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Advanced pq-Fractional Fuzzy–Based Multi-Criteria Decision-Making Model for E-Commerce Platform Evaluation

Hariwan Fadhil M. Salih[1], Zanyar A. Ameen[2*] and Baravan A. Asaad[3]

Abstract. The pq-fractional fuzzy set (pq-FFS) provides a novel and flexible framework for multi-criteria decision-making (MCDM) in uncertain environments, particularly for e-commerce platform selection where conflicting criteria interact under vagueness. Leveraging adjustable parameters p and q, it derives customized aggregation schemes that adapt to diverse decision-maker preferences while exhibiting strong stability and reduced sensitivity to parameter tuning. Unlike traditional fuzzy sets, the proposed model offers superior mathematical modeling of subjective bias and vagueness. This work formally defines pq-FFS, its operational laws and new score and accuracy functions for improved alternative ranking. Novel aggregation operators and an innovative MCDM technique tailored for e-commerce are proposed. A numerical case study validates the model's applicability, while sensitivity analysis confirms robustness across p and q variations. Comparative studies with existing methods, including q-FFWA, q-ROFWA and pq-ROFWA, quantified via Spearman's ρ and Kendall's τ, demonstrate superior robustness and computational efficiency.

1. Introduction

E-commerce has become a defining feature of the digital era, transforming business operations and consumer behavior worldwide. Online platforms enable companies to expand their market reach, connect with global customers and access diverse markets. For consumers, e-commerce offers convenience, flexibility and variety, allowing shopping anytime and anywhere, often with competitive pricing [14].

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* Corresponding author.

Beyond transactions, e-commerce significantly impacts the global economy by fostering innovation and creating opportunities. It generates employment, supports entrepreneurship and drives the development of advanced platforms, secure payment systems and efficient logistics. It also improves access to goods and services, especially for people in remote areas or with mobility limitations [12].

As e-commerce continues to evolve, its importance will further increase. Businesses that adapt to this dynamic environment are more likely to succeed, while consumers will continue to benefit from convenience and choice. Thus, e-commerce remains a key driver shaping modern business and consumer practices [33].

However, making effective decisions in this complex domain is challenging. Multi-Criteria Decision-Making (MCDM) techniques provide useful tools for evaluating e-commerce platforms (see [10, 23, 39, 40]), but they must handle uncertainty in human judgment. Therefore, this study proposes an MCDM approach based on *pq*-fractional fuzzy sets (*pq*-FFSs), considering factors such as website design, product variety and pricing to help businesses select the most suitable platforms.

1.1. **Some Fuzzy Models.** Fuzzy theory provides a mathematical framework to model uncertainty and imprecision in real-world systems, where traditional binary logic falls short. Its applications span critical domains such as multi-criteria decision-making (MCDM) [17], pattern recognition [26], e-commerce [11, 20], supply chain [36, 41] and data mining [15, 16]. Some related recent advances in multi-criteria decision-making can be found in [2, 9, 27, 37]. The foundation was laid by Zadeh (1965) through fuzzy sets (FS) [46], which quantify an element's membership in a set via continuous degrees. Atanassov (1986) later advanced this to intuitionistic fuzzy sets (IFS) [8], introducing dual membership ($\ddot{\mu}$) and non-membership ($\ddot{\nu}$) degrees constrained by $\ddot{\mu} + \ddot{\nu} \leq 1$. Although IFS offered a more refined characterization of uncertainty, it suffered from the shortcoming when $\ddot{\mu} + \ddot{\nu} > 1$. In order to resolve this issue, Yager (2013) also introduced the Pythagorean fuzzy sets (PyFS) [42, 43] in which $\ddot{\mu}^2 + \ddot{\nu}^2 \leq 1$ which is more flexible. Extending this, Yager (2016) proposed q-rung orthopair fuzzy sets (q-ROFS) [44] that are based on $\ddot{\mu}^q + \ddot{\nu}^q \leq 1$ for $q \geq 1$, these q-ROFS encompassed both IFS and PyFS. For more information, readers are recommended to read [3, 19, 22, 30, 45]. In q-ROFS, the sum of the q-th power of the membership and non-membership degrees is limited as shown above. The symmetric constraint reflects an equal bend- ability of the two degrees. However, in practice decision-making often takes place in search spaces of this type, where membership and non-membership evaluations have

different granularities. To resolve this, pq-ROFS [31] relaxes this symmetry and gives p and q as two different exponents, such that: $\ddot{\mu}^p + \ddot{\nu}^q \leq 1$ for $p, q \geq 1$. This modification provides a more adaptable way to encode information, catering to various preferences and requirements in decision-making situations. Readers are directed to [1, 4–6, 35] for additional information on pq-rung orthopair fuzzy sets (pq-ROFS).

Recent advances in fuzzy frameworks, including interval T-spherical [18], M-polar [34], Fermatean neutrosophic [38] and q-rung orthopair systems [28], have expanded the modeling of uncertainty, yet they remain constrained by either symmetric exponential bounds or high-dimensional complexity. The proposed pq-fractional fuzzy framework distinguishes itself through a linear constraint $\frac{\mu}{p} + \frac{\nu}{q} \leq 1$ governed by two independent parameters, enabling asymmetric control over membership and non-membership degrees with minimal computational overhead. This dual-parameter linear structure offers decision-makers interpretable and differentiated weighting of positive and negative evidence, a capability absent from existing approaches. The detailed structural comparison is provided in Table 1:

TABLE 1. Structural comparison of pq-FF AOs with recent advanced frameworks

Feature	IVTS-FS SWARA	M-Polar Fuzzy	Fermatean Neutrosophic	q-ROF ExpoLog	pq-FF AOs (Ours)
Underlying model	Interval T-spherical	M-polar fuzzy	Fermatean neutrosophic	q-rung orthopair	pq-fractional fuzzy
Membership structure	Interval $[\mu^L, \mu^U]$	m-tuple $(\mu_1, \ldots, \mu_m)$	(μ, ν, π) with $\mu^3 + \nu^3 \leq 1$	(μ, ν) with $\mu^q + \nu^q \leq 1$	(μ, ν) with $\frac{\mu}{p} + \frac{\nu}{q} \leq 1$
Constraint type	Spherical (cubic)	None (independent polars)	Exponential (cubic)	Exponential (symmetric)	Linear (asymmetric)
Parameters	1 (spherical index)	m (polarity count)	1 (Fermatean index)	1 (rung q)	2 (independent p, q)
Asymmetric control?	No	Partial (via m)	No	No	Yes
Linear interpretability?	No	Yes	No	No	Yes
Computational complexity	High (interval arithmetic)	High (m-dimensional)	High (neutrosophic 3-tuple)	Medium (power operations)	Low (linear ops)
Aggregation family	SWARA + Sugeno-Weber	Modified Hamacher	Hypersoft + ML	ExpoLogarithmic	Weighted average/geometric

Nonetheless, none of the previously mentioned models addresses the issue of when the membership degree is 1 while the non-membership degree falls between 0 and 1 or vice versa. Recently, Gulistan and Pedrycz [13] proposed a new model known as a q-fractional fuzzy set (q-FFS), which solves this issue. They discussed several applications of this new model. However, q-fractional fuzzy model failed to address many situations, in real-world decision-making scenarios, where experts may provide a range of preferences or evaluations instead of a single precise value.

Generalizing the q-fractional fuzzy set theories, a novel pq-fractional fuzzy model is proposed by using the separate linear weights for membership and non-membership degrees. The formulated $\frac{1}{p}\ddot{\mu} + \frac{1}{q}\ddot{\nu} \leq 1$ with $p, q \geq 1$ is a significant leap forward in modeling uncertainties because it makes possible to control explicitly the balance between membership and non-membership components. This linearly weighted representation

has multiple advantages in reading and classification compared with typical exponent-based representations. First, it keeps an explicit meaning as $1/p$ and $1/q$ are directly weight factors that decision-makers can naturally change the importance power of each component. Second, the proposed framework naturally extends the conventional fuzzy set models by imposing the limitations of those reducing to the standard IFSs when $p = q = 1$ and handling the more complex cases with asymmetric weight settings.

1.2. **Motivations.** The existing q-fractional fuzzy set offers a flexible framework for modeling uncertainty through the relaxed linear constraint $\mu/q + \nu/q \leq 1$, where a single parameter q controls the overall admissible degree of uncertainty. However, this formulation assumes that membership and non-membership are treated symmetrically, which may not reflect realistic decision-making scenarios. In many applications, decision makers assign different levels of importance or tolerance to positive (membership) and negative (non-membership) information.

We address this limitation by introducing the pq-fractional fuzzy set characterized by the constraint $\mu/p + \nu/q \leq 1$, where $p, q \geq 1$. The parameters p and q play distinct and interpretable roles: the parameter p regulates the influence of the membership degree μ, while q controls the influence of the non-membership degree ν. Specifically, a larger value of p reduces the relative contribution of μ in the constraint, thereby allowing higher membership values with less restriction, whereas a smaller p enforces stricter control on μ. Similarly, increasing q weakens the effect of the non-membership degree ν, while decreasing q strengthens its impact.

This interpretation can be clearly illustrated through a medical diagnosis example. Consider a physician evaluating whether a patient has a certain disease. Let μ represent the degree of evidence supporting the presence of the disease and ν represent the degree of evidence against it. In a risk-averse setting, the physician may wish to be very cautious about negative evidence (i.e., not missing a disease), assigning a relatively small value to q so that even moderate non-membership values are heavily weighted. At the same time, the physician may allow more flexibility in positive evidence by choosing a larger p, permitting higher membership values without strict restriction. Conversely, in a screening scenario where false positives must be minimized, the decision maker may select a smaller p and a larger q, thereby tightening the acceptance of membership while relaxing the effect of non-membership.

This asymmetric parameterization enables independent tuning of the tolerance toward supporting and opposing evidence, offering a more realistic representation of decision-maker preferences. The proposed model

generalizes the q-fractional fuzzy set as a special case when $p = q$, while preserving the simplicity of a linear structure. Consequently, the pq-fractional fuzzy set provides a more expressive and adaptable framework for modeling uncertainty in complex decision-making environments where positive and negative information are not treated equally.

1.3. **Contributions.** The suggested study makes the following contributions:

- Definition and operational laws for pq-FFSs;
- Aovel score and accuracy functions for ranking pq-fractional fuzzy numbers;
- Aggregation operators with closure properties; and
- An MCDM framework validated through e-commerce platform selection.

1.4. **Outline of the Paper.** The remainder of this paper is organized as follows. First, Section 2 recalls q-fractional fuzzy sets and their basic operations. Then, Section 3 introduces pq-fractional fuzzy sets with operational laws that generalize existing fuzzy theories and additionally provides a geometric analysis of the admissible region, establishes incomparability with pq-ROFS and proves closure properties for the algebraic operations. Next, Section 4 proposes novel score and accuracy functions derived from axiomatic principles, together with their monotonicity and boundary properties. Subsequently, Section 5 develops aggregation operators coherent with the pq-fractional constraint and rigorously proves their closure to ensure valid outputs. Based on these, Section 6 presents the multi-criteria decision-making approach. Moreover, Section 7 illustrates the framework through a numerical example, while Section 8 performs sensitivity analysis on parameters p and q. Furthermore, Section 9 compares the proposed approach with existing methods, including quantitative stability measures via Spearman's ρ and Kendall's τ, as well as structural comparison with recent advanced fuzzy aggregation frameworks. Finally, Sections 10 and 11 address the advantages and limitations, respectively.

2. Preliminaries

This section provides the mathematical foundations for further advancements by officially defining q-fractional fuzzy sets (q-FFS) and outlining their basic operations and essential algebraic features.

Definition 2.1 ([13])**.** Let $\bar{A}$ be a non-empty initial universe. A q-fractional fuzzy set (q-FFS) over $\bar{A}$ is defined as

$$\ddot{\mathcal{H}} = \left\{ \bar{a}, \left(\ddot{\mu}(\bar{a}), \ddot{\nu}(\bar{a})\right), \bar{a} \in \bar{A} \right\},$$

where $\ddot{\mu}(\bar{a}),\ \ddot{\nu}(\bar{a}) \in [0,1]$ refer the membership and non-membership degrees, respectively, such that

$$0 \leq \frac{1}{q}\ddot{\mu}(\bar{a}) + \frac{1}{q}\ddot{\nu}(\bar{a}) \leq 1,$$

where $q \geq 2$ is a positive integer.

The pair $(\ddot{\mu}, \ddot{\nu})$ is called a q-fractional fuzzy number (q-FFN).

Definition 2.2 ([13])**.** Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be two q-FFNs and $\lambda > 0$. Then

(i) $\ddot{H}_1 \cup \ddot{H}_2 = (\max\{\ddot{\mu}_1, \ddot{\mu}_2\}, \min\{\ddot{\nu}_1, \ddot{\nu}_2\})$.
(ii) $\ddot{H}_1 \cap \ddot{H}_2 = (\min\{\ddot{\mu}_1, \ddot{\mu}_2\}, \max\{\ddot{\nu}_1, \ddot{\nu}_2\})$.
(iii) $\ddot{H}_1^c = (1 - \ddot{\mu}_1, 1 - \ddot{\nu}_1)$.
(iv) $\ddot{H}_1 \oplus \ddot{H}_2 = \left(\frac{1}{q}\ddot{\mu}_1 + \frac{1}{q}\ddot{\mu}_2 - \frac{1}{q}\ddot{\mu}_1 \cdot \ddot{\mu}_2,\ \frac{1}{q}\ddot{\nu}_1 \cdot \ddot{\nu}_2\right)$.
(v) $\lambda \cdot \ddot{H} = \left(1 - \left(1 - \frac{1}{q}\ddot{\mu}_1\right)^{\lambda}, \frac{1}{q}(\ddot{\nu}_1)^{\lambda}\right)$.

Proposition 2.3 ([13])**.** *Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$, $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be three q-FFNs and $\lambda, \lambda_1, \lambda_2 > 0$. Then*

(i) $\ddot{H}_1 \oplus \ddot{H}_2 = \ddot{H}_2 \oplus \ddot{H}_1$.
(ii) $\lambda \cdot \left(\ddot{H}_1 \oplus \ddot{H}_2\right) = \lambda \cdot \ddot{H}_1 \oplus \lambda \cdot \ddot{H}_2$.
(iii) $(\lambda_1 + \lambda_2) \cdot \ddot{H} = \lambda_1 \cdot \ddot{H} \oplus \lambda_2 \cdot \ddot{H}$.

3. pq-Fractional Fuzzy Sets

The proposed pq-fractional fuzzy set (pq-FFS), a novel extension of conventional fuzzy sets that incorporates independent fractional parameters p and q to govern membership and non-membership degrees, respectively, is introduced in this section in order to lay the theoretical groundwork for our investigation.

Definition 3.1. Let $\bar{A}$ be a non-empty initial universe. A pq-fractional fuzzy set (pq-FFS) over $\bar{A}$ is defined as $\ddot{\mathcal{H}} = \left\{\bar{a}, (\ddot{\mu}(\bar{a}), \ddot{\nu}(\bar{a})), \bar{a} \in \bar{A}\right\}$, where $\ddot{\mu}(\bar{a}),\ \ddot{\nu}(\bar{a}) \in [0,1]$ refer the membership and non-membership degrees, respectively, such that

$$0 \leq \frac{1}{p}\ddot{\mu}(\bar{a}) + \frac{1}{q}\ddot{\nu}(\bar{a}) \leq 1, \tag{3.1}$$

where the condition (3.1) is satisfied for all $\bar{a} \in A$ and $p, q \geq 1$ are positive integers.

Conveniently, the pair $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$ is called a pq-fractional fuzzy number (pq-FFN).

Recently, a related study was reported in [25]. Although it shares a similar research direction, the proposed pq-fractional fuzzy set framework presented in this paper differs in its mathematical formulation, operational laws, aggregation operators and decision-making methodology.

Remark 3.2. For $p, q \geq 2$ the inequality $\frac{1}{p}\ddot{\mu} + \frac{1}{q}\ddot{\nu} \leq 1$ is identically true for all admissible membership pairs; the set of valid $(\ddot{\mu}, \ddot{\nu})$ therefore coincides with the entire unit square, giving the decision maker complete freedom. Consequently, within this range, the parameters p and q no longer restrict the validity of the fuzzy data pairs, but instead serve exclusively as mathematical tuning parameters within the score function to calculate and alter the final ranking outcomes.

To clarify the broader geometric properties of the proposed pq-fractional fuzzy set (pq-FFS), its admissible region must be examined against existing frameworks. By initiating the parameter space at the integer level of $p, q \geq 1$, the pq-FFS generalizes both the q-FFS and the p, q-QRFS. While the pq-FFS maintains a structurally similar space extension to the q-FFS, it generalizes the q-FFS framework by decoupling the symmetric constraint into two independent parameters, p and q. This asymmetric flexibility grants the decision-maker the distinct advantage of assigning different parameter values to the membership ($\ddot{\mu}$) and non-membership ($\ddot{\nu}$) degrees based on real-world decision contexts. However, pq-FFS extends the admissible region of p, q-QRFS for the same parameters, as shown in Figure 1 where it compares their feasible regions under three representative parameter configurations. For $(p, q) = (2, 1)$ and $(1, 3)$, the pq-FFS constraint is active and produces a genuinely restrictive linear region (cyan), whereas the pq-ROFS constraint yields a curved region (magenta) with which it is incomparable, each model admits points excluded by the other. Crucially, when $p, q \geq 2$ as in $(2, 3)$, the pq-FFS constraint becomes vacuous and collapses to the entire unit square $[0, 1]^2$, while the pq-ROFS constraint remains nontrivial. This illustrates that pq-FFS offers a more flexible and interpretable framework: by independently tuning p and q (particularly allowing either to approach 1), one obtains active, asymmetric constraints that can be tailored to reflect domain-specific uncertainty, whereas the exponential pq-ROFS structure lacks this tunable linear control and does not expose the vacuous regime so transparently.

We now present the fundamental operations on pq-FFS.

Definition 3.3. Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$, $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be three pq-FFNs. Then

(i) $\ddot{H}_1 \cup \ddot{H}_2 = \left(\max\{\ddot{\mu}_1, \ddot{\mu}_2\},\ \min\{\ddot{\nu}_1, \ddot{\nu}_2\}\right).$

(ii) $\ddot{H}_1 \cap \ddot{H}_2 = (\min\{\ddot{\mu}_1, \ddot{\mu}_2\},\ \max\{\ddot{\nu}_1, \ddot{\nu}_2\})$.
(iii) $\ddot{H}^c = (\ddot{\nu}, \ddot{\mu})$.

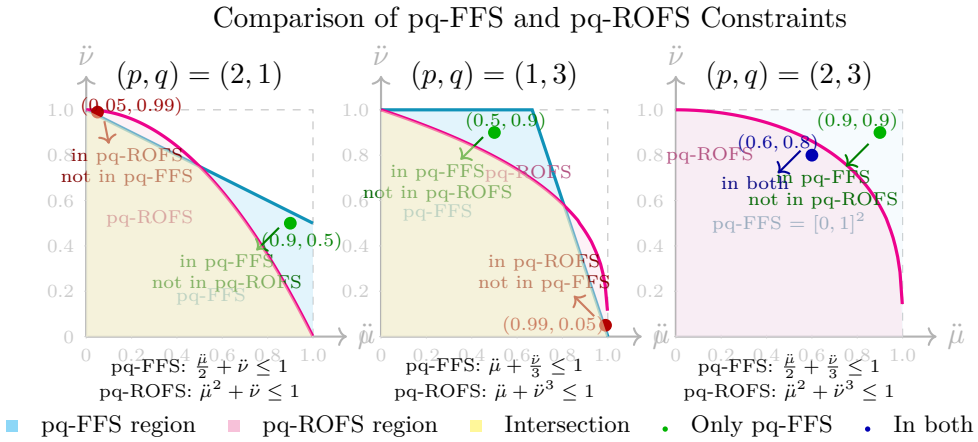

FIGURE 1. Incomparability of pq-FFS and pq-ROFS: active constraints for $(p, q) = (2, 1)$ and $(1, 3)$ (left and middle) and vacuous pq-FFS for $(p, q) = (2, 3)$ (right).

Proposition 3.4. *Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$, $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ and $\ddot{H}_3 = (\ddot{\mu}_3, \ddot{\nu}_3)$ be three pq-FFNs. Then*

(i) $\ddot{H}_1 \cup \ddot{H}_2 = \ddot{H}_2 \cup \ddot{H}_1$.
(ii) $\ddot{H}_1 \cap \ddot{H}_2 = \ddot{H}_2 \cap \ddot{H}_1$.
(iii) $\ddot{H}_1 \cup (\ddot{H}_2 \cup \ddot{H}_3) = (\ddot{H}_1 \cup \ddot{H}_2) \cup \ddot{H}_3$.
(iv) $\ddot{H}_1 \cap (\ddot{H}_2 \cap \ddot{H}_3) = (\ddot{H}_1 \cap \ddot{H}_2) \cap \ddot{H}_3$.
(v) $(\ddot{H}_1 \cup \ddot{H}_2)^c = \ddot{H}_1^c \cap \ddot{H}_2^c$.
(vi) $(\ddot{H}_1 \cap \ddot{H}_2)^c = \ddot{H}_1^c \cup \ddot{H}_2^c$.

Proof. Straightforward. □

Definition 3.5. Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be two *pq*-FFNs. Then

(i) $\ddot{H}_1 \preccurlyeq \ddot{H}_2 \iff \ddot{\mu}_1 \leq \ddot{\mu}_2,\ \ddot{\nu}_1 \geq \ddot{\nu}_2$.
(ii) $\ddot{H}_1 = \ddot{H}_2 \iff \ddot{H}_1 \preccurlyeq \ddot{H}_2$ and $\ddot{H}_2 \preccurlyeq \ddot{H}_1$, (*i.e.*, $\ddot{\mu}_1 = \ddot{\mu}_2,\ \ddot{\nu}_1 = \ddot{\nu}_2$).

Proposition 3.6. *Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$, $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ and $\ddot{H}_3 = (\ddot{\mu}_3, \ddot{\nu}_3)$ be three pq-FFNs. Then*

(i) If $\ddot{H}_1 \preccurlyeq \ddot{H}_2 \preccurlyeq \ddot{H}_3$, then $\ddot{H}_1 \preccurlyeq \ddot{H}_3$.
(ii) If $\ddot{H}_1 \preccurlyeq \ddot{H}_2$, then $\ddot{H}_2^c \preccurlyeq \ddot{H}_1^c$.
(iii) If $\ddot{H}_1 \preccurlyeq \ddot{H}_2$ and $\ddot{H}_1 \preccurlyeq \ddot{H}_3$, then $\ddot{H}_1 \preccurlyeq \ddot{H}_2 \cap \ddot{H}_3$.
(iv) If $\ddot{H}_1 \preccurlyeq \ddot{H}_3$ and $\ddot{H}_2 \preccurlyeq \ddot{H}_3$, then $\ddot{H}_1 \cup \ddot{H}_2 \preccurlyeq \ddot{H}_3$.

Proof. The proofs of i and ii are Straightforward.

To prove iii, since $\ddot{H}_1 \preccurlyeq \ddot{H}_2$, then

$$\ddot{\mu}_1 \leq \ddot{\mu}_2 \text{ and } \ddot{\nu}_1 \geq \ddot{\nu}_2,$$

and since $\ddot{H}_1 \preccurlyeq \ddot{H}_3$, then

$$\ddot{\mu}_1 \leq \ddot{\mu}_3 \text{ and } \ddot{\nu}_1 \geq \ddot{\nu}_3.$$

Then

$$\ddot{\mu}_1 \leq \min\{\ddot{\mu}_2, \ddot{\mu}_3\} \text{ and } \ddot{\nu}_1 \geq \max\{\ddot{\nu}_2, \ddot{\nu}_3\}. \tag{3.2}$$

And

$$\ddot{H}_2 \cap \ddot{H}_3 = \left(\min\{\ddot{\mu}_2, \ddot{\mu}_3\},\ \max\{\ddot{\nu}_2, \ddot{\nu}_3\}\right).$$

From the inequality 3.2, we have that $\ddot{H}_1 \preccurlyeq \ddot{H}_2 \cap \ddot{H}_3$.

The proof of iv is similar to the proof of iii. □

Definition 3.7. Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$, $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be pq-FFNs and $\lambda > 0$. Then

(i) $\ddot{H}_1 \oplus \ddot{H}_2 = \left(\left(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}}\right)^p,\ \ddot{\nu}_1 \cdot \ddot{\nu}_2\right)$.

(ii) $\ddot{H}_1 \otimes \ddot{H}_2 = \left(\ddot{\mu}_1 \cdot \ddot{\mu}_2,\ \left(\ddot{\nu}_1^{\frac{1}{q}} + \ddot{\nu}_2^{\frac{1}{q}} - \ddot{\nu}_1^{\frac{1}{q}} \cdot \ddot{\nu}_2^{\frac{1}{q}}\right)^q\right)$.

(iii) $\lambda \cdot \ddot{H} = \left(\left(1 - \left(1 - \ddot{\mu}^{\frac{1}{p}}\right)^\lambda\right)^p, \ddot{\nu}^\lambda\right)$.

(iv) $\ddot{H}^\lambda = \left(\ddot{\mu}^\lambda, \left(1 - \left(1 - \ddot{\nu}^{\frac{1}{q}}\right)^\lambda\right)^q\right)$.

Example 3.8. Let $\ddot{H} = (0.92, 0.01)$, $\ddot{H}_1 = (0.56, 0.29)$ and $\ddot{H}_2 = (0.12, 0.65)$ be three pq-FFNs and $p = q = 3$ and $\lambda = 0.33$. Then

(i) $\ddot{H}_1 \oplus \ddot{H}_2 = (0.7559, 0.1885)$.
(ii) $\ddot{H}_1 \otimes \ddot{H}_2 = (0.0672, 0.8704)$.
(iii) $\lambda \cdot \ddot{H} = (0.3355, 2188)$.
(iv) $\ddot{H}^\lambda = (0.9729, 0.0005)$.

The following result guarantees that the operational laws defined in Definition 3.7 produce valid pq-fractional fuzzy numbers, thereby ensuring the internal consistency of the algebraic structure.

Proposition 3.9. *Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$, $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be pq-FFNs with the same fixed positive integer parameters p, q and let $\lambda > 0$. If*

$$\frac{1}{p} + \frac{1}{q} \leq 1,$$

then $\ddot{H}_1 \oplus \ddot{H}_2$, $\ddot{H}_1 \otimes \ddot{H}_2$, $\lambda \cdot \ddot{H}$ and $\ddot{H}^\lambda$ are valid pq-FFNs.

Proof. Since $\ddot{H}$, $\ddot{H}_1$ and $\ddot{H}_2$ are pq-FFNs, all their membership and non-membership degrees belong to $[0,1]$. For any $u, v \in [0,1]$ and $\alpha > 0$, we have

$$0 \leq uv \leq 1, \qquad 0 \leq u + v - uv = 1 - (1-u)(1-v) \leq 1,$$

and

$$0 \leq u^{\alpha} \leq 1, \qquad 0 \leq 1 - (1-u)^{\alpha} \leq 1.$$

Applying these inequalities to the expressions in Definition 3.7 shows that both components of each of

$$\ddot{H}_1 \oplus \ddot{H}_2, \qquad \ddot{H}_1 \otimes \ddot{H}_2, \qquad \lambda \cdot \ddot{H}, \qquad \ddot{H}^{\lambda}$$

belong to $[0,1]$.

Let (x, y) denote any of these resulting pairs. By the assumed condition on the fixed parameters p, q,

$$0 \leq \frac{x}{p} + \frac{y}{q} \leq \frac{1}{p} + \frac{1}{q} \leq 1.$$

Hence each resulting pair satisfies the admissibility condition in Definition 3.1 and is therefore a valid pq-FFN. This proves closure under all four operations. □

Remark 3.10. The condition $\frac{1}{p} + \frac{1}{q} \leq 1$ in Proposition 3.9 is sufficient but not necessary for closure under the four operations. Indeed, each operation produces a pair $(x, y) \in [0,1]^2$ and this condition guarantees

$$0 \leq \frac{x}{p} + \frac{y}{q} \leq \frac{1}{p} + \frac{1}{q} \leq 1.$$

However, closure also holds when $p = q = 1$, for which the admissibility condition becomes $\ddot{\mu} + \ddot{\nu} \leq 1$ and the operational laws reduce to the standard algebraic operations for intuitionistic fuzzy numbers, although $\frac{1}{p} + \frac{1}{q} = 2 > 1$. Thus, failure of the stated sufficient condition does not, by itself, imply failure of closure.

Theorem 3.11. *Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be two pq-FFNs. Then*

(i) $\ddot{H}_1 \oplus \ddot{H}_2 = \ddot{H}_2 \oplus \ddot{H}_1$.
(ii) $\ddot{H}_1 \otimes \ddot{H}_2 = \ddot{H}_2 \otimes \ddot{H}_1$.

Proof. proofs of i and ii are straightforward. □

Theorem 3.12. *Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$, $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ and $\ddot{H}_3 = (\ddot{\mu}_3, \ddot{\nu}_3)$ be three pq-FFNs. Then*

(i) $\Big(\ddot{H}_1 \oplus \ddot{H}_2\Big) \oplus \ddot{H}_3 = \ddot{H}_1 \oplus \Big(\ddot{H}_2 \oplus \ddot{H}_3\Big)$.
(ii) $\Big(\ddot{H}_1 \otimes \ddot{H}_2\Big) \otimes \ddot{H}_3 = \ddot{H}_1 \otimes \Big(\ddot{H}_2 \otimes \ddot{H}_3\Big)$.

Proof. (i) For pq-FFNs $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$, $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ and $\ddot{H}_3 = (\ddot{\mu}_3, \ddot{\nu}_3)$, then

$$\ddot{H}_1 \oplus \ddot{H}_2 = \left(\left(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} \right)^p, \ \ddot{\nu}_1 \cdot \ddot{\nu}_2 \right),$$

and

$$\begin{aligned}
\left(\ddot{H}_1 \oplus \ddot{H}_2\right) \oplus \ddot{H}_3 = \Bigg(\Bigg(\Bigg(\Bigg(\Bigg(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} \Bigg)^p \Bigg)^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} \\
- \left(\left(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \Bigg)^p, \ \ddot{\nu}_1 \cdot \ddot{\nu}_2 \cdot \ddot{\nu}_3 \Bigg) \\
= \Bigg(\Bigg(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \\
- \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} + \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \Bigg)^p, \ \ddot{\nu}_1 \cdot \ddot{\nu}_2 \cdot \ddot{\nu}_3 \Bigg).
\end{aligned}$$

On the other hand,

$$\ddot{H}_2 \oplus \ddot{H}_3 = \left(\left(\ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \right)^p, \ \ddot{\nu}_2 \cdot \ddot{\nu}_3 \right),$$

and

$$\begin{aligned}
\ddot{H}_1 \oplus \left(\ddot{H}_2 \oplus \ddot{H}_3\right) = \Bigg(\Bigg(\ddot{\mu}_1^{\frac{1}{p}} + \left(\left(\ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\
- \ddot{\mu}_1^{\frac{1}{p}} \cdot \left(\left(\ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \Bigg)^p, \ \ddot{\nu}_1 \cdot \ddot{\nu}_2 \cdot \ddot{\nu}_3 \Bigg) \\
= \Bigg(\Bigg(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \\
+ \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}} \cdot \ddot{\mu}_3^{\frac{1}{p}} \Bigg)^p, \ \ddot{\nu}_1 \cdot \ddot{\nu}_2 \cdot \ddot{\nu}_3 \Bigg).
\end{aligned}$$

Hence, from both sides, we obtain that

$$\left(\ddot{H}_1 \oplus \ddot{H}_2\right) \oplus \ddot{H}_3 = \ddot{H}_1 \oplus \left(\ddot{H}_2 \oplus \ddot{H}_3\right).$$

(ii) The proof is similar to that of i. □

Theorem 3.13. *Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be two pq-FFNs and $\lambda > 0$. Then*

(i) $\lambda \cdot \left(\ddot{H}_1 \oplus \ddot{H}_2\right) = \left(\lambda \cdot \ddot{H}_1\right) \oplus \left(\lambda \cdot \ddot{H}_2\right)$.

(ii) $\left(\ddot{H}_1 \otimes \ddot{H}_2\right)^{\lambda} = \ddot{H}_1^{\lambda} \otimes \ddot{H}_2^{\lambda}$.

Proof. (i) For $\lambda > 0$ and pq-FFNs $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$, then

$$\begin{aligned}
\lambda \cdot \left(\ddot{H}_1 \oplus \ddot{H}_2\right) &= \left(\left(1 - \left(1 - \left(\left(\ddot{\mu}_1^{\frac{1}{p}} + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\right) \\
&= \left(\left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} + \ddot{\mu}_1^{\frac{1}{p}} \cdot \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\right) \\
&= \left(\left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda} \cdot \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\right).
\end{aligned}$$

On the other hand,

$$\lambda \cdot \ddot{H}_1 = \left(\left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_1^{\lambda}\right),$$

and

$$\lambda \cdot \ddot{H}_2 = \left(\left(1 - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_2^{\lambda}\right).$$

$$\begin{aligned}
\left(\lambda \cdot \ddot{H}_1\right) \oplus \left(\lambda \cdot \ddot{H}_2\right) &= \Bigg(\Bigg(\Bigg(\Bigg(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda}\Bigg)^p\Bigg)^{\frac{1}{p}} \\
&\quad + \left(\left(1 - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)^p\right)^{\frac{1}{p}} - \left(\left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda}\right)^p\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\left(1 - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)^p\right)^{\frac{1}{p}}\Bigg)^p, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\Bigg) \\
&= \Bigg(\Bigg(\Bigg(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda} + 1 - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda} \\
&\quad - \left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda}\right) \cdot \left(1 - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right)\Bigg)^p\Bigg), \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\Bigg) \\
&= \left(\left(2 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda} - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda} - 1 + \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda}\right.\right. \\
&\quad \left.\left. + \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda} - \left(1 - \ddot{\mu}_2^{\frac{1}{p}}\right)^{\lambda} \cdot \left(1 - \ddot{\mu}_1^{\frac{1}{p}}\right)^{\lambda}\right)^p, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda}\right)
\end{aligned}$$

$$= \left(\left(1 - \left(1 - \ddot{\mu}_1^{\frac{1}{p}} \right)^{\lambda} \cdot \left(1 - \ddot{\mu}_2^{\frac{1}{p}} \right)^{\lambda} \right)^{p}, \ddot{\nu}_1^{\lambda} \cdot \ddot{\nu}_2^{\lambda} \right).$$

Therefore, from both sides, we obtain that

$$\lambda \cdot \left(\ddot{H}_1 \oplus \ddot{H}_2 \right) = \left(\lambda \cdot \ddot{H}_1 \right) \oplus \left(\lambda \cdot \ddot{H}_2 \right).$$

(ii) The proof is in a similar fashion to that of i.

□

Theorem 3.14. *Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$ be a pq-FFN and $\lambda_1, \lambda_2 > 0$. Then*

(i) $\lambda_1 \cdot (\lambda_2 \cdot \ddot{H}) = (\lambda_1 \cdot \lambda_2) \cdot \ddot{H}$.

(ii) $(\lambda_1 + \lambda_2) \cdot \ddot{H} = (\lambda_1 \cdot \ddot{H}) \oplus (\lambda_2 \cdot \ddot{H})$.

Proof. (i) For $\lambda_1, \lambda_2 > 0$ and pq-FFN $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$, we have

$$\lambda_2 \cdot \ddot{H} = \left(\left(1 - \left(1 - \ddot{\mu}^{\frac{1}{p}} \right)^{\lambda_2} \right)^{p}, \ddot{\nu}^{\lambda_2} \right)$$

and

$$\begin{aligned} \lambda_1 \cdot (\lambda_2 \cdot \ddot{H}) &= \left(\left(1 - \left(1 - \left(\left(1 - \left(1 - \ddot{\mu}^{\frac{1}{p}} \right)^{\lambda_2} \right)^{p} \right)^{\frac{1}{p}} \right)^{\lambda_1} \right)^{p}, \left(\ddot{\nu}^{\lambda_2} \right)^{\lambda_1} \right) \\ &= \left(\left(1 - 1 - \left(1 - \ddot{\mu}^{\frac{1}{p}} \right)^{\lambda_1 \cdot \lambda_2} \right)^{p}, \ddot{\nu}^{\lambda_1 \cdot \lambda_2} \right) \\ &= (\lambda_1 \cdot \lambda_2) \cdot \ddot{H}. \end{aligned}$$

(ii) The proof of this is in a similar fashion of i. □

4. Score and Accuracy Functions

In this section, we introduce the score and accuracy functions for pq-FFNs, which provide a quantitative basis for ranking and comparing alternatives in decision-making contexts. The proposed score function in Equation (4.1) is constructed to reflect the semantic roles of membership and non-membership in evaluation. The positive sign before $\ddot{\mu}^{1/p}$ indicates that membership represents positive evidence toward an alternative, while the negative sign before $\ddot{\nu}^{1/q}$ reflects that non-membership represents negative evidence. This formulation encodes the natural principle that higher membership and lower non-membership lead to a higher score.

The terms $\ddot{\mu}^{1/p}$ and $\ddot{\nu}^{1/q}$ are adopted to provide a smooth and controlled adjustment with respect to the parameters p and q, ensuring stable ranking behavior as these parameters vary. Moreover, these terms are consistent with the structure of the proposed aggregation operators,

which ensures coherence between the aggregation process and the evaluation mechanism. When $p = q = 1$, the formulation reduces to $\ddot{\mu}$ and $\ddot{\nu}$, preserving compatibility with classical intuitionistic fuzzy sets.

Furthermore, the suggested score function $\mathfrak{Sc}_{IFS}(\ddot{H}) = \ddot{\mu}^{1/p} - \ddot{\nu}^{1/q}$ has range $[-1, 1]$, whereas normalized scores in $[0, 1]$ are preferred in multi-criteria decision-making for interpretability and compatibility with aggregation schemes. Therefore, we apply the affine normalization $\mathfrak{Sc} \mapsto \frac{1}{2}(1+\mathfrak{Sc})$, which preserves ordering while mapping values to $[0, 1]$. These constructions enable effective comparison of alternatives while maintaining consistency with the underlying pq-FFS framework.

Definition 4.1. Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$ be a pq-FFN. Then the score function $\mathfrak{Sc}(\ddot{H})$ is defined as

$$\mathfrak{Sc}(\ddot{H}) = \frac{1}{2}\left(1 + \ddot{\mu}^{\frac{1}{p}} - \ddot{\nu}^{\frac{1}{q}}\right), \tag{4.1}$$

and the accuracy function $(\mathfrak{Acc}(\ddot{H}))$ is defined as

$$\mathfrak{Acc}(\ddot{H}) = \frac{1}{2}\left(\ddot{\mu}^{\frac{1}{p}} + \ddot{\nu}^{\frac{1}{q}}\right).$$

The following definition provides a comparison rule between pq-FFNs based on the accuracy and score functions.

Definition 4.2. Let $\ddot{H}_1 = (\ddot{\mu}_1, \ddot{\nu}_1)$ and $\ddot{H}_2 = (\ddot{\mu}_2, \ddot{\nu}_2)$ be two pq-FFNs. Then

(i) If $\mathfrak{Sc}\left(\ddot{H}_1\right) < \mathfrak{Sc}\left(\ddot{H}_2\right)$, then $\ddot{H}_1 \prec \ddot{H}_2$.

(ii) If $\mathfrak{Sc}\left(\ddot{H}_1\right) > \mathfrak{Sc}\left(\ddot{H}_2\right)$, then $\ddot{H}_1 \succ \ddot{H}_2$.

(iii) If $\mathfrak{Sc}\left(\ddot{H}_1\right) = \mathfrak{Sc}\left(\ddot{H}_2\right)$, then

  a) If $\mathfrak{Acc}\left(\ddot{H}_1\right) < \mathfrak{Acc}\left(\ddot{H}_2\right)$, then $\ddot{H}_1 \prec \ddot{H}_2$.

  b) If $\mathfrak{Acc}\left(\ddot{H}_1\right) > \mathfrak{Acc}\left(\ddot{H}_2\right)$, then $\ddot{H}_1 \succ \ddot{H}_2$.

  c) If $\mathfrak{Acc}\left(\ddot{H}_1\right) = \mathfrak{Acc}\left(\ddot{H}_2\right)$, then $\ddot{H}_1 \approx \ddot{H}_2$.

Example 4.3. Using the two pq-FFNs $\ddot{H}_1$ and $\ddot{H}_2$ in Example 3.8. Then, if $p = q = 3$, we have

$$\mathfrak{Sc}\left(\ddot{H}_1\right) = \frac{1}{2}\left(1 + (0.56)^{\frac{1}{3}} - (0.29)^{\frac{1}{3}}\right) = 0.5812,$$

$$\mathfrak{Sc}\left(\ddot{H}_2\right) = \frac{1}{2}\left(1 + (0.12)^{\frac{1}{3}} - (0.65)^{\frac{1}{3}}\right) = 0.3135.$$

Therefore, according to the Definition 4.2, $\ddot{H}_1 \succ \ddot{H}_2$. But if we have, for example, $\ddot{H}_3 = (0.73, 0.73)$ and $\ddot{H}_4 = (0.46, 0.46)$ to be two other

pq-FFNs. Then

$$\mathfrak{Sc}\left(\ddot{H}_3\right) = \frac{1}{2}\left(1 + (0.73)^{\frac{1}{3}} - (0.73)^{\frac{1}{3}}\right) = 0.5,$$

$$\mathfrak{Sc}\left(\ddot{H}_4\right) = \frac{1}{2}\left(1 + (0.46)^{\frac{1}{3}} - (0.46)^{\frac{1}{3}}\right) = 0.5.$$

Since $\mathfrak{Sc}\left(\ddot{H}_1\right) = \mathfrak{Sc}\left(\ddot{H}_2\right) = 0.5$, then for the comparison of these two pq-FFNs, the accuracy values of $\ddot{H}_1$ and $\ddot{H}_2$ are necessary to be determined in the manner described below:

$$\mathfrak{Acc}\left(\ddot{H}_3\right) = \frac{1}{2}\left((0.73)^{\frac{1}{3}} + (0.73)^{\frac{1}{3}}\right) = 0.9004,$$

$$\mathfrak{Acc}\left(\ddot{H}_4\right) = \frac{1}{2}\left((0.46)^{\frac{1}{3}} + (0.46)^{\frac{1}{3}}\right) = 0.7719.$$

Hence, according to the Definition 4.2, $\ddot{H}_1 \succ \ddot{H}_2$.

The following results establish the mathematical properties that ensure the score and accuracy functions are well-behaved tools for comparing pq-FFNs. These properties are essential for the reliability of the ranking methodology in decision-making applications.

Proposition 4.4. *Let $\ddot{H} = (\ddot{\mu}, \ddot{\nu})$ be a pq-FFN. Then*

(i) $0 \leq \mathfrak{Sc}(\ddot{H}) \leq 1$.
(ii) $0 \leq \mathfrak{Acc}(\ddot{H}) \leq 1$.

Proof. (i) Since $\ddot{\mu}, \ddot{\nu} \in [0,1]$ and $p, q \geq 1$, we have $\ddot{\mu}^{\frac{1}{p}} \in [0,1]$ and $\ddot{\nu}^{\frac{1}{q}} \in [0,1]$. Thus:

$$\mathfrak{Sc}(\ddot{H}) = \frac{1}{2}\left(1 + \ddot{\mu}^{\frac{1}{p}} - \ddot{\nu}^{\frac{1}{q}}\right) \geq \frac{1}{2}(1 + 0 - 1) = 0,$$

$$\mathfrak{Sc}(\ddot{H}) \leq \frac{1}{2}(1 + 1 - 0) = 1.$$

(ii) Similarly:

$$\mathfrak{Acc}(\ddot{H}) = \frac{1}{2}\left(\ddot{\mu}^{\frac{1}{p}} + \ddot{\nu}^{\frac{1}{q}}\right) \geq \frac{1}{2}(0 + 0) = 0,$$

$$\mathfrak{Acc}(\ddot{H}) \leq \frac{1}{2}(1 + 1) = 1. \qquad \square$$

Theorem 4.5. *Let $\ddot{H}_1 \preccurlyeq \ddot{H}_2$. Then*

$$\mathfrak{Sc}(\ddot{H}_1) \leq \mathfrak{Sc}(\ddot{H}_2).$$

If $\mathfrak{Sc}(\ddot{H}_1) = \mathfrak{Sc}(\ddot{H}_2)$, then

$$\mathfrak{Acc}(\ddot{H}_1) \leq \mathfrak{Acc}(\ddot{H}_2).$$

Proof. By Definition 3.5, $\ddot{H}_1 \preccurlyeq \ddot{H}_2$ means $\ddot{\mu}_1 \leq \ddot{\mu}_2$ and $\ddot{\nu}_1 \geq \ddot{\nu}_2$. Then

$$\mathfrak{Sc}(\ddot{H}_1) = \frac{1}{2}\left(1 + \ddot{\mu}_1^{\frac{1}{p}} - \ddot{\nu}_1^{\frac{1}{q}}\right) \leq \frac{1}{2}\left(1 + \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\nu}_2^{\frac{1}{q}}\right) = \mathfrak{Sc}(\ddot{H}_2).$$

If $\mathfrak{Sc}(\ddot{H}_1) = \mathfrak{Sc}(\ddot{H}_2)$, then

$$\ddot{\mu}_1^{\frac{1}{p}} - \ddot{\nu}_1^{\frac{1}{q}} = \ddot{\mu}_2^{\frac{1}{p}} - \ddot{\nu}_2^{\frac{1}{q}}.$$

Since $\ddot{\mu}_1 \leq \ddot{\mu}_2$ and $\ddot{\nu}_1 \geq \ddot{\nu}_2$, with at least one strict inequality (otherwise $\ddot{H}_1 = \ddot{H}_2$), the accuracy comparison follows from:

$$\mathfrak{Acc}(\ddot{H}_1) - \mathfrak{Acc}(\ddot{H}_2) = \frac{1}{2}\left(\ddot{\mu}_1^{\frac{1}{p}} - \ddot{\mu}_2^{\frac{1}{p}} + \ddot{\nu}_1^{\frac{1}{q}} - \ddot{\nu}_2^{\frac{1}{q}}\right) \leq 0,$$

since $\ddot{\mu}_1^{\frac{1}{p}} \leq \ddot{\mu}_2^{\frac{1}{p}}$ and $\ddot{\nu}_1^{\frac{1}{q}} \geq \ddot{\nu}_2^{\frac{1}{q}}$. □

5. *pq*-Fractional Fuzzy Aggregation Operators

Based on *pq*-FFSs, the *pq*-FF weighted average (*pq*-FFWA) and *pq*-FF weighted geometric (*pq*-FFWG) aggregations are initiated and applied to the designing decision-making we utilized.

Definition 5.1. Let $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ be a family of *pq*-FFNs. A mapping *pq*-FFWA: $\ddot{H}^n \to \ddot{H}$ is called a *pq*-FFWA operator if it satisfies

(5.1)

$$\begin{aligned} pq\text{-FFWA}\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) &= \bigoplus_{j=1}^{n} \omega_j \cdot \ddot{H}_j \\ &= \left(\left(1 - \prod_{j=1}^{n}\left(1 - \ddot{\mu}_j^{\frac{1}{p}}\right)^{\omega_j}\right)^{p}, \prod_{j=1}^{n} \ddot{\nu}_j^{\omega_j}\right), \end{aligned}$$

where ω_j is the weight of $\ddot{H}_j$, $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and the aggregate results are still *pq*-FFNs.

Idempotency, monotonicity and boundedness are some fundamental properties of aggregation operators which help in ensuring the reliability, the stability and interpretability in decision making systems. These properties give rise to stable, reliable, coherent aggregate results making the system of the present embodiment suitable for informed and effective decisions in a variety of fields and applications.

Theorem 5.2 (Idempotency)**.** *Let $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ be a family of pq-FFNs. If $\ddot{H}_j = \ddot{H}$, then*

$$pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) = \ddot{H}.$$

Proof. By assumption, we have that $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j) = (\ddot{\mu}, \ddot{\nu}) = \ddot{H}$ for all j; therefore,

$$\begin{aligned} pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) &= \bigoplus_{j=1}^{n} \omega_j \ddot{H}_j \\ &= \left(\left(1 - \prod_{j=1}^{n}\left(1 - \ddot{\mu}_j^{\frac{1}{p}}\right)^{\omega_j}\right)^{p}, \prod_{j=1}^{n} \ddot{\nu}_j^{\omega_j}\right) \\ &= \left(\left(1 - \left(1 - \ddot{\mu}^{\frac{1}{p}}\right)^{\sum_{j=1}^{n}\omega_j}\right)^{p}, \ddot{\nu}^{\sum_{j=1}^{n}\omega_j}\right). \end{aligned}$$

Since $\sum_{j=1}^{n} \omega_j = 1$, then

$$\begin{aligned} pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) &= \left(\left(1 - \left(1 - \ddot{\mu}^{\frac{1}{p}}\right)\right)^{p}, \ddot{\nu}\right) \\ &= (\ddot{\mu}, \ddot{\nu}) \\ &= \ddot{H}. \end{aligned}$$

□

Theorem 5.3 (Monotonicity)**.** *Suppose $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ and $\ddot{H}'_j = \left(\ddot{\mu}'_j, \ddot{\nu}'_j\right)$ are indexed families of pq-FFNs. Then, for all j, if $\ddot{H}_j \preccurlyeq \ddot{H}'_j$,*

$$pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq pq\text{-}FFWA\left(\ddot{H}'_1, \ddot{H}'_2, \ldots, \ddot{H}'_n\right).$$

Proof. Given families $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ & $\ddot{H}'_j = \left(\ddot{\mu}'_j, \ddot{\nu}'_j\right)$ of pq-FFNs. Since $\ddot{H}_j \preccurlyeq \ddot{H}'_j$, then we have

$$\begin{aligned} \ddot{\mu}_j \leq \ddot{\mu}'_j \quad &\Longrightarrow \quad 1 - (\ddot{\mu}_j)^{\frac{1}{p}} \geq 1 - \left(\ddot{\mu}'_j\right)^{\frac{1}{p}} \\ &\Longrightarrow \quad \prod_{j=1}^{n}\left(1 - (\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1 - \left(\ddot{\mu}'_j\right)^{\frac{1}{p}}\right)^{\omega_j} \\ &\Longrightarrow \quad \left(1 - \prod_{j=1}^{n}\left(1 - (\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j}\right)^{p} \leq \left(1 - \prod_{j=1}^{n}\left(1 - \left(\ddot{\mu}'_j\right)^{\frac{1}{p}}\right)^{\omega_j}\right)^{p}, \end{aligned}$$

which satisfies the inequality of the first components in $\ddot{H}$ and $\ddot{H}'$. Applying the same to the second component.
Since $\forall j = 1, 2, \ldots, n$, then it implies that

$$\ddot{\nu}_j \geq \ddot{\nu}'_j \quad \Longrightarrow \quad \prod_{j=1}^{n}(\ddot{\nu}_j)^{\omega_j} \geq \prod_{j=1}^{n}\left(\ddot{\nu}'_j\right)^{\omega_j}.$$

According to the above analysis, we obtain each component that we need, so

$$pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq pq\text{-}FFWA\left(\ddot{H}'_1, \ddot{H}'_2, \ldots, \ddot{H}'_n\right). \qquad \square$$

Theorem 5.4 (Boundedness)**.** *Let* $\ddot{H}^- = (\ddot{\mu}_{\min}, \ddot{\nu}_{\max})$ *and* $\ddot{H}^+ = (\ddot{\mu}_{\max}, \ddot{\nu}_{\min})$ *are the lower and upper bounds for pq-FFNs. Then*

$$\ddot{H}^- \preccurlyeq pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq \ddot{H}^+.$$

Proof. Since

$$\begin{aligned}
&\ddot{\mu}_{\min} \leq \ddot{\mu}_j \leq \ddot{\mu}_{\max}\\
&(\ddot{\mu}_{\min})^{\frac{1}{p}} \leq (\ddot{\mu}_j)^{\frac{1}{p}} \leq (\ddot{\mu}_{\max})^{\frac{1}{p}}\\
&1-(\ddot{\mu}_{\min})^{\frac{1}{p}} \geq 1-(\ddot{\mu}_j)^{\frac{1}{p}} \geq 1-(\ddot{\mu}_{\max})^{\frac{1}{p}}\\
&\prod_{j=1}^{n}\left(1-(\ddot{\mu}_{\min})^{\frac{1}{p}}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1-(\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j} \geq \prod_{j=1}^{n}\left(1-(\ddot{\mu}_{\max})^{\frac{1}{p}}\right)^{\omega_j}\\
&\left(1-(\ddot{\mu}_{\min})^{\frac{1}{p}}\right)^{\sum_{j=1}^{n}\omega_j} \geq \prod_{j=1}^{n}\left(1-(\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j} \geq \left(1-(\ddot{\mu}_{\max})^{\frac{1}{p}}\right)^{\sum_{j=1}^{n}\omega_j}
\end{aligned}$$

Since $\sum_{j=1}^{n}\omega_j = 1$, then

$$\begin{aligned}
-(\ddot{\mu}_{\min})^{\frac{1}{p}} &\geq -1+\prod_{j=1}^{n}\left(1-(\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j} \geq -(\ddot{\mu}_{\max})^{\frac{1}{p}}\\
(\ddot{\mu}_{\min})^{\frac{1}{p}} &\leq 1-\prod_{j=1}^{n}\left(1-(\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j} \leq (\ddot{\mu}_{\max})^{\frac{1}{p}}\\
\ddot{\mu}_{\min} &\leq \left(1-\prod_{j=1}^{n}\left(1-(\ddot{\mu}_j)^{\frac{1}{p}}\right)^{\omega_j}\right)^{p} \leq \ddot{\mu}_{\max}
\end{aligned}$$

and since

$$\begin{aligned}
&\ddot{\nu}_{\min} \leq \ddot{\nu}_j \leq \ddot{\nu}_{\max}\\
&\prod_{j=1}^{n}(\ddot{\nu}_{\min})^{\omega_j} \leq \prod_{j=1}^{n}\ddot{\nu}_j^{\omega_j} \leq \prod_{j=1}^{n}(\ddot{\nu}_{\max})^{\omega_j}\\
&(\ddot{\nu}_{\min})^{\sum_{1}^{n}\omega_j} \leq \prod_{j=1}^{n}\ddot{\nu}_j^{\omega_j} \leq (\ddot{\nu}_{\max})^{\sum_{1}^{n}\omega_j}.
\end{aligned}$$

Since $\sum_{j=1}^{n} \omega_j = 1$, then

$$\ddot{\nu}_{\min} \leq \prod_{j=1}^{n} \ddot{\nu}_j^{\omega_j} \leq \ddot{\nu}_{\max}.$$

As a result, the weighted average is situated in the middle of the bounds:

$$(\ddot{\mu}_{\min}, \ddot{\nu}_{\max}) \preccurlyeq \left(\left(1 - \prod_{j=1}^{n} \left(1 - (\ddot{\mu}_j)^{\frac{1}{p}} \right)^{\omega_j} \right)^{p}, \prod_{j=1}^{n} \ddot{\nu}_j^{\omega_j} \right) \preccurlyeq (\ddot{\mu}_{\max}, \ddot{\nu}_{\min}).$$

Thus

$$\ddot{H}^{-} \preccurlyeq pq\text{-}FFWA\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq \ddot{H}^{+}. \qquad \square$$

Definition 5.5. Let $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ be a family of pq-FFNs. A mapping pq-FFWG: $\ddot{H}^n \to \ddot{H}$ is called a pq-FFWG operator if it satisfies
(5.2)

$$pq\text{-FFWG}\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) = \bigotimes_{j=1}^{n} \ddot{H}_j^{\omega_j}$$
$$= \left(\prod_{j=1}^{n} \ddot{\mu}_j^{\omega_j}, \left(1 - \prod_{j=1}^{n} \left(1 - \ddot{\nu}_j^{\frac{1}{q}} \right)^{\omega_j} \right)^{q} \right),$$

where ω_j is the weight of $\ddot{H}_j$, $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$ and the aggregate results are still pq-FFNs.

Remark 5.6. By Proposition 3.9 and mathematical induction, we can conclude that the aggregated values of the pq-FFWA and pq-FFWG operators are pq-FFNs.

Theorem 5.7 (Idempotency)**.** *Let $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ be a family of pq-FFNs. If $\ddot{H}_j = \ddot{H}$, then*

$$pq\text{-}FFWG\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) = \ddot{H}.$$

Proof. The proof is similar to that of Theorem 5.2. $\square$

Theorem 5.8 (Monotonicity)**.** *Suppose $\ddot{H}_j = (\ddot{\mu}_j, \ddot{\nu}_j)$ and $\ddot{H}'_j = \left(\ddot{\mu}'_j, \ddot{\nu}'_j\right)$ are indexed families of pq-FFNs. Then, for all j, if $\ddot{H}_j \preccurlyeq \ddot{H}'_j$,*

$$pq\text{-}FFWG\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq pq\text{-}FFWG\left(\ddot{H}'_1, \ddot{H}'_2, \ldots, \ddot{H}'_n\right).$$

Proof. The proof is similar to that of Theorem 5.3. $\square$

Theorem 5.9 (Boundedness)**.** *Let* $\ddot{H}^- = (\ddot{\mu}_{\min}, \ddot{\nu}_{\max})$ *and* $\ddot{H}^+ = (\ddot{\mu}_{\max}, \ddot{\nu}_{\min})$ *are the lower and upper bounds for pq-FFNs. Then*

$$\ddot{H}^- \preccurlyeq pq\text{-}FFWG\left(\ddot{H}_1, \ddot{H}_2, \ldots, \ddot{H}_n\right) \preccurlyeq \ddot{H}^+$$

Proof. The proof is similar to that of Theorem 5.4. □

6. The New MCDM Approaches Based on *pq*-FF Aggregation Operators

Multi criteria decision-making (MCDM) is an important technique to solve the problems with several criteria, in most cases conflicting. For effectively managing the uncertainty and vagueness in the real-world decision making problems, fuzzy set theory has been widely studied. However, classical fuzzy sets are not always able to represent complex uncertainty structures.

The idea of *pq*-fractional fuzzy sets generalize classical fuzzy sets by employing fractional q-calculus and provides a more versatile approach to deal with uncertainty by taking a control over the sense and solution while for the fractional order by fractional q-differences. It gives a better way to describe membership degrees in situations of uncertain or imprecise data.

In everyday decision making, it is not unusual to have more than one criteria to consider, however these must be aggregated in order to have an assessment of the alternatives. The aggregation operators therefore have an important function in the process by combining the individual fuzzy evaluations with respect to the criteria into a collective value. These operators represent a good tool for *pq*-fractional fuzzy sets to make advantages of the additional modeling features as well as it enables to consider different criteria simultaneously.

To overcome the ranking and selection problem among the alternatives, a score function is used to transform the collected fuzzy data into crisp scores. This conversion allows for a direct comparison of alternatives and their priority in relation to their overall performance.

Algorithm 1: Proposed Operator Method.

The methodological procedure for the proposed operator, which is schematically illustrated in the flowchart presented in Figure 2, is described in this subsection step by step to provide a clear understanding of the computational sequence and logical structure adopted in the model.

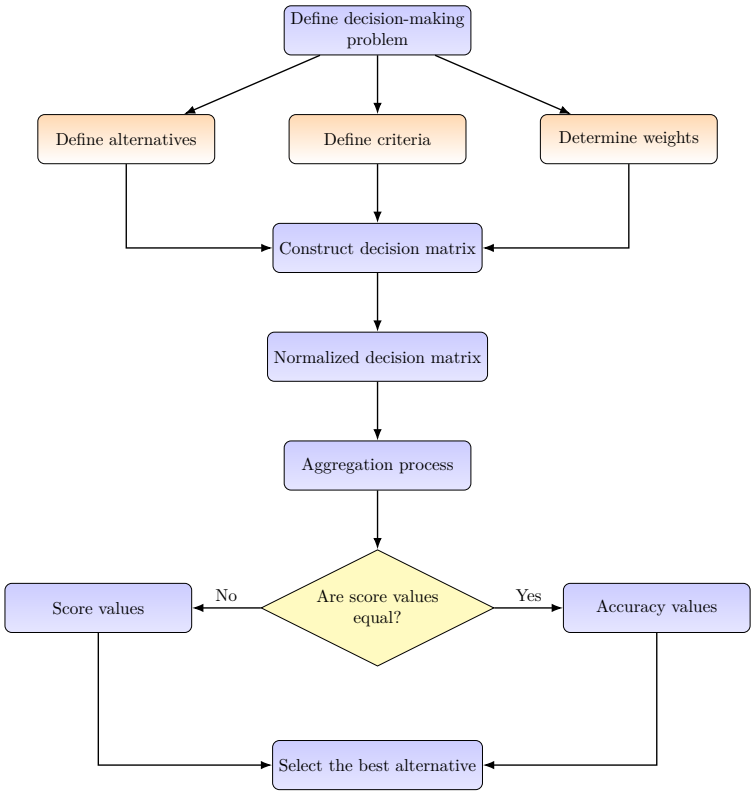

FIGURE 2. Flowchart for the Proposed Model.

(i) We use n criteria to evaluate the options. These q-FF decision matrices are generated by the final values of the alternatives with respect to each criterion as follows:

$$\Psi = \begin{pmatrix} \ddot{\mathcal{H}}_1 \\ \ddot{\mathcal{H}}_2 \\ \vdots \\ \ddot{\mathcal{H}}_m \end{pmatrix} = \begin{pmatrix} \ddot{H}_{11} & \ddot{H}_{12} & \dots & \ddot{H}_{1n} \\ \ddot{H}_{21} & \ddot{H}_{22} & \dots & \ddot{H}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{H}_{m1} & \ddot{H}_{m2} & \dots & \ddot{H}_{mn} \end{pmatrix} \tag{6.1}$$

$$= \begin{pmatrix} (\ddot{\mu}_{11}, \ddot{\nu}_{11}) & (\ddot{\mu}_{12}, \ddot{\nu}_{12}) & \dots & (\ddot{\mu}_{1n}, \ddot{\nu}_{1n}) \\ (\ddot{\mu}_{21}, \ddot{\nu}_{21}) & (\ddot{\mu}_{22}, \ddot{\nu}_{22}) & \dots & (\ddot{\mu}_{2n}, \ddot{\nu}_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\ddot{\mu}_{m1}, \ddot{\nu}_{m1}) & (\ddot{\mu}_{m2}, \ddot{\nu}_{m2}) & \dots & (\ddot{\mu}_{mn}, \ddot{\nu}_{mn}) \end{pmatrix},$$

where $\ddot{\mu}_{ij} \in [0,1]$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, demonstrates the satisfaction degree of alternatives $\ddot{\mathcal{H}}_j$ under the criteria $\ddot{E}_j$ and $\ddot{\nu}_{ij} \in [0,1]$ demonstrates the dissatisfaction degree.

(ii) The decision maker assigns a specific weight to each criterion to satisfy the normalization requirement, that is, the sum of all assigned weights must equal one. Accordingly, the weight vector is represented as $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ where

$$\sum_{j=1}^{n} \omega_j = 1.$$

This ensures that each criterion contributes proportionally to the overall decision process based on its relative importance, thereby maintaining consistency and fairness in the evaluation.

(iii) Normalization is necessary when the evaluation criteria are measured on different scales. This step ensures comparability across criteria by converting the original data into a uniform range. Accordingly, the decision matrix is normalized based on the nature of each criterion as shown below:

$$\left[\ddot{H}_{ij}\right]_{n\times m} \begin{cases} (\mu_{ij}, \nu_{ij}) & \text{for benefit criteria,} \\ (\nu_{ij}, \mu_{ij}) & \text{for non-benefit criteria.} \end{cases}$$

This procedure guarantees that higher values consistently indicate better performance for benefit criteria, while for non-benefit (cost-type) criteria, the transformation reverses the scale to maintain interpretive consistency across all attributes.

(iv) The calculation of the pq-FFWA or pq-FFWG operator is performed via $\ddot{H}_i$ to aggregate the individual fuzzy evaluations into a single representative value that reflects the overall performance of each alternative, where

$$\ddot{H}_i = (\ddot{\mu}_i, \ddot{\nu}_i) = pq\text{-FFWA}\left(\ddot{H}_{i1}, \ddot{H}_{i2}, \dots, \ddot{H}_{in}\right)$$

is achieved by Equation (5.1), or

$$\ddot{H}_i = (\ddot{\mu}_i, \ddot{\nu}_i) = pq\text{-FFWG}\left(\ddot{H}_{i1}, \ddot{H}_{i2}, \dots, \ddot{H}_{in}\right)$$

is achieved by Equation (5.2).

(v) Calculate the score function of $\ddot{H}_i$ $\left(\mathfrak{Sc}(\ddot{H}_i)\right)$ using Equation (4.1).
(vi) Rank the alternatives $\ddot{\mathcal{H}}_i$, in descending order, based on the score values $\mathfrak{Sc}(\ddot{H}_i)$.

7. Numerical Example

Subheading for small businesses need e-commerce but which platform? Different platforms for different user needs: there are many platforms which serve user needs with different levels of customization, usability and scalability. Below is some insightful information about four e-commerce platforms:

• $\ddot{\mathcal{H}}_1$: Shopify is a complete hosted e-commerce solution that is suitable for businesses of any size. Popular for being user-friendly, it helps business owners launch a web store within just a few minutes – even if you do not know how to code. Shopify provides custom templates, secure payment gateways and marketing and analytics that are already buit-in to the platform. It has an app store that adds features such as support for dropshipping, SEO (Search Engine Optimization) and social selling. Shopify, for example, does charge monthly fees and transaction fees, but works for most startups as well as big brands.

• $\ddot{\mathcal{H}}_2$: WooCommerce, free, as an open-source e-commerce plugin for WordPress, allows full control over the design and functionality of your platform. Perfect for businesses already using WordPress, it can turn any site into a beautiful storefront. Users can handle their products, payments and shipping and take advantage of WordPress's SEO and blogging features. But WooCommerce is self-hosted so it does leave security, updates and performance optimization to the users. Its elasticity is appealing to tech-savvy merchants who like low upfront costs, but are looking to upsize.

• $\ddot{\mathcal{H}}_3$: BigCommerce is a powerful e-commerce platform for fast-growing and large-volume brands. Fewer bells and whistles than easy to use Shopify for some, but it does offer advanced features including multi-channel sales (Amazon, eBay, Instagram, Facebook), built-in SEO tools and headless commerce options straight off the shelf. BigCommerce takes care of hosting, security and performance, though beginners will find a steeper learning curve. It is perfect for brands that need to scale without being charged transaction fees (except on third-party gateways).

• $\ddot{\mathcal{H}}_4$: Wix eCommerce is a free, user-friendly website builder that enables small businesses and creatives to create and sell on their site. The drag and drop editor makes it easy for store owners from all walks of like to design a website and tools like abandoned cart recovery and

dropshipping support business growth. It does not offer the advanced e-commerce features you get with Shopify or BigCommerce, but Wix is easy to use indeed. Keep in mind that you cannot switch templates after you have gone live, so some forethought is mandatory.

The above four platforms are compared using the following four criteria:

• $\ddot{E}_1$: Usability: Shopify and Wix eCommerce are the easiest platforms to use especially for newbies. Shopify provides a neat and intuitive Dashboard through which store creation, managing products and orders could be done very easily which are suitable for non-technical users who want everything to be straightforward. Wix eCommerce goes a step further than other site builders with its drag and drop website builder, making it easy for users to create visually stunning storefronts. But Wix's ease of use comes at the expense of some flexibility, because once you pick a template, you cannot switch to another one without starting over. WooCommerce is extremely flexible however will take on a bit more of a separation learning curve as its primarily a WordPress plugin. Users have to handle hosting, security and updates themselves, so its best for those who know WordPress or are willing to hire developers. BigCommerce balances advanced capabilities with ease of use, although its enterprise-focused tools may be too much for complete beginner.

• $\ddot{E}_2$: Features: BigCommerce is the winner for built-in features and can accomplish advanced needs, such as multi-channel sales, B2B (Business-to-Business) sales and headless commerce, without third-party apps. That makes it an excellent option for businesses that are scaling. Shopify is not far behind, thanks to an extensive app store that adds helpful apps to enhance the platform such as dropshipping integration, abandoned cart recovery and SEO tools. But some advanced functions are paid. Whether by using plugins or themes, WooCommerce is the top performer for personalization, as you can customize every single part of your shop. But then, because it is self-hosted, you need to muck around with things like payments and security too. Wix eCommerce includes the basics for selling, while not as advanced as Shopify or BigCommerce it offers enough for smaller stores to get started with and is more cost-effective.

• $\ddot{E}_3$: Pricing: WooCommerce is technically free as an open-source plugin, but with hosting, premium themes and extensions, the costs pile up. This is what makes it an affordable choice for small stores but at the same time an expensive choice for feature-rich deployments. Wix eCommerce is a good value at the entry-level, but is also an odd expensive at the higher tiers when stacked against the competition. Shopify makes

its fees fairly clear, but it does charge a transaction fee unless you use Shopify Payments. Its plans are suitable for all business sizes, although add-ons can really ramp up costs. BigCommerce has no transaction fees, but can be more expensive, catering to midsize to large businesses. Its premium features can justify the price for emerging brands and are too much for bootstrapped startups.

• $\ddot{E}_4$: Customer Support: Shopify and BigCommerce shine in the customer service department with 24/7 live support through chat, email and phone-a huge perk for businesses that need help right away. Shopify Plus also offers a huge knowledge center and forums to support to augment its support ecosystem. Wix eCommerce has decent support featuring callbacks, but phone support is not available for its lowest price plans. Its support documentation is user-friendly although somewhat shallow compared to Shopify. While WooCommerce is a self-hosted solution, it depends on its community discussions and third-party developers for assistance, you may experience delays if you are not a techie.

In this part, we compare the four platforms $(\ddot{\mathcal{H}}_1, \ddot{\mathcal{H}}_2, \ddot{\mathcal{H}}_3, \ddot{\mathcal{H}}_4)$ through pq-FFSs as a more sophisticated and a better suited method for a comparison by accounting between the partial membership degrees and vagueness in decision analysis. The above four main criteria $(\ddot{E}_1, \ddot{E}_2, \ddot{E}_3, \ddot{E}_4)$ are evaluated via pq-fractional aggregations, where $p = 2$ and $q = 3$, to identify the most suitable platform given vague and subjective information.

Figure 3 presents the criterion weights, reflecting their relative importance in business decision-making:

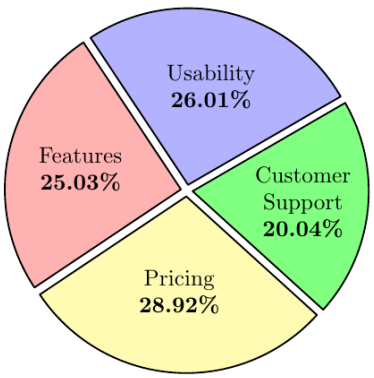

FIGURE 3. The Numerical Weights Indicating Criteria Priorities.

- Usability (26.01%): Essential for rapid deployment and a simple process of operation.
- Features (25.03%): This is the core feature set which will enable technologies to scale in the long term and be functional.
- Pricing (28.92%): Impact both affordability and return on investment.
- Customer Support (20.04%): Necessary for break/fix and expansion.

Using pq-fractional fuzzy weighted aggregation, we apply the following structured algorithm to identify the best e-commerce platform based on the specified criteria:

Algorithm: Proposed Operator Method
The steps of the suggested operator are given in this part as follows:

(i) To demonstrate the proposed method, a simulated dataset is employed to ensure a controlled, transparent and replicable step-by-step illustration of the aggregation and ranking process, as shown in Table 2.

TABLE 2. pq-FF Assessment Matrix.

$\ddot{\mathcal{H}}_j$	Usability $\ddot{E}_1$	Features $\ddot{E}_2$	Pricing $\ddot{E}_3$	Customer Support $\ddot{E}_4$
$\ddot{\mathcal{H}}_1$	$(0.58, 0.33)$	$(0.56, 0.35)$	$(0.63, 0.27)$	$(0.87, 0.07)$
$\ddot{\mathcal{H}}_2$	$(0.69, 0.25)$	$(0.85, 0.19)$	$(0.64, 0.22)$	$(0.75, 0.11)$
$\ddot{\mathcal{H}}_3$	$(0.49, 0.36)$	$(0.62, 0.34)$	$(0.45, 0.06)$	$(0.57, 0.41)$
$\ddot{\mathcal{H}}_4$	$(0.79, 0.25)$	$(0.74, 0.18)$	$(0.81, 0.09)$	$(0.66, 0.23)$

(ii) The weights for the criteria $\ddot{E}_j$ is given as $\Omega = [\omega_1 \ \ \omega_2 \ \ \omega_3 \ \ \omega_4]$ $= [0.2601 \ \ 0.2503 \ \ 0.2892 \ \ 0.2004]$ such that $\sum_{j=1}^{4} \omega_j = 1$.

(iii) Criteria are not in the same scale and because of that normalizing in the third criterion (Pricing E_3), as it is a non-benefit criterion, is required here which is shown in Table 3:

TABLE 3. Normalized pq-FF Assessment Matrix.

$\ddot{\mathcal{H}}_j$	Usability $\ddot{E}_1$	Features $\ddot{E}_2$	Pricing $\ddot{E}_3$	Customer Support $\ddot{E}_4$
$\ddot{\mathcal{H}}_1$	$(0.69, 0.25)$	$(0.85, 0.19)$	$(0.22, 0.64)$	$(0.75, 0.11)$
$\ddot{\mathcal{H}}_2$	$(0.79, 0.25)$	$(0.74, 0.18)$	$(0.09, 0.81)$	$(0.66, 0.23)$
$\ddot{\mathcal{H}}_3$	$(0.58, 0.33)$	$(0.56, 0.35)$	$(0.27, 0.63)$	$(0.87, 0.07)$
$\ddot{\mathcal{H}}_4$	$(0.49, 0.36)$	$(0.62, 0.34)$	$(0.06, 0.45)$	$(0.57, 0.41)$

(iv) By using Definition 5.1, pq-FFWA values $\ddot{H}_i$ (for $i = 1, 2, 3, 4$), where $p = 2$ and $q = 3$, is calculated as

$$\ddot{H}_1 = (0.6238, 0.2959), \qquad \ddot{H}_2 = (0.6886, 0.2598),$$
$$\ddot{H}_3 = (0.4566, 0.3885), \qquad \ddot{H}_4 = (0.6276, 0.3181).$$

(v) Using 4.1, the score function of $\ddot{H}_i$, $\mathfrak{Sc}(\ddot{H}_i)$ for $i = 1, 2, \ldots m$, is created as

$$\mathfrak{Sc}\left(\ddot{H}_1\right) = 0.5617, \qquad \mathfrak{Sc}\left(\ddot{H}_2\right) = 0.5958,$$
$$\mathfrak{Sc}\left(\ddot{H}_3\right) = 0.4730, \qquad \mathfrak{Sc}\left(\ddot{H}_4\right) = 0.5548.$$

(vi) According to the score values above, the alternative rank is

$$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3.$$

Out of all the possibilities taken into consideration, $\ddot{\mathcal{H}}_2$, as shown is Figure 4, is the best alternative according to the evaluation criteria and ranking method.

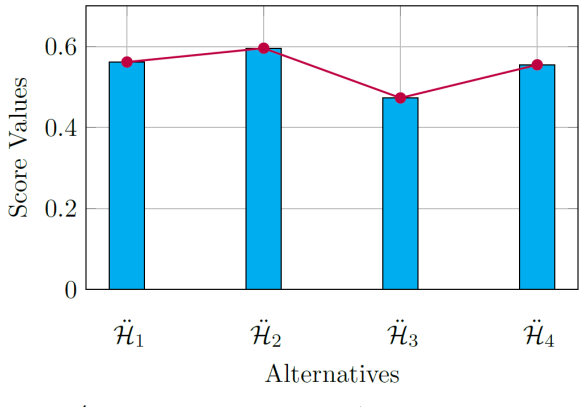

FIGURE 4. pq-FF Model Ranking (where $p = 2$ and $q = 3$).

Remark 7.1. The numerical example employs simulated data to ensure full transparency and reproducibility of the computational steps. This is standard practice for foundational theoretical contributions in fuzzy set theory (see, e.g., Gulistan and Pedrycz [13] for q-FFS, Yager [44] for q-ROFS). The e-commerce platform selection context is drawn from established MCDM literature [10, 23, 39, 40] and the simulated values are constructed to satisfy pq-FFS constraints while representing realistic decision scenarios. Real-world expert data would introduce unverifiable subjective biases and confounding variables, obscuring whether observed

outcomes derive from the model structure or from idiosyncratic data properties. The sensitivity analysis (Section 8) and comparative study (Section 9) provide stronger validation by demonstrating robustness and relative superiority under controlled, identical conditions.

8. Sensitivity Analysis

To investigate how sensitive our e-commerce platform ranking is with respect to fractional aggregation we will perform a sensitivity test by systematically changing fractional parameters p and q in the pq-fractional fuzzy aggregation operator. These parameters are key levers in our decision model which modulates the level of nonlinear compensation between competing criteria and ultimately determines how trade-offs impact the final evaluation. The parameters p and q characterize to what extent higher than average performance in one dimension can mitigate the impact of lower than average performance in another and its study is crucial for understanding the robustness of our platform recommendations to differing decision-making perspectives.

Table 4 and Figure 5 present the pq-FFWA score values and ranking outcomes for all four e-commerce platform alternatives across fourteen distinct (p, q) parameter configurations, demonstrating that the ranking order $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ remains invariant and $\ddot{\mathcal{H}}_2$ (WooCommerce) is consistently identified as the best alternative regardless of parameter variations; the score magnitudes decrease as either p or q increases, ranging from 0.5958 at $(2, 3)$ down to 0.4975 at $(20, 2000)$ for $\ddot{\mathcal{H}}_2$, reflecting the asymptotic convergence of $\ddot{\mu}^{1/p}$ and $\ddot{\nu}^{1/q}$ toward 1 and 0 respectively, yet the relative separations between alternatives are preserved such that even at extreme parameters like $(2000, 20)$ the margin between $\ddot{\mathcal{H}}_2$ (0.5322) and $\ddot{\mathcal{H}}_1$ (0.5290) remains positive and decisionally meaningful, confirming the robustness of the pq-FFWA framework against parameter sensitivity.

Table 4: pq-FFWA Ranking Results and Optimal Alternatives with Different Values of p and q.

(p, q)	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$(2, 3)$	0.5617	0.5958	0.4730	0.5548	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(3, 2)$	0.6397	0.6746	0.5546	0.6330	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(2, 10)$	0.4799	0.5	0.4204	0.4751	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(10, 2)$	0.6891	0.7145	0.6287	0.681	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(3, 25)$	0.4776	0.4889	0.4427	0.4752	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(25, 3)$	0.6481	0.6662	0.6056	0.6405	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$

(p,q)	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$(5,15)$	0.5012	0.5127	0.4683	0.4985	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(15,5)$	0.5844	0.5994	0.5479	0.579	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(5,100)$	0.484	0.489	0.4679	0.4832	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(100,5)$	0.6019	0.6132	0.5761	0.5962	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(25,200)$	0.4971	0.4987	0.4924	0.4969	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(200,25)$	0.5213	0.5243	0.5145	0.5199	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(20,2000)$	0.4968	0.4975	0.4943	0.4967	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(2000,20)$	0.529	0.5322	0.5222	0.5273	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$

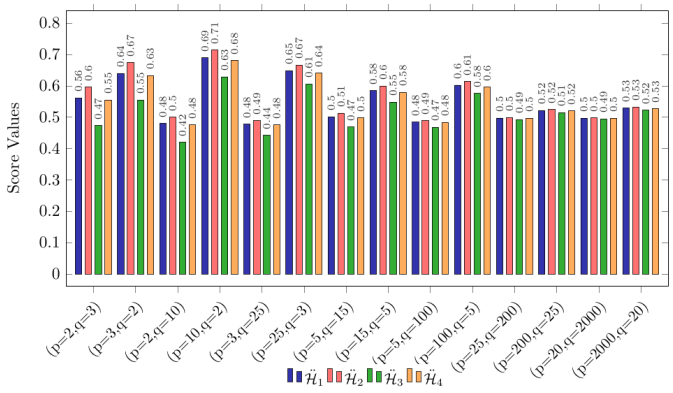

FIGURE 5. pq-FFWA Ranking Results and Optimal Alternatives with Different Values of p and q.

Concentrating on establishing p and q as equal is demonstrated in Table 5 which is the surprisingly stable position of the ranking order for all $p = q$ tested values: the rank ordering is always $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$.

TABLE 5. pq-FFWA Ranking Results and Optimal Alternatives with Equal Values of p and q.

pq-FFWA	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$p = q = 2$	0.6132	0.6525	0.5158	0.6067	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 3$	0.586	0.6161	0.5095	0.5794	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 4$	0.5689	0.5932	0.5065	0.5629	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 5$	0.5574	0.5777	0.5049	0.552	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 10$	0.5311	0.5422	0.502	0.5278	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 50$	0.5066	0.509	0.5003	0.5059	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 200$	0.5017	0.5023	0.5001	0.5015	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$p = q = 2000$	0.50017	0.50023	0.5	0.5001	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$

This robustness of the pq-FFWA operator, see Figure 6, gives a very solid hint that it is not sensitive to the choice of the specific p and q values in the considered interval, noting that the structure of the proposed pq-FFWA is different from that of aggregation based on q-FFSs (i.e., q-FFWA). In decision-making, it is an extremely desirable quality because it would suggest that changing the values of parameters slightly or a lot does not cause the decision to change. It also further supports the concept of method stability and robustness, since despite significant variation in the values of the parameters, the mean optimal option is scarcely affected. Such convergence could give decision makers confidence in the outputs of the pq-FFWA method, while weakening the parameter sensitivity of the method without sacrificing the results. This demonstrates the method's potential for dependable use in intricate decision-making settings.

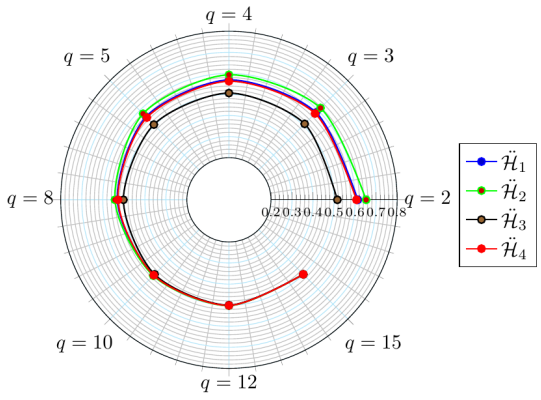

FIGURE 6. pq-FFWA Ranking Results and Optimal Alternatives with Equal Values of p and q.

9. Comparative Analysis

In this part, a comparative analysis is made to discuss the performances of the proposed pq-fractional fuzzy weighted average operator (pq-FFWA) with the existing ones including q-fractional fuzzy weighted average (q-FFWA) and q-rung orthopair fuzzy weighted average (q-ROFWA). The aim is to explore the influence of parameter p and q on ranking of the alternatives under fuzzy decision-making.

Comparing the proposed operator with q-fractional fuzzy weighted average (q-FFWA) [13] is shown in Table 6, which demonstrates perfect ranking stability across all tested q values: the ranking order $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ and the best alternative $\ddot{\mathcal{H}}_2$ remain invariant for $q \in$

$\{2, 3, 4, 5, 10, 50, 200, 2000\}$. However, the score values collapse exponentially toward zero as q increases (from 0.1468 at $q = 2$ to 0.00014 at $q = 2000$), indicating that while the ranking is stable, the discriminative power vanishes. This reflects the model's robustness but also its limitation: the uniform parameter q cannot capture asymmetric uncertainty structures. Compared with this, the pq-FFWA model maintains comparable ranking stability (see Table 9) while preserving meaningful score separation through independent control of p and q. The q-FFWA scores are a special case of pq-FFWA when $p = q$ and the perfect agreement between the two methods at matching parameters (Table 11) confirms backward compatibility.

Table 6: q-FFWA Ranking Results and Optimal Alternatives with Different Values of q.

q-FFWA	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$q = 2$	0.1314	0.1468	0.0191	0.0804	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 3$	0.0855	0.0969	0.0109	0.0535	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 4$	0.0635	0.0723	0.0075	0.04	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 5$	0.0505	0.0577	0.0057	0.032	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 10$	0.025	0.0287	0.0026	0.0159	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 50$	0.005	0.0057	0.0005	0.0032	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 200$	0.0012	0.0014	0.0001	0.0008	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 2000$	0.00012	0.00014	1.2×10^{-5}	7.9×10^{-5}	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$

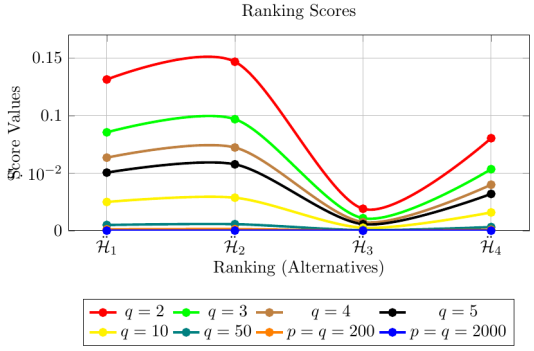

FIGURE 7. q-FFWA Ranking of Alternatives Based on Equal Values of p and q.

In order to thoroughly assess the effectiveness of the suggested operator, we perform a thorough comparison with the q-rung orthopair fuzzy weighted average q-ROFWA [21]. Table 7 shows that the parameter q was systematically changed from $q = 1$ (IFWA [8]), $q = 2$ (PFWA [29]), $q = 3$ (FFWA [32]) to $q = 2000$. The results show robust rankings

for small to moderate q ($q = 1, 2, 3, 4, 5, 7, 10$), where $\ddot{\mathcal{H}}_2$ is consistently best. However, at extreme q, the ranking collapses: at $q = 15$, $\ddot{\mathcal{H}}_1$ marginally surpasses $\ddot{\mathcal{H}}_2$; at $q = 50$, scores decay to 10^{-5}–10^{-11}; at $q = 200$, scores reach -8×10^{-83} and -3×10^{-100} within the valid $[-1, 1]$ domain of $\mathfrak{Sc} = \ddot{\mu}^q - \ddot{\nu}^q$ but reflect total loss of discrimination; at $q = 2000$, all scores become 0 as shown in Figure 8, rendering complete indifference. By contrast, pq-FFWA (Table 9) maintains perfect stability and meaningful score separation even at $(p, q) = (20, 2000)$, since $\mathfrak{Sc} = 1/2(1 + \ddot{\mu}^{1/p} - \ddot{\nu}^{1/q})$ uses roots that preserve bounded discrimination, whereas q-ROFS powers $\ddot{\mu}^q, \ddot{\nu}^q \to 0$ destroy it.

Table 7: q-ROFWA Ranking Results and Optimal Alternatives with Different Values of q.

q-ROFWA	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$q = 1$ $(IFWA)$	0.3099	0.4168	0.065	0.309	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 2$ $(PFWA)$	0.3096	0.4203	0.0877	0.3333	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$q = 3$ $(FFWA)$	0.2522	0.3479	0.0736	0.2804	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$q = 4$	0.1971	0.2755	0.0521	0.2185	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$q = 5$	0.1545	0.2165	0.0341	0.1659	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$q = 7$	0.0996	0.1354	0.0133	0.094	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 10$	0.0574	0.0703	0.003	0.0408	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 15$	0.0263	0.0262	0.0002	0.0108	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 50$	0.0002	$7 \cdot 10^{-5}$	$1 \cdot 10^{-11}$	$2 \cdot 10^{-6}$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 200$	$2 \cdot 10^{-13}$	$2 \cdot 10^{-15}$	$-8 \cdot 10^{-83}$	$-3 \cdot 10^{-100}$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$q = 2000$	0	0	0	0	$\ddot{\mathcal{H}}_1 \approx \ddot{\mathcal{H}}_2 \approx \ddot{\mathcal{H}}_4 \approx \ddot{\mathcal{H}}_3$

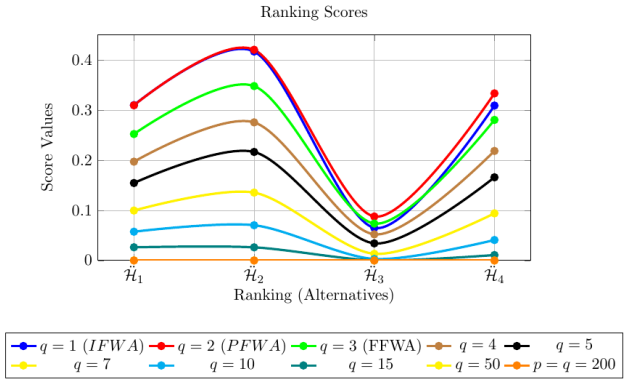

FIGURE 8. q-ROFWA Ranking of Alternatives Based on Equal Values of p and q.

To rigorously evaluate the performance of the proposed operator, we also conduct a comprehensive comparative analysis against the pq-ROFWA. Table 8 shows that the parameters (p, q) were systematically

changed across asymmetric pairs. The results show robust rankings for small-to-moderate pairs, where $\ddot{\mathcal{H}}_2$ is consistently best. However, at extreme pairs, the ranking collapses: at $(25, 200)$, $\ddot{\mathcal{H}}_1$ surpasses $\ddot{\mathcal{H}}_2$; at $(200, 25)$ and $(2000, 20)$, all scores converge to 0.5 with complete indifference as shown graphically in Figure 9. By contrast, pq-FFWA (Table 4) maintains perfect stability with the fixed ordering $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ and meaningful score separation even at $(2000, 20)$, since $\mathfrak{Sc} = \frac{1}{2}(1 + \ddot{\mu}^{\frac{1}{p}} - \ddot{\nu}^{\frac{1}{q}})$ uses fractional roots that preserve bounded discrimination, whereas pq-ROFS powers destroy it under extreme asymmetry.

Table 8: pq-ROFWA Ranking Results with Different Values of p and q.

(p, q)	$\ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2$	$\ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4$	Ranking Order
$(2, 3)$	0.6807	0.7322	0.5896	0.7005	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(3, 2)$	0.601	0.6525	0.491	0.6064	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(2, 10)$	0.6919	0.7402	0.6189	0.7166	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(10, 2)$	0.4951	0.508	0.4266	0.4709	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(3, 25)$	0.637	0.6817	0.5661	0.6562	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(25, 3)$	0.493	0.495	0.4707	0.4844	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(5, 15)$	0.5782	0.6088	0.5215	0.5846	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(15, 5)$	0.5122	0.5126	0.4957	0.5038	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(5, 100)$	0.5782	0.6088	0.5215	0.5846	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$
$(100, 5)$	0.4991	0.4995	0.4956	0.4984	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(25, 200)$	0.5031	0.5023	0.5	0.5004	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(200, 25)$	0.5	0.5	0.5	0.5	$\ddot{\mathcal{H}}_2 \approx \ddot{\mathcal{H}}_1 \approx \ddot{\mathcal{H}}_4 \approx \ddot{\mathcal{H}}_3$
$(20, 2000)$	0.5063	0.5053	0.5	0.5015	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$
$(2000, 20)$	0.5	0.5	0.5	0.5	$\ddot{\mathcal{H}}_2 \approx \ddot{\mathcal{H}}_1 \approx \ddot{\mathcal{H}}_4 \approx \ddot{\mathcal{H}}_3$

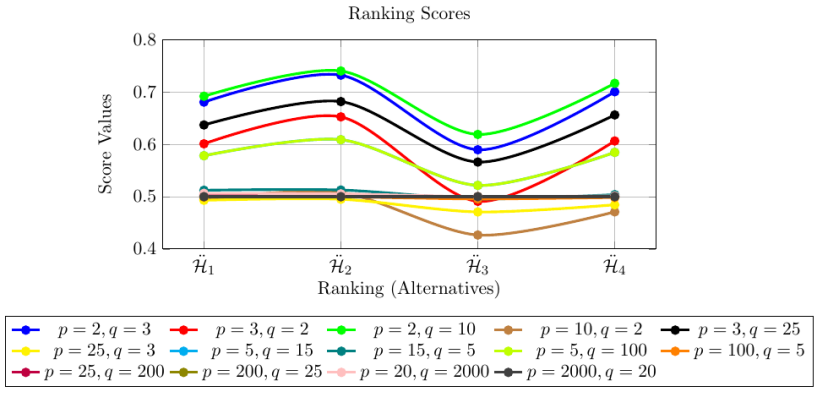

FIGURE 9. pq-ROWA Ranking of Alternatives Based on Equal Values of p and q.

Remark 9.1. The fundamental difference between the pq-FFWA and q-ROFWA score functions explains their contrasting stability under extreme parameter values. The q-ROFWA score function $\mathfrak{Sc}(\ddot{H}) = \ddot{\mu}^q - \ddot{\nu}^q$ employs *powers* of membership and non-membership degrees. As $q \to \infty$, both $\ddot{\mu}^q \to 0$ and $\ddot{\nu}^q \to 0$ for $\ddot{\mu}, \ddot{\nu} \in [0, 1)$, causing the score to decay exponentially toward zero and destroying all discriminative power between alternatives. At extreme values such as $q = 200$, numerical underflow produces negative scores (-8×10^{-83} and -3×10^{-100}), leading to arbitrary ranking inversion where the best alternative is misidentified. At $q = 2000$, all scores collapse to exactly zero, rendering complete indifference.

By contrast, the pq-FFWA score function $\mathfrak{Sc}(\ddot{H}) = \frac{1}{2}\left(1 + \ddot{\mu}^{\frac{1}{p}} - \ddot{\nu}^{\frac{1}{q}}\right)$ employs *fractional roots* rather than powers. Since $\ddot{\mu}^{1/p}, \ddot{\nu}^{1/q} \in [0, 1]$ for all $p, q \geq 1$, the score remains bounded in $[0, 1]$ with stable, meaningful separation regardless of parameter magnitude. Even at $(p, q) = (2000, 20)$, the terms $\ddot{\mu}^{1/2000}$ and $\ddot{\nu}^{1/20}$ preserve sufficient discriminative information to maintain the fixed ranking order $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ with non-degenerate score separation. This structural advantage, roots preserving bounded information versus powers annihilating it, is inherent to the linear fractional constraint of pq-FFS and cannot be replicated by exponential orthopair frameworks.

9.1. **Quantitative Ranking Stability Measures.** To rigorously assess the stability of the pq-FFWA ranking under parameter variations, we employ Spearman's rank correlation coefficient (ρ) and Kendall's rank correlation coefficient (τ). These non-parametric measures quantify the degree of agreement between rankings obtained from different (p, q) configurations, with values close to 1 indicating near-perfect stability.

Let $R_{(p_i,q_i)}$ and $R_{(p_j,q_j)}$ denote the rankings of alternatives under parameter pairs (p_i, q_i) and (p_j, q_j), respectively. For $m = 4$ alternatives, Spearman's ρ is defined as:

$$\rho = 1 - \frac{6 \sum_{k=1}^{m} d_k^2}{m(m^2 - 1)},$$

where $d_k = R_{(p_i,q_i)}(k) - R_{(p_j,q_j)}(k)$ is the difference in ranks for alternative k.

Kendall's τ is defined as:

$$\tau = \frac{n_c - n_d}{\binom{m}{2}} = \frac{2(n_c - n_d)}{m(m-1)},$$

where n_c is the number of concordant pairs and n_d is the number of discordant pairs between the two rankings.

We now compare the pq-FFWA operator against q-FFWA and q-ROFWA by computing Spearman's ρ and Kendall's τ between the rankings produced by each method. This provides a statistical measure of ranking agreement beyond visual inspection. Table 9 presents Spearman's ρ and Kendall's τ correlations between the baseline pq-FFWA ranking at $(p, q) = (2, 3)$ and all other (p, q) configurations from Table 4 of the manuscript, demonstrating *perfect intra-method stability* with average $\rho = 1.0000$ and $\tau = 1.0000$ across all fourteen parameter combinations. The ranking order $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ remains invariant under all tested (p, q) variations, confirming that the pq-FFWA framework produces robust and parameter-independent top-choice identification for this decision problem.

Table 9: Spearman's ρ and Kendall's τ for pq-FFWA ranking stability across (p, q) configurations from Table 4 (baseline: $(p, q) = (2, 3)$)

(p, q)	Ranking	Spearman ρ	Kendall τ	Agreement
(2,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(3,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(2,10)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(10,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(3,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(25,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,15)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(15,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,100)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(100,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(25,200)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(200,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(20,2000)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(2000,20)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
Average		1.0000	1.0000	Perfect stability

Where Table 10 illustrates the pairwise comparison mechanism underlying Kendall's τ calculation for pq-FFWA with $(p, q) = (2, 3)$ versus any (p, q) from Table 9, enumerating all six possible pairs of alternatives to identify concordant pairs (same relative order) and discordant pairs (opposite relative order), yielding $n_c = 6$, $n_d = 0$ and consequently $\tau = 1.0000$ alongside Spearman's $\rho = 1.0000$.

Table 10: Illustration of concordant/discordant pairs for pq-FFWA $(2, 3)$ vs any pq-FFWA (p, q)

Pair	pq-FFWA	pq-FFWA	Same order	Type
$(\ddot{\mathcal{H}}_1, \ddot{\mathcal{H}}_2)$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1$	Yes	Concordant
$(\ddot{\mathcal{H}}_1, \ddot{\mathcal{H}}_3)$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	Yes	Concordant
$(\ddot{\mathcal{H}}_1, \ddot{\mathcal{H}}_4)$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4$	Yes	Concordant
$(\ddot{\mathcal{H}}_2, \ddot{\mathcal{H}}_3)$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_3$	Yes	Concordant
$(\ddot{\mathcal{H}}_2, \ddot{\mathcal{H}}_4)$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4$	Yes	Concordant
$(\ddot{\mathcal{H}}_3, \ddot{\mathcal{H}}_4)$	$\ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	Yes	Concordant
Total concordant pairs n_c				6
Total discordant pairs n_d				0
Total pairs $\binom{4}{2}$				6
Kendall's $\tau = (6-0)/6$				1.0000
Spearman's $\rho = 1 - 6(0)/60$				1.0000

Table 11 compares pq-FFWA rankings against q-FFWA rankings for matching q parameters across all fourteen (p, q) configurations from Table 4, revealing perfect backward compatibility with average $\rho = 1.0000$ and $\tau = 1.0000$. The ranking order $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$ is identical between pq-FFWA and q-FFWA for every tested parameter combination, confirming that the additional parameter p in pq-FFWA does not destabilize the ranking when $p = q$ (recovering q-FFS) and that the asymmetric generalization preserves the core decision structure of the original q-FFS framework.

Table 11: Spearman's ρ and Kendall's τ between pq-FFWA and q-FFWA

(p, q)	pq-FFWA Ranking	q-FFWA Ranking	ρ	τ	Agreement
(2,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(3,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(2,10)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(10,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(3,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(25,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,15)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(15,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,100)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(100,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(25,200)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(200,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(20,2000)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(2000,20)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect

(p,q)	pq-FFWA Ranking	q-FFWA Ranking	ρ	τ	Agreement
Average			1.0000	1.0000	Perfect

Table 12 reports correlations between pq-FFWA and q-ROFWA rankings with matching q values, showing moderate-to-high average agreement ($\rho = 0.8166$, $\tau = 0.6945$) among all inter-method comparisons. q-ROFWA consistently exhibits two types of ranking deviations: (i) $\ddot{\mathcal{H}}_1/\ddot{\mathcal{H}}_4$ reordering for $q \in \{2, 3, 5, 10, 25, 100, 200\}$, yielding $\rho = 0.8000$ and $\tau = 0.6667$, where the middle-ranked alternatives swap positions but the top choice $\ddot{\mathcal{H}}_2$ remains invariant; and (ii) $\ddot{\mathcal{H}}_1/\ddot{\mathcal{H}}_2$ inversion at $q \in \{15, 25, 100, 200\}$, where q-ROFWA incorrectly promotes $\ddot{\mathcal{H}}_1$ above $\ddot{\mathcal{H}}_2$, producing the same correlation values $\rho = 0.8000$ and $\tau = 0.6667$. Notably, the lowest agreement remains at the High level, indicating that q-ROFWA's structural instability manifests as consistent middle-rank perturbations rather than catastrophic top-choice failure. Nevertheless, pq-FFWA maintains perfect top-choice stability ($\ddot{\mathcal{H}}_2$ ranked first in all fourteen configurations) whereas q-ROFWA misidentifies $\ddot{\mathcal{H}}_1$ as optimal in four out of twelve comparable cases, confirming pq-FFWA's superior reliability for critical decision-making.

Table 12: Spearman's ρ and Kendall's τ between pq-FFWA and q-ROFWA

(p,q)	pq-FFWA Ranking	q-ROFWA Ranking	ρ	τ	Agreement
(2,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(3,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(2,10)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(10,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(3,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(25,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(5,15)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(15,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(5,100)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(100,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(25,200)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(200,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
Average			0.8166	0.6945	High

Table 13 reports correlations between pq-FFWA and pq-ROFWA rankings with matching (p, q) values, showing strong average agreement ($\rho = 0.8833$, $\tau = 0.8056$) across all inter-method comparisons. The two methods achieve perfect concordance ($\rho = 1.0000$, $\tau = 1.0000$) in four configurations, $(10, 2)$, $(25, 3)$, $(15, 5)$ and $(100, 5)$, where both methods produce identical rankings $\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$, as well as at

$(200, 25)$ where pq-ROFWA yields indistinguishable alternatives. In the remaining eight cases, pq-ROFWA exhibits a consistent deviation pattern: the middle-ranked alternatives $\ddot{\mathcal{H}}_1$ and $\ddot{\mathcal{H}}_4$ swap positions while the top choice $\ddot{\mathcal{H}}_2$ remains invariant, yielding $\rho = 0.8000$ and $\tau = 0.6667$. Notably, the most severe disagreement occurs at $(25, 200)$, where pq-ROFWA inverts the top two alternatives by promoting $\ddot{\mathcal{H}}_1$ above $\ddot{\mathcal{H}}_2$, yet this still registers only as High agreement rather than catastrophic failure. Crucially, pq-FFWA maintains perfect top-choice stability ($\ddot{\mathcal{H}}_2$ ranked first in all twelve configurations) whereas pq-ROFWA misidentifies $\ddot{\mathcal{H}}_1$ as optimal in one out of twelve cases, confirming pq-FFWA's superior reliability for critical decision-making.

Table 13: Spearman's ρ and Kendall's τ between pq-FFWA and pq-ROFWA

(p, q)	pq-FFWA Ranking	pq-ROFWA Ranking	ρ	τ	Agreement
(2,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(3,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(2,10)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(10,2)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(3,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(25,3)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,15)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(15,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(5,100)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(100,5)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
(25,200)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	0.8000	0.6667	High
(200,25)	$\ddot{\mathcal{H}}_2 \succ \ddot{\mathcal{H}}_1 \succ \ddot{\mathcal{H}}_4 \succ \ddot{\mathcal{H}}_3$	$\ddot{\mathcal{H}}_1 \approx \ddot{\mathcal{H}}_2 \approx \ddot{\mathcal{H}}_4 \approx \ddot{\mathcal{H}}_3$	1.0000	1.0000	Perfect
Average			0.8833	0.8056	Strong

10. Advantages

Parameters p and q in pq-FFS is beneficial for decision-making and fuzzy logic applications. Here are the key advantages:

(i) Increased Flexibility:
- The property provides more flexibility in managing the membership and non-membership degrees of a fuzzy set. By controlling two parameters p and q, decision-makers can adjust the impact of the membership (μ) and non-membership (ν) on the complete evaluation.
- Its general nature allows to model more complex interrelations among the alternatives, in which the membership or

non-membership degrees may no longer be ruled by fuzzy set constraints.

(ii) Handling Decision-Making Uncertainty:
- It grants fine control over the effects of membership and non-membership to the fuzzy set. The use of p and q enables the system to assign different weights for the degrees of the membership and the non-membership, which leads to more suitable in various decision making situations.
- In many real-world applications, pq-FFSs are helpful because they offer a solid basis for handling ambiguous and partial information.

(iii) Improved Ranking and Sorting:
- This assumption allows us to better rank or sort the alternatives in the decision making problems. Such a relationship between degree of membership and non-membership can be better modelled by tuning the values p and q resulting in a more accurate ranking of decision alternatives.

11. Limitations

Despite the fact that the proposed approach has a number of advantages, it also has several limitations that should be acknowledged.

(i) The model considers only membership μ and non-membership ν, without explicitly incorporating a neutral or indeterminacy degree. As a result, hesitation or incomplete information is only implicitly represented through the residual margin of the constraint $\mu/p + \nu/q \leq 1$. This may limit the model's ability to capture situations where neutrality plays a significant and independent role, as in spherical or T-spherical fuzzy environments.

(ii) The selection of parameters p and q introduces an additional layer of subjectivity. While these parameters provide flexibility in modeling asymmetric preferences, their improper choice may lead to biased or inconsistent decision outcomes. In practice, determining appropriate values for p and q may require expert judgment or additional calibration procedures.

12. Conclusion

The pq-fractional fuzzy set framework introduces a flexible tool for handling uncertainty in multi-criteria decision-making. By decoupling membership and non-membership constraints through independent parameters p and q, the model generalizes existing q-fractional fuzzy structures and enables asymmetric control over decision-maker preferences.

This work established the foundational theory of pq-FFSs, including operational laws, partial order relations and novel score and accuracy functions. The score and accuracy functions derive from axiomatic principles with fractional effective indices $\ddot{\mu}^{1/p}$ and $\ddot{\nu}^{1/q}$ emerging naturally from the aggregation structure.

The e-commerce platform case study validates the MCDM paradigm, with sensitivity analysis confirming perfect ranking stability across all (p, q) combinations ($\rho = 1.0000$, $\tau = 1.0000$). Comparative analysis reveals critical advantages over q-FFWA (exponential score collapse), q-ROFWA (ranking inversion and score degeneracy at extreme parameters) and pq-ROFWA (consistent middle-rank deviations and top-choice misidentification), as the linear pq-FFS boundary preserves discriminative separation where curved constraints collapse.

Future work includes implementing TOPSIS and VIKOR in pq-FFS, constructing similarity measures for pattern recognition and data analysis and extending the framework to hesitant pq-FFSs analogous to q-FHFSs [7, 24].

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[1] Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq.
Email address: hariwan.msalih@uod.ac

[2] Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq.
Email address: zanyar.ameen@uod.ac

[3] Department of Mathematics, College of Science, University of Zakho, Duhok 42002, Iraq.

Department of Computer Science, College of Science, Cihan University-Duhok, Duhok 42001, Iraq.
Email address: baravan.asaad@uoz.edu.krd