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Critical Friedmann-Lemaître-Robertson-Walker Metrics

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Abstract. This paper investigates the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime's critical metrics in relation to variational problems. We derive the explicit conditions under which FLRW metric satisfies these critical equations after reviewing the general framework of $\mathcal{F}_u$-critical and $\mathcal{S}$-critical metrics. According to our analysis, the critical of these metrics is highly dependent on how the scale factor and spatial curvature behave. The findings contribute to the understanding of variational structures in homogeneous cosmological models by precisely characterizing the point at which FLRW spacetimes become $\mathcal{F}_u$-critical or $\mathcal{S}$-critical.

1. Introduction

The study of critical metrics over a family of (pseudo-)Riemannian manifolds is an intriguing area of differential geometry with obvious applications in mathematical physics. Let M^n be a closed oriented manifold with a family $\mathcal{M}_1$ of (pseudo-)Riemannian metrics of volume one. Finding critical metrics from $\mathcal{M}_1$ that are extremum of a given curvature functional is the difficult part.

The Einstein metrics are one of the most well-known families of critical metrics. $\varrho = \lambda g$ for some real constant λ coincides with the Euler-Lagrange system associated with the Einstein-Hilbert functional $g \mapsto \int_M \tau \, \mathrm{d}vol_g$. Here, ϱ and τ stand for the scalar curvature and the Ricci tensor, respectively.

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Curvature functionals, especially those based on scalar quadratic curvature invariants, have been a topic of great interest to the mathematical community. As introduced by [3], this intriguing journey started in the Riemannian setting and has since been investigated by many distinguished scholars (for more information, see [5, 6, 8, 9, 12]). While there are several dimensions in which quadratic curvature functionals can be studied, dimension 4 (the framework of space-times) attracts the most attention. Furthermore, the theory of gravitational fields naturally contains some quadratic curvature functionals [11].

To examine the quadratic curvature invariants, we observe that they are based on $\{\Delta\tau, \tau^2, \|\varrho\|^2, \|\mathcal{R}\|^2\}$. Thus, the following is the expression for a generic quadratic curvature functional:

$$g \mapsto \int_M \left(a\tau^2 + b\|\varrho\|^2 + c\|\mathcal{R}\|^2\right) \mathrm{d}vol_g, \tag{1.1}$$

where a, b, c are real constants, and R is the curvature tensor.

The Gauss-Bonnet Theorem for a four-dimensional manifold M offers:

$$32\pi^2\chi(M) = \int_M \left(\tau^2 - 4\|\varrho\|^2 + \|\mathcal{R}\|^2\right) \mathrm{d}vol_g.$$

This suggests that the critical points of the functional $4\|\varrho\|^2+\tau^2$ coincide with the functional critical points associated with the curvature tensor. The functional in equation (1.1) is therefore equal to:

$$g \mapsto \int_M \left((a-c)\tau^2 + (4c+b)\|\varrho\|^2\right) \mathrm{d}vol_g.$$

The manifold (M, g) is therefore critical for the functional (1.1) whenever it is critical for the functionals $\mathcal{S}$ and $\mathcal{F}_u$:

$$\mathcal{F}_u : g \mapsto \int_M \left(t\tau^2 + \|\varrho\|^2\right) \mathrm{d}vol_g, \quad u \in \mathbb{R},$$
$$\mathcal{S} : g \mapsto \int_M \tau^2 \, \mathrm{d}vol_g.$$

Non-Einstein critical cones exist for all quadratic curvature functionals, according to [7], which studied critical metrics for all of these functionals in detail. Remarkably, for quadratic curvature functionals involving scalar and Ricci curvature, Einstein metrics are always critical. [17] provides an interesting partial converse for Bach-flat Riemannian metrics. Although the majority of research focuses on Riemannian settings, some intriguing findings for pseudo-Riemannian cases have surfaced. In [8], for example, all quadratic curvature functionals were examined on semi-direct extensions of the Heisenberg group.

For all quadratic curvature functionals $\mathcal{F}_u$, $u \in \mathbb{R}$, Einstein metrics are critical (see [4]). Weyl metrics with vanishing scalar curvature are

critical for both functionals $\mathcal{F}_{-1/4}$ and $\mathcal{S}$, while Bach-flat metrics are critical for the functional $\mathcal{F}_{-1/3}$, according to known results in the Riemannian setting. Critical metrics in various signatures for quadratic curvature functionals are compellingly considered as a result of these established findings.

The Euler-Lagrange equations for quadratic curvature functionals are well-established and computed in Riemannian settings [3, 13]. Since the signature of the base metric has no effect on the arguments, these results can be easily extended into pseudo-Riemannian settings [1].

For the functionals $\mathcal{F}_u : g \mapsto \int_M \left(u\tau^2 + \|\varrho\|^2\right) \mathrm{d}\,\mathrm{vol}_g$ and $\mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}\,\mathrm{vol}_g$, the gradients can be expressed as follows:

$$\begin{aligned}(\nabla\mathcal{S})_{ij} &= 2\nabla^2_{ij}\tau - 2(\Delta\tau)g_{ij} - 2\tau\varrho_{ij} + \frac{1}{2}\tau^2 g_{ij},\\ (\nabla\mathcal{F}_u)_{ij} &= -\Delta\varrho_{ij} + (1+2t)\nabla^2_{ij}\tau - \frac{1+4u}{2}(\Delta\tau)g_{ij} - 2u\tau\varrho_{ij}\\ &\qquad - 2\varrho_{kl}\mathcal{R}_{ikjl} + \frac{1}{2}\left(u\tau^2 + \|\varrho\|^2\right) g_{ij}.\end{aligned}$$

Note that if $(\nabla\mathcal{F}_u) = cg$ for some real constant c, then g is critical for $\mathcal{F}_u$ and vice versa. By tracing the equation above, we get:

$$(n-4)\left(u\tau^2 + \|\varrho\|^2\right) - (n + 4(n-1)u)\Delta\tau = 2nc.$$

Thus, g is critical for $\mathcal{F}_u$ if and only if:

$$\begin{aligned}-\Delta\varrho_{ij} + (1+2u)\nabla^2_{ij}\tau - \frac{2u}{n}(\Delta\tau)g_{ij} - 2\varrho_{kl}\mathcal{R}_{ikjl} - 2u\tau\varrho_{ij}\\ + \frac{2}{n}\left(u\tau^2 + \|\varrho\|^2\right) g_{ij}\\ = 0,\end{aligned} \tag{1.2}$$

and

$$(n-4)\left(u\tau^2 + \|\varrho\|^2 - \lambda\right) = (n + 4(n-1)u)\Delta\tau, \tag{1.3}$$

where $\lambda = \mathcal{F}_u(g)$ (see [9]). It follows that for any value of u, Einstein metrics are critical for $\mathcal{F}_u$. Furthermore, the relation of $\nabla\mathcal{S}$ indicates that metrics with either vanishing scalar curvature or Einstein are critical points for $\mathcal{S}$.

In the case of four-dimensional spaces with constant scalar curvature (particularly homogeneous spaces), the above Euler-Lagrange equations simplify remarkably. In this context, $(\nabla\mathcal{S})_{ij} = 2\tau\left(\frac{1}{4}\tau g_{ij} - \varrho_{ij}\right)$, equation (1.3) is automatically satisfied, and equation (1.2) reduces to:

$$\Delta\varrho + 2\mathcal{R}[\varrho] + 2u\tau\varrho - \frac{1}{2}\left(\|\varrho\|^2 + u\tau^2\right) g = 0, \tag{1.4}$$

where $\mathcal{R}[\varrho]$ is the tensor field defined by the components $\varrho_{kl}\mathcal{R}_{ikjl}$. We will focus on equation (1.4) and its solutions in the upcoming sections.

The universe is described as a homogeneous, isotropic, and expanding space by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which is an exact solution of Einstein equations in general relativity. This metric, which is frequently used in discussions of gravitational collapse or expansion, defines a spherically symmetric space-time. In this article, we examine the non-Einstein critical FLRW metrics.

2. The Friedmann-Lemaître-Robertson-Walker Metrics

A fundamental component of modern cosmology, the Friedmann-Lemaître-Robertson-Walker metric (FLRW) offers the mathematical underpinnings for characterizing the large-scale dynamics and structure of the universe. Assuming that the universe is homogeneous (the same in every location) and isotropic (the same in every direction) on cosmological scales, it is a solution to Einstein's field equations of general relativity. Astronomical observations, such as the large-scale distribution of galaxies and the cosmic microwave background radiation's near-uniformity, support these hypotheses [14, 15, 18].

A dynamic universe whose geometry changes over time is described by the FLRW metric. It is presented in the form

$$ds^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{2.1}$$

Here, c is the speed of light, t is cosmic time, and (r, θ, ϕ) are comoving spatial coordinates. The scale factor, represented by the function $a(t)$, represents the expansion or contraction of the universe as a function of time. The spatial curvature of the universe is represented by the constant k, which can have values of $1, 0$, or -1, which stand for closed (spherical), flat (Euclidean), or open (hyperbolic) spatial geometries, respectively.

Several important individuals contributed to the development of the FLRW metric: Georges Lemaître, who suggested an expanding universe model; Howard Robertson and Arthur G. Walker (as we know Walker manifolds have been considered from different aspects, see for example [2, 10, 16]), who later formalized the metric in the mathematical foundation for what is now known as the standard model of cosmology; and Alexander Friedmann (1922), who initially derived non-static solutions to Einstein's equations.

The Friedmann equations, which control the evolution of the scale factor $a(t)$, are obtained by introducing the FLRW metric into Einstein's field equations and assuming a perfect fluid energy-momentum tensor.

These formulas link the curvature, energy content (matter, radiation, and dark energy), and expansion rate of the universe (as measured by the Hubble parameter). Important cosmological phenomena like the Hubble expansion, the cosmic microwave background, nucleosynthesis, and large-scale structure formation are explained by the resulting dynamics.

In conclusion, the FLRW metric is a strong and sophisticated instrument that captures what we currently know about the large-scale behavior of the universe. It connects geometry to the physical components of the universe and provides the foundation for a large portion of theoretical and observational cosmology. Studying these metrics is crucial because of this.

Starting with (2.1), we can express the curvature for the FLRW metric in terms of the coordinate basis $\{\partial_i\}$, followed by the Levi-Civita connection ∇. The following are the specific elements of the Levi-Civita connection:

$$\begin{aligned}
&\nabla_{\partial_1}\partial_1 = 0, && \nabla_{\partial_1}\partial_2 = \tfrac{a'(t)}{a(t)}\partial_2,\\
&\nabla_{\partial_1}\partial_3 = \tfrac{a'(t)}{a(t)}\partial_3, && \nabla_{\partial_1}\partial_4 = \tfrac{a'(t)}{a(t)}\partial_4,\\
&\nabla_{\partial_2}\partial_1 = \tfrac{a'(t)}{a(t)}\partial_2, && \nabla_{\partial_2}\partial_2 = \tfrac{a(t)a'(t)}{c^2(1-kr^2)}\partial_1 + \tfrac{kr}{(1-kr^2)}\partial_2,\\
&\nabla_{\partial_2}\partial_3 = \tfrac{1}{r}\partial_3, && \nabla_{\partial_2}\partial_4 = \tfrac{1}{r}\partial_4,\\
&\nabla_{\partial_3}\partial_1 = \tfrac{a'(t)}{a(t)}\partial_3, && \nabla_{\partial_3}\partial_2 = \tfrac{1}{r}\partial_3,\\
&\nabla_{\partial_3}\partial_3 = \tfrac{a(t)a'(t)r^2}{c^2}\partial_1 - (1-kr^2)r\partial_2, && \nabla_{\partial_3}\partial_4 = \tfrac{\cos\theta}{\sin\theta}\partial_4,\\
&\nabla_{\partial_4}\partial_1 = \tfrac{a'(t)}{a(t)}\partial_4, && \nabla_{\partial_4}\partial_2 = \tfrac{1}{r}\partial_4,\\
&\nabla_{\partial_4}\partial_3 = \tfrac{\cos\theta}{\sin\theta}, && \nabla_{\partial_4}\partial_4 = \tfrac{a(t)a'(t)r^2\sin^2\theta}{c^2}\partial_1\\
& && \quad -(1-kr^2)r\sin\theta\partial_2 - \sin\theta\cos\theta\partial_3.
\end{aligned}$$

The Riemann curvature tensor $\mathcal{R}$ now has the following non-zero components:

$$\begin{aligned}
&\mathcal{R}_{\partial_1\partial_2}\partial_2 = \tfrac{a(t)a''(t)}{c^2(1-kr^2)}\partial_1, && \mathcal{R}_{\partial_1\partial_3}\partial_3 = \tfrac{a(t)a''(t)r^2}{c^2}\partial_1,\\
&\mathcal{R}_{\partial_1\partial_4}\partial_4 = \tfrac{a(t)a''(t)r^2\sin^2\theta}{c^2}\partial_1, && \mathcal{R}_{\partial_1\partial_2}\partial_1 = \tfrac{a''(t)}{a(t)}\partial_2,\\
&\mathcal{R}_{\partial_2\partial_3}\partial_3 = \tfrac{kc^2a'(t)r^2}{c^2}\partial_2, && \mathcal{R}_{\partial_2\partial_4}\partial_4 = \tfrac{\sin^2\theta(kc^2+a'(t)r^2)}{c^2}\partial_2,\\
&\mathcal{R}_{\partial_1\partial_3}\partial_1 = \tfrac{a''(t)}{a(t)}\partial_3, && \mathcal{R}_{\partial_2\partial_3}\partial_2 = -\tfrac{kc^2+a'^2(t)}{c^2(1-kr^2)}\partial_3,\\
&\mathcal{R}_{\partial_1\partial_4}\partial_1 = \tfrac{a''(t)}{a(t)}\partial_4, && \mathcal{R}_{\partial_2\partial_4}\partial_2 = -\tfrac{kc^2+a'^2(t)}{c^2(1-kr^2)}\partial_4,\\
&\mathcal{R}_{\partial_3\partial_4}\partial_4 = \tfrac{r^2}{c^2}\Big(kc^2\left(\cos^2\theta - 1\right) & &\\
&\qquad +a'^2(t)(\cos\theta - 1)\Big)\partial_3. & &
\end{aligned}$$

Therefore, the Ricci tensor's non-zero components are provided by

$$\varrho_{11} = -\frac{3a''(t)}{a(t)},$$

$$\varrho_{22} = \frac{2kc^2 + 2a'^2(t) + a(t)a''(t)}{c^2(1-kr^2)},$$
$$\varrho_{33} = \frac{r^2\left(2kc^2 + 2a'^2(t) + a(t)a''(t)\right)}{c^2},$$
$$\varrho_{44} = -\frac{r^2}{c^2}\left(2kc^2(\cos\theta - 1) + \left(a'^2(t) + a(t)a''^2(t)\right)\left(\cos^3 r - 1\right)\right).$$

Moreover, $\tau = -\frac{6\left(kc^2+a'^2(t)+a(t)a''(t)\right)}{c^2a^2(t)}$ is the scalar curvature.

3. Critical metrics on Friedmann-Lemaître-Robertson-walker metrics

This section looks at whether FLRW metrics meet the previously introduced critical conditions for the curvature functional $\mathcal{F}_u$. The variational equations greatly simplify due to the high symmetry of these metrics, which enables us to reduce the issue to the behavior of the scale factor $a(t)$. Our objective is to ascertain the circumstances in which FLRW metrics are $\mathcal{F}_u$-critical. Deeper variational structures in cosmological spacetimes may be revealed by this analysis, which connects the dynamical evolution of these cosmological models with geometric optimality conditions.

To begin the computations, we first determine the Ricci tensor's covariant derivative. This will enable us to examine the curvature characteristics that are pertinent to our research. A simple calculation reveals that $\Delta\varrho$'s non-zero components are provided by

$$\begin{aligned}
\Delta\varrho_{11} = {} & \frac{3}{c^2a^4(t)}\Big(a'^2(t)(-4\left(kc^2 + a'^2(t)\right) + 5a(t)a''(t)) \\
& + a'^2(t)(a'(t)a^{(3)}(t) + a''^2(t) + a^{(4)}(t)a(t))\Big), \\
\Delta\varrho_{22} = {} & \frac{1}{c^2a^2(t)(1-kr^2)}\Big(- 4kc^2(a'^2(t) + a(t)a''(t)) \\
& + 5a'(t)a(t)(a(t)a^{(3)}(t) - a(t)a''(t))\Big) \\
& \Big(+a^2(t)\left(3a''(t) + a^{(4)}(t)a(t)\right)\Big), \\
\Delta\varrho_{33} = {} & \frac{r^2}{c^2a^2(t)}\Big(- 4kc^2(a'^2(t) + a(t)a''(t)) \\
& + 5a'(t)a(t)(a(t)a^{(3)}(t) - a(t)a''(t))\Big) \\
& \Big(+a^2(t)\left(3a''(t) + a^{(4)}(t)a(t)\right)\Big),
\end{aligned}$$

$$\begin{aligned}\Delta\varrho_{44} = \frac{r^2(cos^2\theta-1)}{c^2a^2(t)}\Big(&a'^2(t)\left(4kc^2\left(1+a'^2(t)\right)+5a(t)a''(t)\right)\\&+a(t)a''(t)\left(4kc^2-3a''(t)\right)\Big)\\&\Big(-5a'(t)a^2(t)a^{(3)}(t)-a^3(t)a^{(4)}(t)\Big).\end{aligned}$$

Additionally, the following are $\mathcal{R}[\varrho]$'s non-zero components:

$$\begin{aligned}\mathcal{R}[\varrho]_{11} &= \frac{3a''(t)}{c^2a^3(t)}\left(2kc^2+2a'^2(t)+a(t)a''(t)\right),\\ \mathcal{R}[\varrho]_{22} &= -\frac{1}{c^4a^2(t)(1-kx_2^2)}\Big(2\left(kc^2+a(t)a''(t)\right)\left(kc^2+a'^2(t)\right)\\&\quad+3a''^2(t)a^2(t)+4a'^4(t)\Big),\\ \mathcal{R}[\varrho]_{33} &= -\frac{r^2}{c^4a^2(t)}\Big(2\left(kc^2+a(t)a''(t)\right)\left(kc^2+a'^2(t)\right)\\&\quad+3a''^2(t)a^2(t)+4a'^4(t)\Big),\\ \mathcal{R}[\varrho]_{44} &= -\frac{r^2\sin^2\theta}{c^4a^2(t)}\Big(2\left(kc^2+a(t)a''(t)\right)\left(kc^2+a'^2(t)\right)\\&\quad+3a''^2(t)a^2(t)+4a'^4(t)\Big).\end{aligned}$$

And we have

$$\|\varrho\|^2 = \frac{12}{c^4a^4(t)}\left(\left(kc^2+a(t)a''(t)\right)\left(kc^2+a'^2(t)\right)+a''(t)a^2(t)+a'^4(t)\right).$$

The critical equations for the functional $\mathcal{F}_u$ reduce to the following system of equations based on the calculations performed for the FLRW metrics:

(3.1)

$$\begin{aligned}&-6c^4k^2u-12a'^2(t)c^2ku+6a^2(t)a''^2(t)u-6a'^4(t)u-2c^4k^2\\&\quad-2a(t)a''(t)c^2k-8a'(t)^2c^2k+a^3(t)a^{(4)}(t)+5a^2(t)a'(t)a^{(3)}(t)\\&\quad+3a^2(t)a''(t)^2-3a(t)a'^2(t)a''(t)-6a'^4(t)\\&=0,\\&6c^4k^2u+12a'^2(t)c^2ku-6a^2(t)a''^2(t)u+6a'^4(t)u+2c^4k^2\\&\quad-2a(t)a''(t)c^2k+8a'^2(t)c^2k+a^3(t)a^{(4)}(t)+a^2(t)a'(t)a^{(3)}(t)\\&\quad-a^2(t)a''^2(t)-7a(t)a''(t)a'^2(t)+6a'^4(t)\\&=0.\end{aligned}$$

We can add the equations together to determine a relationship that is independent of u. Let's start by classifying each equation's terms according to the parameter u. The first equation can be expressed as follows: $u \cdot P(t) + Q(t) = 0$, where

(3.2)
$$\begin{aligned} P(t) &= -6c^4k^2 - 12a'^2(t)c^2k + 6a^2(t)a''^2(t) - 6a'^4(t), \\ Q(t) &= -2c^4k^2 - 2a(t)a''(t)c^2k - 8a'(t)^2c^2k + a^3(t)a^{(4)}(t) \\ &\quad + 5a^2(t)a'(t)a^{(3)}(t) + 3a^2(t)a''(t)^2 - 3a(t)a'^2(t)a''(t) - 6a'^4(t). \end{aligned}$$

Also, the second equation can be written as $u \cdot R(t) + S(t) = 0$, where

(3.3)
$$\begin{aligned} R(t) &= 6c^4k^2 + 12a'^2(t)c^2k - 6a^2(t)a''^2(t) + 6a'^4(t), \\ S(t) &= 2c^4k^2 - 2a(t)a''(t)c^2k + 8a'^2(t)c^2k + a^3(t)a^{(4)}(t) \\ &\quad + a^2(t)a'(t)a^{(3)}(t) - a^2(t)a''^2(t) - 7a(t)a''(t)a'^2(t) + 6a'^4(t). \end{aligned}$$

We observe the crucial relationship that $R(t) = -P(t)$. Therefore, by adding the two equations, we obtain

$$Q(t) + S(t) = 0. \tag{3.4}$$

We can state the following theorem in light of the above arguments.

Theorem 3.1. *Consider the Friedmann-Lemaître-Robertson-Walker space time (M, g), which has a spatial curvature $k \in \{-1, 0, 1\}$ and a scale factor $a(t)$. For the functional $\mathcal{F}_u$, g is a critical metric if and only if $u \cdot P(t) + Q(t) = 0$, $-u \cdot P(t) + S(t) = 0$ and the consistency equation $Q(t) + S(t) = 0$ satisfy. Where $P(t)$, $Q(t)$ and $S(t)$ are presented in the equations* (3.2) *and* (3.3) *respectively.*

With further examination of the equations presented above, we can derive the following proposition.

Proposition 3.2. *Let (M, g) be a Friedmann-Lemaître-Robertson-Walker space time with a spatial curvature $k \in \{-1, 0, 1\}$ and a scale factor $a(t)$. For the metric g to be critical of the functional $\mathcal{F}_u$ the following statements are valid.*

(i) *$a(t) = mt+b$ is any linear function and u must take on a specific value determined by the system parameters and the slope m:*

$$u = -\frac{c^2k + 3m^2}{3c^2k + 3m^2}.$$

(ii) *$a(t)$ can't be an exponential or trigonometric function.*

Proof. Let $a(t)$ is a linear solution, $a(t) = mt + b$, the derivatives are: $a'(t) = m$, and $a''(t) = a^{(3)}(t) = a^{(4)}(t) = 0$. Then,

$$\begin{aligned} P(t) &= 6c^4k^2 - 12m^2c^2k - 6m^4, \\ Q(t) &= -2c^4k^2 - 8m^2c^2k - 6m^4. \end{aligned}$$

Again, both are constants. The value for u is given by

$$\begin{aligned} u &= -\frac{-2c^4k^2 - 8m^2c^2k - 6m^4}{-6c^4k^2 - 12m^2c^2k - 6m^4} \\ &= -\frac{c^2k + 3m^2}{3c^2k + 3m^2}. \end{aligned}$$

Therefore, if $a(t)$ is any linear function, u must be a specific constant whose value depends on the parameters c, k, and the slope m.

If $a(t)$ is an exponential solution, $a(t) = Ae^{\lambda t}$, the derivatives follow the pattern $a^{(n)}(t) = \lambda^n a(t)$.

$$\begin{aligned} P(t) &= -6c^4k^2 - 12(\lambda a(t))^2c^2k + 6a(t)^2\left(\lambda^2 a(t)\right)^2 - 6(\lambda a(t))^4 \\ &= -6c^4k^2 - 12\lambda^2c^2ka(t)^2 + 6\lambda^4a(t)^4 - 6\lambda^4a(t)^4 \\ &= -6c^4k^2 - 12\lambda^2c^2k\,a(t)^2, \\ Q(t) &= -2c^4k^2 - 2a(t)\left(\lambda^2a(t)\right)c^2k - 8(\lambda a(t))^2c^2k + a^3(t)(\lambda^4a(t)) \\ &\quad + 5a^2(t)(\lambda a(t))\left(\lambda^3a(t)\right) + 3a(t)^2\left(\lambda^2a(t)\right)^2 \\ &\quad - 3a(t)\left(\lambda^2a(t)\right)(\lambda a(t))^2 - 6(\lambda a(t))^4 \\ &= -2c^4k^2 + \left(-2\lambda^2 - 8\lambda^2\right)c^2ka(t)^2 \\ &\quad + \left(\lambda^4 + 5\lambda^4 + 3\lambda^4 - 3\lambda^4 - 6\lambda^4\right)a(t)^4 \\ &= -2c^4k^2 - 10\lambda^2c^2k\,a(t)^2. \end{aligned}$$

The ratio for u is given by

$$\begin{aligned} u &= -\frac{-2c^4k^2 - 10\lambda^2c^2k\,a(t)^2}{-6c^4k^2 - 12\lambda^2c^2k\,a(t)^2} \\ &= \frac{2c^4k^2 + 10\lambda^2c^2k\,A^2e^{2\lambda t}}{6c^4k^2 + 12\lambda^2c^2k\,A^2e^{2\lambda t}}. \end{aligned}$$

This expression is a function of t. The ratio of the constant parts is $\frac{1}{3}$, while the ratio of the t-dependent parts is $\frac{5}{6}$. Since $\frac{1}{3} \neq \frac{5}{6}$, the expression cannot be simplified to a constant.

So, a non-trivial exponential function cannot be a solution for any constant value of u.

If $a(t)$ is a trigonometric solution, $a(t) = A\cos(\omega t)$, the derivatives involve both $\cos(\omega t)$ and $\sin(\omega t)$.

$$a'(t) = -A\omega \sin(\omega t),$$
$$a''(t) = -A\omega^2 \cos(\omega t) = -\omega^2 a(t).$$

Substituting these into $P(t)$ and $Q(t)$ will produce complex trigonometric polynomials involving terms like $\cos^2(\omega t)$, $\sin^2(\omega t)$, $\cos^4(\omega t)$, etc. For example, a term in $P(t)$ is

$$6a^2(t)a''^2(t) = 6(A\cos(\omega t))^2 \left(-\omega^2 A\cos(\omega t)\right)^2 = 6A^4\omega^4 \cos^4(\omega t).$$

Another term is

$$-12a'^2(t)c^2k = -12(-A\omega\sin(\omega t))^2 c^2 k = -12A^2\omega^2 c^2 k \sin^2(\omega t).$$

The resulting expressions for $P(t)$ and $Q(t)$ will be functions of t that are not proportional to each other.

Hence, a non-trivial trigonometric function cannot be a solution for any constant value of u. □

Having established the conditions under which the FLRW metrics are $\mathcal{F}_u$-critical, we now turn our attention to the analysis of $\mathcal{S}$-critical metrics. The study of $\mathcal{S}$-critical provides a complementary perspective, as it involves a different variational framework while still being closely related to the same geometric setting. In the following theorem, we present the precise conditions under which the FLRW metrics become $\mathcal{S}$-critical.

Theorem 3.3. *Let (M, g) be a Friedmann-Lemaître-Robertson-Walker space time with scale factor $a(t)$ and spatial curvature $k \in \{-1, 0, 1\}$. Then g is a critical metric for the functional $\mathcal{S}$ if and only if one of the following conditions holds:*

(i) *If the scalar curvature τ vanishes, then the scale factor $a(t)$ is of the form*

$$a(t) = \pm\sqrt{-kc^2t^2 - 2tc_1 + 2c_2},$$

where c_1 and c_2 are real constants.

(ii) *If g is Einstein, then $\lambda = -\frac{3}{c_1c_2}$, and the scale factor takes $a(t)$ one of the following forms*

$$a(t) = \frac{t^2c_1}{2e^{\frac{3c_2}{t}}} + \frac{kc^2c_1}{2e^{\frac{c_2}{t}}},$$
$$a(t) = \frac{c_1}{2}e^{\frac{3c_2}{t}} + \frac{kc^2c_1}{2}e^{\frac{c_2}{t}},$$

where c_1 and c_2 are real constants.

Proof. For the FLRW metric to be $\mathcal{S}$-critical, it must either have vanishing scalar curvature or be Einstein. We analyze these two cases separately. If the scalar curvature τ vanishes, then considering that $\tau = -\frac{6\left(kc^2+a'^2(t)+a(t)a''(t)\right)}{c^2a^2(t)}$, we have $a(t) = \pm\sqrt{-kc^2t^2-2tc_1+2c_2}$, where c_1 and c_2 are real constants.

If the metric is Einstein, then the Einstein equations take the form $\varrho = \lambda g$, which, in the FLRW metric setting, lead to the following equations

$$\frac{x^2}{c^2}\left(\lambda c^2a^2(t)+2kc^2+2a'^2(t)+a(t)a''(t)\right)=0,$$
$$\frac{1}{a(t)}\left(\lambda c^2a(t)+3a''(t)\right)=0,$$
$$\frac{x^2}{c^2}\left(-\lambda c^2a^2(t)\left(\cos^2 y-1\right)+2\cos^2 y\left(a'(t)-kc^2\right)-2a'^2(t)-a(t)a''(t)\right)=0,$$
$$\frac{1}{c^2(-1+kx^2)}\left(\lambda c^2a^2(t)+2kc^2+2a'^2(t)+a(t)a''(t)\right)=0.$$

Solving the above equations, the proof is complete. □

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