and T be a ψ-ε-contraction ($\varepsilon > 0$) or a ψ-contraction. Then T is continuous.

Proof. (i) Assume that for some $0 < t \leq \varepsilon$, $\psi(t) \geq t$. Since ψ is nondecreasing, using induction, we find $\psi^n(t) \geq t > 0$ for all $n \in \mathbb{N}$, which is in contradiction to the requirement (4.14). Hence, $\lim_{n\to+\infty} \psi^n(t) \leq \lim_{n\to+\infty} \psi^n(\varepsilon) = 0$ for all $0 < t \leq \varepsilon$

(ii) If $s(x_n, x) \to 0$, then $\exists N \in \mathbb{N}$ such that $s(x_n, x) < \varepsilon, \forall n \geq N$. Thus, $s(Tx_n, Tx) \leq \psi(s(x_n, x)), \forall n \geq N$, which shows that T is a continuous self-map. □

Theorem 4.12. *Suppose (X, s) is a complete symmetric space that satisfies W3 and BSP. Assume that T is a ψ-contraction, where $\psi \in \Psi$. Then T is a PO.*

Proof. Consider the Picard sequence $x_n = T^n(x_0)$, $x_0 \in X$. From [25, part (ii)], x_n is an AR sequence. Suppose x_n is not Cauchy. By BSP, there exist $K \in \mathbb{N}$, constants $\epsilon, \varepsilon > 0$ (depending on x_n) and subsequences $x_{r_k}, x_{s_k} \ll x_n$ such that for all $k \geq K$ and some $i_k \geq k$:

$$\epsilon \leq s\left(x_{r_k+i_k}, x_{s_k+i_k}\right) \quad \text{and} \quad s\left(x_{r_k}, x_{s_k}\right) \leq \varepsilon.$$

Now apply the ψ-contraction property (4.15) repeatedly from index i_k down to 0:

$$\begin{aligned} \epsilon &\leq s\left(x_{r_k+i_k}, x_{s_k+i_k}\right) \leq \psi\left(s\left(x_{r_k+i_k-1}, x_{s_k+i_k-1}\right)\right) \\ &\leq \cdots \leq \psi^{i_k}\left(s\left(x_{r_k}, x_{s_k}\right)\right) \leq \psi^{i_k}(\varepsilon) \leq \psi^k(\varepsilon), \end{aligned} \tag{4.16}$$

because $i_k \geq k$ and ψ is nondecreasing, where $\psi(t) < t$ for $0 < t$.

Taking the limit as $k \to \infty$, since $\psi \in \Psi$, we have $\psi^k(\varepsilon) \to 0$. Thus we obtain

$$\epsilon \leq \liminf_{k \to +\infty} \psi^k(\varepsilon) = 0,$$

a contradiction. Therefore x_n is Cauchy.

Completeness of (X, s) gives $x_n = T^n(x_0) \to u$ for some $u \in X$. By Lemma 4.11, T is continuous, so $u = \lim T^{n+1}(x_0) = Tu$. Uniqueness follows from $\psi(t) < t$ for $t > 0$: if u, v are fixed points, then

$$s(u, v) = s(Tu, Tv) \leq \psi(s(u, v)),$$

which forces $s(u, v) = 0$. Hence $u = v$. Thus T is a PO. □

Theorem 4.13. *Suppose (X, s) is a complete symmetric space satisfying W3 and* $\mathrm{UBSP}_{\mathcal{D}}$ *with parameters (δ, η). Assume that s satisfies*

$$\begin{aligned} &\text{if } s\left(x_n, x_0\right) \to 0 \text{ and } s\left(y_n, x_n\right) \to 0 \text{ then} \\ &\exists N_0; \forall n > N_0,\ s\left(y_n, x_0\right) \leq \eta. \end{aligned} \tag{4.17}$$

$$\begin{aligned} &\text{if } s\left(y_n, y\right) \to 0 \text{ then} \\ &\forall x \in X,\ s\left(y, x\right) \leq \limsup_{n \to +\infty} s\left(y_n, x\right). \end{aligned} \tag{4.18}$$

Suppose T is a ψ-η-contraction, where $\psi \in \Psi(\eta)$. Choose $\ell \in \mathbb{N}$ such that $\psi^\ell(\eta) \leq \delta$, then for every $x_0 \in X$ with $s\left(x_0, Tx_0\right), s\left(x_0, T^\ell x_0\right) \leq \eta$, the Picard sequence $x_n = T^n(x_0)$ converges to a fixed point of T.

Proof. Let $x_n = T^n(x_0)$. Since $s(x_0, x_1) \leq \eta$ and T is a ψ-η-contraction,

$$s\left(x_n, x_{n+1}\right) \leq \psi^n\left(s\left(x_0, T\left(x_0\right)\right)\right) \leq \psi^n(\eta) \to 0,$$

so $x_n \in \mathcal{A}$.

Since $\psi \in \Psi(\eta)$, there exists $\ell \in \mathbb{N}$ such that $\psi^\ell(\eta) \leq \delta$. Set $y_n := x_{n\ell}$.

$$s\left(y_n, y_{n+1}\right) = s\left(x_{n\ell}, x_{(n+1)\ell}\right) \leq \psi^{n\ell}\left(s\left(x_0, T^\ell\left(x_0\right)\right)\right) \leq \psi^{n\ell}(\eta) \to 0.$$

Thus, $y_n \in \mathcal{A}$. We show $y_n \in \mathcal{D}$. If $s(y_m, y_n) \leq \eta$, then

$$s(y_{m+1}, y_{n+1}) = s\left(T^\ell y_m, T^\ell y_n\right) \leq \psi^\ell(s(y_m, y_n)) < s(y_m, y_n) \leq \eta.$$

If $s(y_{m+n}, y_n) \leq \eta$, then

$$s(y_{m+n+1}, y_{n+1}) \leq \psi^\ell(s(y_{m+n}, y_n)) \leq \psi^\ell(\eta) \leq \delta.$$

Thus $y_n \in \mathcal{D}$.

Assume y_n is not Cauchy. By $\mathrm{UBSP}_{\mathcal{D}}$, there exist subsequences y_{r_k}, y_{s_k} and $i_k \geq k$ such that

$$\epsilon \leq s(y_{r_k+i_k}, y_{s_k+i_k}) \quad \text{and} \quad s(y_{r_k}, y_{s_k}) \leq \eta.$$

Applying the contraction i_k times, similar to inequality (4.16) a contradiction obtains. Hence y_n is Cauchy. Completeness gives $y_n \to u \in X$.

Since $y_{n+1} = T^\ell(y_n)$ and T^ℓ is continuous,

$$T^\ell(u) = \lim_{n\to\infty} T^\ell(y_n) = \lim_{n\to\infty} y_{n+1} = u.$$

Now $s(y_n, T(y_n)) = s(x_{n\ell}, x_{n\ell+1}) \leq \psi^{n\ell}(\eta) \to 0$. Moreover, we have $s(T(y_n), T(u)) \to 0$ by continuity of T. Using (4.17) and (4.18),

$$s(u, T(u)) \leq \limsup_{n\to\infty} s(y_n, T(u)) \leq \eta.$$

Thus $T(u) \in B_\eta(u) := \{v \in X : s(u, v) \leq \eta\}$.

Since $T^\ell(u) = u$ and $T^\ell(T(u)) = T(u)$ (because, $T^\ell(T(u)) = T^{\ell+1}(u) = T(u)$), both u and $T(u)$ are fixed points of T^ℓ in $B_\eta(u)$. But T^ℓ is a ψ_ℓ-contraction on $B_\eta(u)$, so it has a unique fixed point in $B_\eta(u)$. Hence $T(u) = u$.

For each $r = 0, 1, \ldots, \ell - 1$, by continuity of T^r,

$$\lim_{q\to\infty} x_{q\ell+r} = \lim_{q\to\infty} T^r(y_q) = T^r(u) = u.$$

Thus all subsequences $x_{q\ell+r}$ converge to the same limit u, so $x_n \to u$. □

Remark 4.14. Condition (4.18) is weaker than lower semicontinuity of $x \mapsto s(x, y)$; indeed, full l.s.c. would require the lim inf, while (4.18) only uses the lim sup. This mild assumption is needed only for local contractions ($\psi \in \Psi(\eta)$) to ensure $s(u, T(u)) \leq \eta$ and apply local uniqueness. If T is a global ψ-contraction ($\psi \in \Psi$), then (4.17) and (4.18) are superfluous and can be omitted.

Theorem 4.15. *The results of Theorems 4.12 and 4.13 hold true if T is continuous and the ψ-contraction condition* (4.15) *is replaced by*

$$s(Tu, Tv) \leq \psi(s_3(u, v)), \quad \forall u, v \in X, \tag{4.19}$$

where

$$s_3(u, v) = \max\{s(u, v), s(u, Tu), s(v, Tv)\}. \tag{4.20}$$

Proof. Theorems 4.12 and 4.13 could be applied to (X, s^*) from points i-iii, where $s^* : X \times X \to [0, \infty)$ is as follows: For $x = y$, $s^*(x, y) = 0$; otherwise, $s^*(x, y) = s_3(x, y)$.

(i) The complete symmetric space (X, s^*) can be considered similarly to [3, Proof of Theorem 16], where condition (4.19) is substituted with

$$s_3(Tu, Tv) \leq \psi(s_3(u, v)), \quad \forall u, v \in X. \tag{4.21}$$

Observe that (4.19) leads to (4.21).

(ii) Suppose (X, s) has UBSP$_{\mathcal{D}}$ with parameters (δ, η). Let x_n be any sequence which belongs to $\mathcal{D}$ with respect to (X, s) and is not Cauchy in (X, s^*). From (4.20) we have $s_3(x_n, x_{n+1}) = \max\{s(x_n, x_{n+1}), s(x_{n+1}, x_{n+2})\}$, hence $s(x_n, x_{n+1}) \to 0$.
Now, if x_n were Cauchy in (X, s), then for every $\varepsilon > 0$ there exists N such that $s(x_n, x_m) < \varepsilon$ and $s(x_n, x_{n+1}) < \varepsilon$ for all $m, n \geq N$. From (4.20), this would imply $s_3(x_n, x_m) < \varepsilon$ for all $m, n \geq N$, contradicting the fact that x_n is not Cauchy in (X, s^*). Therefore, x_n is not Cauchy in (X, s).

Since $x_n \in \mathcal{D}$ in (X, s) and is not Cauchy, by UBSP$_{\mathcal{D}}$ (Definition 4.7) applied to (X, s), there exist $K' \in \mathbb{N}$, subsequences $x_{r_k}, x_{s_k} \ll x_n$ and $i_k \geq k$ such that

$$\epsilon \leq s(x_{r_k+i_k}, x_{s_k+i_k}) \quad \text{and} \quad s(x_{r_k}, x_{s_k}) \leq \eta,$$

for all $k \geq K'$.

Since $s \leq s^*$, the lower bound transfers:

$$\epsilon \leq s(x_{r_k+i_k}, x_{s_k+i_k}) \leq s^*(x_{r_k+i_k}, x_{s_k+i_k}).$$

For the upper bound, note that $s(x_{r_k}, x_{r_k+1}) \to 0$ and $s(x_{s_k}, x_{s_k+1}) \to 0$. Hence, for sufficiently large k,

$$s(x_{r_k}, x_{r_k+1}) \leq 1, \quad s(x_{s_k}, x_{s_k+1}) \leq 1.$$

Define $\eta' := \max\{\eta, 1\}$ (or simply $\eta + 1$). Then

$$s^*(x_{r_k}, x_{s_k}) = \max\{s(x_{r_k}, x_{s_k}), s(x_{r_k}, x_{r_k+1}), s(x_{s_k}, x_{s_k+1})\} \leq \eta',$$

for all sufficiently large k. Thus the upper bound is uniform with parameter η'. Therefore, for every sequence $x_n \in \mathcal{D}$ with respect to (X, s) which is not Cauchy in (X, s^*), the required bounded shift estimates hold in (X, s^*) with the uniform parameter η'.

(iii) Similar to the previous part, one can prove that BSP on (X, s) results in (X, s^*). □

As demonstrated in [3, Example 18], the continuity of T in Theorem 4.15 is essential and cannot be ignored.

4.4. **Some Applications.** Some previous main theorems in the literature can be concluded and improved (or unified) from Sec 4:

Corollary 4.16 (see [20, Theorem 13])**.** *If (X,s) is a complete Φ-symmetric or γ-symmetric space, $\psi\in\Psi$ and T is a ψ-contraction satisfying* (4.15) *(or* (4.19) *and T continuous), then T is a PO.*

Proof. Follows from Theorems 4.3, 4.4, 4.12 and 4.15. □

Corollary 4.17. *If (X,s) is a complete b-metric space, $\psi\in\Psi$ and T is a ψ-contraction satisfying* (4.15) *(or* (4.19) *and T continuous), then T is a PO.*

Proof. Follows from Theorems 4.3, 4.4, 4.12 and 4.15. □

The following generalizes the works of Jachymski et al. [12, Theorem 1], Arandelović and Kečkić [4, Theorem 3] and Alshehri et al. [3, Theorem 16] on Ψ. Since $\Psi\subsetneq\Psi(\eta)$, this corollary extends those theorems to $\Psi(\eta)$.

Corollary 4.18. *Let (X,s) be a complete symmetric JMS-space with parameters (δ,η) satisfying W3 and let $T:X\to X$ be a self-mapping for which there exists $\psi\in\Psi(\eta)$ satisfying* (4.15) *(or* (4.19) *and T continuous). Assume additionally that s satisfies* (4.17) *(this condition is superfluous when $\psi\in\Psi$, i.e., when the contraction is global). Choose $\ell\in\mathbb{N}$ such that $\psi^{\ell}(\eta)\leq\delta/2$, then for every $x_0\in X$ with $s(x_0,Tx_0),s\left(x_0,T^{\ell}x_0\right)\leq\eta$, the Picard sequence $x_n=T^n(x_0)$ converges to a fixed point of T.*

Proof. By Theorem 4.8, (X,s) has $\mathrm{UBSPD}_{\mathcal{D}}$ with parameters $(\delta/2,\eta)$. Apply Theorem 4.13 with $(\delta/2,\eta)$ in place of its abstract (δ,η); the remaining hypotheses follow from Theorems 4.15 and Remark 4.14. □

The subsequent example illustrates how the previous corollary differs from and is a proper extension of Corollaries 4.16 (and Theorem 4.12) and Jachymski et al. [12, Theorem 1]. Furthermore, T becomes PO if the assumption $\psi\in\Psi$ is added to Corollary 4.18; nevertheless, this example also demonstrates that T can be a non-WPO.

Example 4.19. Examine the complete metric space (X,d), which is defined as follows: $X=[0,+\infty)$, $d(x,y)=\max\{x,y\},x\neq y\in X$ and $d(x,x)=0,x\in X$. (X,d) is a complete metric space. Define $T(x)=x^2$ and ψ as $\psi(t)=t^2,t>0$. We have $\psi\in\Psi\left(\frac{3}{4}\right)$ and notice that $\psi\notin\Psi$ since $\psi(t)\geq t$ for all $t\geq 1$. We get $d(Tx,Ty)=\psi(d(x,y))$, for all $x,y\in X$. Also, for $x_0=\frac{1}{2}$ we get $d(x_0,T(x_0))\leq\frac{3}{4}$, thus, the conditions of Corollary 4.18 are satisfied. Notice that $\lim_n T^n\left(\frac{1}{2}\right)=0$ and $\lim_n T^n(2)=+\infty$, thus T is not a WPO.

Remark 4.20. The concepts of BSP and UBSP$_{\mathcal{D}}$ (Definitions 4.1 and 4.7) are new and the author is not aware of any equivalent formulations in the literature. The logical interrelations between these properties and the classical BWP (Definition 2.1), as well as the relationships among the classes of Φ-symmetric, γ-symmetric and JMS spaces, have not been investigated here. Such a systematic study is beyond the scope of this paper and is left for future work.

5. Conclusion

In this work, we extended the foundational idea of Boyd and Wong to symmetric spaces. Through the introduction of properties such as the BWP, BSP and UBSP$_{\mathcal{D}}$, we captured the behaviour of AR sequences that are not necessarily Cauchy. These properties enabled us to prove the existence and uniqueness of fixed points under ζ-contractions, ψ-contractions and generalized simulation functions in a variety of symmetric spaces, including b-metric, Φ-symmetric and γ-symmetric spaces. A notable aspect of our approach is the systematic use of structural properties (such as BSP and UBSP$_{\mathcal{D}}$) to complement the contractive assumptions, thereby obtaining convergence of Picard sequences.

The reducing of the standard presumption $\psi \in \Psi$ to $\psi \in \Psi(\varepsilon)$ is one of the paper's major findings, since it enables the use of fixed point results in broader situations, see [23]. We demonstrated that under suitable contractive conditions, fixed point theorems previously restricted to metric and b-metric spaces can be successfully extended to symmetric spaces. Several corollaries and examples illustrated the theoretical results and highlighted their generality. This approach may be applied to different metrical structures, see [16], (and contractions similar to (4.19)), such as quasi-metric, partial metric, G-metric, w-distance, interpolative metric spaces (see Theorem 3.8), metric-like spaces, (b)-suprametric spaces, etc., as it supports the usage of Boyd and Wong's idea in symmetric spaces. Overall, our results provide a unified framework for fixed point theory in symmetric spaces, highlighting the interplay between structural properties and contractive conditions.

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