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Local Existence and Uniqueness for a Nonlinear Kinetic Equation via Energy Methods

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Abstract. We study a nonlinear kinetic equation with time-dependent external force and quadratic collision operator in a bounded spatial domain. Under suitable Sobolev regularity assumptions and structural conditions on the transport field, we establish local-in-time existence and uniqueness of solutions using high-order energy methods. The proof relies on a Riccati-type differential inequality for the Sobolev energy functional. An explicit lower bound for the lifespan of solutions is derived in terms of the initial Sobolev norm. Uniqueness follows from a linearized energy estimate for the difference of two solutions. The analysis requires only bilinear Sobolev bounds on the collision operator and is inspired by classical energy methods for nonlinear PDEs and kinetic equations.

1. Introduction

Nonlinear kinetic equations play a central role in statistical mechanics, plasma physics and rarefied gas dynamics. Their mathematical structure combines conservative transport mechanisms with nonlinear interaction terms, typically quadratic in the unknown distribution function. The Boltzmann equation is a paradigmatic example [2, 4, 8].

The local well-posedness theory for nonlinear kinetic equations has been developed using two principal approaches:

(i) Fixed-point arguments in low-regularity spaces [7],
(ii) High-order energy methods in Sobolev spaces [4, 5].

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The energy method is particularly robust for quadratic nonlinearities. It allows one to derive a priori bounds without requiring global continuity of the nonlinear operator. Instead, it suffices to assume bilinear Sobolev-type estimates, as commonly used in nonlinear PDE theory [6].

In this work we consider a nonlinear kinetic equation with time-dependent force. Our strategy follows the classical energy framework used for the Boltzmann equation and related models [4, 8]. We establish:

(i) Local-in-time existence in H^s,
(ii) An explicit lower bound for the lifespan,
(iii) Uniqueness via energy estimates on differences.

A key structural observation is that although the equation is quadratic nonlinear, the difference of two solutions satisfies a linear equation in the difference variable. This phenomenon is typical in nonlinear PDE theory and underlies the Grönwall-type uniqueness argument [3].

1. Local Existence and Uniqueness via the Energy Method

We consider the nonlinear kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \tilde{F}(x, \nabla_v f) \cdot \nabla_v f = Q(f, f)$$

with initial datum $f(0, x, v) = f_0(x, v)$, where $\tilde{F} = F + t\partial_t F$. Assume:

(i) $f_0 \in H^s(\mathbb{R}^n_x \times \mathbb{R}^n_v)$, with $s > n$.
(ii) The transport field satisfies

$$\operatorname{div}_{x,v}\left(v, \tilde{F}\right) = 0$$

and its derivatives up to order s are bounded.
(iii) The bilinear operator Q satisfies the Sobolev estimate

$$\|Q(f, g)\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}.$$

Since $s > n$ the space H^s is an algebra.

Theorem 1.1 (Local Existence and Uniqueness)**.** *Under the above assumptions, there exists a time*

$$T^* = \frac{2}{C_1} \ln\left(1 + \frac{\sqrt{2}C_1}{C_2 \|f_0\|_{H^s}}\right)$$

such that the equation admits a unique solution $f \in C([0, T^*], H^s)$.

Proof. We divide the proof into two parts
Part I: Existence via Energy Method. Let us begin by defining the energy functional

$$E_s(t) = \frac{1}{2} \|f(t)\|^2_{H^s}$$

$$=\frac{1}{2}\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_v}|D^\alpha f|^2\,dxdv$$

Differentiate the energy expression in time,

$$\frac{d}{dt}E_s(t)=\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_v}D^\alpha f D^\alpha\left(\partial_t f\right).$$

Substituting the equation gives

$$\frac{d}{dt}E_s(t)=\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_v}D^\alpha f D^\alpha\left(-v\cdot\nabla_x f-\widetilde{F}\cdot\nabla_u f+Q(f,f)\right).$$

We now expand term by term

$$\frac{d}{dt}E_s(t)=T_1(t)+T_2(t)+N(t),$$

where $T_1(t)$ is the first transport term (free transport), given by

$$T_1(t)=-\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_v}D^\alpha f D^\alpha\left(v\cdot\nabla_x f\right).$$

Using Leibniz rule, $D^\alpha\left(v\cdot\nabla_x f\right)=v\cdot\nabla_x D^\alpha f$, since v is independent of x, thus

$$\begin{aligned}T_1(t)&=-\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_x\times\mathbb{R}^n_v}D^\alpha f v\cdot\nabla_x D^\alpha f\\&=-\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n_v}\left(\int_{\mathbb{R}^n_x}D^\alpha f(x,v)v\cdot\nabla_x D^\alpha f(x,v)\right)dv.\end{aligned}$$

For fixed v, define $g(x)=D^\alpha f(x,v)$. Integrating by parts in x yields

$$\int_{\mathbb{R}^n_x}g(x)v\cdot\nabla_x g(x)dx=\frac{1}{2}\int_{\mathbb{R}^n_x}v\cdot\nabla_x\left|g(x)\right|^2dx.$$

This follows from

$$gv\cdot\nabla_x g=\frac{1}{2}v\cdot\nabla_x\left(|g|^2\right).$$

Now, given that

$$v\cdot\nabla_x|g|^2=\mathrm{div}_x\left(v\,|g|^2\right)-\left(\mathrm{div}_x v\right)|g|^2,$$

an application of the divergence theorem in $\mathbb{R}^n_x$ leads to

$$\int_{\mathbb{R}^n_x}v\cdot\nabla_x|g|^2\,dx=\int_{\mathbb{R}^n_x}\mathrm{div}_x\left(v\,|g|^2\right)dx-\int_{\mathbb{R}^n_x}\left(\mathrm{div}_x v\right)\left|g^2\right|dx.$$

Notice if:

(I) f decays sufficiently fast at infinity or

(II) $f \in H^s\left(\mathbb{R}^{2n}\right)$ with $s > n$ (ensuring sufficient decay), then the boundary term vanishes,

$$\int_{\mathbb{R}^n_x} \mathrm{div}_x \left(v\,|g|^2\right) dx = 0.$$

This is because it becomes a flux over the sphere at infinity, which tends to zero. Since v does not depend on x, then $\mathrm{div}_x v = 0$. Thus

$$\int_{\mathbb{R}^n_x} v \cdot \nabla_x |g|^2 \, dx = 0.$$

Hence, returning to the full phase space we have

$$\int_{\mathbb{R}^n_x} D^\alpha f(x,v) v \cdot \nabla_x D^\alpha f(x,v) dx = 0,$$

for every v. Integrating over v gives

$$\int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} D^\alpha f v \cdot \nabla_x D^\alpha f dx dv = 0$$

therefore the free transport term vanishes, i.e. $T_1(t) = 0$. Now, consider the transport term (force transport),

$$T_2(t) = -\sum_{|\alpha| \le s} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} D^\alpha f D^\alpha \left(\widetilde{F} \cdot \nabla_v f\right).$$

One more time, using Leibniz rule,

$$D^\alpha \left(\widetilde{F} \cdot \nabla_v f\right) = \widetilde{F} \cdot \nabla_v D^\alpha f + \sum_{\beta < \alpha} c_{\alpha_\beta} \left(D^{\alpha-\beta} \widetilde{F}\right) \cdot \nabla_v D^\beta f.$$

Thus

$$T_2(t) = -\sum_{|\alpha| \le s} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} D^\alpha f \widetilde{F} \cdot \nabla_v D^\alpha f.$$

Now, we shall write

$$T_2(t) = A(t) + B(t).$$

Functions $A(t)$ and $B(t)$ are called (respectively) the *principal part* and the *commutator part* of $T_2(t)$, that is

$$A(t) = -\sum_{|\alpha| \le s} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} D^\alpha f \widetilde{F} \cdot \nabla_v D^\alpha f$$

and commutator part

$$B(t) = -\sum_{|\alpha| \le s} \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n_x \times \mathbb{R}^n_v} D^\alpha f D^\beta \widetilde{F} \cdot \nabla_v D^{\alpha-\beta} f.$$

We now treat them separately. For a fixed α, consider

$$I_\alpha = \int_{\mathbb{R}^{2n}} D^\alpha f \widetilde{F} \cdot \nabla_v D^\alpha f,$$

since $g\widetilde{F} \cdot g = \frac{1}{2}\widetilde{F} \cdot \nabla_v \left(g^2\right)$, by taking $g = D^\alpha f$, we get

$$I_\alpha = \frac{1}{2} \int_{\mathbb{R}^{2n}} \widetilde{F} \cdot \nabla_v \left|D^\alpha f\right|^2,$$

now, by using the identity

$$\widetilde{F} \cdot \nabla_v \phi = \mathrm{div}_v \left(\widetilde{F}\phi\right) - \left(\mathrm{div}_v \widetilde{F}\right)\phi,$$

we obtain

$$I_\alpha = \frac{1}{2} \int_{\mathbb{R}^{2n}} \mathrm{div}_v \left(\widetilde{F}\left|D^\alpha f\right|^2\right) - \frac{1}{2} \int_{\mathbb{R}^{2n}} \left(\mathrm{div}_x \widetilde{F}\right)\left|D^\alpha f\right|^2.$$

Now, we justify that the first integral above vanishes. Recall that:

(I) $f \in H^s\left(\mathbb{R}^{2n}\right)$

(II) $s > n$ implies sufficient decay and $\widetilde{F}$ is bounded, the divergence term integrates to zero over $\mathbb{R}^n_v$:

$$\int_{\mathbb{R}^n_v} \mathrm{div}_v \left(\widetilde{F}\left|D^\alpha f\right|^2\right) dv = 0.$$

Thus

$$I_\alpha = -\frac{1}{2} \int_{\mathbb{R}^{2n}} \left(\mathrm{div}_v \widetilde{F}\right)\left|D^\alpha f\right|^2.$$

We assume $\mathrm{div}_{x,v}\left(v, \widetilde{F}\right) = 0$. But

$$\mathrm{div}_{x,v}\left(v, \widetilde{F}\right) = \mathrm{div}_x(v) + \mathrm{div}_v\left(\widetilde{F}\right),$$

since v does not depend on x, $\mathrm{div}_x(v) = 0$. Hence $\mathrm{div}_v\left(\widetilde{F}\right) = 0$. Therefore $I_\alpha = 0$. Summing over $|\alpha| \leq s$, we conclude $A(t) = 0$, so the principal force transport term vanishes.

Now we shall consider the commutator term, i.e.

$$B(t) = -\sum_{|\alpha|\leq s} \sum_{0<\beta<\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^{2n}} D^\alpha f D^\beta \widetilde{F} \cdot \nabla_v D^{\alpha-\beta} f.$$

Thus for each pair (α, β) we shall apply the Hölder inequality, that is

$$\left|\int_{\mathbb{R}^{2n}} D^\alpha f D^\beta \widetilde{F} \nabla_v D^{\alpha-\beta} f\right| \leq \left\|D^\alpha f\right\|_{L^2} \left\|D^\beta \widetilde{F}\right\|_{L^\infty} \left\|\nabla_v D^{\alpha-\beta} f\right\|_{L^2}.$$

since $|\alpha| \leq s$ and $0 < \beta \leq \alpha$, we have $|\alpha - \beta| + 1 \leq s$, thus

$$\left\|\nabla_v D^{\alpha-\beta} f\right\|_{L^2} \leq \|f\|_{H^s}.$$

Also $\|D^\alpha f\|_{L^2} \leq \|f\|_{H^s}$, assuming $F \in W^{s,\infty}$, we have

$$\left\|D^\beta \widetilde{F}\right\|_{L^\infty} \leq \left\|\widetilde{F}\right\|_{W^{s,\infty}}.$$

Finally, summing up over finitely many multi-indices, it gives

$$|B(t)| \leq C \left\|\widetilde{F}\right\|_{W^{s,\infty}} \|f\|_{H^s}^2,$$

but $\|f\|_{H^s}^2 = 2E_s(t)$, we obtain $B(t) \leq C_1 E_s(t)$. And so

$$|T_2(t)| \leq C_1 E_s(t).$$

Now, we shall consider the nonlinear term

$$N(t) = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^{2n}} D^\alpha f D^\alpha Q(f,f).$$

Using the bilinear Sobolev estimate

$$\|Q(f,f)\|_{H^s} \leq C \|f\|_{H^s}^2,$$

therefore

$$|N(t)| \leq C \|f\|_{H^s}^3 = C \left(E_s(t)\right)^{\frac{3}{2}}.$$

To obtain an explicit formula, we put everything together on

$$\begin{aligned} \frac{dE_s(t)}{dt} =& T_1(t) + T_2(t) + N(t), \\ |T_1(t)| =& 0, \\ |T_2(t)| \leq& C_1 E_s(t), \\ |N(t)| \leq& C_2 \left(E_s(t)\right)^{\frac{3}{2}}, \end{aligned}$$

we arrived at

$$\frac{d}{dt} E_s(t) \leq C_1 E_s(t) + C_2 \left(E_s(t)\right)^{\frac{3}{2}}. \tag{1.1}$$

Now we shall solve (1.1). Note that

$$\frac{d}{dt}\left(E_s(t)\right) \leq C_1 E_s(t) + C_2 \left(E_s(t)\right)^{\frac{3}{2}},$$

can be written as

$$\left(E_s(t)\right)^{-\frac{3}{2}} \frac{d}{dt}\left(E_s(t)\right) \leq C_1 \left(E_s(t)\right)^{-\frac{1}{2}} + C_2.$$

Then

$$-\frac{1}{2}\left(E_s(t)\right)^{-\frac{3}{2}} \frac{dE_s(t)}{dt} \geq -\frac{1}{2}\left(E_s(t)\right)^{-\frac{1}{2}} + C_2$$

so making the change of variable $U = \left(E_s(t)\right)^{-\frac{1}{2}}$ then

$$\frac{dU}{dt} = -\frac{1}{2}\left(E_s(t)\right)^{-\frac{3}{2}} \frac{dE_s(t)}{dt}$$

and so

$$\frac{dU}{dt} \geq -\frac{C_1}{2}U - \frac{C_2}{2}$$

thus

$$\frac{dU}{dt} + \frac{C_1}{2}U \geq -\frac{C_2}{2}.$$

Multiplying by $e^{C_1\frac{t}{2}}$ leads to

$$e^{C_1\frac{t}{2}}\frac{dU}{dt} + \frac{C_1}{2}e^{C_1\frac{t}{2}}U \geq -\frac{C_2}{2}e^{C_1\frac{t}{2}}$$

integrating from 0 to t

$$\int_0^t \frac{d}{ds}\left(Ue^{C_1\frac{s}{2}}\right) \geq -\frac{C_2}{2}\int_0^t e^{C_1\frac{s}{2}}ds.$$

Thus

$$U(t)e^{C_1\frac{t}{2}} - U(0) \geq -\frac{C_1}{C_2}\left(e^{\frac{C_1 t}{2}} - 1\right),$$

solving for $U(t)$ and reverting to $E_s(t)$ we obtain

$$E_s(t) \leq \left[\frac{e^{-C_1\frac{t}{2}}}{\sqrt{E_s(0)}} - \frac{C_2}{C_1}\left(1 - e^{-C_1\frac{t}{2}}\right)\right].$$

To find T_1^* we set the expression equal to zero

$$\frac{e^{-C_1\frac{T^*}{2}}}{E_s(0)} - \frac{C_2}{C_1}\left(1 - e^{-C_1\frac{T^*}{2}}\right) = 0,$$

after algebraic manipulations, we get

$$\frac{C_1}{C_2\sqrt{E_s(0)}} + 1 = e^{C_1\frac{T^*}{2}},$$

and so

$$T^* = \frac{2}{C_1}\ln\left(1 + \frac{C_1}{C_2\sqrt{E_s(0)}}\right).$$

Notice $E_s(0) = \frac{1}{2}\|f\|_{H^s}^2$. Hence

$$T^* = \frac{2}{C_1}\ln\left(1 + \frac{C_1\sqrt{2}}{C_2\|f_0\|_{H^s}}\right).$$

Finally, taking into account that $E_s(t) = \|f(t)\|_{H^s}^2$ and also, given that the term $C_1E_s(t)$ in (1.1) is negligible, we obtain

$$\frac{d}{dt}E_s(t) \leq C\left(E_s(t)\right)^{\frac{3}{2}}. \tag{1.2}$$

We already know $f \in L^\infty\left([0,T^*], H^s\right)$, hence $0 \leq E_s(t) \leq M$ for some $M < \infty$.

By using (1.2) and the fact that $E_s(t) \leq M$

$$\left| \frac{d}{dt} E_s(t) \right| \leq C M^{\frac{3}{2}} = K.$$

Thus $\frac{d}{dt} E_s \in L^\infty([0, T^*])$. Hence

$$|E_s(t) - E_s(t_0)| \leq K|t - t_0|,$$

this means that E_s in continuous on $[0, T^*]$, i.e. $E_s \in C([0, T^*])$. This implies the continuity of $\|f(t)\|_{H^s} = \sqrt{E_s(t)}$, that is, the map $t \mapsto \|f\|_{H^s}$ is continuous on $[0, T^*]$. Hence $f \in C([0, T^*], H^s)$.

Now let J_ϵ be a standard mollifier. Define the smooth approximate equation

$$\begin{aligned} &\partial_t f^\epsilon + J_\epsilon (U(f^\epsilon)) \cdot \nabla f^\epsilon = J_\epsilon Q(f^\epsilon, f^\epsilon), \\ &f^\epsilon(0) = J_\epsilon f_{0,\epsilon}. \end{aligned} \tag{1.3}$$

Smoothness of terms involved above, guarantees that this is an *ODE* in H^s. Hence, there exists a solution $f^\epsilon \in C^1([0, T_\epsilon], H^s)$ of (1.3), so we have existence of solutions for each ϵ. Define

$$E_s^\epsilon(t) = \|f^\epsilon(t)\|_{H^s}^2,$$

using the same energy computation as before

$$\frac{d}{dt} E_s^\epsilon \leq C (E_s^\epsilon)^{\frac{3}{2}},$$

so we get

$$E_s^\epsilon(t) \leq \frac{E_s^\epsilon(0)}{\left(1 - Ct\sqrt{E_s^\epsilon(0)}\right)^2}.$$

Since $E_s^\epsilon(0) = \|J_\epsilon f_0\|_{H^s} \leq \|f_0\|_{H^s}$ there exists $T > 0$ independent of ϵ such that $\sup_{0 \leq t \leq T} \|J^\epsilon(t)\|_{H^s} \leq C_0$.

From the equation

$$\partial_t f^\epsilon = -\left[J_\epsilon (U(f^\epsilon))\right] \cdot \nabla f^\epsilon + J_\epsilon Q(f^\epsilon, f^\epsilon), \tag{1.4}$$

we obtain

$$\|\partial_t f^\epsilon\|_{H^{s-1}} \leq C \|f^\epsilon\|_{H^s}^2 \leq C. \tag{1.5}$$

Inequality (1.5) warrants further elaboration.

From (1.4) and given that the mollifier J_ϵ constitutes a bounded linear operator on every Sobolev space,

$$\|J_\epsilon g\|_{H^r} \leq \|g\|_{H^r},$$

it suffices to estimate each term on the right-hand side of (1.4).

We first address the transport term

$$\|J_\epsilon (U(f^\epsilon)) \cdot \nabla f^\epsilon\|_{H^{s-1}}.$$

Since J_ϵ is a contraction,

$$\|J_\epsilon\left(U(f^\epsilon)\right)\cdot\nabla f^\epsilon\|_{H^{s-1}} \leq \|U(f^\epsilon)\cdot\nabla f^\epsilon\|_{H^{s-1}}\,.$$

Now, assume that $s > \frac{n}{2}+1$. Then the Sobolev embedding theorem furnishes

$$H^s\left(\mathbb{R}^{2n}\right) \hookrightarrow W^{1,\infty}\left(\mathbb{R}^{2n}\right),$$

and H^{s-1} is a Banach algebra. Consequently, the Moser product estimate yields

$$\begin{aligned}&\|U(f^\epsilon)\cdot\nabla f^\epsilon\|_{H^{s-1}}\\ &\quad\leq C\left(\|U(f^\epsilon)\|_{L^\infty}\|\nabla f^\epsilon\|_{H^{s-1}} + \|U(f^\epsilon)\|_{H^{s-1}}\|\nabla f^\epsilon\|_{L^\infty}\right).\end{aligned}$$

Moreover, since $\|\nabla f^\epsilon\|_{H^{s-1}} = \|f^\epsilon\|_{H^s}$ and $\|\nabla f^\epsilon\|_{L^\infty} \leq C\|f^\epsilon\|_{H^s}$, in conjunction with the hypothesis

$$\|U(f)\|_{H^{s-1}} + \|U(f)\|_{L^\infty} \leq C\|f\|_{H^s}\,,$$

it follows that

$$\|U(f^\epsilon)\cdot\nabla f^\epsilon\|_{H^{s-1}} \leq C\|f^\epsilon\|_{H^s}^2\,.$$

We now turn to the collision term. Again, by the boundedness of J_ϵ,

$$\|J_\epsilon Q(f^\epsilon,f^\epsilon)\|_{H^{s-1}} \leq \|Q(f^\epsilon,f^\epsilon)\|_{H^{s-1}}\,.$$

By virtue of the assumed bilinear Sobolev estimate

$$\|Q(f,g)\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}\,,$$

and the continuous embedding $H^s \hookrightarrow H^{s-1}$, we immediately obtain

$$\|Q(f^\epsilon,f^\epsilon)\|_{H^{s-1}} \leq \|Q(f^\epsilon,f^\epsilon)\|_{H^s} \leq C\|f^\epsilon\|_{H^s}^2\,.$$

Consequently,

$$\|J_\epsilon Q(f^\epsilon,f^\epsilon)\|_{H^{s-1}} \leq C\|f^\epsilon\|_{H^s}^2\,.$$

We are now in a position to reach the desired conclusion. Applying the triangle inequality,

$$\begin{aligned}\|\partial_t f^\epsilon\|_{H^{s-1}} &\leq \|J_\epsilon\left(U(f^\epsilon)\right)\cdot\nabla f^\epsilon\|_{H^{s-1}} + \|J_\epsilon Q(f^\epsilon,f^\epsilon)\|_{H^{s-1}}\\ &\leq C\|f^\epsilon\|_{H^s}^2 + C\|f^\epsilon\|_{H^s}^2\\ &= C\|f^\epsilon\|_{H^s}^2\,.\end{aligned}$$

Finally, by the uniform energy estimate established previously,

$$\sup_{0\leq t\leq T}\|f^\epsilon(t)\|_{H^s} \leq C_0,$$

we conclude that

$$\begin{aligned}\|\partial_t f^\epsilon\|_{H^{s-1}} &\leq C\|f^\epsilon\|_{H^s}^2\\ &\leq C\left(\sup_{0\leq t\leq T}\|f^\epsilon(t)\|_{H^s}\right)^2\end{aligned}$$

$$\leq CC_0^2,$$

Thus

$$\|\partial_t f^\epsilon\|_{H^{s-1}} \leq C,$$

and inequality (1.5) is established. Thus we have uniform bounds:

$$f^\epsilon \quad \text{bounded in} \quad L^\infty\left([0,T], H^{s-1}\right), \tag{1.6}$$

$$\partial_t f^\epsilon \quad \text{bounded in} \quad L^\infty\left([0,T], H^{s-1}\right), \tag{1.7}$$

we use the compact embeddings $H^s \hookrightarrow H^{s-1} \hookrightarrow H^{s-2}$. Since (1.6) and (1.7) hold, Aubin-Lions implies

$$f^\epsilon \to f \quad \text{strongly in} \quad C\left([0,T], H^{s-1}\right).$$

Thus we extract a subsequence (still denoted f^ϵ) such that

$$f^\epsilon \to f \quad \text{strongly in} \quad H^{s-1}.$$

Since convergence is strong in H^{s-1} and $s-1 > \frac{n}{2}$, we have convergence in L^∞. Therefore

(I) $U(f^\epsilon) \to U(f)$
(II) $Q(f^\epsilon, f^\epsilon) \to Q(f,f)$

Hence we may pass to the limit in (1.3). The limit f satisfies (1.3). From weak-$*$ compactness $f \in L^\infty\left([0,T], H^s\right)$, and from the time derivative bound, $f \in C\left([0,T], H^s\right)$. So, energy continuity upgrades the above result to $f \in C\left([0,T], H^s\right)$. Summing up, energy methods give:

$$\|f^\epsilon\|_{L^\infty \to H^s} \leq C,$$

and

$$\|\partial_t f\|_{L^\infty \to H^s} \leq C.$$

Compactness gives:

$$f^\epsilon \to f \quad \text{strongly}$$

limit passage gives existence and part I is proven.
Part II: Uniqueness. Let f and g be two solutions with the same initial data. Define the difference: $h = f - g$. Then h satisfies

$$\partial_t h + v \cdot \nabla_x h + \widetilde{F} \cdot \nabla_v h = Q(f,f) - Q(g,g).$$

Bilinearity of Q gives

$$Q(f,f) - Q(g,g) = Q(f, f-g) + Q(f-g, g).$$

But $h = f - g$, then

$$Q(f,f) - Q(g,g) = Q(f,h) + Q(h,g).$$

Thus

$$\partial_t h + \text{transport} = Q(f,h) + Q(h,g).$$

Define

$$H_s(t) = \frac{1}{2} \|h(t)\|_{H^s}^2 .$$

Differentiating

$$\frac{d}{dt} H_s(t) = \sum_{|\alpha| \le s} \int_{\mathbb{R}_x^n \times \mathbb{R}_v^n} D^\alpha h D^\alpha \left(\partial_t h\right).$$

Transport terms vanish as before. Using bilinear estimate:

$$\|Q(f,h)\|_{H^s} \le C \|f\|_{H^s} \|h\|_{H^s} ,$$

and an analogous estimate holds for $Q(h,g)$. Hence,

$$\frac{d}{dt} \|h\|_{H^s}^2 \le C \left(\|f\|_{H^s} + \|g\|_{H^s}\right) \|h\|_{H^s}^2 .$$

Let $M = \sup_{t \le T^*} \left(\|f\|_{H^s} + \|g\|_{H^s}\right)$. Then

$$\frac{d}{dt} \|h(t)\|_{H^s}^2 \le CM \|h(t)\|_{H^s}^2 .$$

Applying Grönwall's inequality

$$\|h(t)\|_{H^s}^2 \le \|h(0)\|_{H^s}^2 e^{CMt}.$$

But $h(0) = 0$, then the above inequality implies $\|h(t)\|_{H^s} = 0$. Thus $h = 0$ and the solution is unique. □

It is worth to mention that an alternative approach to proving uniqueness for the Boltzmann equation was introduced by the third named author in [1]. There, the authors established the uniqueness of local solutions in L^1 by means of a fixed-point theorem of Hardy–Rogers type. In their analysis, the collision operator is treated as a possibly discontinuous bilinear operator and an associated integral operator is constructed such that its fixed point corresponds to a solution of the Boltzmann equation.

The method adopted in the present work is fundamentally different. Rather than relying on a fixed-point argument as in [1], we employ high-order Sobolev energy estimates in H^s. This energy-based framework allows us to derive explicit a priori bounds and to obtain a quantitative lower bound on the lifespan of solutions. Together, these two approaches illustrate complementary techniques for addressing quadratic nonlinearities in kinetic equations.

2. Conclusion

The energy method yields local well-posedness under minimal structural assumptions. The energy method allows us to bypass the continuity requirement for Q by controlling the growth of the Sobolev norms.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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