Sahand Communications in Mathematical Analysis (SCMA) Vol. 24 No. 1 (2027), 95-116
http://scma.maragheh.ac.ir
DOI: https://doi.org/10.22130/scma.2026.2088341.2637

Hermite-Hadamard Type Inequality for Pseudo-Integrals

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Abstract. In this paper, we define $(\oplus, \otimes)$-convex set and $(\oplus, \otimes)$-convex functions and generalize them for s-p-convex and p-convex. We also generalize Hermite-Hadamard type inequalities for s-p-convex functions and p-convex functions for pseudo-integrals in two of semirings: one, when the pseudo-operations $\otimes$ and $\oplus$ are defined by a comonotone and continuous generator function α, the second semiring $([\eta, \theta], sup, \otimes)$, where the multiplication operator is generated. Also, we give further discussion about it.

1. Introduction

Pseudo-analysis and pseudo-integrals are generalizations of classical analysis and Sugeno integrals, where the semiring $([\eta, \theta], \oplus, \otimes)$ is defined on real interval $[\eta, \theta] \subset [-\infty, +\infty]$ instead of $\mathbb{R}$ (see [7, 8, 13, 14]), and the pseudo-integral with $\oplus = \max$ and $\otimes = \min$ yields the fuzzy integral as special case when the fuzzy measure is Max-additive. Based on this structure there are developed the concepts of pseudo-additive measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The pseudo-analysis uses many mathematical tools form different fields, usually nonlinear and under uncertainty, such as system theory, optimization, decision making, control theory, differential equations and difference equations, economics, game theory, risk management, etc.
In recent years, much attention has been given by many scientists to convex function for its great utility in various fields of pure and applied sciences, such as architecture, management sciences, industrial, arts and

2020 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Hermite-Hadamard's inequality, Fuzzy integral inequality, Semiring, Pseudo-Integral, Pseudo-Operation, s-p-convex, p-convex.
Received: 2026-06-15, Accepted: 2026-09-09.
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pharmaceutical research and many more. In convexity a very interesting and applicable inequality is Hermite-Hadamard type inequality. Hermite-Hadamard type inequalities provide estimates of the mean value of a nonnegative and ordinary convex function and so has great utility in integral calculations. There are many various generalizations of this inequality in literature. It provides a necessary and sufficient condition for a function to be convex. Some mathematicians have investigated convexity, see [6, 12]. Recently, H. Agahi, by using a pseudo-average value of a function over a simplex, presented some pseudo-inequalities over simplices. These developments are in line with the ongoing research on integral inequalities in nonlinear and uncertain settings, which motivates our study on fuzzy integral inequalities for s-convex functions of the first kind (see [2]). In [1, 10], the classical Hermite-Hadamard inequality was extended to Sugeno integrals within the framework of convex functions. Subsequently, [11] generelized these results for p-convex functions, while [16] further extended the theory to the class of s-p-convex functions. Extending Hermite-Hadamard inequalities to the pseudo-integral setting addresses the limitations of classical linear tools in nonlinear and uncertain enviroments. This reformulation for s-p-convex functions provides a unified framework that captures both function curvature and nonadditive aggregation, making it particularly useful in decision-making, risk analysis, nonlinear optimization and fuzzy modeling. Our results thus bridge theoretical pseudo-analysis with practical applications in economics, engineering and machine learning under uncertainly. The classical or the usual convex function is defined as follows:

Let $\rho : \Lambda \to \mathbb{R}$ be a function defined on the interval Λ of real numbers, then ρ is called convex if

$$\rho(\omega\nu + (1-\omega)\tau) \leq \omega\rho(\nu) + (1-\omega)\rho(\tau) \tag{1.1}$$

for all $\nu, \tau \in \Lambda$ and $\omega \in [0,1]$. We say that ρ is concave if $(-\rho)$ is convex. For $p \in \mathbb{R}\backslash\{0\}$ a function $\rho : \Lambda \subset [0,\infty) \to [0,\infty)$ is said to be the p-convex function, if

$$\rho\left([\omega\nu^p + (1-\omega)\tau^p]^{\frac{1}{p}}\right) \leq \omega\rho(\nu) + (1-\omega)\rho(\tau) \tag{1.2}$$

for every $\nu, \tau \in \Lambda$ and $\omega \in [0,1]$. If we choose $p = 1$ in (1.2), we get ordinary convex function [4, 5].

Also for $s \in (0,1], p \in \mathbb{R}\backslash\{0\}$ a function $\rho : \Lambda \subset [0,\infty) \to [0,\infty)$ is said to be the s-p-convex function in the first kind, if

$$\rho\left([\omega\nu^p + (1-\omega)\tau^p]^{\frac{1}{p}}\right) \leq \omega^s\rho(\nu) + (1-\omega^s)\rho(\tau) \tag{1.3}$$

for every $\nu, \tau \in \Lambda$ and $\omega \in [0,1]$. If $p = 1$ in (1.3), we obtain an s-convex function of the first kind and we say that ρ belongs to the first kind class.

As a remark from the reference [3], we bring a general example of functions of the first kind as follows:

Example 1.1. For $s \in (0,1]$ and $\eta, \theta, \beta \in \mathbb{R}$, if $\rho : [0,\infty) \to \mathbb{R}$ is defined by

$$\rho(\nu) = \begin{cases} \eta, & \nu = 0, \\ \theta\nu^s + \beta, & \nu > 0, \end{cases}$$

then $\rho \in k_s^1$, where $\theta \geq 0$ and $\beta \leq \eta$.

If $s = 1$ in (1.3), we obtain the p-convex function and if $s = p = 1$ in (1.3), we obtain ordinary convex function [3].

In the following, we have Hermite-Hadamard type inequality[3]:

$$\rho\left(\frac{\eta+\theta}{2}\right) \leq \frac{1}{\theta-\eta}\int_\eta^\theta \rho(\nu)d\nu \leq \frac{\rho(\eta)+\rho(\theta)}{2} \tag{1.4}$$

where $\rho : \Lambda \to \mathbb{R}$ is a convex function defined on the interval $\Lambda \subset \mathbb{R}$ and $\eta, \theta \in \Lambda$, with $\eta < \theta$. For a concave function ρ, both inequalities hold in the reverse order.

In [3] have presented some new Hermite-Hadamard type inequalities for s-p-convex functions as follows:

Let $\rho : \Lambda \to \mathbb{R}\backslash\{0\}$ be an s-p-convex function of the first kind, then the following inequalities hold:

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}}d\nu \leq \frac{\rho(\eta)+s\rho(\theta)}{s+1}. \tag{1.5}$$

If $s = 1$ in (1.5), we obtain the inequality for a p-convex function and we have,

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}}d\nu \leq \frac{\rho(\eta)+\rho(\theta)}{2}. \tag{1.6}$$

If $s = p = 1$ in (1.5), we obtain ordinary Hermite-Hadamard type inequality in (1.4).

In this paper, we generalize Hermite-Hadamard type inequalities for s-p-convex functions for pseudo-integrals in two cases of the semiring in Section 3 and delve into this generalization as a further discussion in this Section. In Section 4, we generalize Hermite-Hadamard type inequalities for p-convex functions for pseudo-integrals in two cases of the semiring and provide further discussion on it. Section 5 is devoted to conclusion of the paper.

2. Preliminaries

Here, we bring some definitions and concepts of the preliminaries relevance to our main results. Everywhere in this article, we consider the Lebesgue measure by μ.

2.1. **Pseudo-Integrals.** In the following, we give definitions of the operations $\oplus$ and $\otimes$ that are named pseudo-addition and pseudo-multiplication, respectively. Let $[\eta, \theta]$ be an interval of $[-\infty, \infty]$. Here $\preceq$ is a total order on $[\eta, \theta]$ and $[\eta, \theta]_+ = \{\nu | \nu \in [\eta, \theta], \mathbf{0} \preceq \nu\}$.

The operations $\oplus$ and $\otimes$ are defined in [13]. There are three cases for these operations as following:

I. pseudo-addition is idempotent, i.e., $\nu \oplus \nu = \nu$ and pseudo-multiplication is not idempotent.
II. Generated pseudo-operations by a continuous function α. In this case we will consider $\alpha : [\beta, \gamma] \to [0, \infty)$ such that $\alpha(0) = 0$, $u \oplus v = \alpha^{-1}(\alpha(u) + \alpha(v))$ and $u \otimes v = \alpha^{-1}(\alpha(u)\alpha(v))$.
III. Both operations are idempotent, i.e, $\oplus = \vee$ and $\otimes = \wedge$ or $\oplus = \wedge$ and $\otimes = \vee$.

From case II (see [13]), when $\alpha : [\beta, \gamma] \to [0, \infty)$ and function $\rho : [\eta, \theta] \to [\beta, \gamma]$ are defined, we have

$$\int_{[\eta,\theta]}^{\oplus} \rho(\nu) d\nu = \alpha^{-1} \int_{\eta}^{\theta} \alpha(\rho(\nu)) d\nu.$$

From case I (see [13]), if $\nu \oplus \tau = \nu \vee \tau$ and operation $\otimes$ is generated, then for a function $\rho : \mathbb{R} \to [\beta, \gamma]$,

$$\int_{\mathbb{R}}^{\sup} \rho \otimes dm = \sup_{\nu \in \mathbb{R}} (\rho(\nu) \otimes \psi(\nu)),$$

where the function $\psi : \mathbb{R} \to [\beta, \gamma]$ defines a sup-measure m and the following equalities

$$\begin{aligned} \int^{\sup} \rho \otimes dm &= \lim_{\lambda \to \infty} \int^{\oplus_\lambda} \rho \otimes dm_\lambda \\ &= \lim_{\lambda \to \infty} \left(\alpha^\lambda\right)^{-1} \left(\int \alpha^\lambda(\rho(\nu)) d\nu \right) \end{aligned} \tag{2.1}$$

hold.

2.2. **$(\oplus, \otimes)$-Convex Sets and $(\oplus, \otimes)$-Convex Functions.** In this section, we shall introduce the notion of $(\oplus, \otimes)$-convex sets and $(\oplus, \otimes)$-convex functions with respect to pseudo-operations on $(\oplus, \otimes)$-convex sets, respectively (see [18]).

Let $([\eta, \theta], \oplus, \otimes)$ be a semiring and $[\eta, \theta]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, +\infty]$. A subset Λ

of $[\eta,\theta]$ is said to be $(\oplus,\otimes)$-convex if and only if for all $\nu,\tau\in\Lambda$ and $\omega\in[0,1]$,

$$\omega\otimes\nu\oplus(1-\omega)\otimes\tau\in\Lambda. \tag{2.2}$$

Suppose that $([\eta,\theta],\oplus,\otimes)$ is a semiring and $\Lambda\subset[\eta,\theta]$ is a $(\oplus,\otimes)$-convex set. A function $\rho:\Lambda\to[\beta,\gamma]$ is said to be $(\oplus,\otimes)$-convex if for all $\nu,\tau\in\Lambda$ and $\omega\in[0,1]$, the following inequality

$$\rho(\omega\otimes\nu\oplus(1-\omega)\otimes\tau)\preceq\omega\otimes\rho(\nu)\oplus(1-\omega)\otimes\rho(\tau), \tag{2.3}$$

holds. We say that ρ is $(\oplus,\otimes)$-concave if $(-\rho)$ is $(\oplus,\otimes)$-convex.

3. Main Results

In this section, we state the inequality (2.3) for s-p-convex and p-convex.

Definition 3.1. Let $([0,\infty),\oplus,\otimes)$ be a semiring and $\Lambda\subset[0,\infty)$ be a $(\oplus,\otimes)$-convex set. A function $\rho:\Lambda\subset[0,\infty)\to[0,\infty)$ is said to be $(\oplus,\otimes)$-s-p-convex in first kind, if

$$\rho\bigg((\omega\otimes\nu^p\oplus(1-\omega)\otimes\tau^p)^{\frac{1}{p}}\bigg)\preceq\omega^s\otimes\rho(\nu)\oplus(1-\omega^s)\otimes\rho(\tau), \tag{3.1}$$

for all $\nu,\tau\in\Lambda$, $\omega\in[0,1]$, $p\in\mathbb{R}\backslash\{0\}$ and $s\in(0,1]$.

Definition 3.2. Let $([0,\infty),\oplus,\otimes)$ be a semiring and $\Lambda\subset[0,\infty)$ be a $(\oplus,\otimes)$-convex set. A function $\rho:\Lambda\subset[0,\infty)\to[0,\infty)$ is said to be $(\oplus,\otimes)$-p-convex , if

$$\rho\bigg((\omega\otimes\nu^p\oplus(1-\omega)\otimes\tau^p)^{\frac{1}{p}}\bigg)\preceq\omega\otimes\rho(\nu)\oplus(1-\omega)\otimes\rho(\tau), \tag{3.2}$$

for all $\nu,\tau\in\Lambda$, $\omega\in[0,1]$ and $p\in\mathbb{R}\backslash\{0\}$.

Now we state and prove the inequality (1.5) for pseudo-integral.

Theorem 3.3. *Let $\rho:[\eta,\theta]\to[\beta,\gamma]$ be a $(\oplus,\otimes)$-s-p-convex function in the first kind and $\alpha:[\beta,\gamma]\to[0,\infty)$ be a continuous and increasing function. Then for any $p\in\mathbb{R}\backslash\{0\}$ and $s\in(0,1]$, the following inequalities*

$$\begin{aligned}\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)&\preceq\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\\&\preceq\frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta))\end{aligned} \tag{3.3}$$

hold.

Proof. From inequality (1.5) we know that

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}}d\nu. \tag{3.4}$$

Since α is increasing, we apply it to both sides of (3.4). However, α is not necessarily linear, so we can not move it inside the integral. Instead, we apply inequality (3.4) to the composed function $\alpha o\rho$. This is legitimate because $\alpha o\rho$ is again $(\oplus,\otimes)$-s-p-convex:

$$\alpha\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \leq \frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu. \tag{3.5}$$

Applying α^{-1} (which is also increasing) to both sides of (3.5) gives

$$\alpha^{-1}\left(\alpha\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right) \leq \alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right).$$

Thus,

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right).$$

Now, by the definition of the pseudo-integral, we have

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm.$$

Also we know that

$$\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}}d\nu \leq \frac{\rho(\eta)+s\rho(\theta)}{s+1}.$$

Similarly, we have

$$\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu \leq \frac{\alpha(\rho(\eta))+s\alpha(\rho(\theta))}{s+1}. \tag{3.6}$$

Applying α^{-1} (which is also increasing) to both sides of (3.6) gives

$$\alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right) \leq \alpha^{-1}\left(\frac{\alpha(\rho(\eta))+s\alpha(\rho(\theta))}{s+1}\right).$$

Thus,

$$\frac{p}{\theta^p-\eta^p}\otimes\alpha^{-1}\int_\eta^\theta \frac{\alpha(\rho(\nu))d\nu}{\nu^{1-p}} \preceq \frac{1}{s+1}\otimes\alpha^{-1}(\alpha(\rho(\eta))+s\alpha(\rho(\theta))).$$

So, we have

$$\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\preceq\frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta)).$$

This completes the proof. □

Example 3.4. Let $\rho(\nu)=\nu^2$ and $[\eta,\theta]=[0,1]$, $\alpha(\nu)=\nu^\chi$ and $\chi\in[1,\infty)$. Then we have,

$$\nu\oplus\tau=\alpha^{-1}(\alpha(\nu)+\alpha(\tau))=\alpha^{-1}(\nu^\chi+\tau^\chi)=(\nu^\chi+\tau^\chi)^{\frac{1}{\chi}},$$
$$\nu\otimes\tau=\alpha^{-1}(\alpha(\nu)\alpha(\tau))=\alpha^{-1}(\nu^\chi\tau^\chi)=(\nu^\chi\tau^\chi)^{\frac{1}{\chi}}=\nu\tau.$$

If $\eta=0,\theta=1$ we have, $\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right)=\rho\left((\frac{1}{2})^{\frac{1}{p}}\right)=\left(\frac{1}{2}\right)^{\frac{2}{p}}$ and

$$\begin{aligned}\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm&=\alpha^{-1}\left(\int_\eta^\theta\frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right)\\&=\alpha^{-1}\left(\int_\eta^\theta\frac{\nu^{2\chi}}{\nu^{1-p}}d\nu\right)\\&=\alpha^{-1}\left(\frac{1}{2\chi+p}\right),\end{aligned}$$

on the other hand,

$$\frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta))=\frac{1}{s+1}\otimes\left(\eta^2\oplus s\otimes\theta^2\right)=\frac{1}{s+1}\otimes(0\oplus s\otimes1),$$

since $\rho(0)=0,\rho(1)=1$, we have, $0\oplus s\otimes1=(0^\chi+(s.1)^\chi)^{\frac{1}{\chi}}=s$.
And we have,

$$\frac{1}{s+1}\otimes s=\left((\frac{1}{s+1})^\chi.s^\chi\right)^{\frac{1}{\chi}}=\frac{s}{s+1}.$$

Thus,

$$\left(\frac{1}{2}\right)^{\frac{2}{p}}=\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right)\leq p\left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}}\leq\frac{s}{s+1}.$$

If we get $\omega\in[0,1]$, $p\in\mathbb{R}\backslash\{0\}$ and $s\in(0,1]$, then we have,

$$\left(\frac{1}{2}\right)^{\frac{2}{p}}\leq p\left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}}\leq\frac{s}{s+1}.$$

Theorem 3.5. *Let $\rho : [\eta, \theta] \to [\beta, \gamma]$ be a $(\oplus, \otimes)$-s-p-convex function in the first kind and $\alpha : [\beta, \gamma] \to [0, \infty)$ be a continuous and increasing function and $([0, 1], \sup, \otimes)$ be a semiring and m be the same as equality (2.1). Then for any $p \in \mathbb{R}\backslash\{0\}$ and $s \in (0, 1]$, the following inequalities* (3.7)

$$\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm \preceq \frac{1}{s+1} \otimes (\rho(\eta) \oplus s \otimes \rho(\theta))$$

hold.

Proof. By inequality (1.5) we know that

$$\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta^p - \eta^p} \int_{\eta}^{\theta} \frac{\rho(\nu)}{\nu^{1-p}} d\nu, \tag{3.8}$$

The sup-integral (with respect to $\otimes$) is defined as the following:

$$\int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm = \lim_{\lambda \to \infty} \left(\alpha^{-\lambda}\left(\frac{p}{\theta^p - \eta^p} \alpha^{\lambda}\left(\int_{\eta}^{\theta} \frac{(\rho(\nu))}{\nu^{1-p}} d\nu\right)\right)\right).$$

Since α^{λ} is increasing, applying it to inequality (3.8),we have

$$\alpha^{\lambda}\left(\rho\left(\left(\frac{\eta^p + \theta^p}{2}\right)^{\frac{1}{p}}\right)\right) \leq \alpha^{\lambda}\left(\frac{p}{\theta^p - \eta^p} \int_{\eta}^{\theta} \frac{\rho(\nu)}{\nu^{1-p}} d\nu\right),$$

then we have

$$\lim_{\lambda \to \infty}\left(\alpha^{\lambda}\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right) \leq \lim_{\lambda \to \infty}\left(\frac{p}{\theta^p - \eta^p} \int_{\eta}^{\theta} \frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}} d\nu\right).$$

It follows that

$$\lim_{\lambda \to \infty}\left(\alpha^{-\lambda}\left(\alpha^{\lambda}\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right)\right) \leq \lim_{\lambda \to \infty}\left(\alpha^{-\lambda}\left(\frac{p}{\theta^p - \eta^p} \int_{\eta}^{\theta} \frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}} d\nu\right)\right).$$

Thus,

$$\begin{aligned}\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right) &\leq \lim_{\lambda \to \infty}\left(\alpha^{-\lambda}\left(\frac{p}{\theta^p - \eta^p} \alpha^{-\lambda} \alpha^{\lambda} \int_{\eta}^{\theta} \frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}} d\nu\right)\right) \\ &= \alpha^{-\lambda}\left(\frac{p}{\theta^p - \eta^p} \alpha^{\lambda}\left(\int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right)\right)\end{aligned}$$

$$= \frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm.$$

Also, from the right side of the inequality (1.5) we know that

$$\frac{p}{\theta^p - \eta^p} \int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}} d\nu \leq \frac{\rho(\eta) + s\rho(\theta)}{s+1},$$

then we have

$$\lim_{\lambda\to\infty} \left(\frac{p}{\theta^p - \eta^p} \int_\eta^\theta \frac{\alpha^\lambda(\rho(\nu))}{\nu^{1-p}} d\nu \right) \leq \lim_{\lambda\to\infty} \left(\frac{1}{s+1} \left(\alpha^\lambda(\rho(\eta) + s\rho(\theta)) \right) \right).$$

It follows that

$$\begin{aligned} &\lim_{\lambda\to\infty} \left(\alpha^{-\lambda}\alpha^\lambda \frac{p}{\theta^p - \eta^p} \alpha^\lambda \alpha^{-\lambda} \int_\eta^\theta \frac{\alpha^\lambda(\rho(\nu))}{\nu^{1-p}} d\nu \right) \\ &\leq \lim_{\lambda\to\infty} \left(\alpha^{-\lambda}\alpha^\lambda \frac{1}{s+1} \alpha^\lambda \alpha^{-\lambda} \left(\alpha^\lambda(\rho(\eta)) + s\alpha^\lambda(\rho(\theta)) \right) \right) \end{aligned}$$

thus,

$$\begin{aligned} &\lim_{\lambda\to\infty} \left(\frac{p}{\theta^p - \eta^p} \alpha^{-\lambda} \int_\eta^\theta \frac{\alpha^\lambda(\rho(\nu))}{\nu^{1-p}} d\nu \right) \\ &\preceq \lim_{\lambda\to\infty} \left(\frac{1}{s+1} \otimes \alpha^{-\lambda} \left(\alpha^\lambda(\rho(\eta)) + s\alpha^\lambda(\rho(\theta)) \right) \right). \end{aligned}$$

So we have

$$\frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm \preceq \frac{1}{s+1} \otimes (\rho(\eta) \oplus s \otimes \rho(\theta)).$$

Thereby, the proof is completed. □

Example 3.6. Let $\alpha^\lambda(\nu) = e^{\lambda\nu}$ and $\rho(\nu) = \nu^2$, $[\eta, \theta] = [0, 1]$. Then we have,

$$\nu \oplus_\lambda \tau = \alpha^{-\lambda} \left(\alpha^\lambda(\nu) + \alpha^\lambda(\tau) \right) = \alpha^{-\lambda} \left(e^{\lambda\nu} + e^{\lambda\tau} \right) = \frac{1}{\lambda} \ln(\nu + \tau) = \max(\nu, \tau)$$

$$\nu \otimes_\lambda \tau = \alpha^{-\lambda} \left(\alpha^\lambda(\nu)\alpha^\lambda(\tau) \right) = \alpha^{-\lambda} \left(e^{\lambda\nu} e^{\lambda\tau} \right) = \frac{1}{\lambda} \ln \left(e^{\lambda\nu} e^{\lambda\tau} \right) = \nu + \tau.$$

We know that

$$\int_\nu^\oplus \rho \otimes dm = \sup_{\nu\in[0,1]} (\rho(\nu) \otimes \psi(\nu)) = \sup_{\nu\in[0,1]} (\rho(\nu) + \psi(\nu)).$$

Thus,

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p}\left(\sup_{\nu\in[0,1]}(\rho(\nu)+\psi(\nu))\right)$$
$$\preceq \frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta)).$$

Now we generalize Hermite-Hadamard inequality for s-p-convex functions.

Theorem 3.7. *Let $\rho:[\eta,\theta]\to[\beta,\gamma]$ be a $(\oplus,\otimes)$-s-p-convex function in the first kind and α be an increasing generator for $\oplus$ and $\otimes$. Let $\phi:[\beta,\gamma]\to[\beta,\gamma]$ be a continuous function such that commutes with $\otimes$. Then for any $p\in\mathbb{R}\backslash\{0\}$ and $s\in(0,1]$, the following inequalities*

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{s+1}\right)\otimes\phi((\rho(\eta)\oplus s\otimes\rho(\theta)))$$

hold, where ϕ is a strictly increasing function.

Proof. From the inequality (3.3), we know that

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm$$
$$\preceq \frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta)).$$

From hypothesis, ϕ and $\phi o\rho$ are convex functions. Applying Theorem 3.3, we have

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{s+1}\otimes(\rho(\eta)\oplus s\otimes\rho(\theta))\right).$$

Now, we use the crucial commutativity property of ϕ with the operation $\otimes$. Since ϕ commutes with $\otimes$, for any elements η,θ in the domain, we have

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$

$$\preceq \phi\left(\frac{1}{s+1}\right) \otimes \phi((\rho(\eta) \oplus s \otimes \rho(\theta))).$$

From the above inequalities the proof is complete. □

Example 3.8. Let $\rho(\nu) = \nu^2$ and $[\eta, \theta] = [0, 1]$, $\alpha(\nu) = \nu^{\chi}$, $\chi \in [1, \infty)$ and $\phi(\nu) = \nu^s$, $s \in [1, \infty)$. Then we have,

$$\nu \oplus \tau = \alpha^{-1}(\alpha(\nu) + \alpha(\tau)) = \alpha^{-1}(\nu^{\chi} + \tau^{\chi}) = (\nu^{\chi} + \tau^{\chi})^{\frac{1}{\chi}}$$

$$\nu \otimes \tau = \alpha^{-1}(\alpha(\nu)\alpha(\tau)) = \alpha^{-1}(\nu^{\chi}\tau^{\chi}) = (\nu^{\chi}\tau^{\chi})^{\frac{1}{\chi}} = \nu\tau$$

and we have, $\rho\left(\left(\frac{\eta^p + \theta^p}{2}\right)^{\frac{1}{p}}\right) = \rho\left((\frac{1}{2})^{\frac{1}{p}}\right) = \left(\frac{1}{2}\right)^{\frac{2}{p}}$ and

$$\begin{aligned}\int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm &= \alpha^{-1}\left(\int_{\eta}^{\theta} \frac{\alpha(\rho(\nu))}{\nu^{1-p}} d\nu\right) \\ &= \alpha^{-1}\left(\int_{\eta}^{\theta} \frac{\nu^{2\chi}}{\nu^{1-p}} d\nu\right) \\ &= \alpha^{-1}\left(\frac{1}{2\chi + p}\right) = \left(\frac{1}{2\chi + p}\right)^{\frac{1}{\chi}}.\end{aligned}$$

Thus,

$$\phi\left(\rho\left(\left(\frac{\eta^p + \theta^p}{2}\right)^{\frac{1}{p}}\right)\right) = \phi\left(\left(\frac{1}{2}\right)^{\frac{2}{p}}\right) = \left(\frac{1}{2}\right)^{\frac{2s}{p}}$$

and

$$\phi\left(\int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right) = \left(\frac{1}{2\chi + p}\right)^{\frac{s}{\chi}}.$$

Hence,

$$\left(\frac{1}{2}\right)^{\frac{2s}{p}} \preceq \phi\left(\frac{p}{\theta^p - \eta^p}\right) \otimes \phi\left(\int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right).$$

Since $\alpha(\nu) = \nu^{\chi}$ and $\otimes$ is ordinary multiplication, we have:

$$\phi\left(\frac{p}{\theta^p - \eta^p}\right) = \left(\frac{p}{\theta^p - \eta^p}\right)^s.$$

Therefore,

$$\left(\frac{1}{2}\right)^{\frac{2s}{p}} \leq \left(\frac{p}{\theta^p - \eta^p}\right)^s \cdot \left(\frac{1}{2\chi + p}\right)^{\frac{s}{\chi}}$$

$$\preceq \phi\left(\frac{1}{s+1}\right) \otimes \phi((\rho(\eta) \oplus s \otimes \rho(\theta)))$$

and we have

$$\phi\left(\frac{1}{s+1}\right) \otimes \phi((\rho(\eta) \oplus s \otimes \rho(\theta))) = \left(\frac{1}{s+1}\right)^{s} . ((\rho(\eta)^{\chi} + s^{\chi}\rho(\theta)^{\chi})^{\frac{s}{\chi}}$$

thus,

$$\begin{aligned}\left(\frac{1}{2}\right)^{\frac{2s}{p}} &\leq \left(\frac{p}{\theta^p - \eta^p}\right)^{s} . \left(\frac{1}{2\chi + p}\right)^{\frac{s}{\chi}} \\ &\leq \left(\frac{1}{s+1}\right)^{s} . ((\rho(\eta)^{\chi} + s^{\chi}\rho(\theta)^{\chi})^{\frac{s}{\chi}} .\end{aligned}$$

In the second class, when $\nu \oplus \tau = \max(\nu, \tau)$ and $\nu \otimes \tau = \alpha^{-1}(\alpha(\nu)\alpha(\tau))$, the Theorem 3.3, will be as following theorem.

Theorem 3.9. *Let $\rho : [\eta, \theta] \to [\beta, \gamma]$ be a $(\oplus, \otimes)$-s-p-convex function in the first kind and $\otimes$ be generated by α. If ϕ is the same as in Theorem 3.7, then for any $p \in \mathbb{R}\backslash\{0\}$ and $s \in (0, 1]$, the following inequalities*

$$\begin{aligned}\phi\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right) &\preceq \phi\left(\frac{p}{\theta^p - \eta^p}\right) \otimes \phi\left(\int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right) \\ &\preceq \phi\left(\frac{1}{s+1}\right) \otimes \phi\left((\rho(\eta) \oplus s \otimes \rho(\theta))\right)\end{aligned}$$

hold.

Proof. From Theorem 3.5, we have

$$\begin{aligned}\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right) &\preceq \frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm \\ &\preceq \frac{1}{s+1} \otimes (\rho(\eta) \oplus s \otimes \rho(\theta)).\end{aligned}$$

Under the assumptions on ϕ, $\phi o \rho$ is a convex function. Now, applying the Theorem 3.5, we have

$$\begin{aligned}\phi\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right) &\preceq \phi\left(\frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right) \\ &\preceq \phi\left(\frac{1}{s+1} \otimes (\rho(\eta) \oplus s \otimes \rho(\theta))\right).\end{aligned}$$

Since ϕ commutes with $\otimes$, so we have

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{s+1}\right)\otimes\phi\left(\left(\rho(\eta)\oplus s\otimes\rho(\theta)\right)\right).$$

This completes the proof. □

4. Generalization of Hermite-Hadamard Type Inequality for p-Convex Functions for Pseudo-Integrals

Now we state and prove the inequality (1.6) for pseudo-integral.

Theorem 4.1. *Let $\rho:[\eta,\theta]\to[\beta,\gamma]$ be a $(\oplus,\otimes)$-p-convex function and $\alpha:[\beta,\gamma]\to[0,\infty)$ be a continuous and increasing function. Then for any $p\in\mathbb{R}\backslash\{0\}$, the following inequalities*

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm \tag{4.1}$$
$$\preceq \frac{1}{2}\otimes(\rho(\eta)\oplus\rho(\theta))$$

hold.

Proof. From inequality (1.6) we know that

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\theta^p-\eta^p}\int_\eta^\theta\frac{\rho(\nu)}{\nu^{1-p}}d\nu. \tag{4.2}$$

Since α is increasing, we apply it to both sides of (4.2). However, α is not necessarily linear, so we can not move it inside the integral. Instead, we apply inequality (4.2) to the composed function $\alpha o\rho$. This is legitimate because $\alpha o\rho$ is again $(\oplus,\otimes)$-s-p-convex:

$$\alpha\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\leq\frac{p}{\theta^p-\eta^p}\int_\eta^\theta\frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu. \tag{4.3}$$

Applying α^{-1} (which is also increasing) to both sides of (4.3) gives

$$\alpha^{-1}\left(\alpha\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right)\leq\alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta\frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right).$$

Thus,

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right).$$

Now, by the definition of the pseudo-integral, we have

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm.$$

Also we know that

$$\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\rho(\nu)}{\nu^{1-p}}d\nu \leq \frac{\rho(\eta)+\rho(\theta)}{2}.$$

Similarly, we have

$$\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu \leq \frac{\alpha(\rho(\eta))+\alpha(\rho(\theta))}{2}. \tag{4.4}$$

Applying α^{-1} (which is also increasing) to both sides of (4.4) gives

$$\alpha^{-1}\left(\frac{p}{\theta^p-\eta^p}\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right) \leq \alpha^{-1}\left(\frac{\alpha(\rho(\eta))+\alpha(\rho(\theta))}{2}\right).$$

Thus,

$$\frac{p}{\theta^p-\eta^p}\otimes\alpha^{-1}\int_\eta^\theta \frac{\alpha(\rho(\nu))d\nu}{\nu^{1-p}} \preceq \frac{1}{2}\otimes\alpha^{-1}(\alpha(\rho(\eta))+\alpha(\rho(\theta))).$$

So, we have

$$\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm \preceq \frac{1}{2}\otimes(\rho(\eta)\oplus\rho(\theta)).$$

This completes the proof. □

Example 4.2. Let $\rho(\nu)=\nu^2$, $\alpha(\nu)=\nu^\chi$, $[\eta,\theta]=[0,1]$ and $\chi\in[1,\infty)$. Then we have,

$$\nu\oplus\tau=\alpha^{-1}(\alpha(\nu)+\alpha(\tau))=\alpha^{-1}(\nu^\chi+\tau^\chi)=(\nu^\chi+\tau^\chi)^{\frac{1}{\chi}}$$

$$\nu\otimes\tau=\alpha^{-1}(\alpha(\nu)\alpha(\tau))=\alpha^{-1}(\nu^\chi\tau^\chi)=(\nu^\chi\tau^\chi)^{\frac{1}{\chi}}=\nu\tau$$

If $\eta=0,\theta=1$ we have, $\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right)=\rho\left((\frac{1}{2})^{\frac{1}{p}}\right)=\left(\frac{1}{2}\right)^{\frac{2}{p}}$ and

$$\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm=\alpha^{-1}\left(\int_\eta^\theta\frac{\alpha(\rho(\nu))}{\nu^{1-p}}d\nu\right)$$

$$= \alpha^{-1}\left(\int_{\eta}^{\theta} \frac{\nu^{2\chi}}{\nu^{1-p}} d\nu\right)$$
$$= \alpha^{-1}\left(\frac{1}{2\chi+p}\right)$$
$$= \left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}}.$$

We know that $\frac{p}{\theta^p-\eta^p} = p, p \otimes \left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}} = p.\left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}}$.

Since $\otimes$ is ordinary multiplication, we have

$$\frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta))) = \frac{1}{2}.$$

So, we have

$$\left(\frac{1}{2}\right)^{\frac{2}{p}} \leq p\left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}} \leq \frac{1}{2}.$$

Theorem 4.3. *Let $\rho : [\eta, \theta] \to [\beta, \gamma]$be a $(\otimes, \oplus)$-p-convex function and $\alpha : [\beta, \gamma] \to [0, \infty)$ be a continuous and increasing function and $([\eta, \theta], \sup, \otimes)$ be a semiring and m be the same as equality (2.1). Then for any $p \in \mathbb{R}\backslash\{0\}$, the following inequalities*

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \preceq \frac{p}{\theta^p-\eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm \tag{4.5}$$
$$\preceq \frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta))$$

hold.

Proof. From inequality (1.6) we know that

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\theta^p-\eta^p} \int_{\eta}^{\theta} \frac{\rho(\nu)}{\nu^{1-p}} d\nu, \tag{4.6}$$

The sup-integral (with respect to $\otimes$) is defined as the following:

$$\int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm = \lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\left(\frac{p}{\theta^p-\eta^p}\alpha^{\lambda}\left(\int_{\eta}^{\theta} \frac{(\rho(\nu))}{\nu^{1-p}} d\nu\right)\right)\right).$$

Since α^{λ} is increasing, applying it to inequality (4.6),we have

$$\alpha^{\lambda}\left(\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right)\right) \leq \alpha^{\lambda}\left(\frac{p}{\theta^p-\eta^p}\int_{\eta}^{\theta} \frac{\rho(\nu)}{\nu^{1-p}} d\nu\right),$$

then we have

$$\lim_{\lambda\to\infty}\left(\alpha^{\lambda}\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right)\leq\lim_{\lambda\to\infty}\left(\frac{p}{\theta^p-\eta^p}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right).$$

It follows that

$$\begin{aligned}&\lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\left(\alpha^{\lambda}\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\right)\right)\\&\quad\leq\lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\left(\frac{p}{\theta^p-\eta^p}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)&\leq\lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\left(\frac{p}{\theta^p-\eta^p}\alpha^{-\lambda}\alpha^{\lambda}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right)\right)\\&=\alpha^{-\lambda}\left(\frac{p}{\theta^p-\eta^p}\alpha^{\lambda}\left(\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)\right)\\&=\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm.\end{aligned}$$

Also, from the right side of the inequality (1.6) we know that

$$\frac{p}{\theta^p-\eta^p}\int_{\eta}^{\theta}\frac{\rho(\nu)}{\nu^{1-p}}d\nu\leq\frac{\rho(\eta)+\rho(\theta)}{2},$$

then we have

$$\lim_{\lambda\to\infty}\left(\frac{p}{\theta^p-\eta^p}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right)\leq\lim_{\lambda\to\infty}\left(\frac{1}{2}\left(\alpha^{\lambda}(\rho(\eta)+\rho(\theta))\right)\right).$$

It follows that

$$\begin{aligned}&\lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\alpha^{\lambda}\frac{p}{\theta^p-\eta^p}\alpha^{\lambda}\alpha^{-\lambda}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right)\\&\quad\leq\lim_{\lambda\to\infty}\left(\alpha^{-\lambda}\alpha^{\lambda}\frac{1}{2}\alpha^{\lambda}\alpha^{-\lambda}\left(\alpha^{\lambda}(\rho(\eta))+\alpha^{\lambda}(\rho(\theta))\right)\right),\end{aligned}$$

thus,

$$\lim_{\lambda\to\infty}\left(\frac{p}{\theta^p-\eta^p}\alpha^{-\lambda}\int_{\eta}^{\theta}\frac{\alpha^{\lambda}(\rho(\nu))}{\nu^{1-p}}d\nu\right)$$

$$\preceq \lim_{\lambda\to\infty}\left(\frac{1}{2}\otimes\alpha^{-\lambda}\left(\alpha^{\lambda}(\rho(\eta))+\alpha^{\lambda}(\rho(\theta))\right)\right).$$

So we have

$$\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\preceq\frac{1}{2}\otimes(\rho(\eta)\oplus\rho(\theta)).$$

Thereby, the proof is completed. □

Example 4.4. Let $\alpha(\nu)=e^{\nu}$, $[\eta,\theta]=[0,1]$ and $\rho(\nu)=\nu^2$. Then we have,

$$\nu\oplus_{\lambda}\tau=\alpha^{-\lambda}\left(\alpha^{\lambda}(\nu)+\alpha^{\lambda}(\tau)\right)=\alpha^{-\lambda}(e^{\nu}+e^{\tau})=\ln(\nu+\tau)=\max(\nu,\tau)$$

$$\nu\otimes_{\lambda}\tau=\alpha^{-\lambda}\left(\alpha^{\lambda}(\nu)\alpha^{\lambda}(\tau)\right)=\alpha^{-\lambda}(e^{\nu}e^{\tau})=\ln(e^{\nu}e^{\tau})=\nu+\tau.$$

We know that,

$$\int_{[0,1]}^{\oplus}\rho\otimes dm=\sup_{\nu\in[0,1]}(\rho(\nu)\otimes\psi(\nu))=\sup_{\nu\in[0,1]}(\rho(\nu)+\psi(\nu)).$$

Thus,

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\preceq\left(\frac{p}{\theta^p-\eta^p}\right)\left(\sup_{\nu\in[0,1]}(\rho(\nu)+\psi(\nu))\right)$$
$$\preceq\frac{1}{2}\otimes(\rho(\eta)\oplus\rho(\theta)).$$

Now we generalize Hermite-Hadamard inequality for p-convex functions.

Theorem 4.5. *Let $\rho:[\eta,\theta]\to[\beta,\gamma]$ be a $(\oplus,\otimes)$-p-convex function and α be an increasing generator for $\oplus$ and $\otimes$. Let $\phi:[\beta,\gamma]\to[\beta,\gamma]$ be a continuous function such that commutes with $\otimes$. Then for any $p\in\mathbb{R}\backslash\{0\}$, the following inequalities*

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\preceq\phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$
$$\preceq\phi\left(\frac{1}{2}\right)\otimes\phi((\rho(\eta)\oplus\rho(\theta)))$$

hold where ϕ is a strictly increasing function.

Proof. From the inequality (4.1), we know that

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\preceq\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm$$

$$\preceq \frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta)).$$

By hypothesis, ϕ and $\phi o \rho$ are convex functions. Applying Theorem 4.1, we have

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p} \otimes \int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta))\right)$$

Now, we use the crucial commutativity property of ϕ with the operation $\otimes$. Since ϕ commutes with $\otimes$, for any elements η, θ in the domain, we have

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p-\eta^p}\right) \otimes \phi\left(\int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{2}\right) \otimes \phi((\rho(\eta) \oplus \rho(\theta))).$$

From the above inequalities the proof is complete. □

Example 4.6. Let $\rho(\nu) = \nu^2$ and $[\eta, \theta] = [0, 1]$, $\alpha(\nu) = \nu^\chi$, $\chi \in [1, \infty)$ and $\phi(\nu) = \nu^s$, $s \in [1, \infty)$. Then we have,

$$\nu \oplus \tau = \alpha^{-1}(\alpha(\nu) + \alpha(\tau)) = \alpha^{-1}(\nu^\chi + \tau^\chi) = (\nu^\chi + \tau^\chi)^{\frac{1}{\chi}}$$
$$\nu \otimes \tau = \alpha^{-1}(\alpha(\nu)\alpha(\tau)) = \alpha^{-1}(\nu^\chi \tau^\chi) = (\nu^\chi \tau^\chi)^{\frac{1}{\chi}} = \nu\tau$$

and we have, $\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right) = \rho\left((\frac{1}{2})^{\frac{1}{p}}\right) = \left(\frac{1}{2}\right)^{\frac{2}{p}}$ and

$$\int_{[\eta,\theta]}^{\oplus} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm = \alpha^{-1}\left(\int_\eta^\theta \frac{\alpha(\rho(\nu))}{\nu^{1-p}} d\nu\right)$$
$$= \alpha^{-1}\left(\int_\eta^\theta \frac{\nu^{2\chi}}{\nu^{1-p}} d\nu\right)$$
$$= \alpha^{-1}\left(\frac{1}{2\chi+p}\right)$$
$$= \left(\frac{1}{2\chi+p}\right)^{\frac{1}{\chi}}$$

Thus,

$$\phi\left(\rho\left(\left(\frac{\eta^p+\theta^p}{2}\right)^{\frac{1}{p}}\right)\right)=\phi\left(\left(\frac{1}{2}\right)^{\frac{2}{p}}\right)=\left(\frac{1}{2}\right)^{\frac{2s}{p}}$$

and

$$\phi\left(\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)=\left(\frac{1}{2\chi+p}\right)^{\frac{s}{\chi}}.$$

Hence,

$$\left(\frac{1}{2}\right)^{\frac{2s}{p}}\preceq\phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\oplus}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right).$$

Since $\alpha(\nu)=\nu^{\chi}$ and $\otimes$ is ordinary multiplication, we have:

$$\phi\left(\frac{p}{\theta^p-\eta^p}\right)=\left(\frac{p}{\theta^p-\eta^p}\right)^{s}.$$

Therefore,

$$\left(\frac{1}{2}\right)^{\frac{2s}{p}}\leq\left(\frac{p}{\theta^p-\eta^p}\right)^{s}.\left(\frac{1}{2\chi+p}\right)^{\frac{s}{\chi}}\preceq\phi\left(\frac{1}{2}\right)\otimes\phi((\rho(\eta)\oplus\rho(\theta)))$$

and we have

$$\phi\left(\frac{1}{2}\right)\otimes\phi((\rho(\eta)\oplus\rho(\theta)))=\left(\frac{1}{2}\right)^{s}.((\rho(\eta)^{\chi}+\rho(\theta)^{\chi})^{\frac{s}{\chi}}$$

thus,

$$\left(\frac{1}{2}\right)^{\frac{2s}{p}}\leq\left(\frac{p}{\theta^p-\eta^p}\right)^{s}.\left(\frac{1}{2\chi+p}\right)^{\frac{s}{\chi}}\leq\left(\frac{1}{2}\right)^{s}.((\rho(\eta)^{\chi}+\rho(\theta)^{\chi})^{\frac{s}{\chi}}.$$

Theorem 4.7. *Let $\rho:[\eta,\theta]\to[\beta,\gamma]$ be a $(\oplus,\otimes)$-p-convex function and $\otimes$ be generated by α. If ϕ is the same as in Theorem 4.5, then for any $p\in\mathbb{R}\backslash\{0\}$, the following inequalities*

$$\phi\left(\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\right)\preceq\phi\left(\frac{p}{\theta^p-\eta^p}\right)\otimes\phi\left(\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm\right)$$
$$\preceq\phi\left(\frac{1}{2}\right)\otimes\phi\left((\rho(\eta)\oplus\rho(\theta))\right)$$

hold.

Proof. From Theorem 4.3, we have

$$\rho\left(\left[\frac{\eta^p+\theta^p}{2}\right]^{\frac{1}{p}}\right)\preceq\frac{p}{\theta^p-\eta^p}\otimes\int_{[\eta,\theta]}^{\sup}\frac{1}{\nu^{1-p}}\otimes\rho\otimes dm$$

$$\preceq \frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta)).$$

From assumptions of ϕ, $\phi o \rho$ is convex function. Now, applying Theorem 4.3, we have

$$\phi\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p - \eta^p} \otimes \int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{2} \otimes (\rho(\eta) \oplus \rho(\theta))\right).$$

Since ϕ commutes with $\otimes$, so we have

$$\phi\left(\rho\left(\left[\frac{\eta^p + \theta^p}{2}\right]^{\frac{1}{p}}\right)\right) \preceq \phi\left(\frac{p}{\theta^p - \eta^p}\right) \otimes \phi\left(\int_{[\eta,\theta]}^{\sup} \frac{1}{\nu^{1-p}} \otimes \rho \otimes dm\right)$$
$$\preceq \phi\left(\frac{1}{2}\right) \otimes \phi\left((\rho(\eta) \oplus \rho(\theta))\right).$$

This completes the proof. □

5. Conclusion

We have generalized Hermite-Hadamard's inequality for s-p-convex and p-convex function for pseudo-integrals in two cases of the semiring. The first one is on α-semiring, when the operations $\oplus$ and $\otimes$ are generated by a comonotone and continuous function α. The second type of this inequality is obtained on semiring $([\eta, \theta], \sup, \otimes)$ where $\otimes$ is generated. We further discussed the generalized version of this inequality for convex functions. In the future, it will be investigated how to extend the obtained results to abstract spaces. Future work includes extending results to set-valued s-p-convex functions for interval-valued optimization and fuzzy differential equations, developing discrete analogues for numerical applications, exploring connections with fractional calculus for non-stationary systems and investigating applications in machine learning for robust loss functions and adversarial training under data uncertainty.

Acknowledgment. The authors would like to express their sincere thanks to the editor(s) and anonymous reviewer(s) for insightful comments and constructive suggestions, which have significantly improved the quality and clarity of this manuscript. Their meticulous review and expert feedback helped us to enhance the presentation of our results and strengthen the overall contribution of this work.

Conflict of Interest. "The authors declare no conflict of interest."

References

1. S. Abbaszadeh and A. Ebadian, *Nonlinear integrals and Hadamard-type inequalities*, Soft. Comput., 22 (9) (2018), pp. 2843-2849.
2. H. Agahi, *On pseudo-average value on graphical simplices and pseudo-inequalities*, J. FSS., 467 (2023), 108504.
3. A. Arshad and A.R. Khan, *Hermite-Hadamard-Fejer type inequalities for s-p-convex functions of several senses*, Transylv. J. Math. Mech., 11 (2019), pp. 25-40.
4. E.F. Beckenbach, *Convex functions*, Bull. Am. Math. Soc., 54 (1948), pp. 439-480.
5. M. Bilal and A.R. Khan, *New generalized Hermite-Hadamard inequalities for p-convex functions in the mixed kind*, Eur. J. Pure Appl. Math.., 14 (3) (2021), pp. 863-880.
6. J. Caballero and k. Sadarangani, *Hermite-Hadamard inequality for fuzzy integrals*, J. Appl. Math. Comput., 215 (2009), pp. 2134-2138.
7. B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets Syst., 194 (2012), pp. 90-96.
8. B. Daraby, *Study on some integral inequalities for pseudo-integrals*, Sahand Commun. Math. Anal., 19 (1) (2022), pp. 139-159.
9. A. Flores-Franulič and H. Román-Flores, *A Chebyshev type inequality for fuzzy integrals*, Appl. Math. Comput., 190 (2007), pp. 1178-1184.
10. M. Hosseini, A. Babakhani, H. Agahi and S.H. Rasouli, *On pseudo-fractional integral inequalities related to Hermite-Hadamard type*, Soft Comput., 20 (7) (2016), pp. 2521-2529.
11. P. Jain, *Classical inequalities for pseudo-integral*, Georgian Math. J., 29 (3) (2022), pp. 373-385.
12. D-O. Li , X-Q. Song and T. Yue, *Hermite-Hadamard type inequality for Sugeno integrals*, Applied Mathematics and Computation, 237 (2014), pp. 632-638.
13. R. Mesiar and E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets Syst., 160 (2009), pp. 58-64.
14. E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Inf. Sci., 180 (2010), pp. 543-548.
15. M. Sugeno, *Theory of fuzzy integrals and its applications* (ph. D. thesis), Tokyo Institute of Technology, 1974.
16. S. Turhan, A. Kilic and I. Iscan, *New integral inequalities involving p-convex and s-p-convex functions*, Commun. Fac. Sci. Univ. Ank., Sér. A1, Math. Stat, 74 (1) (2025), pp. 138-149.
17. Z. Wang and G.J. Klir, *Fuzzy Measure Theory*, Springer Science & Business Media, 1992.

18. D. Zhang and E. Pap, *Jensen's inequalities for pseudo-integrals*, Iran. J. Fuzzy Syst., 18 (3) (2021), pp. 99-100.

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