Sahand Communications in Mathematical Analysis (SCMA) Vol. 24 No. 1 (2027), 27-43
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2066192.2262

A Note on Gabor and Wilson Systems

Rekha Rani[1] and Shah Jahan[2*]

Abstract. In this article, we analyse the Gabor and the Wilson systems using the concept of the Fourier transform. Firstly, we characterize the Gabor system $\{\zeta_{pc}\eta_{qd}t\}$ in terms of the family of the translated functions $\{\eta_{qd+w_0}t\}$ which gives necessary and sufficient conditions under which the Gabor system will be the Gabor frame. Also, the sufficient conditions for the Wilson system $\{\psi_{m,q}t\}$ and the necessary and sufficient condition for the system $\{\mathfrak{F}\psi_{m,q}t\}$, to be a Wilson-Bessel sequence are established. To illustrate the results, examples and counter-examples are given. We discussed the necessary and sufficient conditions for which both the Wilson system and its Fourier transform are Wilson-Bessel sequences. Furthermore, the sufficient conditions for the system $\{\psi_{m,q}t\}$ to be a Wilson frame are given.

1. Introduction

Signals play a crucial role in a society where connections are becoming more frequent. There are different approaches to analyse the signal properties for communication theory. But, the main problem observed in signal processing is decomposing a signal as a superposition of fundamental atoms. The approach of signal decomposition was introduced in 1946 by Gabor [19]. This decomposition displays the local content of time and frequency of the signal with the help of translation and the modulation operator. Based on this development, in 1952, the concept of frames was introduced by Duffin and Schaeffer [15] for Hilbert spaces. Although the frames are more flexible, yet the idea did not generate much interest.

2020 *Mathematics Subject Classification.* 42A38, 42C15, 46B15.
Key words and phrases. Frames, Gabor system, Wilson system, Fourier transform, Gabor frames, Wilson frame
Received: 2025-07-17, Accepted: 2026-05-25.
* Corresponding author.

After three decades, the concept of frame was reintroduced in an abstract setting and used in the context of the non-harmonic Fourier series by Young in 1980 [36]. Later, Daubechies et al. [13] observed that the concept of frame can be used to express functions in $L^2(\mathbb{R})$ in series expansions. Since the theory of frames and Gabor systems are interrelated, Gabor analysis serves as the foundation for a number of frame theory issues. In signal and image processing, Gabor frames have several uses. But by the Balian-Low theorem [2] for Gabor frames, these frames do not provide proper localization of time and frequency simultaneously in $L^2(\mathbb{R})$ [3]. Therefore, a more reliable system that concurrently enhanced the localization qualities of time-frequency was required to obtain more precise information about the signal.

In 1987, Wilson introduced the generalized Wannier functions centered around the similar order positive and negative frequencies. Further, Daubechies et al. [13] used the idea of Wilson and formalized the definition of the Wilson system. They introduced the Wilson system consisting of functions:

$$\psi_j^k(x) = \begin{cases} \epsilon_k \cos(2k\pi x)\, \omega\left(x - \frac{j}{2}\right), & \text{if } j \text{ is even,} \\ 2\sin(2(k+1)\pi x)\, \omega\left(x - \frac{j+1}{2}\right), & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\epsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0, \\ 2, & \text{if } k \in \mathbb{N}. \end{cases}$$

These bases overcome the limitations in the Balian-Low theorem [19]. Separate from Daubechies [14] work, Malvar [27] developed orthonormal local trigonometric bases made up of the functions, whose window functions are assumed to have compact support. Additionally, only two windows that are right next to one other may have overlapping support. In order to have greater flexibility in selecting window functions, the basis functions introduced by Malvar were generalized.

Wilson bases developed by Daubechies et al. were taken into consideration by Bittner [4] using trigonometric functions. Feichtinger et al. [16–18] proved that the Wilson bases can not be a basis for all modulation spaces. Wilson bases were examined by Kutynoik and Strohmer [26] for generic time-frequency lattices. Additionally, non-symmetrical window functions were used to study discrete-time Wilson frames with generic lattices.

Panwar et al. [24] examined Bittner's generalized system. In 2014, Jarrah and Panwar [23] obtained the conditions under which the Gabor and the Wilson systems and their Hilbert transforms enjoyed some common properties.

Wojdyllo [33–35] introduced Wilson bases and discussed the Wilson system for triple redundancy. Okoudjou et al. [6] characterized the bimodal Wilson systems as Parseval frames for $L^2(\mathbb{R})$. Panwar and Kohli [25] analyzed the Bittner [5] system in 2022 and described the Wilson frames on the W-H frame set created by Cassaza and Kalton [7].

In 2022, Casazza et al. [8] worked on piecewise scalable frames. In 2023, Casazza et al. [9] worked on the properties of Grassmannian frame. Poumai and Jahan [29] worked on an atomic system of different operators. In 2023, Andreolli and Grochenig [1] presented the notion of variable bandwidth in the context of Wilson expansion.

As a result, frame theory has recently drawn a lot of interest, particularly in the fields of signal analysis, image processing, engineering and physics, including voice recognition, optical visualization and crystallography using X-rays.

Motivated by the work, we observed that in practical life, we need to study the signal's behaviour in different domains and the Hilbert transform can only represent the function in the same domain. It is unable to alter the function's domain. However, we can additionally require the signal behavior in the frequency domain to get more effective outcomes. To represent the spectral behaviour of a function, the Fourier transform is required. In this article we attempted to work on the Gabor and the Wilson systems.

Outlines of the paper: In this paper, we use the Fourier operator on the Gabor and the Wilson system and obtain some results. Section 2 includes basic notations and preliminaries. Section 3 contains the main results of the paper. Lemma 3.1 shows an equivalence result of the Gabor system by using the Fourier transform. Theorem 3.2 shows the conditions for the Gabor system to be a Gabor frame. Corollary 3.6 is all about the uniqueness of the frame operator. Theorem 3.7 shows that the frame operator associated with the Gabor system and the Fourier transform are non-commutative. In section 4, we work on the Wilson system. In Lemma 4.1, the Fourier transform is applied to the Wilson system, which helps in Theorem 4.2 to find the necessary and the sufficient condition for $\{\mathfrak{F}\psi_{m,q}t\}$ to be a Bessel sequence. Theorem 4.4 gives the sufficient conditions for the system $\{\psi_{m,q}t\}$ to be a Wilson Bessel sequence. Finally, sufficient conditions for the Wilson system to be a Wilson frame are given.

2. Preliminaries

Here are the basic definitions and the fundamental results which will be used throughout this manuscript. We will use the following notations in this article:

$\mathfrak{F}$ denotes the Fourier transform, η_a and ζ_b denote the translation and the modulation operators with their corresponding parameters a and b. $L^2(\mathbb{R})$ is the space of all measurable square integrable functions over $\mathbb{R}$.

Definition 2.1 ([10])**.** Let J be a countable index set and $\mathcal{H}$ stands for the Hilbert space. The family of vectors $\{h_j\}_{j\in J}$, is referred to be a frame for $\mathcal{H}$ if constants C and D exist with $0 < C \leq D < \infty$ such that

$$C\|h\|^2 \leq \sum_{j\in J} |\langle h, h_j\rangle|^2 \leq D\|h\|^2,$$

holds $\quad \forall\ h \in \mathcal{H}$.

This inequality is called the frame inequality and C, D are known as the frame lower and upper bound. If only the above inequality is satisfied, then $\{h_j\}_{j\in J}$ is a Bessel sequence with Bessel bound D. Also, if $C = D$ are equal, then $\{h_j\}_{j\in J}$ will be a tight frame and if $C = D = 1$, then $\{h_j\}_{j\in J}$ is a Parseval frame.

For more information and recent developments in frame theory, one can refer [20–22, 28, 30, 31].

Definition 2.2 ([12])**.** The Fourier transform for a function $f_1 \in L^2(\mathbb{R})$ is defined as:

$$\mathfrak{F}\{f_1(x)\} = \hat{f}_1(\omega) = \int_{-\infty}^{\infty} f_1(x)\, e^{-\iota\omega x} dx,$$

and the inverse Fourier transform of a function $\hat{f}_1(\omega)$ is defined as:

$$f_1(x) = \mathfrak{F}^{-1}\mathfrak{F}\left(\hat{f}_1(\omega)\right) = \int_{-\infty}^{\infty} \hat{f}_1(\omega)\, e^{\iota\omega x} d\omega.$$

Also, the convolution of two functions $f_1(x)$ and $f_2(x)$, is defined as:

$$h(x) = (f_1 \star f_2)(x) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(x-\tau)\, d\tau.$$

Definition 2.3 ([11])**.** Gabor system is defined as the collection $\{\zeta_{pc}\eta_{qd}t\}_{p,q\in\mathbb{Z}}$ where $t \in L^2(\mathbb{R})$ and $c, d > 0$ and if it satisfies the frame inequality then it is called a Gabor frame.

Definition 2.4 ([32])**.** The Wilson system with respect to the function $t \in L^2(\mathbb{R})$ is a sequence of functions $\{\psi_{m,q}t\}$ where $m \in \mathbb{Z}, q \in \mathbb{N}_0$ in $L^2(\mathbb{R})$ given by

$$\psi_{m,q}t = (-1)^{mq}\, c_q \left(\zeta_q \eta_{\frac{m}{2}} t + (-1)^{m+q}\, \zeta_{-q}\eta_{\frac{m}{2}} t\right),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, c_0 = \frac{1}{2}, c_q = \frac{1}{\sqrt{2}}, q \in \mathbb{N}$.
If $\{\psi_{m,q}t\}$ satisfies the frame inequality then it is called a Wilson frame.

We will use the following Lemmas throughout the article.

Lemma 2.5 ([24]). *If u, v, w are in $L^2(\mathbb{R})$, then*

$$|\langle u, v\rangle|^2 + |\langle u, w\rangle|^2 = |\langle u, v + iw\rangle|^2 + 2Im\left(\langle u, v\rangle \langle w, u\rangle\right).$$

Lemma 2.6 ([24]). *Let $u, v \in L^2(\mathbb{R})$ and $(\eta_k u)(x) = u(x-k)$. Then,*

$$\sum_{m,k\in\mathbb{Z}} Im\left\{\langle u, \cos(2m\pi(\cdot))\,\eta_k g(\cdot)\rangle\, \overline{\langle u, \sin(2m\pi(\cdot))\,\eta_k g(\cdot)\rangle}\right\} = 0,$$

where

$$(\cos(2m\pi(\cdot))\,\eta_k g(\cdot))(x) = \cos(2m\pi x)\, g(x-k), \quad x \in \mathbb{R},$$
$$(\sin(2m\pi(\cdot))\,\eta_k g(\cdot))(x) = \sin(2m\pi x)\, g(x-k), \quad x \in \mathbb{R}.$$

Lemma 2.7 ([10]). *For any $a \neq 0 \in \mathbb{R}$ and $f \in {}^2(\mathbb{R})$, we have $\mathfrak{F}(\eta_a f) = \zeta_{-a}\mathfrak{F}(f)$ and $\mathfrak{F}(\zeta_a f) = \eta_a \mathfrak{F}(f)$.*

3. Main Results

In this section, we are evaluating the Fourier transform of the Gabor system and characterizing the Gabor system in terms of the system of translated functions. Also, we are providing the necessary and sufficient conditions for the Gabor system so that it constitutes a Gabor frame. We give some examples to illustrate our results.

To prove the forthcoming theorem we need the following Lemma.

Lemma 3.1. *Let $t \in L^2(\mathbb{R})$ and $\{\zeta_{pc}\eta_{qd}t\}_{p,q\in\mathbb{Z}}$ be the corresponding Gabor system and $\mathfrak{F}$ denotes the Fourier operator on $L^2(\mathbb{R})$ then*

$$\mathfrak{F}\{\zeta_{pc}\eta_{qd}t\} = \mathfrak{F}\{\eta_{qd+w_0}t\} = \eta_{pc}\zeta_{-qd}\mathfrak{F}t,$$

where $w_0 = 2\pi pc$.

Proof. Consider the function $t \in L^2(\mathbb{R})$ and the corresponding Gabor system $\{\zeta_{pc}\eta_{qd}t\}$. Then, using the Fourier transform, we have

$$\begin{aligned}\mathfrak{F}\{\zeta_{pc}\eta_{qd}t\} &= \mathfrak{F}\left[(\cos(2\pi pcy) + \iota\sin(2\pi pcy))\,\eta_{qd}t(y)\right],\\ &= \frac{1}{2\pi}\left\{\mathfrak{F}\left[(\cos(2\pi pcy) + \iota\sin(2\pi pcy))\right] * \mathfrak{F}\eta_{qd}t(y)\right\}.\end{aligned}$$

We know that

$$\mathfrak{F}(\sin w_0 x) = -\iota\pi\left[\delta(w - w_0) - \delta(w + w_0)\right],$$

and

$$\mathfrak{F}(\cos w_0 x) = \pi\left[\delta(w - w_0) + \delta(w + w_0)\right],$$

where δ denotes the Dirac delta function.

Thus,

$$\mathfrak{F}\cos(2\pi pcy) + \iota\mathfrak{F}(\sin 2\pi pcy) = 2\pi\delta(w - 2\pi pc).$$

This gives

$$\begin{aligned}\mathfrak{F}\left\{\zeta_{pc}\eta_{qd}t\right\} &= \frac{2\pi\delta\left(w-2\pi pc\right)}{2\pi} * \mathfrak{F}\eta_{qd}t\left(y\right), \\ &= \delta\left(w-2\pi pc\right) * \mathfrak{F}\eta_{qd}t\left(y\right).\end{aligned}$$

Now,

$$\begin{aligned}\delta\left(w-2\pi pc\right) * \mathfrak{F}\left(\eta_{qd}t\left(w\right)\right) &= \int_{\mathbb{R}} \mathfrak{F}\left(\left(\eta_{qd}t\left(\tau\right)\right)\delta\left(w-2\pi pc-\tau\right)\right)d\tau \\ &= \int_{\mathbb{R}} \mathfrak{F}\left(\left(\eta_{qd}t\left(\tau\right)\right)\delta\left(\tau-\left(w-2\pi pc\right)\right)\right)d\tau \\ &= \mathfrak{F}\left(\eta_{qd}t\left(w-2\pi pc\right)\right) = \mathfrak{F}\left(\eta_{qd+2\pi pc}t\right).\end{aligned}$$

Thus,

$$\mathfrak{F}\left\{\zeta_{pc}\eta_{qd}t\right\} = \mathfrak{F}\left\{\eta_{qd+2\pi pc}t\right\}.$$

Applying inverse Fourier transform, we have

$$\zeta_{pc}\eta_{qd}t = \eta_{qd+2\pi pc}t.$$

Also,

$$\mathfrak{F}\left\{\zeta_{pc}\eta_{qd}t\right\} = \eta_{pc}\mathfrak{F}\left(\eta_{qd}t\right) = \eta_{pc}\zeta_{-qd}\mathfrak{F}t.$$

Thus,

$$\mathfrak{F}\left\{\zeta_{pc}\eta_{qd}t\right\} = \mathfrak{F}\left\{\eta_{qd+w_0}t\right\} = \eta_{pc}\zeta_{-qd}\mathfrak{F}t,$$

where $w_0 = 2\pi pc$. □

The equality obtained in the above result will help to determine the equivalence conditions for both the families to be a frame.

Theorem 3.2. *Consider the Gabor system $\{\zeta_{pc}\eta_{qd}t\}$ associated with $t \in L^2(\mathbb{R})$. Then, $\{\zeta_{pc}\eta_{qd}t\}$ is a Gabor frame iff $\{\eta_{qd+2\pi pc}t\}$ is a frame with the same bounds. Furthermore, the family $\{\eta_{pc}\zeta_{-qd}\mathfrak{F}t\}$ will also be a frame having the same frame bounds.*

Proof. One can prove the result easily. □

Following are examples that illustrate Theorem 3.2.

Example 3.3. The function

$$w(x) = \begin{cases} 2, & \text{if } x \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

is a tight Gabor frame with frame bound 4.

Example 3.4. The function $m\left(y\right) = \sqrt{2}e^{-y}\kappa_{[0,\infty)}\left(y\right)$ (where $\kappa_{[0,\infty)}\left(y\right)$ stands for the characteristics function) satisfies the above result with lower bound $\frac{1}{2}$ and upper bound 1.

Example 3.5. The function $p(y) = e^{-y^2}$ also satisfies the above result with its frame bounds $\frac{\sqrt{\pi}}{\sqrt{2}}$ and $\sqrt{\pi}$ respectively.

The above theorem leads to the corollary that the frame operator associated with the two families $\{\zeta_{pc}\eta_{qd}t\}_{p,q\in\mathbb{Z}}$ and $\{\eta_{qd+2\pi pc}t\}_{p,q\in\mathbb{Z}}$ is unique.

Corollary 3.6. *If $\{\zeta_{pc}\eta_{qd}t\}_{p,q\in\mathbb{Z}}$ be the Gabor Bessel sequence. Then, the family $\{\eta_{qd+2\pi pc}t\}_{p,q\in\mathbb{Z}}$ is a Bessel sequence with the same frame operator S.*

Proof. Let A be the frame operator associated with $\{\eta_{qd+w_0}t\}_{p,q\in\mathbb{Z}}$, where $w_0 = 2\pi pc$.

Then,

$$\begin{aligned} Af &= \sum \langle f, \eta_{qd+w_0}t\rangle \, \eta_{qd+w_0}t, \quad \forall f \in L^2(\mathbb{R}) \\ &= \sum \langle f, \zeta_{pc}\eta_{qd}t\rangle \, \zeta_{pc}\eta_{qd}t = Sf, \quad \forall f \in L^2(\mathbb{R}). \end{aligned}$$

Thus,

$$Af - Sf = 0, \quad \forall f \in L^2(\mathbb{R}).$$

Therefore, $A = S$. Hence the result. □

In the upcoming result, we will examine the commutativity of the Fourier and the Frame operator.

Theorem 3.7. *The Fourier and the frame operators are non-commutative unless*

$$\mathfrak{F}(\zeta_{pc}\eta_{qd}t) = \zeta_{pc}\eta_{qd}t.$$

Proof. Let S be a frame operator. Therefore, we have

$$Sf = \sum \langle f, \zeta_{pc}\eta_{qd}t\rangle \, \zeta_{pc}\eta_{qd}t, \quad \forall f \in L^2(\mathbb{R}).$$

Applying the Fourier transform

$$\begin{aligned} \mathfrak{F}Sf &= \sum \langle f, \zeta_{pc}\eta_{qd}\rangle \, \mathfrak{F}\zeta_{pc}\eta_{qd}t \\ &= \sum \langle f, \zeta_{pc}\eta_{qd}t\rangle \, \mathfrak{F}\eta_{qd+w_0}t, \quad \text{where } w_0 = 2\pi pc. \end{aligned}$$

Now,

$$\begin{aligned} S\mathfrak{F}f &= \sum \langle \mathfrak{F}f, \zeta_{pc}\eta_{qd}t\rangle \, \zeta_{pc}\eta_{qd}t \\ &= \sum \left\langle f, \mathfrak{F}^{-1}\zeta_{pc}\eta_{qd}t\right\rangle \zeta_{pc}\eta_{qd}t. \end{aligned}$$

Thus,

$$\mathfrak{F}S \neq S\mathfrak{F}.$$

□

4. Wilson System

In this section, we will work on the Wilson system by applying the Fourier operator to find the sufficient conditions for the systems $\{\mathfrak{F}\psi_{m,q}t\}$ and $\{\psi_{m,q}t\}$ to be a Wilson Bessel sequence.

The following Lemma will be used in the upcoming theorem.

Lemma 4.1. *Let $\{\psi_{m,q}t\}, m \in \mathbb{Z}, q \in \mathbb{N}_0$ be the Wilson system. Then,*

$$\mathfrak{F}\{\psi_{m,q}t(y)\} = \begin{cases} 2(-1)^{mq} c_q e^{-\pi\iota m}[\cos(2\pi qy)]\mathfrak{F}t(y), & \text{if } m+q \text{ even,} \\ -2(-1)^{mq} c_q e^{-\pi\iota m}[\iota\sin(2\pi qy)]\mathfrak{F}t(y), & \text{if } m+q \text{ odd .} \end{cases}$$

Proof. Using the Fourier transform on the Wilson system. We have

$$\begin{aligned} \mathfrak{F}\{\psi_{m,q}t(y)\} &= \mathfrak{F}\left[(-1)^{mq} c_q\left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q}\zeta_{-q}\eta_{\frac{m}{2}}t\right)\right] \\ &= (-1)^{mq} c_q\left[\mathfrak{F}\zeta_q\eta_{\frac{m}{2}}t + (-1)^{m+q}\mathfrak{F}\zeta_{-q}\eta_{\frac{m}{2}}t\right] \\ &= c_q(-1)^{mq}\left[\mathfrak{F}\left(\eta_{\frac{m}{2}+q}t + (-1)^{m+q}\mathfrak{F}\eta_{(\frac{m}{2}-q)}t\right)\right]. \end{aligned}$$

So,

$$\begin{aligned} \mathfrak{F}\{\psi_{m,q}t(y)\} &= (-1)^{mq} c_q\left[\zeta_{-(\frac{m}{2}+q)}\mathfrak{F}t + (-1)^{m+q}\zeta_{-(\frac{m}{2}-q)}\mathfrak{F}t\right] \\ &= (-1)^{mq} c_q\left[e^{-(\frac{m}{2}+q)2\pi\iota y}\mathfrak{F}t + (-1)^{m+q}e^{-(\frac{m}{2}-q)2\pi\iota y}\mathfrak{F}t\right] \\ &= (-1)^{mq} c_q\left[e^{-\pi\iota y m}e^{-2\pi\iota qy}\mathfrak{F}t + (-1)^{m+q}e^{-\pi\iota m y}e^{2\pi\iota qy}\mathfrak{F}t\right] \\ &= (-1)^{mq} c_q e^{-\pi\iota y m}\left[(-1)^{m+q}e^{2\pi\iota qy} + e^{-2\pi\iota qy}\right]\mathfrak{F}t. \end{aligned}$$

$$\begin{aligned} \mathfrak{F}\{\psi_{m,q}t(y)\} &= \begin{cases} (-1)^{mq} c_q e^{-\pi\iota m y}\left[e^{2\pi\iota qy} + e^{-2\pi\iota qy}\right]\mathfrak{F}t, & m+q \text{ even,} \\ (-1)^{mq} c_q e^{-\pi\iota m y}\left[-e^{2\pi\iota qy} + e^{-2\pi\iota qy}\right]\mathfrak{F}t, & m+q \text{ odd.} \end{cases} \\ &= \begin{cases} (-1)^{mq} c_q e^{-\pi\iota m y}\left[e^{2\pi\iota qy} + e^{-2\pi\iota qy}\right]\mathfrak{F}t, & m+q \text{ even,} \\ -(-1)^{mq} c_q e^{-\pi\iota m y}\left[e^{2\pi\iota qy} - e^{-2\pi\iota qy}\right]\mathfrak{F}t, & m+q \text{ odd.} \end{cases} \\ &= \begin{cases} (-1)^{mq} c_q e^{-\pi\iota m y}[2\cos(2\pi qy)]\mathfrak{F}t, & m+q \text{ even,} \\ -(-1)^{mq} c_q e^{-\pi\iota m y}[2\iota\sin(2\pi qy)]\mathfrak{F}t, & m+q \text{ odd.} \end{cases} \end{aligned}$$

Therefore,

$$\mathfrak{F}\{\psi_{m,q}t(y)\} = \begin{cases} 2(-1)^{mq} c_q e^{-\pi\iota m y}[\cos(2\pi qy)]\mathfrak{F}t, & m+q \text{ even,} \\ -2(-1)^{mq} c_q e^{-\pi\iota m y}[\iota\sin(2\pi qy)]\mathfrak{F}t, & m+q \text{ odd.} \end{cases}$$

□

The following theorem gives the necessary and sufficient condition for the system $\mathfrak{F}\{\psi_{m,q}t\}$ to be a Bessel sequence.

Theorem 4.2. *Let $t \neq 0 \in L^2(\mathbb{R})$ and $\{\psi_{m,q}t\}$ be the respective Wilson system. Then, the necessary and sufficient condition for the system $\{\mathfrak{F}\psi_{m,q}t\}$ to be a Bessel sequence is $q = 0$.*

Proof. By using Lemma 4.1, we have

$$
\begin{aligned}
&|\langle s, \mathfrak{F}(\psi_{m,q}t)\rangle|^2 \\
&= \begin{cases} |\langle s, 2(-1)^{mq} c_q e^{-\pi\iota my}\cos(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ even}, \\ |\langle s, -2\iota(-1)^{mq} c_q e^{-\pi\iota my}\sin(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ odd}, \end{cases} \\
&= \begin{cases} |2c_q(-1)^{mq}|^2\,|\langle s, e^{-\pi\iota my}\cos(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ even}, \\ |-2\iota c_q(-1)^{mq}|^2\,|\langle s, e^{-\pi\iota my}\sin(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ odd}, \end{cases} \\
&= \begin{cases} 4|c_q|^2\,|\langle s, e^{-\pi\iota my}\cos(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ even}, \\ 4|c_q|^2\,|\langle s, e^{-\pi\iota my}\sin(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ odd}. \end{cases}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 &= \begin{cases} \sum_{m\in\mathbb{Z}} 4c_q^2 |\langle s, e^{-\pi\iota my}\cos(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ even}, \\ \sum_{m\in\mathbb{Z}} 4c_q^2 |\langle s, e^{-\pi\iota my}\sin(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ odd}, \end{cases} \\
&= \begin{cases} \sum_{m\in\mathbb{Z}} 4c_q^2 |\langle e^{\pi\iota my}, s\cos(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ even}, \\ \sum_{m\in\mathbb{Z}} 4c_q^2 |\langle e^{\pi\iota my}, s\sin(2\pi qy)\,\mathfrak{F}t\rangle|^2, & m+q \text{ odd}. \end{cases}
\end{aligned}
$$

Using some fundamental results, we have

$$
\sum_{m\in\mathbb{Z}} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq \begin{cases} 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2, & m+q \text{ even}, \\ 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2, & m+q \text{ odd}. \end{cases}
$$

So,

$$
\sum_{m\in\mathbb{Z}} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2, \quad \forall m+q \in \mathbb{Z}. \tag{4.1}
$$

Therefore, from inequality (4.1), we have

$$
\sum_{m\in\mathbb{Z}, q\in\mathbb{N}_0} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2 \leq \sum_{q\in\mathbb{N}_0} 4c_q^2\|s\|^2\|t\|^2, \quad \forall\, m+q \in \mathbb{Z}. \tag{4.2}
$$

Now, for $q = 0$, inequality (4.2) gives

$$\sum_{m\in\mathbb{Z},q=0} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2 \leq \|s\|^2\|t\|^2 \leq \mathfrak{A}\|s\|^2,$$

where $\mathfrak{A} = \|t\|^2 > 0$.
Conversely, assume that for $q \neq 0$, $\{\mathfrak{F}\psi_{m,q}t\}$ is a Bessel sequence with a finite Bessel bound $\mathfrak{A} > 0$. Then, for some non-zero function $s \in L^2(\mathbb{R})$, we have

$$\sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq 4c_q^2\|s\|^2\|\mathfrak{F}t\|^2 \leq \|s\|^2\|t\|^2 \leq \mathfrak{A}\|s\|^2 \leq \mathfrak{A}\|s\|^2.$$

When $m + q$ is even, then again using Lemma 4.1, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z},q\in\mathbb{N}} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 &= \sum_{m\in\mathbb{Z},q\in\mathbb{N}} 4c_q^2\left|\left\langle s, e^{-\pi\iota my}\cos(2\pi qy)\,\mathfrak{F}t\right\rangle\right|^2 \\ &= \sum_{q\in\mathbb{N}} 4c_q^2\|s\cos(2\pi qy)\,\mathfrak{F}t\|^2.\end{aligned} \tag{4.3}$$

Now, for $y = 0$, it gives

$$\sum_{m\in\mathbb{Z},q\in\mathbb{N}} 4c_q^2\|s\mathfrak{F}t\|^2 = \|s\mathfrak{F}t\|^2\sum_{q\in\mathbb{N}} 4c_q^2 \leq \mathfrak{A}\|s\|^2,$$

which contradicts the assumption. One can easily prove for the case "$m + q$ is odd". Hence, q must be 0. □

Now the above theorem, states the condition under which the Wilson system corresponding to the function $t \in L^2(\mathbb{R})$ becomes a Wilson Bessel sequence in the frequency domain. The following example shows that if the condition of the Theorem 4.2 is not satisfied, then the family of functions $\{\mathfrak{F}\psi_{m,q}t\}$ will not be a Wilson Bessel sequence.

Example 4.3. The function $t \in L^2(\mathbb{R})$ defined as:

$$t = \begin{cases} 1+p, & \text{when } p \in [1,2), \\ p, & \text{when } p \in [2,3), \\ 0, & \text{otherwise,} \end{cases}$$

be such that the Wilson system $\{\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ is well defined. Then, $\{\mathfrak{F}\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ is not a Wilson Bessel sequence. But, for $q = 0$, we have

$$\sum_{m\in\mathbb{Z},q=0} |\langle s, \mathfrak{F}\psi_{m,q}t\rangle|^2 \leq \|s\|^2\|t\|^2 \leq \frac{38}{3}\|s\|^2.$$

Therefore, under the assumption of Theorem 4.2, the above example becomes the Wilson Bessel sequence in frequency domain with its Bessel bound $\frac{38}{3}$.

Now, the question is: "Is it true for each function $t \in L^2(\mathbb{R})$ to generate a Wilson Bessel sequence?". This leads to the following problem.

Problem: Is the sequence consisting functions $\{\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ is a Wilson Bessel sequence for each function $t \in L^2(\mathbb{R})$.

The answer to the problem is not affirmative.

In the next result we will obtain the sufficient conditions for the Wilson system $\{\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ to be a Wilson Bessel sequence.

Theorem 4.4. *Let $t \in L^2(\mathbb{R})$ and $\{\psi_{m,q}t\}$ be the associated Wilson system. Then, the sufficient conditions for $\{\psi_{m,q}t\}$ to be a Wilson Bessel sequence are $q = 0$ and $\left\{\eta_{\frac{m}{2}}t\right\}$ is a Bessel sequence with the same Bessel bound.*

Proof. Let $t \in L^2(\mathbb{R})$ and $\{\psi_{m,q}t\}$ be the Wilson system associated with t defined as:

$$\psi_{m,q}t = (-1)^{mq} c_q \left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q} \zeta_{-q}\eta_{\frac{m}{2}}t\right),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, c_0 = \frac{1}{2}, c_q = \frac{1}{\sqrt{2}}, q \in \mathbb{N}$.

$$\begin{aligned}
|\psi_{m,q}t| &= \left|(-1)^{mq} c_q \left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q} \zeta_{-q}\eta_{\frac{m}{2}}t\right)\right| \\
&= |(-1)^{mq} c_q| \left|\left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q} \zeta_{-q}\eta_{\frac{m}{2}}t\right)\right| \\
&\leq c_q \left[\left|\zeta_q \eta_{\frac{m}{2}}t\right| + \left|(-1)^{m+q}\right| \left|\zeta_{-q}\eta_{\frac{m}{2}}t\right|\right] \\
&= c_q \left[|\zeta_q| \left|\eta_{\frac{m}{2}}t\right| + |\zeta_{-q}| \left|\eta_{\frac{m}{2}}t\right|\right] = c_q \left[\left|\eta_{\frac{m}{2}}t\right| + \left|\eta_{\frac{m}{2}}t\right|\right] \\
&= 2c_q \left|\eta_{\frac{m}{2}}t\right|.
\end{aligned}$$

So,

$$\begin{aligned}
|\langle f, \psi_{m,q}t\rangle|^2 = |\langle f, |\psi_{m,q}t|\rangle|^2 &\leq \left|\left\langle f, 2c_q \left|\eta_{\frac{m}{2}}t\right|\right\rangle\right|^2 \\
&= 4c_q^2 \left|\left\langle f, \eta_{\frac{m}{2}}t\right\rangle\right|^2.
\end{aligned}$$

Now, for $q = 0$, we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle f, \psi_{m,0}t\rangle|^2 &\leq \sum_{m\in\mathbb{Z}} 4{c_0}^2 \left|\left\langle f, \eta_{\frac{m}{2}}t\right\rangle\right|^2 \\
&= \sum_{m\in\mathbb{Z}} 4\left(\frac{1}{2}\right)^2 \left|\left\langle f, \eta_{\frac{m}{2}}t\right\rangle\right|^2 \\
&= \sum_{m\in\mathbb{Z}} \left|\left\langle f, \eta_{\frac{m}{2}}t\right\rangle\right|^2 \leq D\|f\|^2, \quad \text{where} \quad D > 0.
\end{aligned}$$

The result is obvious now. □

By the previous discussion, one can inquire as to the setting in which the response of this issue is positive. The above discussion leads to the following corollary.

Corollary 4.5. *The conditions for the system* $\{\psi_{m,q}t\}$ *and its Fourier transform to be a Wilson Bessel sequence are* $q = 0, \left\{\eta_{\frac{m}{2}}t\right\}$ *and* $\{\mathfrak{F}t\}$ *are Bessel sequences.*

Proof. The proof is obvious from the above discussion. □

Now, the question arises that is it possible for every function $t \in L^2(\mathbb{R})$ the Wilson system $\{\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ is a Wilson frame.

The answer is **No**.

In the next result we will discuss some sufficient conditions for the Wilson system to be a Wilson frame.

Theorem 4.6. *Let* $t \in L^2(\mathbb{R})$ *and*

$$\psi_{m,q}t = (-1)^{mq}\, c_q \left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q}\, \zeta_{-q}\eta_{\frac{m}{2}}t\right),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $c_0 = \frac{1}{2}$, $c_q = \frac{1}{\sqrt{2}}$, $q \in \mathbb{N}$, *is the Wilson system. Then, the following result holds:*

(i) *When* $m+q$ *is odd, then the sufficient conditions for the Wilson system to be the Wilson frame is that* $\left\{\zeta_q \eta_{\frac{m}{2}}t\right\}_{m\in\mathbb{Z},q\in\mathbb{N}}$ *is a Gabor frame and*

$$\sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \left[\left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2 - 2\mathrm{Im}\left(\left\langle s, \cos(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle}\right)\right] < \epsilon.$$

(ii) *When* $m+q$ *is even, then the sufficient conditions for the Wilson system to be a Wilson frame are* $\left\{\eta_{\frac{m}{2}}t\right\}_{m\in\mathbb{Z}}$ *and* $\left\{\zeta_q \eta_{\frac{m}{2}}t\right\}_{m\in\mathbb{Z},q\in\mathbb{N}}$ *are frames.*

Proof. Since,

$$\psi_{m,q}t = (-1)^{mq}\, c_q \left(\zeta_q \eta_{\frac{m}{2}}t + (-1)^{m+q}\, \zeta_{-q}\eta_{\frac{m}{2}}t\right),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, c_0 = \frac{1}{2}, c_q = \frac{1}{\sqrt{2}}, q \in \mathbb{N}, t \in L^2(\mathbb{R})$.

$$\psi_{m,q}t = \begin{cases} 2\,(-1)^{mq}\, c_q \cos(2\pi qx)\left(\eta_{\frac{m}{2}}t\right), & m+q \text{ even}, \\ \\ 2\,(-1)^{mq}\, c_q \sin(2\pi qx)\left(\eta_{\frac{m}{2}}t\right), & m+q \text{ odd}. \end{cases}$$

$$|\langle s, \psi_{m,q}t\rangle|^2 = \begin{cases} \left|\left\langle s, 2\,(-1)^{mq}\, c_q \cos(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2, & m+q \text{ even}, \\ \\ \left|\left\langle s, 2\,(-1)^{mq}\, c_q \sin(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2, & m+q \text{ odd}. \end{cases}$$

Case 1: When $m+q$ is odd.

$$\begin{aligned} \sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} |\langle s, \psi_{m,q}t\rangle|^2 &= \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left|\left\langle s, 2\,(-1)^{mq}\, c_q \sin(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2, \\ &= \sum_{m\in\mathbb{Z},q\in\mathbb{N}} 2\left|\left\langle s, \sin(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2, \end{aligned}$$

using Lemma 2.5, we have

$$\begin{aligned} &\sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} |\langle s, \psi_{m,q}t\rangle|^2 \\ &= 2\sum_{m\in\mathbb{Z},q\in\mathbb{N}} \Bigg[\left|\left\langle s, (\cos(2\pi qx) + \iota\sin(2\pi qx))\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2 \\ &\quad - \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}}t\right)\right\rangle\right|^2 \\ &\quad -2\mathrm{Im}\left(\left\langle s, \cos(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle\left\langle \sin(2\pi qx)\,\eta_{\frac{m}{2}}t, s\right\rangle\right)\Bigg] \\ &= 2\sum_{m\in\mathbb{Z},q\in\mathbb{N}} \Bigg[\left|\left\langle s, \zeta_q\eta_{\frac{m}{2}}t\right\rangle\right|^2 - \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}}g\right)\right\rangle\right|^2 \\ &\quad -2\mathrm{Im}\left(\left\langle s, \cos(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle\left\langle \sin(2\pi qx)\,\eta_{\frac{m}{2}}t, s\right\rangle\right)\Bigg] \\ &= 2\sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left[\left|\left\langle s, \zeta_q\eta_{\frac{m}{2}}t\right\rangle\right|^2 - \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}}g\right)\right\rangle\right|^2\right] \\ &\quad - 4\sum_{m\in\mathbb{Z},q\in\mathbb{N}} \mathrm{Im}\left(\left\langle s, \cos(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle\overline{\left\langle s, \sin(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle}\right) \\ &\quad + 4\sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \mathrm{Im}\left(\left\langle s, \cos(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle\overline{\left\langle s, \sin(2\pi qx)\,\eta_{\frac{m}{2}}t\right\rangle}\right) \end{aligned}$$

$$
\begin{aligned}
&- 4 \sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \operatorname{Im}\left(\left\langle s, \cos(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle}\right) \\
&= 2 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left[\left|\left\langle s, \zeta_q \eta_{\frac{m}{2}} t\right\rangle\right|^2 - \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}} g\right)\right\rangle\right|^2\right] \\
&\quad - 4 \sum_{m\in\mathbb{Z},q\in\mathbb{Z}} \operatorname{Im}\left(\left\langle s, \cos(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle}\right) \\
&\quad + 4 \sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \operatorname{Im}\left(\left\langle s, \cos(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle}\right) \\
&= 2 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left[\left|\left\langle s, \zeta_q \eta_{\frac{m}{2}} t\right\rangle\right|^2 - \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2\right] \\
&\quad + 4 \sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \operatorname{Im}\left(\left\langle s, \cos(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle}\right) \\
&= 2 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left|\left\langle s, \zeta_q \eta_{\frac{m}{2}} t\right\rangle\right|^2 \\
&\quad - 2 \sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2 \\
&\quad + 4 \sum_{m\in\mathbb{Z},-q\in\mathbb{N}} \operatorname{Im}\left(\left\langle s, \cos(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle \overline{\left\langle s, \sin(2\pi qx)\, \eta_{\frac{m}{2}} t\right\rangle}\right).
\end{aligned}
$$

This gives

$$
\begin{aligned}
\sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} |\langle s, \psi_{m,q} t\rangle|^2 &< 2 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left|\left\langle s, \zeta_q \eta_{\frac{m}{2}} t\right\rangle\right|^2 + \epsilon \\
&= 2 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left|\left\langle s, \zeta_q \eta_{\frac{m}{2}} t\right\rangle\right|^2, \quad as\ \epsilon \longrightarrow 0.
\end{aligned}
$$

Case 2 : When $m+q$ is even.

$$
\begin{aligned}
\sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} |\langle s, \psi_{m,q} t\rangle|^2 &= \sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} \left|\left\langle s, 2(-1)^{mq} c_q \cos(2\pi qx)\left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2 \\
&= \sum_{m\in\mathbb{Z}} \left|\left\langle s, \left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2 \\
&\quad + 4 \sum_{m\in\mathbb{Z},q\in\mathbb{N}} \left|\left\langle s, \cos(2\pi qx)\left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2 \\
&= \sum_{m\in\mathbb{Z}} \left|\left\langle s, \left(\eta_{\frac{m}{2}} t\right)\right\rangle\right|^2. \qquad \square
\end{aligned}
$$

5. Conclusion

The Wilson and the Gabor systems are important frameworks in signal processing, particularly in time-frequency analysis. Various properties of the Wilson and Gabor system are discussed earlier in [1, 3, 25, 32]. Also, in [23], the effect of the Hilbert transform on the properties of Gabor and Wilson systems was derived. However, the major drawback we observe in the Hilbert transform is that it can't change the domain of the function or the signal. In this article, we discuss the Fourier transform in the context of Gabor and Wilson systems. Firstly, we discuss some new necessary and sufficient conditions for the Gabor system $\{\zeta_{pc}\eta_{qd}t\}_{p,q\in\mathbb{Z}}$ to be a Gabor frame by characterizing the system in terms of the family of the translated functions $\{\eta_{qd+w_0}t\}$ using the Fourier transform. The uniqueness of the Fourier and frame operator is explained. We observed that not every square-integrable function can generate a Wilson-Bessel sequence in the frequency domain. The counter-example is given for the problem. Therefore, to resolve this problem, we provide necessary and sufficient conditions for the system $\{\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ and its Fourier transform $\{\mathfrak{F}\psi_{m,q}t\}_{m\in\mathbb{Z},q\in\mathbb{N}_0}$ to be a Wilson Bessel sequence. The results are illustrated with some examples. Finally, we discuss the sufficient conditions for the Wilson system to be a Wilson frame.

Acknowledgment. The first author is thankful to CSIR for fellowship support with Grant No. 09/1152(16189)/2023-EMR-I.

References

1. B. Andreolli and K. Gröchenig, *Variable bandwidth via Wilson bases*, Appl. Comput. Harmon. Anal., 71 (2024), 101641.
2. J.J. Benedetto, C. Heil and D.F. Walnut, *Gabor systems and the Balian-Low theorem*, Gabor Analysis and Algorithms: Theory and Applications, MA: Birkhäuser Boston, (1998), pp. 85-122.
3. J.J. Benedetto and D.F. Walnut, *Gabor frames for L^2 and related spaces*, Wavelets: mathematics and applications, CRC Boca Raton, (1994), pp. 97-162.
4. K. Bittner, Biorthogonal wilson bases, *Wavelet Applications in Signal and Image Processing VII*, SPIE, (1999), pp. 410-421.
5. K. Bittner, *Linear approximation and reproduction of polynomials by Wilson bases*, J. Fourier Anal. Appl., 8(1) (2002), pp. 85-108.
6. M. Bownik, M.S. Jakobsen, J. Lemvig and K.A. Okoudjou, *On Wilson Bases in $L^2(\mathbb{R}^d)$*, SIAM J. Math. Anal., 49(5) (2017), pp. 3999-4023.

7. P. Casazza and N. Kalton, *Roots of complex polynomials and Weyl-Heisenberg frame sets*, Proc. Am. Math. Soc., 130(8) (2002), pp. 2313-2318.
8. P.G. Casazza, I. Campbell and T.T. Tran, *The core of a Grassmannian frame*, J. Fourier Anal. Appl., 29(2) (2023), 16.
9. P.G., Casazza, L. De Carli and T.T. Tran, *Piecewise scalable frames*, Linear Algebra Appl., 694 (2024), pp. 262-282.
10. O. Christensen, *Frames and bases: An introductory course*, MA Birkhäuser Boston, (2008).
11. O. Christensen, *A short introduction to frames, Gabor systems and wavelet systems*, Azerb. J. Math., 4(1) (2014), pp. 25-39.
12. C.K. Chui, *An introduction to wavelets*, Academic press, Inc., MA Boston, (1992).
13. I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27(5) (1986), pp. 1271-1283.
14. I. Daubechies, S. Jaffard and J.L. Journe, *A simple Wilson orthonormal basis with exponential decay*, SIAM J. Math. Anal., 22(2) (1991), pp. 554-573.
15. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc., 72(2) (1952), pp. 341-366.
16. H.G. Feichtinger, K. Gröchenig and D. Walnut, *Wilson bases and modulation spaces*, Math. Nachr., 155(1) (1992), pp. 7-17.
17. H.G. Feichtinger and T. Strohmer, *Gabor analysis and algorithms: Theory and applications*, Springer Science and Business Media, Inc. Boston, MA, (1998).
18. H.G. Feichtinger and T. Strohmer, *Advances in Gabor analysis*, Springer Science and Business Media, Birkhäuser Boston, (2003).
19. D. Gabor, *Theory of communication, Part 1: The analysis of information*, J. Inst. Electr. Eng. Part III, 93(26) (1946), pp. 429-441.
20. S. Jahan, *Approximative K-atomic decompositions and frames in Banach spaces*, Arab J. Math. Sci., 26(1/2) (2020), pp. 153-166.
21. S. Jahan and P.S. Johnson, *Frame operators for frames in Krein spaces*, Eur. J. Math. Anal., 4 (2024), pp. 1-10.
22. S. Jahan and V. Kumar, *On cones associated with Schauder frames*, Jordan J. Math. Stat., 10(4) (2017), pp. 265-280.
23. A.M. Jarrah and S. Panwar, *On Hilbert transform of Gabor and Wilson systems*, Int. J. Wavelets Multiresolution Inf. Process., 12(02) (2014), 1450012.
24. S.K. Kaushik and S. Panwar, *A note on Gabor frames*, Math. Commun., 19(1) (2014), pp. 75-89.
25. T. Kohli and S. Panwar, *Characterizations of Wilson Systems using Roots of Polynomials*, Results Math., 77(2) (2022), 79.

26. G. Kutyniok and T. Strohmer, *Wilson bases for general time-frequency lattices*, SIAM J. Math. Anal., 37(3) (2005), pp. 685-711.
27. H.S. Malvar, *Lapped transforms for efficient transform/subband coding*, IEEE Trans. Acoust., Speech, Signal Process, 38(6) (1990), pp. 969-978.
28. E. Osgooei and A. Rahimi, *An Explicit Structure for Duals of Frames in Krein Spaces*, Sahand Commun. Math. Anal., 22(2) (2025), pp. 143-173.
29. K. T. Poumai and S. Jahan, *Atomic systems for operators*, Int. J. Wavelets Multiresolution Inf. Process., 17(01) (2019), 1850066.
30. A. Rahimi, S. Basati, B. Daraby and F.A. Shah, *Woven(Weaving) frames in Banach spaces*, Kragujevac J. Math., 48(2) (2024), pp. 181-197.
31. A. Rahimi and S. Moayyadzadeh, *Scalable Fusion Frames*, Iran. J. Sci. Technol. Trans. A Sci., 49(3) (2025), pp. 825-831.
32. N. Teofanov, *Wilson bases and ultradistributions*, Axioms, 10(4) (2021), 241.
33. P. Wojdyłło, *Modified Wilson orthonormal bases*, Sampl. Theory Signal Image Process., 6 (2007), pp. 223-235.
34. P. Wojdyłło, *Characterization of Wilson systems for general lattices*, Int. J. Wavelets Multiresolution Inf. Process., 6(02) (2008), pp. 305-314.
35. P. Wojdyłło, *Wilson system for triple redundancy*, Int. J. Wavelets Multiresolution Inf. Process., 9(01) (2011), pp. 151-167.
36. R. M. Young, *An introduction to nonharmonic Fourier series*, Academic press, Inc. New York, 93 (1981).

---

[1] Department of Mathematics, Central University of Haryana, Mohindergarh-123031, India.
Email address: rekha222017@cuh.ac.in

[2] Department of Mathematics, Central University of Haryana, Mohindergarh-123031, India.
Email address: shahjahan@cuh.ac.in