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Modified Inertial Subgradient Extragradient-Type Algorithm for Zeros and Fixed Point Problems in Banach Spaces

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Abstract. In this paper, we introduce a new modified inertial self-adaptive subgradient extragradient-type algorithm for approximating an element in the set of common solutions of pseudomonotone equilibrium problems and common fixed points of a family of Bregman demigeneralized mappings which is also a common zero of the sum of Bregman inverse strongly monotone and maximal monotone operators in the setting of a reflexive Banach space. Strong convergence of the newly proposed scheme to a solutions of the said problems is established. Our results improve and generalize several results announced recently in the literature.

1. Introduction

Let X^* be a dual space of a reflexive real Banach space X. An operator $A : X \to 2^{X^*}$ is known to be $\alpha-$inverse strongly monotone if there exists α, a positive real number, such that for any $x, y \in X$, $u \in Ax, v \in Ay$ it holds that

$$\langle u - v, x - y\rangle \geq \alpha\|u - v\|^2. \tag{1.1}$$

Setting $\alpha = 0$ in (1.1) then the operator A becomes monotone. Let $Gr(A) := \{(x, u) \in X \times X^* : u \in X^*\}$ be the graph of a monotone operator A, then A is maximal monotone if no any other monotone operator say $\hat{A}$ can be obtained such that $Gr(A) \subset Gr\left(\hat{A}\right)$. Monotone operator theory which is known to play a vital role in semi group theory,

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optimization and variational inequality problems among others was independently studied by Kačurovskĭi [23], Minty [25] and Zarantonello [41].

Let A_1 and A_2 be two monotone operators. The problem of finding the zeros of the sum of A_1 and A_2 is to find $x \in X$ such that

$$0 \in (A_1 + A_2)x. \tag{1.2}$$

Denote by $(A_1 + A_2)^{-1}(0)$ the solution set of (1.2). Inclusion problem (1.2) which includes other problems of interest such as fixed point problems(FPP), minimization problems(MP), equilibrium problems(EP) and variational inequality problems(VIP) as special cases, has recently captured the attention of many authors. Many nonlinear problems arising in such areas as machine learning, signal processing and image recovery can be mathematically modeled as problem (1.2) above (see for example [21]) and the references therein. Considerable efforts have therefore been recorded, by several researchers, on approximation methods of solution for a sum of two monotone operators, see [29]. One of the early well known method for solving the Inclusion problem (1.2) is the forward-backward splitting method studied by Passty [29] in the setting of a Hilbert space H which is given as

$$x_1 \in H, \quad x_{n+1} = (I + \gamma B)^{-1}(x_n - \gamma A x_n), \quad n \geq 1, \tag{1.3}$$

where $\gamma > 0$. Other method includes the so-called Douglas-Rachford splitting algorithm [24] presented as

$$x_1 \in H, \quad x_{n+1} = 2J_{\gamma A}(2J_{\gamma B} - I)x_n + (I - 2J_{\gamma B})x_n, \quad n \geq 1, \tag{1.4}$$

where A and B are two maximal monotone operators.

Algorithms (1.3) and (1.4) generally fail to achieve strong convergence for problem (1.2). Because norm convergence is more desirable than weak convergence in certain applications, alternative methods that guarantee strong convergence are required.

Let $A : C \to X^*$ be a nonlinear mapping, a VIP associated with A is to find

$$z \in C \quad \text{such that } \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

Let $X \neq \emptyset$ and $g : C \times C \to \mathbb{R}$ be a bifunction such that C is a nonempty closed convex subset of X. Then the EP for a bifunction g is to find

$$w \in C \quad \text{satisfying } g(w, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

EP (1.6) was studied by Bluem and Otli [3] in the setting of Hilbert space. It generalizes many other important problems such as VIP, MP,

FPP to mention but a few. We denote by $EP(g,C)$ the set of solutions of EP (1.6), that is

$$EP(g,C) = \{w \in C : g(w,y) \geq 0, \quad \forall y \in C\}.$$

Various methods have been used to approximate the solutions of problems (1.6). One such method is the proximal point method in which the convergence analysis is considered when the bifunction g is monotone, see [22]. Moreover, the proximal point method is not applicable when the underlying bifunction g is pseudomonotone, see Wen [38].

Other method used is the extragradient method which involves two strongly convex optimization problems defined over the constrained set say C and the Lipschitz-type condition imposed on the bifunction g. However, solving the two strongly convex problem over the constrained set C in each iteration can sometimes be complicated. In trying to overcome this problem, Censor et al. [13] proposed a method called subgradient extragradient method for approximation of solutions of VIP in a real Hilbert space H, in which one projection was taken over constructed subspace that can easily be computed. Hieu [18] studied the subgradient extragradient method for EP thereby proving strong convergence of the following iterative scheme to the solution of the said problem in the setting of a real Hilbert space;

$$\begin{cases} x_0 \in H, \\ z_n = \arg\min\left\{g(x_n,z) + \frac{1}{2\lambda}\|z-x_n\|^2, z \in C\right\}, \\ y_n = \arg\min\left\{g(z_n,z) + \frac{1}{2\lambda}\|z-x_n\|^2, z \in \Gamma_n\right\}, \\ x_{n+1} = \mu_n x_0 + (1-\mu_n)y_n, n \geq 0, \end{cases} \tag{1.7}$$

where $\Gamma_n = \{w \in H : \langle (x_n - \lambda w_n) - z_n, w - z_n\rangle \leq 0\}$ such that $w_n \in \partial_2 g(x_n, z_n)$ with λ and $\{\mu_n\}$ satisfying

(i) $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and

(ii) $0 < \mu_n < 1$, $\lim_{n\to\infty}\mu_n = 0$, $\sum_{n=1}^{\infty}\mu_n = +\infty$ respectively.

Dadashi et al. [15] used the aforementioned method to approximate solution of pseudomonotone equilibrium problem in a real Hilbert space. One problem with subgradient extragradient algorithm was the computation of Lipschitz-type constants of the bifunction which is sometimes difficult to estimate. Motivated by this, Liu and Yang [39] studied the subgradient extragradient method for pseudomonotone equilibrium problems using an introduced self adaptive step size thereby dropping

the estimation of the Lipschitz constants requirement for the method. Bokodisa, Jolaoso and Aphane [9] also studied self adaptive subgradient extragradient type scheme for equilibrium and fixed point problems of Bregman quasi nonexpansive mappings in the setting of a reflexive Banach space;

$$
\begin{cases}
\text{For } \lambda_1 > 0, \mu \in (0,1) \text{ and } v, x_1 \in X \text{ chosen abitrarily,} \\
y_n = \underset{y \in C}{\arg\min} \left\{ \lambda_n g(x_n, y) + D_f(y, x_n) \right\}, \\
\Gamma_n = \Big\{ y \in X : \langle \nabla f(x_n) - \lambda_n w_n - \nabla f(y_n), y - y_n \rangle \leq 0, \\
\qquad w_n \in \partial_2 g(x_n, y_n) \Big\}, \\
z_n = \underset{y \in \Gamma_n}{\arg\min} \left\{ \lambda_n g(y_n, y) + D_f(y, x_n) \right\}, \\
u_n = \nabla f^* \left(\mu_n \nabla f(v) + (1 - \mu_n) \nabla f(z_n) \right), \\
x_{n+1} = \nabla f^* \left(\beta_{n,0} \nabla f(u_n) + \sum_{s=1}^{\infty} \beta_{n,i} \nabla f(T_j u_n) \right), \\
n \geq 1,
\end{cases} \tag{1.8}
$$

$$
\lambda_{n+1} = \begin{cases}
\min \left\{ \frac{\mu(D_f(y_n, x_n) + D_f(z_n, y_n))}{(g(x_n, z_n) - g(x_n, y_n) - g(y_n, z_n))}, \lambda_n \right\}, \\
\qquad \text{if} (g(x_n, z_n) - g(x_n, y_n) - g(y_n, z_n)) \neq 0, \\
\lambda_n, \qquad \text{otherwise.}
\end{cases}
$$

The study of convergence properties of an iterative procedure has become an area of contemporary interest to many researchers. The inertial method due to Polyak [31] was introduced to speed up the rate of convergence of iterative methods. Motivated by this and the work of Liu and Yang [39], Harbau [17] studied inertial self-adaptive subgradient extragradient method for pseudomonotone equilibrium problem and FPP of quasi$-\phi-$nonexpansive multivalued mapping in the setting of uniformly convex uniformly smooth Banach spaces.

Let $dom(f)$ denote the domain of a proper, convex and lower semicontinuous function $f : X \to (-\infty, +\infty]$. Then

$$dom(f) := \{x \in X : f(x) < +\infty\}.$$

Now for any $u \in \text{int}(domf)$ and $y \in X$, we denote by $f^{'}(u,y)$ the right-hand derivative of f at u in the direction of y, which is defined as

$$f^{'}(u,y) = \lim_{t\to 0} \frac{f(u+ty)-f(u)}{t}. \tag{1.9}$$

The function f is known to be Gâteaux differentiable at u if, for each y, the limit in (1.9) exists. In this regard, the gradient of f at u is a function $\nabla f(u) : X \to (-\infty, +\infty]$ given by

$$\langle \nabla f(u), y\rangle = f^{'}(u,y), \quad \text{for all } y \in X.$$

The function f is said to be Gâteaux differentiable on $\text{int}(domf)$ if it is Gâteaux differentiable at every point $u \in \text{int}(domf)$. In addition, f is said to be Fréchet differentiable at u provided the limit in (1.9) is attained uniformly for any $y \in X$ with $\|y\| = 1$ and it is uniformly Fréchet differentiable on a subset Ω of X if the limit in (1.9) is attained uniformly for $u \in X$ and $\|y\| = 1$.

Let $u \in \text{int}(domf)$, the subdifferential of f at u, $\partial f(u)$, is a convex set defined as

$$\partial f(u) = \{u^* \in X^* : f(u) + \langle u^*, y-u\rangle \leq f(y), \forall y \in X\},$$

and the Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(u^*) = \sup\{\langle u^*, u\rangle - f(u) : u \in X\}, \quad \forall u^* \in X^*. \tag{1.10}$$

Observe that f^* defined by (1.10) above is proper, convex and lower semicontinuous as f is. In addition, $u^* \in \partial f(u)$ if and only if $f(u) + f^*(u^*) = \langle u^*, u\rangle$, see [20].

Let $f : X \to (-\infty, +\infty]$ be convex and Gâteaux differentiable function. The function $D_f : domf \times \text{int}(domf) \to [0, +\infty)$ defined as

$$D_f(x,y) = f(x) - f(y) - \langle \nabla f(y), x-y\rangle, \quad \forall x \in domf, y \in \text{int}(domf) \tag{1.11}$$

is called the Bregman distance with respect to f [12].

Observe that D_f here is not a distance function in the usual sense. In general, D_f neither satisfies symmetry nor triangular inequality. However, for all $x \in domf$ and $y, z \in \text{int}(domf)$, D_f satisfies the so-called three point identity

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle x-y, \nabla f(z) - \nabla f(y)\rangle. \tag{1.12}$$

Let $T : C \to C$ be a map with C a nonempty subset of a Banach space X. A point $\hat{x} \in C$ is called a fixed point of T if $T\hat{x} = \hat{x}$. The set of fixed point of T is denoted by $F(T)$. If C contains a sequence $\{x_n\}$ which converges weakly to $\hat{x}$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\hat{x}$ is called an

asymptotic fixed point of the map T [32]. The set of asymptotic fixed point of T is denoted by $\hat{F}(T)$.

Definition 1.1 ([11])**.** Let C be a nonempty closed convex subset of X. A mapping $T : C \to \text{int}(domf)$ is called

(i) Bregman firmly nonexpansive if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle, \quad \forall x, y \in C.$$

(ii) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall p \in \hat{F}(T).$$

(iii) Bregman quasi-nonexpansive if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C \text{and } \forall p \in F(T).$$

Let $B : X \to 2^{X^*}$ be a maximal monotone operator and $\lambda > 0$. An operator $Res^f_{\lambda B} : X \to 2^X$ defined by

$$Res^f_{\lambda B} := (\nabla f + \lambda B)^{-1} \circ \nabla f$$

is called the resolvent operator of B. It is known that $Res^f_{\lambda B}$ is a Bregman firmly nonexpansive operator, it is also single-valued and $F\left(Res^f_{\lambda B}\right) = B^{-1}(0)$ [23].

A multivalued operator $A : X \to 2^{X^*}$ is called Bregman inverse strongly monotone [28] if for any $x, y \in \text{int}(domf)$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v) \rangle \geq 0, \quad \forall u \in Ax, v \in Ay.$$

Define $A^f : X \to 2^X$ by $A^f := \nabla f^* \circ (\nabla f - A)$, then A^f here is called the antiresolvent operator of A. It was shown in [5] that A is Bregman inverse strongly monotone if and only if A^f is single-valued Bregman firmly nonexpansive and $F\left(A^f\right) = A^{-1}(0)$.

Let X be a reflexive Banach space, C be a nonempty closed convex subset of X and $\eta \in (-\infty, 1)$. Then a map $T : C \to X$ with $F(T) \neq \emptyset$ is called $(\eta, 0)-$Bregman demigeneralized mapping [2] if for any $x \in C$ and $q \in F(T)$

$$\langle x - q, \nabla f(x) - \nabla f(Tx) \rangle \geq (1 - \eta) D_f(x, Tx). \tag{1.13}$$

Due to their useful applications, the problems of finding zeros of sum of maximal monotone and Bregman inverse strongly monotone operators which is also a fixed point of Bregman demigeneralized mapping have been studied by many authors. Very recently, Ali and Auwalu [1] studied

the said problems in the setting of a reflexive Banach space. They established a strong convergence theorem of the following algorithm;

$$
\begin{cases}
v, x_1 \in X \text{ chosen arbitrarily,} \\
z_n^i = \underset{y\in C}{\arg\min}\left\{\lambda_n h_i(x_n, y) + D_f(y, x_n), i = 1, 2, 3, \ldots, N\right\}, \\
y_n^i = \underset{y\in C}{\arg\min}\left\{\lambda_n h_i(z_n^i, y) + D_f(y, x_n), i = 1, 2, 3, \ldots, N\right\}, \\
i_n = \arg\max\left\{D_f\left(y_n^i, x_n\right), i = 1, 2, 3, \ldots, N\right\}, \overline{y}_n = y_n^{i_n}, \\
w_n = \nabla f^*\left(\beta_n \nabla f(\bar{y}_n) + (1-\beta_n)\nabla f\left(U_j \bar{y}_n\right)\right), j = 1, 2, \cdots, m, \\
u_n = \nabla f^*\Big(\alpha_n \nabla f\left(\bar{y}_n\right) + \gamma_n \nabla f\left(x_n\right) + \eta_n \nabla f\left(w_n\right) \\
\qquad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda B_s} \circ A^f_s(w_n\right)\Big), \\
x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n)), n \geq 1,
\end{cases}
\tag{1.14}
$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_{n,s}\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences satisfying some conditions.

In another development, Ali et al. [2], studied a modified inertial subgradient extragradient algorithm in real reflexive Banach space. They proved strong convergence theorem for the sequence generated by the following algorithm to the solutions of VIP and fixed point of Bregman demigeneralized mappings;

$$
\begin{cases}
x_0, x_1 \in X \text{ chosen arbitrarily,} \\
\theta_n = \nabla f^*(\nabla f(x_n) + \sigma_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\
y_n = P_C \nabla f^*(\nabla f(\theta_n) - \lambda A\theta_n), \\
T_n = \{z \in X : \langle z - y_n, \nabla f(\theta_n) - \lambda A\theta_n - \nabla f(y_n)\rangle \leq 0\}, \\
w_n = P_{T_n} \nabla f^*(\nabla f(\theta_n) - \lambda A y_n), \\
z_n = \nabla f^*((1-\alpha_n)\nabla f(w_n) + \alpha_n \nabla f(Tw_n)), \\
x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + \delta_n \nabla f(z_n) + \gamma_n \nabla f(u)), n \geq 0,
\end{cases}
\tag{1.15}
$$

where $\sigma_n \in \left(0, \frac{1}{2}\right)$, $\lambda \in \left(0, \frac{\alpha}{L}\right)$ such that $L > 0$ and $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying some conditions.

Motivated by the above mentioned results, the purpose of this paper is to propose and study a modified inertial self adaptive subgradient extragradient type algorithm for approximating common solution in the set of common fixed point of finite families of Bregman demigeneralized mappings and the set of solution of equilibrium problems of pseudomonotone type which is also a common zero of the sum of maximal monotone and Bregman inverse strongly monotone operators in the setting of reflexive real Banach spaces. The estimation of the Lipschitz-type constant requirement in the well known subgradient extragradient method are dispensed with. Our results extend and improve some recent results announced by some authors in the literature.

2. Preliminaries

We shall, in the rest of this paper, use " $\rightharpoonup$ " and " $\to$ " for weak and strong convergence respectively. However, the following concepts and Lemmas are essential in the proof of our main results.

Definition 2.1 ([4])**.** The function $f : X \to (-\infty, +\infty]$ is known to be:

(1) Essentially smooth if ∂f is locally bounded and single-valued on its domain;
(2) Essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $domf$;
(3) Legendre when it is both essentially smooth and essentially strictly convex.

For a Legendre function f, we have the following properties;

(i) f is Legendre if and only if f^* is Legendre (see [10, Corollary 5.5]);
(ii) $(\partial f)^{-1} = \partial f^*$ (see [10, p.83]);
(iii) ∇f is a bijection and it satisfies $\nabla f = (\nabla f^*)^{-1}$, $ran\nabla f = dom\nabla f^* = \text{int}(domf^*)$ and $ran\nabla f^* = dom\nabla f = \text{int}(domf)$.

Also, if $f : X \to \mathbb{R}$ is a Legendre function which is bounded and uniformly Fréchet differentiable on a bounded subsets of X, then $Res^f_{\lambda B}$ is Bregman strongly nonexpansive and $\hat{F}\left(Res^f_{\lambda B}\right) = F\left(Res^f_{\lambda B}\right)$ [33].

Lemma 2.2 ([37])**.** *Let X be a reflexive Banach space, C be a nonempty convex subset of X, $f : C \to \mathbb{R}$ be a convex and subdifferentiable function. Then f attains its minimum at $x \in C$ if and only if $0 \in \partial f(x) + \mathcal{N}_C(x)$, where $\mathcal{N}_C(x)$ is defined as*

$$\mathcal{N}_C(x) := \{x^* \in X^* : \langle x - z, x^* \rangle \geq 0, \forall z \in C\},$$

which is a normal cone of C at x.

Lemma 2.3 ([14]). *Let X be a reflexive Banach space. Let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be two convex functions such that $domf \cap domg \neq \emptyset$. If f is continuous, then*

$$\partial(f + g) = \partial f(x) + \partial g(x), \quad \forall x \in X.$$

Lemma 2.4 ([2]). *Let C be a nonempty, closed and convex subset of a reflexive Banach space X. Let $f : X \to \mathbb{R}$ be a strongly coercive and Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of X. Let $\eta \in (-\infty, 1)$ and suppose T is an $(\eta, 0)-$Bregman demigeneralized mapping of C onto X. Then $F(T)$ is closed and convex.*

Lemma 2.5 ([2]). *Let C be a closed convex and nonempty subset of a reflexive Banach space X. Let $f : X \to \mathbb{R}$ be a convex and Fréchet differentiable function. Let, for $\eta \in (-\infty, 0]$, $T : C \to X$ be $(\eta, 0)-$Bregman demigeneralized map with $F(T) \neq \emptyset$. Suppose α is a real number in $[0, 1)$ and let $Q = \nabla f^*(\alpha \nabla f + (1 - \alpha)\nabla f(T))$. Then $Q : C \to X$ is quasi-Bregman nonexpansive map.*

Lemma 2.6 ([36]). *Let $A : X \to X^*$ and $B : X \to 2^{X^*}$ be Bregman inverse strongly monotone and maximal monotone operators respectively. Let $R_\lambda x := Res^f_{\lambda B} \circ A^f(x)$ for $x \in X$ and $\lambda > 0$. Then $F(R_\lambda) = (A + B)^{-1}(0)$.*

Lemma 2.7 ([36]). *Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose $A : X \to X^*$ is a Bregman inverse strongly monotone mapping and $B : X \to 2^{X^*}$ is a maximal monotone operator. Then it holds that:*

$$D_f\left(z, Res^f_{\lambda B} \circ A^f(x)\right) + D_f\left(Res^f_{\lambda B} \circ A^f(x), x\right) \leq D_f(z, x),$$

$\forall z \in (A + B)^{-1}(0), \quad x \in X and\ \lambda > 0.$

Lemma 2.8 ([33]). *Let C be a closed convex and nonempty subset of a Banach space X. Let $f : X \to \mathbb{R}$ be a Legendre function, if $T : C \to X$ is a Bregman quasi-nonexpansive operator, then $F(T)$ is closed and convex.*

Lemma 2.9 ([40]). *Suppose f is convex and bounded on bounded subset of a Banach space X. Then f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on a bounded subset of X^*.*

Butnariu and Iusem [6] studied the notion of modulus of convexity of a function. Let X be a Banach space, let $f : X \to (-\infty, +\infty]$ be a

Gâteaux differentiable function. The modulus of total convexity of f at $x \in domf$ is the function $v_f(x,.) : [0,+\infty] \to [0,+\infty]$ defined by

$$v_f(x,t) := \inf \{D_f(x,y) : y \in \text{int}(domf), \|y-x\| = t\}.$$

The function f here is called totally convex at x if $v_f(x,t) > 0$ whenever $t > 0$. It is said to be convex if it is totally convex at any point $x \in \text{int}(domf)$. Let B be bounded and nonempty subset of X. The modulus of total convexity of f on the set B is the function $v_f : \text{int}(domf) \times [0,+\infty) \to [0,+\infty)$ defined by

$$v_f(B,t) := \inf \{v_f(x,t) : x \in B \cap domf\}.$$

The function f is said to be totally convex on a bounded subset if $v_f(B,t) > 0$ for any nonempty and bounded subset B of X and for any $t > 0$.

Lemma 2.10 ([34]). *Suppose $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Lemma 2.11 ([35]). *Let $f : X \to \mathbb{R}$ be Gâteaux differentiable and totally convex function. Suppose $x_0 \in X$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

A function $f : X \to \mathbb{R}$ is called sequentially consistent (see [8]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X for which one is bounded and $\lim_{n\to\infty} D_f(y_n, x_n) = 0$ implies

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.12 ([6]). *If $domf$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Let $f : X \to (-\infty, \infty]$ be convex and Gâteaux differentiable function. Then the Bregman projection [7] with respect to f, $P_C^f : \text{int}(domf) \to C$, of $x \in \text{int}(domf)$ onto nonempty closed convex set $C \subset domf$ is defined to be a unique vector $P_C^f(x) \in C$ such that

$$D_f\left(P_C^f(x), x\right) = \inf \{D_f(y,x) : y \in C\}.$$

Recall the following properties concerning the Bregman projection as studied in [8].

Lemma 2.13 ([8]). *Let X be a reflexive Banach space and C be a nonempty closed convex subset of X, let $f : X \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in X$. Then*

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$, $\forall y \in C$;
(ii) $D_f\left(y, P_C^f(x)\right) + D_f\left(P_C^f(x), x\right) \leq D_f(y, x)$, $\forall x \in X, y \in C$.

Lemma 2.14 ([8]). *Let X be a reflexive Banach space, let $f : X \to \mathbb{R}$ be a strongly coercive Bregman function and define $V_f : X \times X^* \to [0, +\infty]$ by*

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in X, x^* \in X^*.$$

Then it holds that

$$D_f(x, \nabla f^*(x^*)) = V_f(x, x^*), \quad \forall x \in X, x^* \in X^* \tag{2.1}$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \leq V_f(x, x^* + y^*), \quad \forall x \in X, \forall x^*, y^* \in X^*. \tag{2.2}$$

Let $\phi_x : X^* \to \mathbb{R}$ be a map defined by

$$\phi_x(f) = \langle f, x\rangle, \quad \text{for each } x \in X \text{and} f \in X^*.$$

Then, the weak star (denoted by weak^*) topology is the smallest topology on X^* for which the map ϕ_x is continuous.

Lemma 2.15 ([30]). *Let X be a reflexive Banach space, $f : X \to (-\infty, +\infty]$ be a proper and lower semicontinuous function, then $f^* : X^* \to (-\infty, +\infty]$ is a proper weak*, lower semicontinuous and convex function. Moreover, for all $z \in X$, $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$ we have that*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i).$$

Let X be a Banach space, let B and S be respectively a closed unit ball and a unit sphere of X. Let $rB = \{z \in X : \|z\| \leq r\}$ for all $r > 0$. Then the function $f : X \to \mathbb{R}$ is called uniformly convex on bounded subset (see [40]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is a function defined by

$$\rho_r(t) := \inf_{x,y \in rB, \|x-y\| = t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)},$$

$\forall t \geq 0$.

The function ρ_r which is known to be nondecreasing is called the gauge function of uniform convexity of f. Moreover, if f is uniformly convex function then the following result is well known.

Lemma 2.16 ([27]). *Let X be a Banach space and $f : X \to \mathbb{R}$ be a uniformly convex function on bounded subsets of X. Then*

$$f\left(\sum_{k=0}^{n}\alpha_k x_k\right) \leq \sum_{k=0}^{n}\alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(\|x_i - x_j\|), \quad \forall i,j \in \{0,1,2,\ldots,n\},$$

where $r > 0$, $x_k \in rB$, $\alpha_k \in (0,1)$ such that $k = 0,1,2,\ldots,n$ with $\sum_{k=0}^{n}\alpha_k = 1$ and ρ_r a gauge function of uniform convexity of f.

The function f on the other hand is said to be uniformly smooth on bounded subsets [40] if

$$\lim_{t\to 0^-}\frac{\rho_r(t)}{t} = 0, \quad \forall r > 0,$$

where $\rho_r : [0,\infty) \to [0,\infty]$ is defined here by

$$\rho_r(t) = \sup_{x\in rB, y\in S, \alpha\in(0,1)} \frac{\alpha f(x+(1-\alpha)ty) + (1-\alpha)f(x-\alpha ty) - f(x)}{\alpha(1-\alpha)},$$

$\forall t \geq 0$.

A function f is also called strongly coercive if $\lim_{\|x\|\to\infty}\frac{f(x)}{\|x\|} = +\infty$.

Definition 2.17 ([35]). Let C be a closed convex and nonempty subset of a Banach space X. A bifunction $g : C \times C \to \mathbb{R}$ is said to be

(1) Monotone if $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$,
(2) Pseudomonotone if $g(x,y) \geq 0$ implies $g(y,x) \leq 0$.

Observe that every (1) is (2) but not the converse, see [35]. It is assumed, in this paper, a bifunction g satisfies the following properties:

(P1) g is Pseudomonotone on C,
(P2) g is Bregman-Lipschitz type continuous, i.e. $g(x,y)+g(y,z) \geq g(x,z) - c_1 D_f(y,x) - c_2 D_f(z,y), \forall z \in C, x, y \in \text{int}dom(f)$ and for some $c_1, c_2 > 0$, where $f : X \to (-\infty,+\infty]$ is a Legendre function,
(P3) g is weakly continuous on $C \times C$, i.e. if $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and $y \in C$ respectively, then $g(x_n, y_n) \to g(x,y)$,
(P4) $g(x,.) : C \to \mathbb{R}$ is convex, lower semicontinuous and subdifferentiable,
(P5) $\limsup_{t\to 0^-} g(tx + (1-t)y, w) \leq g(y,w)$ for each $x, y, w \in C$.

Lemma 2.18 ([16]). *Let g be a bifunction satisfying $(P1)$, $(P3)-(P5)$. Then $EP(g,C)$ is closed and convex.*

Concerning VIP (1.5), we have the following results.

Lemma 2.19 ([16]). *Let C be a nonempty closed convex subset of a reflexive Banach space X. Let $g : C \times C \to \mathbb{R}$ be bifunction satisfying $g(x, x) = 0$ and suppose $f : X \to \mathbb{R}$ is a Legendre and totally coercive function. Then, a point $\hat{x} \in EP(g, C)$ if and only if $\hat{x}$ solves the following minimization problem*

$$\min \{\lambda g(x, y) + D_f(y, x) : y \in C\}, \quad \text{where } x \in C \text{ and } \lambda > 0.$$

Lemma 2.20 ([19]). *Let C be a nonempty closed convex subset of a real reflexive Banach space X. Let $A : C \to X^*$ be a nonlinear mapping and $f^* : X \to \mathbb{R}$ be a Legendre function. Then*

$$P_C^f(\nabla f^*(\nabla f(x) - \lambda Ay)) = \arg\min_{w \in C} \{\lambda \langle w - y, Ay \rangle + D_f(w, x)\},$$

$\forall x \in X, y \in C and\ \lambda \in (0, +\infty)$.

Lemma 2.21 ([26]). *Let $\{r_n\}$ be a sequence of real numbers such that there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ satisfying $r_{n_j} < r_{n_j+1} \forall j \geq 0$. Let $\{m_k\} \subset \mathbb{N}$ be defined by*

$$m_k = \max \{i \leq k : r_i < r_{i+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence satisfying $\lim_{k \to \infty} m_k = \infty$ and for all $k \geq n_0$, the following two estimates hold;

$$r_{m_k} \leq r_{m_k+1}, \qquad r_k \leq r_{m_k+1}.$$

Lemma 2.22 ([33]). *Let $\{r_n\}$ be a sequence of nonnegative real numbers such that*

$$r_{n+1} \leq (1 - \mu_n) r_n + \mu_n \gamma_n, \quad n \geq 0 \tag{2.3}$$

with $\{\mu_n\} \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\limsup_{n \to \infty} \gamma_n \leq 0$. Then $\lim_{n \to \infty} r_n = 0$.

3. Main Results

In this section, using a constructed half space, we propose and study a modified inertial subgradient extragradient-type algorithm for approximation of solutions of pseudomonotone equilibrium problems and common fixed points of family of Bregman demigeneralized mappings which is also a common zero of the sum of Bregman inverse strongly monotone and maximal monotone operators in the setting of reflexive Banach

space.

(3.1)

$$\begin{cases} \text{For } \lambda_1 > 0, \mu \in (0,1) \text{ and } v, x_1 \in X \text{ chosen abitrarily,} \\ \\ \theta_n = \nabla f^* \left(\nabla f(x_n) + \sigma_n \left(\nabla f(x_{n-1}) - \nabla f(x_n) \right) \right), \\ \\ y_n^i = \underset{y \in C}{\arg\min} \left\{ \lambda_n g_i(\theta_n, y) + D_f(y, \theta_n), i = 1, 2, 3, \ldots, N \right\}, \\ \\ \Gamma_{n,i} = \left\{ z \in X : \left\langle z - y_n^i, \nabla f(\theta_n) - \lambda_n w_n - \nabla f\left(y_n^i\right) \right\rangle \leq 0, \right. \\ \qquad \left. w_n \in \partial_2 g_i\left(\theta_n, y_n^i\right) \right\}, \\ \\ u_n^i = \underset{y \in \Gamma_{n,i}}{\arg\min} \left\{ \lambda_n g_i\left(y_n^i, y\right) + D_f(y, \theta_n), i = 1, 2, 3, \ldots, N \right\}, \\ \\ i_n = \arg\max \left\{ D_f\left(u_n^i, \theta_n\right), i = 1, 2, 3, \ldots, N \right\}, \overline{u}_n = u_n^{i_n}, \\ \\ z_n = \nabla f^* \left(\alpha_n \nabla f(\overline{u}_n) + (1 - \alpha_n) \nabla f\left(S_j \overline{u}_n\right) \right), j = 1, 2, \ldots, m, \\ \\ r_n = \nabla f^* \Big(\beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\ \qquad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n) \right) \Big), \\ \\ x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1 - \mu_n) \nabla f(r_n)), n \geq 1, \end{cases}$$

where $\kappa_{n,i} := \left(g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right) - g_i\left(y_n^i, u_n^i\right) \right)$ is such that

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\left(D_f\left(y_n^i, \theta_n\right) + D_f\left(u_n^i, y_n^i\right)\right)}{\kappa_{n,i}}, \lambda_n \right\}, & \kappa_{n,i} > 0, \\ \\ \lambda_n, & \text{otherwise,} \end{cases}$$

and $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_{n,s}\}$ and $\{\mu_n\}$ are sequences satisfying the following conditions;

(S1) $\{\mu_n\} \subset (0,1)$ such that $\lim_{n \to \infty} \mu_n = 0$ and $\sum_{n=0}^{\infty} \mu_n = \infty$,

(S2) $0 < \gamma \leq \beta_n, \gamma_n, \eta_n, \delta_{n,s} \leq \beta < 1$ and $\beta_n + \gamma_n + \eta_n + \sum_{s=1}^{\overline{N}} \delta_{n,s} = 1$, $\forall n \in \mathbb{N}$,

(S3) $0 < a \leq \alpha_n \leq \min\{1 - k_1, 1 - k_2, \ldots, 1 - k_m\}$ $\forall n$,

(S4) $\{\sigma_n\} \subset [0, \infty)$ such that $\sigma_n = o(\mu_n)$.

Remark 3.1. It is obvious from the iterative scheme (3.1) that $C \subseteq \Gamma_{n,i}$.

Indeed, since $y_n^i = \arg\min_{y \in C} \{\lambda_n g_i(\theta_n, y) + D_f(y, \theta_n)\}$, for each $i = 1, 2, 3, \ldots, N$, we obtain from Lemma 2.2 and Lemma 2.3 that

$$0 \in \lambda_n \partial_2 g_i \left(\theta_n, y_n^i\right) + \nabla_1 D_f \left(y_n^i, \theta_n\right) + \mathcal{N}_C \left(y_n^i\right),$$

that is there exists $w_n \in \lambda_n \partial_2 g_i \left(\theta_n, y_n^i\right)$ such that

$$\nabla f(\theta_n) - \lambda_n w_n - \nabla f \left(y_n^i\right) \in \mathcal{N}_C \left(y_n^i\right).$$

Hence there exists $w \in \mathcal{N}_C \left(y_n^i\right)$ for which $\lambda_n w_n + \nabla f \left(y_n^i\right) - f(\theta_n) + w = 0$. Thus

$$\begin{aligned} \left\langle \nabla f(\theta_n) - \nabla f \left(y_n^i\right), y - y_n^i \right\rangle &= \left\langle \lambda_n w_n, y - y_n^i \right\rangle + \left\langle w, y - y_n^i \right\rangle \\ &\leq \left\langle \lambda_n w_n, y - y_n^i \right\rangle, \quad \forall y \in C, \end{aligned}$$

from which it follows that

$$\left\langle \nabla f(\theta_n) - \lambda_n w_n - \nabla f \left(y_n^i\right), y - y_n^i \right\rangle \leq 0, \quad \forall y \in C.$$

Hence $C \subseteq \Gamma_{n,i}$.

Lemma 3.2. *The sequence $\{\lambda_n\}$ defined as in iterative scheme (3.1) is monotone decreasing having a lower bound $\ell := \min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \lambda_1\right\}$ where c_1, c_2 are constants as in assumption $(P2)$.*

Proof. It is obvious that the sequence $\{\lambda_n\}$ is monotone decreasing.
Claim 1: $\ell := \min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \lambda_1\right\}$ is a lower bound for $\{\lambda_n\}$. Indeed, by property $(P2)$ we get that

$$\begin{aligned} \frac{\mu \left(D_f \left(y_n^i, \theta_n\right) + D_f \left(u_n^i, y_n^i\right)\right)}{\kappa_{n,i}} &\geq \frac{\mu \left(D_f \left(y_n^i, \theta_n\right) + D_f \left(u_n^i, y_n^i\right)\right)}{c_1 D_f \left(y_n^i, \theta_n\right) + c_2 D_f \left(u_n^i, y_n^i\right)} \\ &\geq \frac{\mu}{2\max\{c_1, c_2\}}. \end{aligned}$$

Thus,

$$\begin{aligned} \min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_1\right\} &\leq \min\left\{\frac{\mu(D_f \left(y_n^i, \theta_n\right) + D_f \left(u_n^i, y_n^i\right))}{\kappa_{n,i}}, \lambda_n\right\} \\ &\leq \lambda_n, \end{aligned}$$

verifying Claim 1. Hence the proof. □

Lemma 3.3. *Let X be a reflexive Banach, let C be a nonempty closed convex subset of X and let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. For $i = 1, 2, 3, \ldots, N$, let $g_i : C \times C \to \mathbb{R}$ be bifunction satisfying properties (1), (2), $(P1) - (P4)$. For $\{\lambda_n\} \subset (0, +\infty)$, let $\{x_n\}$ be a sequence generated by algorithm (3.1). Then for any*

$$q \in \Omega = \left\{ x^* \in \cap_{s=1}^{\overline{N}} (B_s + M_s)^{-1}(0) \cap \left(\cap_{j=1}^{m} F\left(S_j\right)\right) \cap \left(\cap_{i=1}^{N} EP(g_i, C)\right) \right\}$$

we have

$$D_f(q, \overline{u}_n) \leq D_f(q, \theta_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right)\left(D_f\left(y_n^i, \theta_n\right) + D_f\left(\overline{u}_n, y_n^i\right)\right).$$

Proof. Let $q \in \Omega$. Then from the definition of u_n^i in algorithm (3.1), Lemma 2.2 and Lemma 2.3, it follows that for each $i = 1, 2, 3, \ldots, N$,

$$0 \in \lambda_n \partial_2 g_i\left(y_n^i, u_n^i\right) + \nabla_1 D_f\left(u_n^i, \theta_n\right) + \mathcal{N}_{\Gamma_{n,i}}\left(u_n^i\right).$$

Hence, $0 = \lambda_n \overline{w}_n + \nabla f\left(u_n^i\right) - \nabla f(\theta_n) + \overline{w}$, for some $\overline{w}_n \in \partial_2 g_i\left(y_n^i, u_n^i\right)$ and $\overline{w} \in \mathcal{N}_{\Gamma_{n,i}}\left(u_n^i\right)$. Thus,

$$\overline{w} = -\nabla f\left(u_n^i\right) - \lambda_n \overline{w}_n + \nabla f(\theta_n). \tag{3.2}$$

By the definition of $\mathcal{N}_{\Gamma_{n,i}}\left(u_n^i\right)$

$$\left\langle y - u_n^i, \overline{w}\right\rangle \leq 0, \forall y \in \Gamma_{n,i}. \tag{3.3}$$

Using (3.2) together with (3.3) we obtain

$$\left\langle y - u_n^i, -\lambda_n \overline{w}_n - \nabla f\left(u_n^i\right) + \nabla f(\theta_n)\right\rangle \leq 0,$$

which implies that

$$\lambda_n \left\langle y - u_n^i, \overline{w}_n\right\rangle \geq \left\langle y - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right)\right\rangle, \quad \forall y \in \Gamma_{n,i}.$$

Since $q \in \Omega \subset \cap_{i=1}^{N} EP(g_i, C) \subset C \subset \Gamma_{n,i}$, we have

$$\lambda_n \left\langle q - u_n^i, \overline{w}_n\right\rangle \geq \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right)\right\rangle. \tag{3.4}$$

On the other hand $\overline{w} \in \partial_2 g_i\left(y_n^i, u_n^i\right)$ implies

$$g_i\left(y_n^i, y\right) - g_i\left(y_n^i, u_n^i\right) \geq \left\langle y - u_n^i, \overline{w}_n\right\rangle, \quad \forall y \in X. \tag{3.5}$$

Therefore, (3.4) and (3.5) give

$$\lambda_n\left(g_i(y_n^i, q) - g_i\left(y_n^i, u_n^i\right)\right) \geq \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right)\right\rangle. \tag{3.6}$$

From property $(P3)$ we get

$$\begin{aligned} -\lambda_n g_i\left(y_n^i, u_n^i\right) &\geq \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right)\right\rangle - \lambda_n g_i(y_n^i, q) \\ &\geq \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right)\right\rangle. \end{aligned} \tag{3.7}$$

Since $w_n \in \partial_2 g_i\left(\theta_n, y_n^i\right)$ then

$$g_i(\theta_n, y) - g_i\left(\theta_n, y_n^i\right) \geq \left\langle y - y_n^i, w_n \right\rangle, \quad \forall y \in X,$$

so that letting $y = u_n^i$ we have

$$g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right) \geq \left\langle u_n^i - y_n^i, w_n \right\rangle.$$

Thus

$$\lambda_n(g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right)) \geq \lambda_n \left\langle u_n^i - y_n^i, w_n \right\rangle. \tag{3.8}$$

As $u_n^i \in \Gamma_{n,i}$ we get

$$\left\langle u_n^i - y_n^i, \nabla f(\theta_n) - \nabla f\left(y_n^i\right) \right\rangle \leq \lambda_n \left\langle u_n^i - y_n^i, w_n \right\rangle. \tag{3.9}$$

Combining (3.7), (3.8), (3.9) and the three point identity (1.12) we get

$$\begin{aligned}
&\lambda_n \left(g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right) - g_i\left(y_n^i, u_n^i\right)\right) \\
&\quad \geq \lambda_n \left\langle u_n^i - y_n^i, w_n \right\rangle + \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right) \right\rangle \\
&\quad \geq \left\langle u_n^i - y_n^i, \nabla f(\theta_n) - \nabla f\left(y_n^i\right) \right\rangle + \left\langle q - u_n^i, \nabla f(\theta_n) - \nabla f\left(u_n^i\right) \right\rangle \\
&\quad = D_f\left(u_n^i, y_n^i\right) + D_f\left(y_n^i, \theta_n\right) - D_f\left(u_n^i, \theta_n\right) + D_f\left(q, u_n^i\right) \\
&\qquad + D_f\left(u_n^i, \theta_n\right) - D_f(q, \theta_n).
\end{aligned}$$

This implies

$$\begin{aligned}
D_f\left(q, u_n^i\right) &\leq D_f(q, \theta_n) - D_f\left(u_n^i, y_n^i\right) - D_f\left(y_n^i, \theta_n\right) \\
&\quad + \frac{\lambda_n}{\lambda_{n+1}} \left[\lambda_{n+1} \left(g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right) - g_i\left(y_n^i, u_n^i\right)\right)\right] \\
&\leq D_f(q, \theta_n) - D_f\left(u_n^i, y_n^i\right) - D_f\left(y_n^i, \theta_n\right) \\
&\quad + \frac{\lambda_n}{\lambda_{n+1}} \left(\mu \left(D_f\left(y_n^i, \theta_n\right) + D_f\left(u_n^i, y_n^i\right)\right)\right) \\
&= D_f(q, \theta_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f\left(y_n^i, \theta_n\right) \\
&\quad - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f\left(u_n^i, y_n^i\right)
\end{aligned}$$

for each $i \in \{1, 2, 3, \ldots, N\}$. Therefore,

$$D_f(q, \overline{u}_n) \leq D_f(q, \theta_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) \left(D_f\left(y_n^i, \theta_n\right) + D_f\left(\overline{u}_n, y_n^i\right)\right),$$

for each $i \in \{1, 2, 3, \ldots, N\}$. □

Theorem 3.4. *Let C be a closed convex and nonempty subset of a reflexive Banach space X, and let $f : X \to \mathbb{R}$, $g_i : C \times C \to \mathbb{R}$ be as in Lemma 3.3. Suppose for $\overline{N} \in \mathbb{N}$ and $s \in \left\{1, 2, \ldots, \overline{N}\right\}$, $B_s : X \to 2^{X^*}$ and $M_s : X \to 2^{X^*}$ are finite families of Bregman inverse strongly*

monotone and maximal monotone operators respectively. Let $S_j : C \to C$ *be finite family of* $(k_j, 0)-$*Bregman demigeneralized mappings such that* $(I - S_j)$ *is demiclosed at zero and* $k_j \in (-\infty, 0)$ *for each* $j = 1, 2, 3, \ldots, m$. *Suppose* $\Omega := \left(\cap_{s=1}^{\overline{N}} F\left(Res^f_{\lambda M_s} \circ B^f_s\right)\right) \cap \left(\cap_{j=1}^{m} F\left(S_j\right)\right) \cap \left(\cap_{i=1}^{N} EP(g_i, C)\right) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to some* $q \in \Omega$.

Proof. The proof is divided into steps as follows;
Step 1: We show the sequence $\{x_n\}$ is bounded. Let $q \in \Omega$ and set $R^\lambda_s := Res^f_{\lambda M_s} \circ B^f_s$. Then from algorithm (3.1), Lemma 2.5 and Lemma 3.3 we have

$$\begin{aligned} D_f(q, \theta_n) &= D_f(q, \nabla f^*(\nabla f(x_n) + \sigma_n(\nabla f(x_{n-1}) - \nabla f(x_n)))) \\ &= D_f(q, \nabla f^*((1-\sigma_n)\nabla f(x_n) + \sigma_n \nabla f(x_{n-1}))) \\ &\leq (1-\sigma_n) D_f(q, x_n) + \sigma_n D_f(q, x_{n-1}) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} D_f(q, z_n) &= D_f\left(q, \nabla f^*\left(\alpha_n \nabla f(\overline{u}_n) + (1-\alpha_n)\nabla f(S_j \overline{u}_n)\right)\right) \\ &= D_f(q, Q\overline{u}_n) \\ &\leq D_f(q, \overline{u}_n) \\ &\leq D_f(q, \theta_n). \end{aligned} \tag{3.11}$$

Using (3.11), Lemma 2.15 and Lemma 2.7 we get

$$\begin{aligned} D_f(q, r_n) &= D_f\Bigg(q, \nabla f^*\Bigg(\beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\ &\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n)\right)\Bigg)\Bigg) \\ &\leq \beta_n D_f(q, \overline{u}_n) + \gamma_n D_f(q, \theta_n) + \eta_n D_f(q, z_n) \\ &\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} D_f\left(q, R^\lambda_s z_n\right) \\ &\leq D_f(q, \theta_n). \end{aligned}$$

It therefore follows from algorithm (3.1), inequality (3.10) and Lemma 2.15 that

$$\begin{aligned} D_f(q, x_{n+1}) &= D_f(q, \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(r_n))) \\ &\leq \mu_n D_f(q, v) + (1-\mu_n) D_f(q, r_n) \\ &\leq \mu_n D_f(q, v) + (1-\mu_n)((1-\sigma_n) D_f(q, x_n) + \sigma_n D_f(q, x_{n-1})) \\ &\leq \operatorname{Max}\left\{D_f(q, v), \operatorname{Max}\left\{D_f(q, x_n), D_f(q, x_{n-1})\right\}\right\} \end{aligned}$$

$$\begin{aligned}
&\vdots \\
&\leq \operatorname{Max}\left\{D_f(q,v), \operatorname{Max}\left\{D_f(q,x_1), D_f(q,x_0)\right\}\right\}.
\end{aligned}$$

Therefore, $\{D_f(q,x_{n+1})\}$ is bounded and by Lemma 2.11 the sequence $\{x_n\}$ is also bounded. Furthermore, $\{r_n\}$, $\{z_n\}$, $\{\overline{u}_n\}$, $\left\{R_s^{\lambda} z_n\right\}$ and $\{S_j \overline{u}_n\}$ for $j = 1, 2, \ldots, m$ are all bounded. However, since f is bounded on bounded subset of X we have that ∇f is bounded on bounded subset of X^* which implies $\{\nabla f(x_n)\}$, $\{\nabla f(r_n)\}$, $\{\nabla f(z_n)\}$, $\{\nabla f(\overline{u}_n)\}$, $\left\{\nabla R_s^{\lambda} z_n\right\}$ and $\{\nabla f(S_j \overline{u}_n)\}$ are all bounded too.

Step 2: We show that $x_n \to q$ as $n \to \infty$. Set $r := \sup_n \{\|\overline{u}_n\|, \|\nabla f(x_n)\|\}$ and let $\rho_r^* : X^* \to \mathbb{R}$ be a gauge function of uniform convexity of the conjugate function f^*. We then obtain using Lemma 2.15 and Lemma 2.16 that

$$\begin{aligned}
D_f(q, r_n) &= D_f\Bigg(q, \nabla f^*\Bigg(\beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\
&\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n)\right)\Bigg)\Bigg) \\
&= V_f\Bigg(q, \beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\
&\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n)\right)\Bigg) \\
&= f(q) - \Bigg\langle q, \beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) \\
&\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n)\right)\Bigg\rangle \\
&\quad + f^*\Bigg(\beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\
&\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f\left(Res^f_{\lambda M_s} \circ B^f_s(z_n)\right)\Bigg) \\
&\leq \beta_n \left(f(q) - \langle q, \nabla f(\overline{u}_n)\rangle + f^*(\nabla f(\overline{u}_n))\right) \\
&\quad + \gamma_n \left(f(q) - \langle q, \nabla f(\theta_n)\rangle + f^*(\nabla f(\theta_n))\right) \\
&\quad + \eta_n \left(f(q) - \langle q, \nabla f(z_n)\rangle + f^*(\nabla f(z_n))\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=1}^{\overline{N}}\delta_{n,s}\Bigg(f(q)-\left\langle q,\nabla f\left(Res^{f}_{\lambda M_s}\circ B^{f}_{s}(z_n)\right)\right\rangle\\
&+f^{*}\left(\nabla f\left(Res^{f}_{\lambda M_s}\circ B^{f}_{s}(z_n)\right)\right)\Bigg)\\
&-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|)\\
&=\beta_n D_f(q,\overline{u}_n)+\gamma_n D_f(q,\theta_n)+\eta_n D_f(q,z_n)\\
&+\sum_{s=1}^{\overline{N}}\delta_{n,s}D_f\left(q,R^{\lambda}_{s}z_n\right)-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|)\\
&\leq D_f(q,\theta_n)-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|).
\end{aligned}
$$

That is

$$
D_f(q,r_n)\leq D_f(q,\theta_n)-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|). \tag{3.12}
$$

Similarly,

$$
D_f(q,r_n)\leq D_f(q,\theta_n)-\beta_n\gamma_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(\theta_n)\|) \tag{3.13}
$$

and

(3.14)

$$
\begin{aligned}
D_f(q,r_n)&\leq D_f(q,\theta_n)-\sum_{s=1}^{\overline{N}}\beta_n\delta_{n,s}\rho_r^{*}\left(\left\|\nabla f(\overline{u}_n)-\nabla f\left(R^{\lambda}_{s}z_n\right)\right\|\right)\\
&\leq D_f(q,\theta_n)-\beta_n\delta_n\rho_r^{*}\left(\left\|\nabla f(\overline{u}_n)-\nabla f\left(R^{\lambda}_{s}z_n\right)\right\|\right)
\end{aligned}
$$

where $\delta_n:=\min\limits_{1\leq s\leq\overline{N}}\delta_{n,s}$.

Using (3.12), (3.13) and (3.14) we respectively obtain

(3.15)

$$
\begin{aligned}
D_f(q,x_{n+1})&\leq\mu_n D_f(q,v)+(1-\mu_n)D_f(q,r_n)\\
&\leq D_f(q,\theta_n)-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|)\\
&\quad-\mu_n[D_f(q,\theta_n)-D_f(q,v)-\beta_n\eta_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(z_n)\|)],
\end{aligned}
$$

(3.16)

$$
\begin{aligned}
D_f(q,x_{n+1})&\leq D_f(q,\theta_n)-\beta_n\gamma_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(\theta_n)\|)\\
&\quad-\mu_n[D_f(q,\theta_n)-D_f(q,v)-\beta_n\gamma_n\rho_r^{*}(\|\nabla f(\overline{u}_n)-\nabla f(\theta_n)\|)]
\end{aligned}
$$

and

(3.17)

$$
D_f(q,x_{n+1})
$$

$$\leq D_f(q,\theta_n) - \beta_n\delta_n\rho_r^*\left(\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right\|\right)$$
$$- \mu_n\left[D_f(q,\theta_n) - D_f(q,v) - \beta_n\delta_n\rho_r^*\left(\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right)\right\|\right].$$

(3.15), (3.16) and (3.17) also imply, respectively,

(3.18)
$$\begin{aligned}
&\beta_n\eta_n\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(z_n)\|)\\
&\quad\leq D_f(q,\theta_n) - D_f(q,x_{n+1})\\
&\qquad - \mu_n[D_f(q,\theta_n) - D_f(q,v) - \beta_n\eta_n\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(z_n)\|)]\\
&\quad\leq D_f(q,x_n) - D_f(q,x_{n+1}) + \sigma_n(D_f(q,x_{n+1} - D_f(q,x_n)\\
&\qquad - \mu_n[D_f(q,\theta_n) - D_f(q,v) - \beta_n\eta_n\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(z_n)\|)],
\end{aligned}$$

(3.19)
$$\begin{aligned}
&\beta_n\gamma_n\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(\theta_n)\|)\\
&\qquad\leq D_f(q,x_n) - D_f(q,x_{n+1}) + \sigma_n(D_f(q,x_{n-1} - D_f(q,x_n)\\
&\qquad\quad - \mu_n[D_f(q,x_n) - D_f(q,v) - \beta_n\gamma_n\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(\theta_n)\|)]
\end{aligned}$$

and

(3.20)
$$\begin{aligned}
&\beta_n\delta_n\rho_r^*\left(\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right\|\right)\\
&\quad\leq D_f(q,x_n) - D_f(q,x_{n+1}) + \sigma_n(D_f(q,x_{n-1} - D_f(q,x_n)\\
&\qquad - \mu_n\left[D_f(q,\theta_n) - D_f(q,v) - \beta_n\delta_n\rho_r^*\left(\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right\|\right)\right].
\end{aligned}$$

Set $t_n = \nabla f^*(\mu_n\nabla f(v) + (1-\mu_n)\nabla f(r_n))$. Then using Lemma 2.14 we have

(3.21)
$$\begin{aligned}
D_f(q,x_{n+1}) &= D_f(q,\nabla f^*(\mu_n\nabla f(v) + (1-\mu_n)\nabla f(r_n)))\\
&= V_f(q,\mu_n\nabla f(v) + (1-\mu_n)\nabla f(r_n))\\
&\leq V_f(q,\mu_n\nabla f(v) + (1-\mu_n)\nabla f(r_n)\\
&\quad - \mu_n(\nabla f(v) - \nabla f(q))) + \mu_n\langle t_n - q, \nabla f(v) - \nabla f(q)\rangle\\
&= V_f(q,\mu_n\nabla f(q) + (1-\mu_n)\nabla f(r_n))\\
&\quad + \mu_n\langle t_n - q, \nabla f(v) - \nabla f(q)\rangle\\
&= \mu_n V_f(q,\nabla f(q)) + (1-\mu_n)V_f(q,\nabla f(r_n))\\
&\quad + \mu_n\langle t_n - q, \nabla f(v) - \nabla f(q)\rangle\\
&= \mu_n D_f(q,q) + (1-\mu_n)D_f(q,r_n)\\
&\quad + \mu_n\langle t_n - q, \nabla f(v) - \nabla f(q)\rangle\\
&\leq (1-\mu_n)D_f(q,\theta_n) + \mu_n\langle t_n - q, \nabla f(v) - \nabla f(q)\rangle.
\end{aligned}$$

On the other hand, as $\{D_f(q, x_{n+1})\}$ is bounded from Step 1, we proceed by the following two cases;

Case 1: Suppose the sequence $\{D_f(q, x_n)\}$ is monotone decreasing, then $\{D_f(q, x_n)\}$ is convergent. Consequently

(3.22)
$$\begin{aligned}\lim_{n\to\infty}(D_f(q, x_n) - D_f(q, x_{n+1})) &= \lim_{n\to\infty}(D_f(q, x_{n-1}) - D_f(q, x_n)) \\ &= 0.\end{aligned}$$

Therefore, using this and the conditions on the parameter sequences β_n, γ_n, η_n, δ_n and σ_n, it follows from (3.18), (3.19), (3.20) that

$$\lim_{n\to\infty}\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(z_n)\|) = 0, \tag{3.23}$$

$$\lim_{n\to\infty}\rho_r^*(\|\nabla f(\overline{u}_n) - \nabla f(\theta_n)\|) = 0 \tag{3.24}$$

$$\lim_{n\to\infty}\rho_r^*\left(\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right\|\right) = 0. \tag{3.25}$$

Thus, by the property of ρ_r^*, we get from (3.23), (3.24) and (3.25) that

(3.26)
$$\lim_{n\to\infty}\|\nabla f(\overline{u}_n) - \nabla f(z_n)\| = 0, \qquad \lim_{n\to\infty}\|\nabla f(\overline{u}_n) - \nabla f(\theta_n)\| = 0$$

and

$$\begin{aligned}&\lim_{n\to\infty}\left\|\nabla f(\overline{u}_n) - \nabla f\left(R_s^\lambda z_n\right)\right\| \\ &\quad= \lim_{n\to\infty}\left\|\nabla f(\overline{u}_n) - \nabla f\left(Res_{\lambda M_s}^f \circ B_s^f(z_n)\right)\right\| \\ &\quad= 0.\end{aligned}$$

Since ∇f^* is norm-to-norm uniformly continuous on bounded subset of X^*, we obtain from (3.26) that

(3.27)
$$\begin{cases}(i)\lim_{n\to\infty}\|\overline{u}_n - z_n\| = 0, \\ (ii)\lim_{n\to\infty}\|\overline{u}_n - \theta_n\| = 0 \\ (iii)\lim_{n\to\infty}\left\|\overline{u}_n - R_s^\lambda z_n\right\| = \lim_{n\to\infty}\left\|\overline{u}_n - Res_{\lambda M_s}^f \circ B_s^f(z_n)\right\| = 0.\end{cases}$$

However,

$$\left\|z_n - Res_{\lambda M_s}^f \circ B_s^f(z_n)\right\| \le \|z_n - \overline{u}_n\| + \left\|\overline{u}_n - Res_{\lambda M_s}^f \circ B_s^f(z_n)\right\|,$$

so that

$$\lim_{n\to\infty}\left\|z_n - R_s^\lambda z_n\right\| = 0, \quad \forall s \in \left\{1, 2, \ldots, \overline{N}\right\}. \tag{3.28}$$

Now, let $k = \underset{1\leq j\leq m}{\text{Max}}\, k_j$ then following inequality (1.13) we obtain

$$\begin{aligned}\langle \overline{u}_n - q, \nabla f(\overline{u}_n) - \nabla f(z_n)\rangle &= \langle \overline{u}_n - q, \nabla f(\overline{u}_n) - \alpha_n \nabla f(\overline{u}_n)\\ &\quad -(1-\alpha_n)\nabla f(S_j\overline{u}_n)\rangle\\ &= \langle \overline{u}_n - q, (1-\alpha_n)\nabla f(\overline{u}_n)\\ &\quad -(1-\alpha_n)\nabla f(S_j\overline{u}_n)\rangle\\ &= (1-\alpha_n)\langle \overline{u}_n - q, \nabla f(\overline{u}_n) - \nabla f(S_j\overline{u}_n)\rangle\\ &\geq (1-\alpha_n)(1-k)D_f(\overline{u}_n, S_j\overline{u}_n).\end{aligned}$$

Hence

$$\begin{aligned}(1-\alpha_n)(1-k)D_f(\overline{u}_n, S_j\overline{u}_n) &\leq \langle \overline{u}_n - q, \nabla f(\overline{u}_n) - \nabla f(z_n)\rangle\\ &\leq \|\overline{u}_n - q\|\,\|\nabla f(\overline{u}_n) - \nabla f(z_n)\|,\end{aligned}$$

thus

$$\lim_{n\to\infty} D_f(\overline{u}_n, S_j\overline{u}_n) = 0, \quad \forall j \in \{1,2,\ldots,m\}. \tag{3.29}$$

As f is totally convex, we have that f is sequentially consistent. Therefore it follows from (3.29) that

$$\lim_{n\to\infty} \|\overline{u}_n - S_j\overline{u}_n\| = 0, \quad \forall j \in \{1,2,\ldots,m\}. \tag{3.30}$$

Since $\{\overline{u}_n\}$ is a bounded sequence in X, X a reflexive Banach space, then there exists a subsequence $\{\overline{u}_{n_l}\}$ of $\{\overline{u}_n\}$ such that $\overline{u}_{n_l} \rightharpoonup p$ as $l \to \infty$. This put together with (3.30) and the fact that $(I - S_j)$ is demiclosed at origin give

$$p \in \cap_{j=1}^{m} F(S_j).$$

From (3.27)(i) we obtain $z_{n_l} \rightharpoonup p$ as $l \to \infty$ which also when combined with (3.28) gives $p \in F\left(R_s^{\lambda} z_n\right))$ for each $s \in \left\{1,2,\ldots,\overline{N}\right\}$ and hence

$$p \in \left(\cap_{j=1}^{m} F(S_j)\right) \cap \left(\cap_{s=1}^{\overline{N}} F\left(R_s^{\lambda} z_n\right)\right).$$

Next we show that $p \in \cap_{i=1}^{N} EP(g_i, p)$. From Lemma 3.2 and the three point identity we have

$$\begin{aligned}\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f\left(y_n^i, \theta_n\right) &\leq D_f(q,\theta_n) - D_f(q,\overline{u}_n)\\ &\leq D_f(q,\theta_n) - D_f(q,\overline{u}_n) + D_f(\theta_n,\overline{u}_n)\\ &= \langle q - \theta_n, \nabla f(\overline{u}_n) - \nabla f(\theta_n)\rangle\\ &\leq \|q - \theta_n\|\,\|\nabla f(\overline{u}_n) - \nabla f(\theta_n)\|.\end{aligned} \tag{3.31}$$

Since $\lim_{n\to\infty}\left(1-\frac{\lambda_n}{\lambda_{n+1}}\mu\right)=1-\mu\geq 0$, we obtain from (3.26), (3.31) that

$$\lim_{n\to\infty} D_f\left(y_n^i,\theta_n\right)=0,$$

hence

$$\lim_{n\to\infty}\left\|y_n^i-\theta_n\right\|=0. \tag{3.32}$$

Now, let $n\geq 1$, then by definition of y_n^i from algorithm (3.1) we obtain in a similar manner as in (3.6) that

$$\lambda_n\left(g_i\left(\theta_n,w\right)-g_i\left(\theta_n,y_n^i\right)\right)\geq\left\langle y_n^i-w,\nabla f\left(y_n^i\right)-\nabla f(\theta_n)\right\rangle,\quad \forall w\in C.$$

Thus,

$$\lambda_{n_l}\left(g_i\left(\theta_{n_l},w\right)-g_i\left(\theta_{n_l},y_{n_l}^i\right)\right)\geq\left\langle y_{n_l}^i-w,\nabla f(y_{n_l}^i)-\nabla f(\theta_{n_l})\right\rangle,\quad \forall w\in C.$$

We also obtain, due to (3.32) and the fact that $\theta_{n_l}\rightharpoonup p$ as $l\to\infty$, that $y_{n_l}^i\rightharpoonup p$ as $l\to\infty$. Since

$$\lim_{l\to\infty}\lambda_{n_l}>\min\left\{\frac{\mu}{\min\{c_1,c_2\}},\lambda_1\right\}>0,$$

it follows that $p\in\cap_{i=1}^N EP(g_i,C)$ and hence $p\in\Omega$.

Set $D_n:=D_f(q,x_{n-1})-D_f(q,x_n)$. Then from (3.22), $D_n\to 0$ as $n\to\infty$. Also, by property $(S4)$ $\exists\ N_0\in\mathbb{N}$ such that $\frac{\sigma_n}{\mu_n}\leq 1\ \forall n\geq N_0$. Now from (3.21), we have that for $n\geq N_0$

(3.33)

$$\begin{aligned}
D_f(q,x_{n+1}) &\leq (1-\mu_n)D_f(q,\theta_n)+\mu_n\left\langle t_n-q,\nabla f(v)-\nabla f(q)\right\rangle\\
&\leq (1-\mu_n)\big((1-\sigma_n)D_f(q,x_n)+\sigma_n D_f(q,x_{n-1})\big)\\
&\quad+\mu_n\left\langle t_n-q,\nabla f(v)-\nabla f(q)\right\rangle\\
&=(1-\mu_n)D_f(q,x_n)+\mu_n\Big(\left\langle t_n-q,\nabla f(v)-\nabla f(q)\right\rangle\\
&\quad-\sigma_n(D_f(q,x_{n-1})-D_f(q,x_n))\\
&\quad+\frac{\sigma_n}{\mu_n}(D_f(q,x_{n-1})-D_f(q,x_n))\Big)\\
&\leq(1-\mu_n)D_f(q,x_n)\\
&\quad+\mu_n\left(\left\langle t_n-q,\nabla f(v)-\nabla f(q)\right\rangle+(1-\sigma_n)D_n\right).
\end{aligned}$$

Claim 2: $\limsup_{n\to\infty}\big(\left\langle t_n-q,\nabla f(v)-\nabla f(q)\right\rangle+(1-\sigma_n)D_n\big)\leq 0$. Let $\{\theta_{n_k}\}$ be a subsequence of $\{\theta_n\}$ such that

$$\limsup_{n\to\infty}\left\langle\theta_n-q,\nabla f(v)-\nabla f(q)\right\rangle=\lim_{k\to\infty}\left\langle\theta_{n_k}-q,\nabla f(v)-\nabla f(q)\right\rangle.$$

As $\{\theta_{n_k}\}$ is a bounded we have that there exists $\left\{\theta_{n_{k_j}}\right\}$, a subsequence of $\{\theta_{n_k}\}$, such that $\theta_{n_{k_j}} \rightharpoonup \hat{v} \in \Omega$ as $j \to \infty$. Assume w.l.o.g. $\theta_{n_k} \to \hat{v}$ as $k \to \infty$. Then it follows from Lemma 2.13(i) that

(3.34)
$$\begin{aligned}\limsup_{n\to\infty} \langle \theta_n - q, \nabla f(v) - \nabla f(q)\rangle &= \lim_{k\to\infty} \langle \theta_{n_k} - q, \nabla f(v) - \nabla f(q)\rangle \\ &= \langle \hat{v} - q, \nabla f(v) - \nabla f(q)\rangle \\ &\leq 0.\end{aligned}$$

Following algorithm (3.1) on the other hand we have

$$\begin{aligned}\|\nabla f(r_n) - \nabla f(\overline{u}_n)\| \leq \beta_n \|\nabla f(\overline{u}_n) - \nabla f(\overline{u}_n)\| &+ \gamma_n \|\nabla f(\theta_n) - \nabla f(\overline{u}_n)\| \\ &+ \eta_n \|\nabla f(z_n) - \nabla f(\overline{u}_n)\| \\ &+ \sum_{s=1}^{\overline{N}} \delta_{n,s} \left\|\nabla f\left(Res^f_{M_s} \circ B^f_s(z_n)\right) - \nabla f(\overline{u}_n)\right\|\end{aligned}$$

from which it implies by (3.26) that

$$\lim_{n\to\infty} \|\nabla f(r_n) - \nabla f(\overline{u}_n)\| = 0.$$

Since ∇f^* is norm to norm uniformly continuous on bounded subset of X^*, we obtain that

$$\lim_{n\to\infty} \|r_n - \overline{u}_n\| = 0. \tag{3.35}$$

From the definition of t_n, we have

$$D_f(r_n, t_n) \leq \mu_n D_f(r_n, v) + (1 - \mu_n) D_f(r_n, r_n),$$

from which it follows by $(S1)$ that

$$\lim_{n\to\infty} D_f(r_n, t_n) = 0.$$

But f being totally convex on bounded subset of X, it implies f is sequentially consistent and hence

$$\lim_{n\to\infty} \|r_n - t_n\| = 0. \tag{3.36}$$

Also,

$$\|t_n - \overline{u}_n\| \leq \|t_n - r_n\| + \|r_n - \overline{u}_n\|$$

and so by (3.35) and (3.36), we get

$$\lim_{n\to\infty} \|t_n - \overline{u}_n\| = 0. \tag{3.37}$$

Similarly,

$$\|t_n - \theta_n\| \leq \|t_n - \overline{u}_n\| + \|\overline{y}_u - \theta_n\|$$

which therefore implies by (3.32) and (3.37) that

$$\lim_{n\to\infty}\|t_n-\theta_n\|=0. \tag{3.38}$$

By (3.34) and (3.38), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle t_n-q,\nabla f(v)-\nabla f(q)\rangle &= \limsup_{n\to\infty}\langle \theta_n-q,\nabla f(v)-\nabla f(q)\rangle\\ &\leq 0,\end{aligned} \tag{3.39}$$

so that

$$\limsup_{n\to\infty}\left(\langle t_n-q,\nabla f(v)-\nabla f(q)\rangle-\sigma_n D_n+\frac{\sigma_n}{\mu_n}D_n\right)\leq 0, \tag{3.40}$$

proving claim 2. We therefore conclude from (3.21), (3.40) and Lemma 2.22 that $x_n\to q$ as $n\to\infty$, completing the proof of Case 1.

Case 2: Suppose on the other hand the sequence $\{D_f(q,x_n)\}$ is not monotone decreasing, then there exists $\{D_f(q,x_{n_j})\}$, a subsequence of $\{D_f(q,x_n)\}$, such that $D_f(q,x_{n_j})\leq D_f(q,x_{n_{j+1}})\ \forall j\geq 1$. Also for an N large enough satisfying $k\geq N$, let $\alpha:\mathbb{N}\to\mathbb{N}$ by

$$\alpha(k)=\max\{j\leq k: D_f(q,x_j)\leq D_f(q,x_{j+1})\}.$$

Then the sequence $\{\alpha(k)\}$, by Lemma 2.22, is nondecreasing sequence satisfying $\alpha(k)\to\infty$ as $k\to\infty$ and that

$$D_f(q,x_{\alpha(k)})\leq D_f(q,x_{\alpha(k)+1})\text{and}D_f(q,x_k)\leq D_f(q,x_{\alpha(k)+1}),\quad \forall k\in N.$$

This combine with (3.18), (3.19) and (3.20) yield as in Case 1, that

$$\begin{aligned}&\beta_{\alpha(k)}\eta_{\alpha(k)}\rho_r^*(\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(z_{\alpha(k)})\|)\\ &\quad\leq -u_{\alpha(k)}\big[D_f(q,\theta_{\alpha(k)})-D_f(q,v)\\ &\qquad-\beta_{\alpha(k)}\eta_{\alpha(k)}\rho_r^*(\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(z_{\alpha(k)})\|)\big],\end{aligned} \tag{3.41}$$

$$\begin{aligned}&\beta_{\alpha(k)}\gamma_{\alpha(k)}\rho_r^*\left(\left\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(\theta_{\alpha(k)})\right\|\right)\\ &\quad\leq -u_{\alpha(k)}\big[D_f(q,\theta_{\alpha(k)})-D_f(q,v)\\ &\qquad-\beta_{\alpha(k)}\gamma_{\alpha(k)}\rho_r^*\left(\left\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(\theta_{\alpha(k)})\right\|\right)\big]\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}&\beta_{\alpha(k)}\delta_{\alpha(k)}\rho_r^*\left(\left\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(R_s^\lambda z_{\alpha(k)})\right\|\right)\\ &\quad\leq -u_{\alpha(k)}\Big[D_f(q,\theta_{\alpha(k)})-D_f(q,v)\\ &\qquad-\beta_{\alpha(k)}\delta_{\alpha(k)}\rho_r^*\left(\left\|\nabla f\left(\overline{u}_{\alpha(k)}\right)-\nabla f(R_s^\lambda z_{\alpha(k)})\right\|\right)\Big]\end{aligned} \tag{3.43}$$

respectively.

Utilizing conditions $(S1)$, $(S2)$ and both the property of ρ_r^* and the fact that ∇f^* is norm-to-norm uniformly continuous on bounded subset of X^* we obtain in a similar manner as in Case 1 above that

$$\lim_{k\to\infty}\|\overline{u}_{\alpha(k)} - z_{\alpha(k)}\| = 0$$
$$\lim_{k\to\infty}\|\overline{u}_{\alpha(k)} - \theta_{\alpha(k)}\| = 0,$$
$$\lim_{k\to\infty}\left\|\overline{u}_{\alpha(k)} - R_s^{\lambda} z_{\alpha(k)}\right\| = 0.$$

We further obtain

$$\limsup_{k\to\infty}\big(\langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle + (1-\sigma_{\alpha(k)})D_{\alpha(k)}\big) \leq 0, \tag{3.44}$$

where $D_{\alpha(k)} := D_f(q, x_{\alpha(k)-1}) - D_f(q, x_{\alpha(k)})$.

Therefore it follows from (3.21) that

(3.45)

$$\begin{aligned}
D_f(q, x_{\alpha(k)+1}) &\leq (1-\mu_{\alpha(k)})D_f(q,\theta_{\alpha(k)}) + \mu_{\alpha(k)}\langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle \\
&\leq (1-\mu_{\alpha(k)})\big((1-\sigma_{\alpha(k)})D_f(q, x_{\alpha(k)}) + \sigma_{\alpha(k)}D_f(q, x_{\alpha(k)-1})\big) \\
&\quad + \mu_{\alpha(k)}\langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle \\
&\leq (1-\mu_{\alpha(k)})D_f(q, x_{\alpha(k)}) + \mu_{\alpha(k)}\Bigg(\langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle \\
&\quad - \sigma_{\alpha(k)}D_{\alpha(k)} + \frac{\sigma_{\alpha(k)}}{\mu_{\alpha(k)}}D_{\alpha(k)}\Bigg) \\
&\leq (1-\mu_{\alpha(k)})D_f(q, x_{\alpha(k)}) + \mu_{\alpha(k)}\Bigg(\langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle \\
&\quad + (1-\sigma_{\alpha(k)})D_{\alpha(k)}\Bigg).
\end{aligned}$$

Since $D_f(x_{\alpha(k)}, q) \leq D_f(x_{\alpha(k)+1}, q)$, we get from (3.45) that

$$D_f(q, x_{\alpha(k)}) \leq \langle t_{\alpha(k)} - q, \nabla f(v) - \nabla f(q)\rangle + (1-\sigma_{\alpha(k)})D_{\alpha(k)}.$$

This together with (3.44) give

$$\lim_{k\to\infty} D_f(q, x_{\alpha(k)}) = 0. \tag{3.46}$$

Moreover, as $D_f(q, x_k) \leq D_f(q, x_{\alpha(k)+1})$ for all $k \in \mathbb{N}$, it follows from (3.46) that $\lim_{k\to\infty} D_f(q, x_k) = 0$, which complete the proof of Case 2.

It is therefore concluded from the two cases, Case 1 and Case 2, that $x_n \to q$ as $n \to \infty$. This completes the proof.

□

Observe that if, in Theorem 3.1, we replace the half space $\Gamma_{n,i}$ by C, a nonempty closed convex subset of X and set $\sigma_n = 0$ then we obtain the following result of Ali and Auwalu [1] as a corollary to our result.

Corollary 3.5. *Assume the hypothesis of Theorem 3.4 and let* $\{x_n\}$ *be a sequence generated by algorithm* (1.14). *Then the sequence* $\{x_n\}$ *converges strongly to some* $q \in \Omega$.

Similarly, setting, in algorithm (3.1), $N = 1$, $\sigma_n = 0$, $\beta_n = \beta_{n,0}$, $\eta_n = \beta_{n,i}$ such that $\beta_{n,0} + \sum_{i=1}^{\overline{y}} \beta_{n,i} = 1$ with B_s and M_s both as zero maps for all s, noting that $\nabla f^*(\alpha_n \nabla f + (1-\alpha_n)\nabla f\,(S_j))\overline{u}_n =: Q_j \overline{u}_n$ where $Q_j : C \to C$ is quasi-Bregman nonexpansive (see Lemma 2.5 above), we have the following corollary obtained from our result which is similar to that of [9] for solving equilibrium and common fixed point problems.

Corollary 3.6. *Let* C *be a nonempty closed convex subset of a reflexive Banach space* X, *and let* $f : X \to \mathbb{R}$ *be as in Lemma 3.3. Suppose* $g : C \times C \to \mathbb{R}$ *is a bifunction satisfying* $(P1)-(P5)$. *Let* $Q_j : C \to C$ *be finite family of quasi-Bregman nonexpansive mappings such that* $(I - Q_j)$ *is demiclosed at zero. For* $\Omega := (\cap_{j=1}^{m} F(Q_j)) \cap EP(g, C) \neq \emptyset$ *the sequence* $\{x_n\}$ *generated by algorithm* (3.47) *converges strongly to some* $q \in \Omega$.

$$
\begin{cases}
\text{For } \lambda_1 > 0, \mu \in (0,1) \text{ and } v, x_1 \in X \text{ chosen abitrarily,} \\
y_n = \underset{y \in C}{\arg\min} \left\{ \lambda_n g(x_n, y) + D_f(y, x_n) \right\}, \\
\Gamma_n = \{ z \in X : \langle z - y_n, \nabla f(x_n) - \lambda_n w_n - \nabla f(y_n) \rangle \leq 0, \\
\qquad w_n \in \partial_2 g(x_n, y_n) \}, \\
u_n = \underset{y \in \Gamma_n}{\arg\min} \left\{ \lambda_n g(y_n, y) + D_f(y, x_n) \right\}, \\
r_n = \nabla f^*(\beta_{n,0} \nabla f(u_n) + \sum_{s=1}^{\overline{N}} \beta_{n,i} \nabla f(Q_j u_n), \\
x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(r_n)), n \geq 1,
\end{cases}
\tag{3.47}
$$

where $\kappa_n := (g(x_n, u_n) - g(x_n, y_n) - g(y_n, u_n))$ *is such that*

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu(D_f(y_n,x_n)+D_f(u_n,y_n))}{\kappa_n}, \lambda_n\right\}, \kappa_n > 0, \\ \\ \lambda_n, \textit{otherwise,} \end{cases}$$

and $\{\mu_n\} \subset (0,1)$ *such that* $\lim\limits_{n \to \infty} \mu_n = 0$ *and* $\sum\limits_{n=0}^{\infty} \mu_n = \infty$.

4. Application of Theorem 3.4 to Finding Solution of Some Variational Inequality Problems

Setting $g(x,y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$, where $A : C \to X^*$ is a nonlinear map on a nonempty, closed, convex subset of a Banach space X. Then the equilibrium problem (1.6) reduces to the VIP (1.5). Therefore, for each $i = 1, 2, \ldots, N$, we have from Lemma 2.19, Lemma 2.20 and algorithm (3.1) that

$$\begin{aligned} &\arg\min_{y \in C} \{\lambda_n g_i(\theta_n, y) + D_f(y, \theta)\} \\ &\quad = \arg\min_{y \in C} \{\lambda_n \langle A_i\theta_n, y - \theta_n\rangle + D_f(y, \theta_n)\} \\ &\quad = P_C^f\left(\nabla f^*(\nabla f(\theta_n) - \lambda_n A_i\theta_n)\right), \end{aligned}$$

and

$$\arg\min_{y \in \Gamma_{n,i}} \left\{\lambda_n g_i\left(y_n^i, y\right) + D_f(y, \theta_n)\right\} = P_{\Gamma_{n,i}}^f\left(\nabla f^*\left(\nabla f(\theta_n) - \lambda_n A_i y_n^i\right)\right)$$

also

$$\begin{aligned} &g_i\left(\theta_n, u_n^i\right) - g_i\left(\theta_n, y_n^i\right) - g_i\left(y_n^i, u_n^i\right) \\ &\quad = \left\langle A\theta_n, u_n^i - \theta_n\right\rangle \\ &\qquad - \left\langle A\theta_n, y_n^i - \theta_n\right\rangle + \left\langle Ay_n^i, u_n^i - y_n^i\right\rangle \\ &\quad = \left\langle A\theta_n, u_n^i - y_n^i\right\rangle - \left\langle Ay_n^i, u_n^i - y_n^i\right\rangle \\ &\quad = \left\langle A\theta_n - Ay_n^i, u_n^i - y_n^i\right\rangle. \end{aligned}$$

Thus, following the above setting, algorithm (3.1) reduces to the following algorithm;

(4.1)

$$\begin{cases} \text{For } \lambda_1 > 0, \mu \in (0,1) \text{ and } v, x_1 \in X \text{ chosen abitrarily,} \\ \\ \theta_n = \nabla f^* \left(\nabla f(x_n) + \sigma_n (\nabla f(x_{n-1}) - \nabla f(x_n)) \right), \\ \\ y_n^i = P_C^f \left(\nabla f^* (\nabla f(\theta_n) - \lambda_n A_i \theta_n) \right), i = 1, 2, 3, \ldots, N, \\ \\ \Gamma_{n,i} = \left\{ z \in X : \left\langle z - y_n^i, \nabla f(\theta_n) - \lambda_n A_i \theta_n - \nabla f\left(y_n^i\right) \right\rangle \leq 0, \right\} \\ \\ u_n^i = P_{\Gamma_{n,i}}^f \left(\nabla f^* \left(\nabla f(\theta_n) - \lambda_n A_i y_n^i \right) \right), i = 1, 2, 3, \ldots, N, \\ \\ i_n = \arg\max \left\{ D_f \left(u_n^i, \theta_n \right), i = 1, 2, 3, \ldots, N \right\}, \overline{u}_n = u_n^{i_n}, \\ \\ z_n = \nabla f^* \left(\alpha_n \nabla f(\overline{u}_n) + (1 - \alpha_n) \nabla f(S_j \overline{u}_n) \right), j = 1, 2, \ldots, m, \\ \\ r_n = \nabla f^* \Big(\beta_n \nabla f(\overline{u}_n) + \gamma_n \nabla f(\theta_n) + \eta_n \nabla f(z_n) \\ \qquad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f \left(Res_{\lambda M_s}^f \circ B_s^f (z_n) \right) \Big), \\ \\ x_{n+1} = \nabla f^* (\mu_n \nabla f(v) + (1 - \mu_n) \nabla f(r_n)), n \geq 1, \end{cases}$$

where $\kappa_{n,i} := \left\langle A\theta_n - Ay_n^i, u_n^i - y_n^i \right\rangle$ is such that

$$\begin{cases} \min \left\{ \frac{\mu(D_f(y_n^i, \theta_n) + D_f(u_n^i, y_n^i))}{\kappa_{n,i}}, \lambda_n \right\}, \kappa_{n,i} > 0, \\ \\ \lambda_n, \text{otherwise,} \end{cases}$$

and $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_{n,s}\}$ and $\{\mu_n\}$ are sequences satisfying conditions $(S1) - (S4)$.

Hence, our result is applicable to solving VIP as follows.

Theorem 4.1. *Let C be a closed convex and nonempty subset of a reflexive Banach space X, and let $f : X \to \mathbb{R}$ be as in Lemma 3.3, let $A_i : C \to X^*$ be nonlinear mappings satisfying $(P1) - (P5)$. Suppose for $\overline{N} \in \mathbb{N}$ and $s \in \left\{1, 2, \ldots, \overline{N}\right\}$, $B_s : X \to 2^{X^*}$ and $M_s : X \to 2^{X^*}$ are finite families of Bregman inverse strongly monotone and maximal monotone operators respectively. Let $S_j : C \to C$ be finite family of*

$(k_j, 0)-$Bregman demigeneralized mappings such that $(I - S_j)$ is demiclosed at zero and $k_j \in (-\infty, 0)$ for each $j = 1, 2, 3, \ldots, m$. Suppose $\Omega := (\cap_{s=1}^{\overline{N}} F\left(Res^f_{\lambda M_s} \circ B^f_s\right)) \cap \left(\cap_{j=1}^{m} F\left(S_j\right)\right) \cap (\cap_{i=1}^{N} VI(A_i, C)) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by algorithm (4.1) converges strongly to some $q \in \Omega$.*

Proof. The proof follows from the proof of Theorem 3.4 with $g(x, y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$, where $A : C \to X^*$ is a nonlinear map on a nonempty, closed, convex subset of X. □

Remark 4.2. The results presented in this paper improve many recent results in the Literature. For instance

(i) The extragradient-type method used in [1] (algorithm (1.14)) requires computation of two projections onto a nonempty closed convex set in each iteration which can sometimes be complicated. While our present method in this paper involves projection onto the hyperplane which makes the computation much easier than as in [1].

Also our present method uses self-adaptive step size and hence it does not require the estimation of the constants of the Lipschitz-type bifunction which makes the computation cumbersome as in the method used in [1].

(ii) In addition to solving fixed point and equilibrium problems as considered in Bokodisa, Jolaoso and Aphane [9], in our results, solutions of zero problems involving sum of maximal monotone and Bregman inverse strongly monotone operators with an inertia term are considered.

(iii) It improve Ali et al.[2] from variational inequality and common fixed point problems to common equilibrium, common fixed point and common zero of sum of maximal monotone and Bregman inverse strongly monotone operators involving self adaptive technique in the setting of a reflexive Banach space.

5. Numerical Example

Numerical example demonstrating Theorem 3.4 of this paper is presented in this section.

Example 5.1. Consider $X = \mathbb{R}$ with $\|.\| = |.|$, $C = [0, 2]$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{5}x^2$, $\forall x \in \mathbb{R}$. Then C and f respectively satisfy all the assumptions of Theorem 3.4. Also, following (1.10), we have that $f^*(x^*) := \sup_{x \in \mathbb{R}} \{x^* x - f(x)\}$ $\forall x^* \in \mathbb{R}$, thus $f^*(\omega) = \frac{5}{4}\omega^2$ and $\nabla f^*(\omega) =$

$\frac{5}{2}\omega$. Let $B_s, M_s : \mathbb{R} \to \mathbb{R}$, $s \in \{1, 2\}$, be defined respectively by

$$B_s(x) = \frac{1}{4}x \qquad \text{and} \qquad M_s(x) = 2x, \quad \forall x \in \mathbb{R}.$$

Then for each $s \in \{1, 2\}$, B_s is Bregman inverse strongly monotone and M_s is maximal monotone with respective resolvent associated with f obtained as follows;
For each $x \in \mathbb{R}$,

$$\begin{aligned} \hat{w} = B_1^f(x) \quad &\Leftrightarrow \quad \hat{w} = (\nabla f^* \circ (\nabla f - B_1))x \\ &\Leftrightarrow \quad \hat{w} = \nabla f^*((\nabla f - B_1)(x)) \\ &\Leftrightarrow \quad \hat{w} = \nabla f^*(\nabla f(x) - B_1(x)) \\ &\Leftrightarrow \quad \hat{w} = \nabla f^*\left(\frac{3}{20}x\right) \\ &\Leftrightarrow \quad \hat{w} = \frac{3}{8}x, \end{aligned}$$

that is $B_1^f(x) = \frac{3}{8}x$ which is the resolvent of B_1. Also,

$$\begin{aligned} w = Res^f_{\lambda M_1}(x) \quad &\Leftrightarrow \quad w = (\nabla f + \lambda M_1)^{-1} \circ \nabla f(x) \\ &\Leftrightarrow \quad (\nabla f + \lambda M_1)w = \nabla f(x) \\ &\Leftrightarrow \quad 2\lambda w = \nabla f(x) - \nabla f(w) \\ &\Leftrightarrow \quad w = \frac{x}{5\lambda + 1}, \end{aligned}$$

that is $Res^f_{\lambda M_1}(x) = \frac{x}{5\lambda+1}$ which is the resolvent of M_1. Thus,

$$Res^f_{\lambda M_1} \circ B_1^f(x) = Res^f_{\lambda M_1}\left(\frac{3}{8}x\right) = \frac{3x}{40\lambda + 8}.$$

Similarly,

$$Res^f_{\lambda M_2} \circ B_2^f(x) = Res^f_{\lambda M_2}\left(\frac{3}{8}x\right) = \frac{3x}{40\lambda + 8}.$$

Define $g_i : C \times C \to \mathbb{R}$ by $g_i(x, y) = y^2 + 6xy - 7x^2$, for each $i = 1, 2$. It is then obvious to verify that $0 \in \cap_{i=1}^2 EP(g_i, C)$ and that each g_i's satisfy assumptions $(P1)$, $(P3)$ and $(P5)$. Moreover, g_i's satisfy assumptions $(P2)$ and $(P4)$ with $c_1 = c_2 = 3$ and $\partial_2 g_i(x, y) = 2y + 6x$ respectively. Indeed for $z \in C$, $x, y \in \text{int}(domf)$ and $D_f(x, y) = (x - y)^2$

$$\begin{aligned} g_i(x, y) + g_i(y, z) &= y^2 + 6xy - 7x^2 + z^2 + 6yz - 7y^2 \\ &= z^2 + 6xz - 7x^2 + 6xy + 6yz - 6xz - 6y^2 \\ &= g_i(x, y) - 3D_f(y, x) - 3D_f(z, y) + 3D_f(z, x) \\ &\geq g_i(x, y) - 3D_f(y, x) - 3D_f(z, y). \end{aligned}$$

Let $S_j : C \to C$ be define by $S_j(x) = \frac{x}{2}\ \forall x \in C$ and for each $j = 1, 2$. Clearly, $0 \in \cap_{j=1}^{2} F(S_j)$ and S_j is Bregman demigeneralized mapping for each $j \in \{1, 2\}$. Moreover, if $\mu = \frac{7}{10}$, $\lambda_n = \frac{1}{25}$, $\sigma_n = \frac{1}{2n^2}$, $\mu_n = \frac{3n+1}{5n^2}$, $\beta_n = \gamma_n = \eta_n = \delta_{n,1} = \delta_{n,2} = \frac{1}{5}$ and $\alpha_n = \frac{1}{2n}$ then $\mu, \lambda_n, \sigma_n, \mu_n, \beta_n, \gamma_n, \eta_n, \delta_{n,i}$ and α_n satisfy all the requirements of Theorem 3.4. Therefore, algorithm (3.1) takes the following form;

$$
\begin{cases}
\theta_n = x_n + \sigma_n(x_{n-1} - x_n), \\
y_n^i = \frac{1+3\lambda_n}{1+\lambda_n}\theta_n, i = 1, 2, \\
\Gamma_{n,i} = \Big\{ z \in \mathbb{R} : \big\langle z - y_n^i, \nabla f(\theta_n) - \lambda_n w_n - \nabla f\left(y_n^i\right)\big\rangle \\
\qquad\qquad \leq 0, w_n \in \partial_2 g_i\left(\theta_n, y_n^i\right) \Big\}, \\
u_n^i = \frac{\theta_n - 3\lambda_n y_n^i}{2(1+\lambda_n)}, i = 1, 2, \\
\overline{u}_n = u_n^i, i = 1, 2, \\
z_n = \frac{(1+2n)}{4n}\overline{u}_n, \\
r_n = \frac{2}{25}\left(\overline{u}_n + \theta_n + \left(1 + \frac{6}{(40\lambda+8)}\right) z_n\right), \\
x_{n+1} = \frac{2(3n+1)}{25n^2} v + \frac{2\left(5n^2-3n-1\right)}{25n^2} r_n, n \in \mathbb{N}.
\end{cases}
\tag{5.1}
$$

Utilizing (5.1) the numerical results, using R2014a MATLAB version, are obtained as shown in the following two cases;

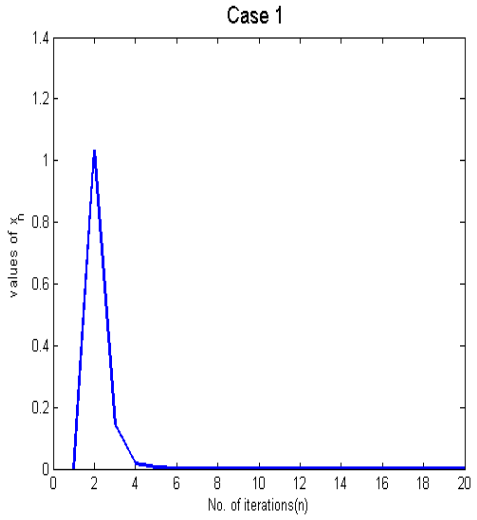

FIGURE 1. Convergence of a sequence $\{x_n\}$ generated by algorithm (5.1) with initial points $x_1 = 5.0$, $x_2 = 20$ and v=0.05.

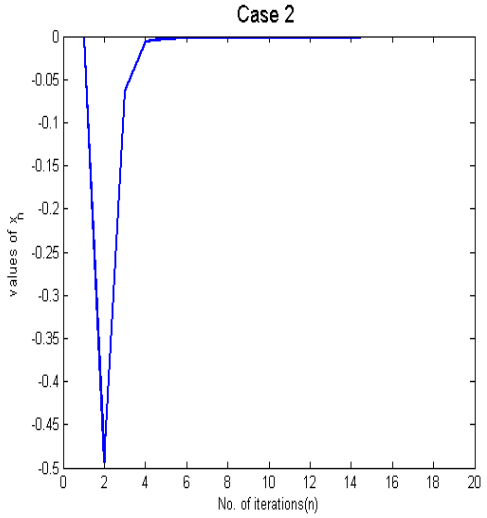

FIGURE 2. Convergence of a sequence $\{x_n\}$ generated by algorithm (5.1) with initial points $x_1 = 3.5$, $x_2 = -15$ and v=-0.025.

6. Conclusion

In this paper, a new modified inertial self-adaptive subgradient extragradient-type iteration was presented. A strong convergence Theorem is proved without prior knowledge of the Lipschtz-type constants (whose computations are generally difficult) as in the well known subgradient extragradient methods, see [14, 39].

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