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On Multiplicative Trigonometric Exponential Convex Function and Its Inequalities via Multiplicative Fractional Calculus with Applications

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Abstract. In this paper, we introduce a new class of multiplicative trigonometric exponential convex ($MTEC$) functions within the framework of multiplicative calculus. We establish their fundamental algebraic properties and rigorously investigate their relationships with several existing notions of convexity. Based on this framework, we derive new Hermite–Hadamard (HH) type inequalities in the setting of multiplicative fractional calculus by employing the multiplicative Riemann–Liouville (RL) fractional integral operator. By considering products and quotients of the proposed multiplicative convex structures, we obtain a variety of novel fractional HH inequalities. In addition, refined midpoint and trapezoidal type inequalities are developed, yielding sharper bounds under the proposed convexity assumptions. It is also shown that, by taking $\alpha = 1$, the obtained results reduce to their corresponding integer-order inequalities. The theoretical findings are supported by graphical illustrations, confirming the accuracy and sharpness of the results. Finally, applications to special functions are provided to demonstrate the effectiveness and applicability of the proposed approach.

1. Introduction

In mathematics and engineering, the concept of convexity is crucial since it offers a thorough foundation for a variety of phenomena. Because

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they may optimize intricate mathematical models and lessen the process's colossal complexity, convex sets and functions are a fundamental component of this theory. Because they only have one global minimum and are crucial to optimization, these issues are therefore very simple to solve. Convexity is also an applied mathematical notion in control theory, finance, optimization, and other areas. It is not just a theoretical subject; engineers must address it in order to improve the stability of systems development and lessen the severity of efficiency issues. In economics it helps to analyze consumer preferences and market behaviors, which makes it easier to decide how best to allocate resources. Convexity theory is one of the most remarkable fields of study because of its various facets, including its connection to integral inequalities. Integral inequalities are applied in various areas of analysis and applied sciences. Convex functions are frequently associated with these inequalities, particularly like the famous HH inequality that give estimations of upper and lower bounds for the averages of the function [11, 21].

Fractional calculus has been utilized widely in computer science, as a useful technique to model systems with memory and inherited features, physics, and engineering [13]. It has recently gained popularity as it has made distinctive contributions to the field of inequality theory especially in potential analysis. Mathematicians have devoted the past decade to the study of various fractional integral inequalities. Thus Sarikaya et al. [26] in 2013 introduced first variant of fractional HH inequalities pertaining RL fractional integrals. Since that time, scholars have made noteworthy advancements in the foundational investigation concerning HH type inequalities related to various fractional integrals, including conformable fractional integrals, Katugampola fractional integrals, generalized RL fractional integrals, RL fractional integrals, Sarikaya fractional integrals, generalized proportional fractional integrals, and (k, p) fractional integrals, among others. Furthermore, several types of fractional inequalities that give sharp and refine bounds for quadrature rules are available in [5, 8, 12, 17, 20].

Recent developments in fractional calculus have shown significant progress in both theoretical and applied directions. For instance, Turab *et al.* [28] analyzed dual-order memory-driven fractional differential equations, emphasizing computational aspects and parameter sensitivity with applications to animal learning processes. Furthermore, Turab *et al.* [27] provided a detailed analytical and numerical study of memory-dependent fractional models, highlighting the role of nonlocal dynamics in behavioral learning systems. These contributions demonstrate the growing importance of fractional operators in modeling complex systems

with memory effects and motivate further investigations in generalized frameworks such as the one proposed in this paper.

In recent years, multiplicative calculus has gained considerable attention, particularly for its role in integral inequality analysis. Because its foundation structure differs than classical calculus by replacing addition and subtartion by multiplication and division rules. Thus it is very effective while tackling problems incorporating growth procedures and quotient system problems. Analyzing inequality of other function classes has been made simple by this structure and several investigations and papers are published recently. These discoveries improve multiplicative calculus's theoretical underpinnings and provide helpful tools for applications involving mathematical analysis and optimization. Further types of integer order multiplicative inequalities have also been extensively studied. For example, Budak et al. [18] constructed inequalities of HH's type for multiplicative *differentiable function. Later Xie et al. [30] expanded such parallel inequalities for multiplicative **differentiable functions. Further details and advancements on multiplicative integer-order inequalities can be found in [7] and its references. These studies demonstrate the versatility of multiplicative calculus in addressing established mathematical issues from a fresh angle and keep expanding the boundaries of this developing field.

Extensive study has been done on integer-order versions of inequalities in the context of multiplicative calculus for example Recent advancements in generalized convexity have introduced new and rich structures within the framework of multiplicative calculus. In particular, Özcan [22] developed the notion of exponential type multiplicatively convex functions, establishing their algebraic properties and deriving refined HH type inequalities for such functions, including results for their products and quotients. Building upon this direction, Özcan et al. [23] introduced the class of multiplicatively trigonometric convex functions, which extends multiplicative convexity by incorporating trigonometric weights and reveals deeper structural properties such as symmetry and connections with other convexity concepts. These contributions not only generalize classical inequalities but also provide a flexible analytical framework that may stimulate further developments in integral inequalities and their applications. Nevertheless, a little amount of research is done on fractional forms of the multiplicative integrals. Budak and Ozçelik's 2020 proposal of HH type inequalities utilizing multiplicative RL fractional integrals in [6] has garnered significant attention. Numerous researchers are now interested in this. The fractional multiplicative variants of HH type inequalities deploying multiplicative tempered

fractional integrals in [10], tempered HH type inequalities for multipilcative $m-$preinvexity and $(s,m)-$preinvexity. In 2022, Peng et al. [25] explored a class of integral operators called multiplicative fractional integrals with exponential kernels, from which some upper bound estimates were obtained due to the proven identity.

Motivation and Novelty. Despite the substantial progress in generalized convexity within multiplicative calculus, the existing literature remains largely confined to isolated extensions such as exponential-type multiplicatively convex functions [22] and multiplicatively trigonometric convex functions [23]. These classes, although significant, treat exponential and trigonometric structures independently and do not capture their combined influence within a unified analytical framework. However, many natural phenomena-particularly those arising in oscillatory growth processes, signal analysis, and exponential-type transformations-simultaneously exhibit both multiplicative exponential behavior and trigonometric modulation. This intrinsic coupling motivates the need for a more comprehensive class of convex functions capable of incorporating these dual characteristics.

In this regard, the present work introduces, for the first time, a new class of multiplicative trigonometric exponential convex ($MTEC$) functions, which unifies and significantly extends the existing notions of multiplicative convexity. Unlike previously studied classes, the proposed $MTEC$ framework integrates exponential growth patterns with trigonometric weighting structures in a multiplicative setting, thereby providing a richer and more flexible analytical tool.

From a theoretical perspective, we establish fundamental algebraic properties of $MTEC$ functions and rigorously investigate their connections with known convexity classes. More importantly, the novelty of this study lies in extending the analysis to the fractional setting via multiplicative RL fractional integral operators, a direction that has remained largely unexplored for such hybrid convex structures. Within this framework, we derive new HH type inequalities, along with refined midpoint and trapezoidal type estimates, yielding sharper bounds compared to existing results. Furthermore, to ensure the reliability and applicability of the theoretical findings, graphical illustrations and numerical validations are provided, which confirm the sharpness and consistency of the obtained inequalities. It is worth emphasizing that, to the best of our knowledge, this is the first study in the literature where applications of such multiplicative fractional inequalities are explicitly demonstrated in the context of special functions. These applications not only highlight the effectiveness of the proposed approach but also open

new avenues for future research in inequality theory, applied analysis, and related scientific disciplines.

2. Preliminaries

We start by reviewing the concepts that are important to the current study.

Definition 2.1 ([24])**.** The function Υ defined on convex set I is convex, if

$$\Upsilon\left(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o\right) \leq \lambda\Upsilon(\mathfrak{u}_o) + (1-\lambda)\Upsilon(\mathfrak{v}_o),$$

$\forall\ \lambda \in [0,1]$ and $\mathfrak{u}_o, \mathfrak{v}_o \in \mathrm{I}$.

The HH inequalities, attributed to Charles Hermite and Jacques Hadam -ard [9], are a crucial outcome in the analysis of convex functions. For a convex function defined on a closed and bounded interval, the HH inequality furnishes sharp lower and upper bounds for its integral average. This inequality is important since it offers an effective way to estimate. Convex functions can be integrated using simply the function's midpoint value together with the endpoint values of the interval. Hermite and Hadamard separately developed this inequality, providing the basis for convex analysis. The HH inequality has become an essential tool in mathematical analysis because of its beautiful structure and applications in a variety of contexts. Beyond its theoretical significance, it is crucial to optimization problems, approximation theory, and numerical integration, all of which depend on an understanding of convex function behavior. In along with enabling accurate estimations, the HH inequality enhances our comprehension of the essential structural aspects of convex functions on bounded intervals by encapsulating basic principles of convexity.

The inequality for convex mapping $\Upsilon : \mathrm{I} \to \mathbb{R}$ on an interval I is:

$$\Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right) \leq \frac{1}{\mathfrak{v}_o - \mathfrak{u}_o}\int_{\mathfrak{u}_o}^{\mathfrak{v}_o} \Upsilon(\zeta)\mathrm{d}\zeta \leq \frac{\Upsilon(\mathfrak{u}_o) + \Upsilon(\mathfrak{v}_o)}{2},$$

for all $\mathfrak{u}_o, \mathfrak{v}_o \in \mathrm{I}$.

Within the HH framework, the midpoint-type inequality delivers a lower estimate of the integral through evaluation at the interval midpoint, whereas the trapezoid-type inequality supplies an upper estimate derived from the function values at the endpoints.

In [14, 15],The class of trigonometrically convex and trigonometrically exponential convex (TEC) functions was introduced by Kadakal, who also obtained HH type inequalities for this family.

Definition 2.2 ([14])**.** A positive function Υ is trigonometrically convex, if

$$\Upsilon(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o)\leq \sin\left(\frac{\pi\lambda}{2}\right)[\Upsilon(\mathfrak{u}_o)]+\cos\left(\frac{\pi\lambda}{2}\right)\Upsilon\left[(\mathfrak{v}_o)\right],$$

$\forall\ \mathfrak{u}_o,\mathfrak{v}_o\in \mathrm{I}$ and $\lambda\in[0,1]$.

Theorem 2.3 ([14])**.** *Let $\Upsilon:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ be a TC function, if $\mathfrak{u}_o<\mathfrak{v}_o$ and $\Upsilon\in L[\mathfrak{u}_o,\mathfrak{v}_o]$, then*

$$\frac{1}{\sqrt{2}}\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\leq\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}\left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))\,\mathrm{d}\zeta\right)\leq\frac{4}{\pi}\left(\frac{\Upsilon(\mathfrak{u}_o)+\Upsilon(\mathfrak{v}_o)}{2}\right).$$

Definition 2.4 ([15])**.** A positive function Υ is TEC, if

$$\Upsilon(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o)\leq\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}[\Upsilon(\mathfrak{u}_o)]+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}[\Upsilon(\mathfrak{v}_o)],$$

$\forall\ \mathfrak{u}_o,\mathfrak{v}_o\in \mathrm{I}$ and $\lambda\in[0,1]$.

Theorem 2.5 ([15])**.** *Let $\Upsilon:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ be a TEC function, if $\mathfrak{u}_o<\mathfrak{v}_o$ and $\Upsilon\in L[\mathfrak{u}_o,\mathfrak{v}_o]$, then*

$$\begin{aligned}\sqrt{\frac{\exp\{1\}}{2}}\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)&\leq\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}\left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))\,\mathrm{d}\zeta\right)\\&\leq\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\left(\Upsilon(\mathfrak{u}_o)+\Upsilon(\mathfrak{v}_o)\right).\end{aligned}$$

Definition 2.6 ([2])**.** A function $\Upsilon:\mathrm{I}\to\mathbb{R}$ is said to be multiplicative convex, if it holds

$$\Upsilon\left(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o\right)\leq[\Upsilon(\mathfrak{u}_o)]^{\lambda}[\Upsilon(\mathfrak{v}_o)]^{(1-\lambda)},$$

$\forall\ \lambda\in[0,1]$ and $\mathfrak{u}_o,\mathfrak{v}_o\in \mathrm{I}$. It readily yield that

$$\Upsilon\left(\lambda\mathfrak{u}_o+(1-\lambda)\,\mathfrak{v}_o\right)\leq\left[\Upsilon\left(\mathfrak{u}_o\right)\right]^{\lambda}\left[\Upsilon\left(\mathfrak{v}_o\right)\right]^{1-\lambda}\leq\lambda\Upsilon\left(\mathfrak{u}_o\right)+(1-\lambda)\,\Upsilon\left(\mathfrak{v}_o\right).$$

Convex functions may not always have multiplicative convexity, but this shows that multiplicatively convex functions have convexity. HH type inequalities for such a function was first proved by Ali et al. [2].

Theorem 2.7 ([2])**.** *Let a function Υ be a multiplicatively convex, then*

$$\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\leq\left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}\right)^{\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}}\leq G(\Upsilon(\mathfrak{u}_o),\Upsilon(\mathfrak{v}_o)),\tag{2.1}$$

where $G(.,.)$ is geometric mean.

Multiplicative derivatives and integrals

In this section, we initiate by revisiting certain definitions, properties, and concepts related to differentiation, along with exploring aspects of multiplicative integration.

Definition 2.8 ([4])**.** Let the function $\Upsilon : \mathbb{R} \to \mathbb{R}$ be positive, then the multiplicative derivative of Υ is represented by Υ^* is defined as follow.

$$\frac{\mathrm{d}^*\Upsilon}{\mathrm{d}\zeta} = \Upsilon^*(\zeta) = \lim_{\mathfrak{h}\to 0}\left(\frac{\Upsilon(\zeta+\mathfrak{h})}{\Upsilon(\zeta)}\right)^{\frac{1}{\mathfrak{h}}}.$$

Remark 2.9. The relation between Υ^* and Υ' is given by:

$$\Upsilon^* = \exp\{(\ln\Upsilon(\zeta))'\} = \exp\left\{\frac{\Upsilon'(\zeta)}{\Upsilon(\zeta)}\right\}.$$

The multiplicative derivative is governed by the following properties:

Proposition 2.10 ([4])**.** *Let Υ and Ω be *differentiable functions and* c *is arbitrary constant. Then functions $c\Upsilon$, $\Upsilon\Omega$, Υ/Ω and $\Upsilon+\Omega$ are *differentiable:*

(i) $(\mathrm{p}\Upsilon)^*(\zeta) = \Upsilon^*(\zeta),\ \mathrm{p}\in\mathrm{R}$
(ii) $(\Upsilon\Omega)^*(\zeta) = \Upsilon^*(\zeta)\Omega^*(\zeta)$,
(iii) $(\Upsilon+\Omega)^*(\zeta) = \Upsilon^*(\zeta)^{\frac{\Upsilon(\zeta)}{\Upsilon(\zeta)+\Omega(\zeta)}}\cdot\Omega^*(\zeta)^{\frac{\Omega(\zeta)}{\Upsilon(\zeta)+\Omega(\zeta)}}$,
(iv) $\left(\frac{\Upsilon}{\Omega}\right)^*(\zeta) = \frac{\Upsilon^*(\zeta)}{\Omega^*(\zeta)}$,
(v) $(\Upsilon^\Omega)^*(\zeta) = \Upsilon^*(\zeta)^{\Omega(\zeta)}\Upsilon(\zeta)^{\Omega'(\zeta)}$,

Below, we describe the relationship between the Riemann and multiplicative integrals [4]:

Proposition 2.11. *If Υ on $[\mathfrak{u}_o,\mathfrak{v}_o]$ is Riemann integrable, then it is *integrable*

$$\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta} = \exp\left\{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\ln(\Upsilon(\zeta))\mathrm{d}\zeta\right\}.$$

Moreover, multiplicative integrable possesses the following properties and results as shown by Bashirov et al. [4]:

Proposition 2.12. *If Υ on $[\mathfrak{u}_o,\mathfrak{v}_o]$ is Riemann integrable, then it is *integrable*

(i) $\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}((\Upsilon(\zeta))^{\mathrm{p}})^{\mathrm{d}\zeta} = \int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left((\Upsilon(\zeta))^{\mathrm{d}\zeta}\right)^{\mathrm{p}}$,p $\in$ R.
(ii) $\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta)\Omega(\zeta))^{\mathrm{d}\zeta} = \int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}\cdot\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Omega(\zeta))^{\mathrm{d}\zeta}$,
(iii) $\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\frac{\Upsilon(\zeta)}{\Omega(\zeta)}\right)^{\mathrm{d}\zeta} = \frac{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}}{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Omega(\zeta))^{\mathrm{d}\zeta}}$,
(iv) $\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta} = \int_{\mathfrak{u}_o}^{\mathrm{c}}(\Upsilon(\zeta))^{\mathrm{d}\zeta}\cdot\int_{\mathrm{c}}^{\mathfrak{v}_o}(f(\zeta))^{\mathrm{d}\zeta}, \mathfrak{u}_o\le\mathrm{c}\le\mathfrak{v}_o$,

(v) $\int_{\mathfrak{u}_o}^{\mathfrak{u}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}=1$ *and* $\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}=\left(\int_{\mathfrak{v}_o}^{\mathfrak{u}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}\right)^{-1}$.

Alternatively, the subsequent multiplicative RL fractional integrals were put forward by Abdeljawed and Grossman [1].

Definition 2.13. The symbol $({}_{\mathfrak{u}_o}\mathrm{I}_*^{\alpha}\Upsilon)(\zeta)$ is a designation of left multiplicative RL fractional integral of order $\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0$ with α as an initial point is given by

$$({}_{\mathfrak{u}_o}\mathrm{I}_*^{\alpha}\Upsilon)(\zeta)=\exp\left\{(\mathrm{J}_{\mathfrak{u}_o+}^{\alpha}(\ln\circ\Upsilon))(\zeta)\right\},$$

and what defines the multiplicative right one is

$$({}_*\mathrm{I}_{\mathfrak{v}_o}^{\alpha}\Upsilon)(\zeta)=\exp\left\{(\mathrm{J}_{\mathfrak{v}_o-}^{\alpha}(\ln\circ\Upsilon))(\zeta)\right\},$$

Here $\mathrm{J}_{\mathfrak{v}_o-}^{\alpha}(\ln\circ\Upsilon)$ and $\mathrm{J}_{\mathfrak{u}_o+}^{\alpha}(\ln\circ\Upsilon)$ describe the right and left RL fractional integrals, given by [16]:

$$\mathrm{J}_{\mathfrak{v}_o-}^{\alpha}(\ln\circ\Upsilon)(\lambda)=\frac{1}{\Gamma(\alpha)}\int_{\zeta}^{\mathfrak{v}_o}(\lambda-\zeta)^{\alpha-1}(\ln\circ\Upsilon)(\lambda)\mathrm{d}\lambda,\quad \mathfrak{v}_o>\zeta,$$

and

$$\mathrm{J}_{\mathfrak{u}_o+}^{\alpha}(\ln\circ\Upsilon)(\lambda)=\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{u}_o}^{\zeta}(\zeta-\lambda)^{\alpha-1}(\ln\circ\Upsilon)(\lambda)\mathrm{d}\lambda,\quad \mathfrak{u}_o<\zeta,$$

accordingly, where $\Gamma(\alpha)=\int_0^{\infty}\exp\{-\lambda\}\lambda^{\alpha-1}\mathrm{d}\lambda$. Here is

$$\mathrm{J}_{\mathfrak{v}_o-}^{0}\Upsilon(\zeta)=\Upsilon(\zeta)=\mathrm{J}_{\mathfrak{u}_o+}^{0}\Upsilon(\zeta).$$

Theorem 2.14 (*Integration by parts [4])**.** *Let* $\Upsilon:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ *and* $\Omega:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ *are* **differentiable and differentiable respectively, therefore* Υ^{Ω} *is* **integrable, then*

$$\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon^*(\zeta)^{\Omega(\zeta)}\right)^{\mathrm{d}\zeta}=\frac{\Upsilon(\mathfrak{v}_o)^{\Omega(\mathfrak{v}_o)}}{\Upsilon(\mathfrak{u}_o)^{\Omega(\mathfrak{u}_o)}}\cdot\frac{1}{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon(\zeta)^{\Omega'(\zeta)}\right)^{\mathrm{d}\zeta}}.$$

Lemma 2.15. *Let* $\Upsilon:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ *and* $\Omega:[\mathfrak{u}_o,\mathfrak{v}_o]\to\mathbb{R}$ *are* **differentiable and differentiable respectively, therefore* Υ^{Ω} *is* **integrable then*

$$\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon^*(\mathfrak{h}(\zeta))^{\mathfrak{h}'(\zeta)\Omega(\zeta)}\right)^{\mathrm{d}\zeta}=\frac{\Upsilon(\mathfrak{h}(\mathfrak{v}_o))^{\Omega(\mathfrak{v}_o)}}{\Upsilon(\mathfrak{h}(\mathfrak{u}_o))^{\Omega(\mathfrak{u}_o)}}\cdot\frac{1}{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon(\mathfrak{h}(\zeta))^{\Omega'(\zeta)}\right)^{\mathrm{d}\zeta}}.$$

3. Fractional Estimates for $MTEC$ Function via RL Fractional Integral Operators

In this section, we offer the definition of $MTEC$ function and using it as a base, we discuss its algebraic properties as well as its associated fractional order inequalities.

Definition 3.1. A positive function $\Upsilon : \mathrm{I} \to \mathbb{R}$ is said to be *MTEC*, if $\forall \mathfrak{u}_o, \mathfrak{v}_o \in \mathrm{I}$ and $\lambda \in [0,1]$,

$$\Upsilon(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) \leq [\Upsilon(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} \cdot [\Upsilon(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}}.$$

Theorem 3.2. *Let $\Upsilon, \Omega : \mathrm{I} \to \mathbb{R}$ be the MTEC functions, then $\Upsilon\Omega$ is MTEC function.*

Proof. Since Υ and Ω be *MTEC* functions. Then

$$\begin{aligned}
(\Upsilon\Omega)(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) &= \Upsilon(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) \cdot \Omega(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) \\
&\leq [\Upsilon(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} \cdot [\Upsilon(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} \\
&\quad \times [\Omega(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} \cdot [\Omega(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} \\
&= [\Upsilon(\mathfrak{u}_o) \cdot \Omega(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} \cdot [\Upsilon(\mathfrak{v}_o) \cdot \Omega(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} \\
&= [(\Upsilon\Omega)(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} . [(\Upsilon\Omega)(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}}.
\end{aligned}$$ □

Theorem 3.3. *Let $\Upsilon : \mathrm{I} \to \mathbb{R}$ is a convex and $\Omega : \mathrm{I} \to \mathbb{R}$ is a MTEC function, then $\Omega \circ \Upsilon : \mathrm{I} \to \mathbb{R}$ is a MTEC function.*

Proof. For $\mathfrak{u}_o, \mathfrak{v}_o \in \mathrm{I}$ and $\lambda \in$ [0,1], one gets

$$\begin{aligned}
(\Omega \circ \Upsilon)(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) &= \Omega(\Upsilon(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o)) \\
&\leq \Omega(\lambda\Upsilon(\mathfrak{u}_o) + (1-\lambda)\Upsilon(\mathfrak{v}_o)) \\
&\leq [\Omega(\Upsilon(\mathfrak{u}_o))]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} . [\Omega(\Upsilon(\mathfrak{v}_o))]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} \\
&= [(\Omega \circ \Upsilon)(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} . [(\Omega \circ \Upsilon)(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} . \square
\end{aligned}$$

Theorem 3.4. *Let $\Upsilon_i : \mathrm{I} \to \mathbb{R}$ be an arbitrary MTEC functions and let $\Upsilon(\zeta) = \sup_i\{\Upsilon_i(\zeta) : i = 1, 2, 3, ...\}$. If $\mathrm{J} = \{r \in [\mathfrak{u}_o, \mathfrak{v}_o] : \Upsilon(\mathrm{r}) < \infty\} = \varnothing$, then J is an interval and Υ is a MTEC function on J.*

Proof. For $\mathfrak{u}_o, \mathfrak{v}_o \in \mathrm{I}$ and $\lambda \in [0, 1]$, one has

$$\begin{aligned}
\Upsilon(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) &= \sup_i\{\Upsilon_i(\mathfrak{u}_o\lambda + (1-\lambda)\mathfrak{v}_o) : i = 1, 2, 3, ...\} \\
&\leq \sup_i \left[[\Upsilon_i(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} . [\Upsilon_i(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} : i = 1, 2, 3, .. \right] \\
&\leq \sup_i [\Upsilon_i(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{\exp\{1-\lambda\}}} . \sup_i [\Upsilon_i(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{\exp\{\lambda\}}} \\
&= [\Upsilon(\mathfrak{u}_o)]^{\frac{\sin\frac{\pi\lambda}{2}}{e^{1-\lambda}}} . [\Upsilon(\mathfrak{v}_o)]^{\frac{\cos\frac{\pi\lambda}{2}}{e^{\lambda}}}
\end{aligned}$$

$$< \infty.$$

Thus Υ is a $MTEC$ function on J. This completes the proof. □

Next we derive the HH's type inequalities pertaining to the $MTEC$ via multiplicative RL integral operators.

Theorem 3.5. *Let Υ be a $MTEC$ function on $[\mathfrak{u}_o, \mathfrak{v}_o]$ with $\alpha > 0$, then*

$$\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o).{}_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}} \leq [\Upsilon(\mathfrak{u}_o).\Upsilon(\mathfrak{v}_o)]^{\frac{\alpha Z}{\sqrt{2\exp\{1\}}}}, \tag{3.1}$$

holds. Where

$$Z = \int_0^1 \lambda^{\alpha-1}\left(\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} + \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\right)\mathrm{d}\lambda,$$

Proof. Since Υ be a $MTEC$ function, then

$$\Upsilon(\zeta\lambda+(1-\lambda)\vartheta) \leq [\Upsilon(\zeta)]^{\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}}.[\Upsilon(\vartheta)]^{\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}}.$$

Setting $\lambda = \frac{1}{2}$

$$\Upsilon\left(\frac{\zeta+\vartheta}{2}\right) \leq [\Upsilon(\zeta)]^{\frac{1}{\sqrt{2\exp\{1\}}}}.[\Upsilon(\vartheta)]^{\frac{1}{\sqrt{2\exp\{1\}}}}.$$

Substituting $\zeta = \lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o$ and $\vartheta = \lambda\mathfrak{v}_o + (1-\lambda)\mathfrak{u}_o$, then

$$\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq [\Upsilon(\lambda\mathfrak{u}_o+(1-\lambda)\mathfrak{v}_o)]^{\frac{1}{\sqrt{2\exp\{1\}}}} \times [\Upsilon(\lambda\mathfrak{v}_o+(1-\lambda)\mathfrak{u}_o)]^{\frac{1}{\sqrt{2\exp\{1\}}}}.$$

It implies that

$$\ln\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \ln\left([\Upsilon(\lambda\mathfrak{u}_o+(1-\lambda)\mathfrak{v}_o)]^{\frac{1}{\sqrt{2\exp\{1\}}}} \times [\Upsilon(\lambda\mathfrak{v}_o+(1-\lambda)\mathfrak{u}_o)]^{\frac{1}{\sqrt{2\exp\{1\}}}}\right)$$
$$= \frac{1}{\sqrt{2\exp\{1\}}}\ln\Upsilon(\lambda\mathfrak{u}_o+(1-\lambda)\mathfrak{v}_o) + \frac{1}{\sqrt{2\exp\{1\}}}\ln\Upsilon(\lambda\mathfrak{v}_o+(1-\lambda)\mathfrak{u}_o).$$

Multiplying $\lambda^{\alpha-1}$ on both sides then integrating the above inequality w.r.t λ on [0, 1], we have

$$\int_0^1 \lambda^{\alpha-1} \ln \Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \mathrm{d}\lambda \leq \frac{\int_0^1 \lambda^{\alpha-1} \ln \Upsilon(\lambda \mathfrak{u}_o+(1-\lambda)\mathfrak{v}_o)\mathrm{d}\lambda}{\sqrt{2\exp\{1\}}} + \frac{\int_0^1 \lambda^{\alpha-1} \ln \Upsilon(\lambda \mathfrak{v}_o+(1-\lambda)\mathfrak{u}_o)\mathrm{d}\lambda}{\sqrt{2\exp\{1\}}}.$$

After simple calculations, we get

$$\frac{1}{\alpha} \ln \Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \frac{1}{\sqrt{2\exp\{1\}}}\left[\int_{\mathfrak{v}_o}^{\mathfrak{u}_o} \left(\frac{\mathfrak{v}_o-\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1} \ln \Upsilon(\zeta) \frac{\mathrm{d}\zeta}{\mathfrak{u}_o-\mathfrak{v}_o} + \int_{\mathfrak{u}_o}^{\mathfrak{v}_o} \left(\frac{\zeta-\mathfrak{u}_o}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1} \ln \Upsilon(\zeta) \frac{\mathrm{d}\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}\right].$$

It implies that

$$\ln \Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \frac{\alpha}{\sqrt{2\exp\{1\}}}\left[\int_{\mathfrak{v}_o}^{\mathfrak{u}_o} \left(\frac{\mathfrak{v}_o-\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1} \ln \Upsilon(\zeta) \frac{\mathrm{d}\zeta}{\mathfrak{v}_o-\mathfrak{u}_o} + \int_{\mathfrak{u}_o}^{\mathfrak{v}_o} \left(\frac{\zeta-\mathfrak{u}_o}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1} \ln \Upsilon(\zeta) \frac{\mathrm{d}\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}\right],$$

or

$$\ln \Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\mathfrak{u}_o+} \ln \Upsilon(\mathfrak{v}_o)+\mathrm{J}^{\alpha}_{\mathfrak{v}_o-} \ln \Upsilon(\mathfrak{u}_o)\right],$$

exponential and use of multiplicative RL, we get

$$\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \leq \left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o).{}_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}. \tag{3.2}$$

The proof of the first inequality of the result (3.1) is now complete. As Υ is $MTEC$ function, therefore we get

$$\Upsilon(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o) \leq [\Upsilon(\mathfrak{u}_o)]^{\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}}.[\Upsilon(\mathfrak{v}_o)]^{\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}}$$

and

$$\Upsilon(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o) \leq [\Upsilon(\mathfrak{v}_o)]^{\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}}.[\Upsilon(\mathfrak{u}_o)]^{\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}}.$$

Taking natural log and summing the resulting expressions, we obtain

$$\ln \Upsilon(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o)+\ln \Upsilon(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o) \leq \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \ln \Upsilon(\mathfrak{u}_o)+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \ln \Upsilon(\mathfrak{v}_o) \tag{3.3}$$

$$+\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\ln\Upsilon(\mathfrak{v}_o)+\frac{\cos\frac{\pi\lambda}{2}}{e^{\lambda}}\ln\Upsilon(\mathfrak{u}_o)$$
$$=\left(\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\right)[\ln\Upsilon(\mathfrak{u}_o)+\ln\Upsilon(\mathfrak{v}_o)].$$

Applying the multiplication by $\lambda^{\alpha-1}$ to both sides of (3.3) and integrating over $[0,1]$ with respect to λ, we derive

$$\int_0^1\lambda^{\alpha-1}\ln\Upsilon(\mathfrak{u}_o\lambda+(1-\lambda)\mathfrak{v}_o)\mathrm{d}\lambda+\int_0^1\lambda^{\alpha-1}\ln\Upsilon(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o)\mathrm{d}\lambda$$
$$\leq\int_0^1\lambda^{\alpha-1}\left(\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\right)\mathrm{d}\lambda[\ln\Upsilon(\mathfrak{u}_o)+\ln\Upsilon(\mathfrak{v}_o)].$$

After simple calculations, we get

$$\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\frac{\mathfrak{v}_o-\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1}\ln\Upsilon(\zeta)\,\frac{d\zeta}{\mathfrak{u}_o-\mathfrak{v}_o}+\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\frac{\zeta-\mathfrak{u}_o}{\mathfrak{v}_o-\mathfrak{u}_o}\right)^{\alpha-1}\ln\Upsilon(\zeta)\,\frac{d\zeta}{\mathfrak{v}_o-\mathfrak{u}_o}$$
$$\leq\mathrm{Z}[\ln\Upsilon(\mathfrak{u}_o)+\ln\Upsilon(\mathfrak{v}_o)],$$

or

$$\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\mathfrak{u}_o+}\ln\Upsilon(\mathfrak{v}_o)+\mathrm{J}^{\alpha}_{\mathfrak{v}_o-}\ln\Upsilon(\mathfrak{u}_o)\right]$$
$$\leq\frac{\alpha\mathrm{Z}}{\sqrt{2\exp\{1\}}}[\ln\Upsilon(\mathfrak{u}_o)+\ln\Upsilon(\mathfrak{v}_o)].$$

Using definitions of multiplicative RL fractional operators, we get

$$\left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o).{}_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}\leq[\Upsilon(\mathfrak{u}_o).\Upsilon(\mathfrak{v}_o)]^{\frac{\alpha\mathrm{Z}}{\sqrt{2\exp\{1\}}}}.\tag{3.4}$$

Combining (3.2) and (3.4), we get the desired result. □

Remark 3.6. By taking $\alpha=1$ in Theorem 3.5, we achieve the subsequent result for $MTEC$ function.

$$\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)^{\frac{1}{\sqrt{2\exp\{1\}}}}\leq\left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}(\Upsilon(\zeta))^{\mathrm{d}\zeta}\right)^{\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}}\leq[\Upsilon(\mathfrak{u}_o).\Upsilon(\mathfrak{v}_o)]^{\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}}.$$

Example 3.7. Figure 1 provides a graphical representation supporting Theorem 3.5.

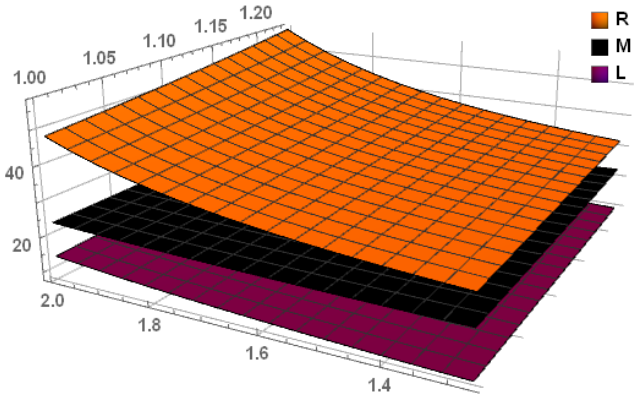

FIGURE 1. Graphical description of Theorem 3.5 for $\mathfrak{u}_o$=[1, 1.23], $\mathfrak{v}_o$=[1.27, 2], and α=0.1 for the function $\Upsilon(\zeta) = exp\left\{\zeta^2\right\}$.

Corollary 3.8. *If Υ and Ω are MTEC functions on $[\mathfrak{u}_o, \mathfrak{v}_o]$, then*

$$
\begin{aligned}
(3.5)\quad & \Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\Omega\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right) \\
&\leq \left[{}_{\mathfrak{u}_o}\mathrm{I}_*{}^{\alpha}\Upsilon(\mathfrak{v}_o).{}_*\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\cdot{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_*\Omega(\mathfrak{v}_o).{}_*\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Omega(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}} \\
&\leq \left[\Upsilon(\mathfrak{u}_o).\Upsilon(\mathfrak{v}_o)\cdot\Omega(\mathfrak{u}_o).\Omega(\mathfrak{v}_o)\right]^{\frac{\alpha Z}{\sqrt{2\exp\{1\}}}},
\end{aligned}
$$

holds.

Remark 3.9. By taking $\alpha = 1$ in Corollary 3.8, we achieve the subsequent result for the product of *MTEC* functions.

$$
\begin{aligned}
&\left[\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right).\Omega\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right]^{\frac{1}{\sqrt{2\exp\{1\}}}} \\
&\leq \left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon\left(\zeta\right)\right)^{d\zeta}.\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Omega\left(\zeta\right)\right)^{d\zeta}\right)^{\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}} \\
&\leq \left[\Upsilon(\mathfrak{u}_o)\cdot\Upsilon(\mathfrak{v}_o)\cdot\Omega(\mathfrak{u}_o)\cdot\Omega(\mathfrak{v}_o)\right]^{\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}}.
\end{aligned}
$$

Example 3.10. Figure 2 provides a graphical representation supporting Corollary 3.8.

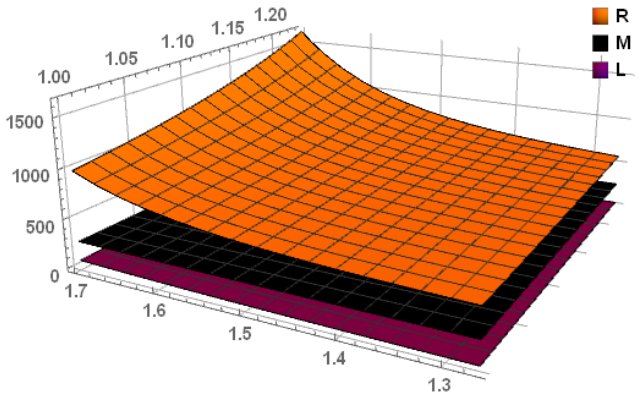

FIGURE 2. Graphical description of Corollary 3.8 for $\mathfrak{u}_o$=[1,1.23], $\mathfrak{v}_o$=[1.27,1.7], and α=0.1 for the function $\Upsilon(\zeta) = exp\left\{\zeta^2\right\}$ and $\Omega(\zeta) = \exp\left\{\zeta^3\right\}$.

Remark 3.11. By taking $\alpha = 1$ in Corollary 3.8, we achieve the subsequent result for the product of $MTEC$ function.

$$\begin{aligned}
&\left[\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\Omega\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right]^{\frac{1}{\sqrt{2\exp\{1\}}}} \\
&\leq \left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon\left(\zeta\right)\right)^{d\zeta}\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Omega(\zeta)\right)^{d\zeta}\right)^{\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}} \\
&\leq \left[\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)\Omega(\mathfrak{u}_o)\Omega(\mathfrak{v}_o)\right]^{\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}}.
\end{aligned}$$

Corollary 3.12. *Let Υ and Ω are $MTEC$ function and MTE concave function on $[\mathfrak{u}_o, \mathfrak{v}_o]$ respectively, then*

$$\begin{aligned}
\frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\Omega\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)} &\leq \left[\frac{{}_{\mathfrak{u}_o}\mathrm{I}*^{\alpha}\Upsilon(\mathfrak{v}_o).*\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)}{{}_{\mathfrak{u}_o}\mathrm{I}*^{\alpha}\Omega(\mathfrak{v}_o).*\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Omega(\mathfrak{u}_o)}\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}} \\
&\leq \left[\frac{\Upsilon(\mathfrak{u}_o)\cdot\Upsilon(\mathfrak{v}_o)}{\Omega(\mathfrak{u}_o)\cdot\Omega(\mathfrak{v}_o)}\right]^{\frac{\alpha\mathrm{Z}}{\sqrt{2\exp\{1\}}}},
\end{aligned} \tag{3.6}$$

holds.

Remark 3.13. By taking $\alpha = 1$ in Corollary 3.12, we achieve the subsequent result for the quotient of $MTEC$ function and MTE concave function.

$$\left[\frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\Omega\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\right]^{\frac{1}{\sqrt{2\exp\{1\}}}} \leq \left[\frac{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Upsilon\left(\zeta\right)\right)^{d\zeta}}{\int_{\mathfrak{u}_o}^{\mathfrak{v}_o}\left(\Omega\left(\zeta\right)\right)^{d\zeta}}\right]^{\frac{1}{\mathfrak{v}_o-\mathfrak{u}_o}}$$

$$\leq \left[\frac{\Upsilon(\mathfrak{u}_o).\Upsilon(\mathfrak{v}_o)}{\Omega(\mathfrak{u}_o).\Omega(\mathfrak{v}_o)}\right]^{\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}}.$$

Example 3.14. Figure 3 provides a graphical representation supporting Corollary 3.12.

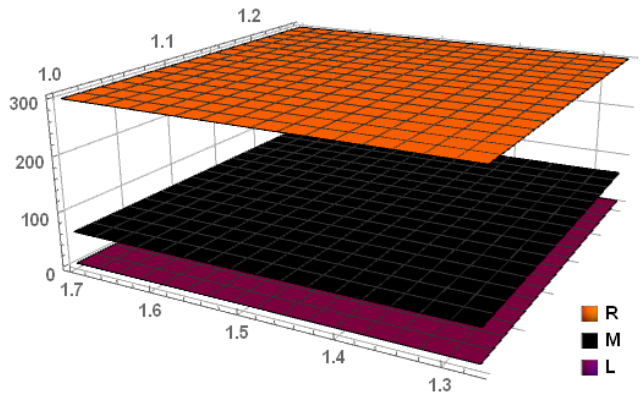

FIGURE 3. Graphical description of Corollary 3.12 for $\mathfrak{u}_o$=[1, 1.23], $\mathfrak{v}_o$=[1.27, 1.7], and α=0.1 by taking the functions $\Upsilon(\zeta) = \exp\left\{\zeta^2\right\}$ and $\Omega(\zeta) = \exp\left\{-\zeta^3\right\}$.

4. Trapezoidal and Midpoint Type Inequalities

This section introduces new trapezoidal and midpoint type estimates in the framework of multiplicative *RL* fractional integrals for *MTEC* function.

Lemma 4.1. *Let the function Υ^* be a *integrable and Υ be a *differentiable function over $[\mathfrak{u}_o, \mathfrak{v}_o]$, then*

$$\frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o).{}_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}} = \left(\int_0^1 \left([\Upsilon^*(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o)]^{(1-\lambda)^{\alpha}-\lambda^{\alpha}}\right)\mathrm{d}\lambda\right)^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}}, \tag{4.1}$$

holds.

Proof. Taking the right side of (4.1), we attain

$$I = \left(\int_0^1 \left([\Upsilon^*(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o)]^{(1-\lambda)^{\alpha}-\lambda^{\alpha}}\right)\mathrm{d}\lambda\right)^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}}$$

$$= \left(\int_0^1 \left([\Upsilon^*(\mathfrak{v}_o\lambda + (1-\lambda)\mathfrak{u}_o)]^{(\mathfrak{u}_o - \mathfrak{v}_o)\left(\frac{\lambda^\alpha - (1-\lambda)^\alpha}{2}\right)} \right) \mathrm{d}\lambda \right).$$

Employing the Lemma 2.15, we attain

$$I = \frac{\Upsilon(\mathfrak{v}_o)^{\frac{1}{2}}}{\Upsilon(\mathfrak{u}_o)^{-\frac{1}{2}}} \times \frac{1}{\int_0^1 \left(\Upsilon(\lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o)^{\frac{\alpha}{2}(\lambda^{\alpha-1} + (1-\lambda)^{\alpha-1})} \right) \mathrm{d}\lambda}. \tag{4.2}$$

Next considering the integral term of the result (4.2), we attain

$$\begin{aligned}
&\int_0^1 \left(\Upsilon(\lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o)^{\frac{\alpha}{2}\left(\lambda^{\alpha-1} + (1-\lambda)^{\alpha-1}\right)} \right)^{\mathrm{d}\lambda} \\
&= \exp\left\{ \int_0^1 \frac{\alpha}{2} \left(\lambda^{\alpha-1} + (1-\lambda)^{\alpha-1} \right) \ln \Upsilon(\lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o) \mathrm{d}\lambda \right\} \\
&= \exp\left\{ \int_0^1 \frac{\alpha}{2} \lambda^{\alpha-1} \ln \Upsilon(\lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o) \mathrm{d}\lambda \right. \\
&\quad \left. + \int_0^1 \frac{\alpha}{2} (1-\lambda)^{\alpha-1} \ln \Upsilon(\lambda\mathfrak{u}_o + (1-\lambda)\mathfrak{v}_o) \mathrm{d}\lambda \right\} \\
&= \exp\left\{ \int_0^1 \frac{\alpha}{2} \left(\frac{\zeta - \mathfrak{u}_o}{\mathfrak{u}_o - \mathfrak{v}_o} \right)^{\frac{\alpha}{k}-1} \ln \Upsilon(\zeta) \mathrm{d}\zeta \right. \\
&\quad \left. + \int_0^1 \frac{\alpha}{2} \left(1 - \frac{\zeta - \mathfrak{u}_o}{\mathfrak{u}_o - \mathfrak{v}_o} \right)^{\frac{\alpha}{k}-1} \ln \Upsilon(\zeta) \mathrm{d}\zeta \right\} \\
&= \exp\left\{ \frac{\alpha}{2(\mathfrak{v}_o - \mathfrak{u}_o)^{\frac{\alpha}{k}}} \int_{\mathfrak{v}_o}^{\mathfrak{u}_o} (\mathfrak{u}_o - \zeta)^{\frac{\alpha}{k}-1} \ln \Upsilon(\zeta) \mathrm{d}\zeta \right. \\
&\quad \left. + \frac{\alpha}{2(\mathfrak{v}_o - \mathfrak{u}_o)^{\alpha}} \int_{\mathfrak{u}_o}^{\mathfrak{v}_o} (\zeta - \mathfrak{u}_o)^{\alpha-1} \ln \Upsilon(\zeta) d\zeta \right\} \\
&= \exp\left\{ \frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o - \mathfrak{u}_o)^{\alpha}} \left[\frac{1}{\Gamma(\alpha)} \int_{\mathfrak{u}_o}^{\zeta} (\zeta - \mathfrak{u}_o)^{\alpha-1} \ln \Upsilon(\zeta) \mathrm{d}\zeta \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_{\zeta}^{\mathfrak{v}_o} (\mathfrak{v}_o - \zeta)^{\alpha-1} \ln \Upsilon(\zeta) \mathrm{d}\zeta \right] \right\} \\
&= \left[{}_{\mathfrak{u}_o}\mathrm{I}_*^{\alpha} \Upsilon(\mathfrak{v}_o) .{}_*\mathrm{I}_{\mathfrak{v}_o}^{\alpha} \Upsilon(\mathfrak{u}_o) \right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o - \mathfrak{u}_o)^{\alpha}}}.
\end{aligned}$$

Substituting this value in (4.2), we get the desired result □

Theorem 4.2. *Considering the same conditions of Lemma 4.1, if* $|\Upsilon^*|$ *is MTEC function, then*

$$\left| \frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}_*^{\alpha} \Upsilon(\mathfrak{v}_o) .{}_*\mathrm{I}_{\mathfrak{v}_o}^{\alpha} \Upsilon(\mathfrak{u}_o) \right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o - \mathfrak{u}_o)^{\alpha}}}} \right| \leq \left[|\Upsilon^*(\mathfrak{u}_o)|^{\mathfrak{A}_1} |\Upsilon^*(\mathfrak{v}_o)|^{\mathfrak{A}_2} \right]^{\frac{(\mathfrak{v}_o - \mathfrak{u}_o)}{2}}. \tag{4.3}$$

Where

$$\mathfrak{A}_1 = \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \mathrm{d}\lambda$$
$$\mathfrak{A}_2 = \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \mathrm{d}\lambda,$$

holds.

Proof. From equation (4.1), we have

$$\begin{aligned}
&\left| \frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}^\alpha_* \Upsilon(\mathfrak{v}_o) .{}_*\mathrm{I}^\alpha_{\mathfrak{v}_o} \Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^\alpha}}} \right| \\
&\quad \leq \exp\left\{ \frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2} \int_0^1 |[(1-\lambda)^\alpha - \lambda^\alpha] \ln \Upsilon^*(\mathfrak{v}_o\lambda + (1-\lambda)\mathfrak{u}_o)| \, d\lambda \right\} \\
&\quad = \exp\left\{ \frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2} \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \, |\ln \Upsilon^*(\mathfrak{v}_o\lambda + (1-\lambda)\mathfrak{u}_o)| \, \mathrm{d}\lambda \right\} \\
&\quad \leq \exp\left\{ \frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2} \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \right. \\
&\qquad \left. \times \ln \left[[\Upsilon(\mathfrak{u}_o)]^{\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}} . [\Upsilon(\mathfrak{v}_o)]^{\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}} \right] \mathrm{d}\lambda \right\} \\
&\quad = \exp\left\{ \frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2} \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \right. \\
&\qquad \left. \times \left[\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \ln|\Upsilon^*(\mathfrak{u}_o)| + \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \ln|\Upsilon^*(\mathfrak{v}_o)| \right] \mathrm{d}\lambda \right\} \\
&\quad = \exp\left\{ \frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2} \left[\ln|\Upsilon^*(\mathfrak{u}_o)| \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \mathrm{dt} \right. \right. \\
&\qquad \left. \left. + \ln|\Upsilon^*(\mathfrak{v}_o)| \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \mathrm{d}\lambda \right] \right\}.
\end{aligned}$$

It is to be noted that

$$\mathfrak{A}_1 = \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \mathrm{d}\lambda$$
$$\mathfrak{A}_2 = \int_0^1 |(1-\lambda)^\alpha - \lambda^\alpha| \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \mathrm{d}\lambda.$$

Substituting these values in above inequality, we get the following expression

$$\begin{aligned}
&\left|\frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o)\cdot_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}}\right| \\
&\quad\leq \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\left[\mathfrak{A}_1\ln|\Upsilon^*(\mathfrak{u}_o)|+\mathfrak{A}_2\ln|\Upsilon^*(\mathfrak{v}_o)|\right]\right\} \\
&\quad= \left[|\Upsilon^*(\mathfrak{u}_o)|^{\mathfrak{A}_1}|\Upsilon^*(\mathfrak{v}_o)|^{\mathfrak{A}_2}\right]^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.3. *Considering the same conditions of Lemma 4.1, if the function* $|\Upsilon^*|^q$ *for* $q>1$ *is MTEC, then*

$$\begin{aligned}
&\left|\frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o)\cdot_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}f(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}}\right| \\
&\quad\leq \left[|\Upsilon^*(\mathfrak{u}_o)\cdot\Upsilon^*(\mathfrak{v}_o)|^q\right]^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2q}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}},
\end{aligned}$$

holds. Where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 4.1 . and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{\sqrt{\Upsilon(\mathfrak{u}_o)\Upsilon(\mathfrak{v}_o)}}{\left[{}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\Upsilon(\mathfrak{v}_o)\cdot_{*}\mathrm{I}^{\alpha}_{\mathfrak{v}_o}\Upsilon(\mathfrak{u}_o)\right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}}\right| \\
&\leq \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\int_0^1|(1-\lambda)^{\alpha}-\lambda^{\alpha}|\,|\ln\Upsilon^*(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o)|\,\mathrm{d}\lambda\right\} \\
&\leq \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\left(\int_0^1|(1-\lambda)^{p\alpha}-\lambda^{p\alpha}|\,\mathrm{d}\lambda\right)^{\frac{1}{p}}\right. \\
&\quad\left.\times\left(\int_0^1|\ln\Upsilon^*(\mathfrak{v}_o\lambda+(1-\lambda)\mathfrak{u}_o)|^q\,\mathrm{d}\lambda\right)^{\frac{1}{q}}\right\} \\
&\leq \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\right. \\
&\quad\left.\times\left[|\ln\Upsilon^*(\mathfrak{u}_o)|^q\int_0^1\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\mathrm{d}\lambda+|\ln\Upsilon^*(\mathfrak{v}_o)|^q\int_0^1\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\mathrm{d}\lambda\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\right.\\
&\quad\left.\times\left[\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)\ln\left[|\Upsilon^*(\mathfrak{u}_o)\cdot\Upsilon^*(\mathfrak{v}_o)|^q\right]\right]^{\frac{1}{q}}\right\}\\
&= \exp\left\{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}\right.\\
&\quad\left.\times\ln\left[|\Upsilon^*(\mathfrak{u}_o)\cdot\Upsilon^*(\mathfrak{v}_o)|^q\right]^{\frac{1}{q}}\right\}\\
&\leq \left[|\Upsilon^*(\mathfrak{u}_o)\cdot\Upsilon^*(\mathfrak{v}_o)|^q\right]^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2q}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}}.
\end{aligned}
$$

This completes the proof of theorem. □

Next part provides the derivation of some new midpoint type inequalities in the setting of multiplicative RL fractional integrals for $MTEC$ function.

Lemma 4.4. *Let Υ be a positive $MTEC$ function, then*

$$
\begin{aligned}
&\frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\left[{}_*\mathrm{I}^{\alpha}_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\Upsilon\left(\mathfrak{u}_o\right)\cdot_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\mathrm{I}^{\alpha}_{*}\Upsilon\left(\mathfrak{v}_o\right)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}}\\
&\quad=\left[\int_0^1\left(\Upsilon^*\left((1-\lambda)\,\mathfrak{u}_o+\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)^{\lambda^{\alpha}}\right)\mathrm{d}\lambda\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}}\\
&\qquad\times\left[\int_0^1\left(\Upsilon^*\left((1-\lambda)\,\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}+\lambda\mathfrak{v}_o\right)^{(\lambda-1)^{\alpha}}\right)\mathrm{d}\lambda\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}},
\end{aligned}
\tag{4.4}
$$

holds.

Proof. Taking the right term of (4.4), we gain

$$
\begin{aligned}
&\left[\int_0^1\left(\Upsilon^*\left((1-\lambda)\,\mathfrak{u}_o+\lambda\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)^{\lambda^{\alpha}}\right)\mathrm{d}\lambda\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}}\\
&\qquad\times\left[\int_0^1\left(\Upsilon^*\left((1-\lambda)\,\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}+\lambda\mathfrak{v}_o\right)^{(\lambda-1)^{\alpha}}\right)\mathrm{d}\lambda\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}},
\end{aligned}
$$

suppose that

$$I_1 = \int_0^1 \left(\Upsilon^* \left((1-\lambda)\,\mathfrak{u}_o + \lambda \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right)^{\left(\frac{\mathfrak{v}_o - \mathfrak{u}_o}{2}\right)\left(\frac{\lambda^\alpha}{2}\right)} \right)^{\mathrm{d}\lambda},$$

$$I_2 = \int_0^1 \left(\Upsilon^* \left((1-\lambda)\,\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} + \lambda \mathfrak{v}_o \right)^{\left(\frac{\mathfrak{v}_o - \mathfrak{u}_o}{2}\right)\left(\frac{(\lambda-1)^\alpha}{2}\right)} \right)^{\mathrm{d}\lambda}.$$

By using Lemma 2.15, we attain

$$I_1 = \frac{\Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)^{\frac{1}{2}}}{\Upsilon(\mathfrak{u}_o)^0} \times \frac{1}{\int_0^1 \left(\Upsilon\left((1-\lambda)\mathfrak{u}_o + \lambda \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right)^{\frac{\alpha}{2}\lambda^{\alpha-1}} \right) \mathrm{d}\lambda},$$

$$I_2 = \frac{\Upsilon(\mathfrak{v}_o)^0}{\Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)^{-\frac{1}{2}}} \times \frac{1}{\int_0^1 \left(\Upsilon\left((1-\lambda)\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} + \lambda \mathfrak{v}_o \right)^{\frac{\alpha}{2}(\lambda-1)^{\alpha-1}} \right) \mathrm{d}\lambda}.$$

After simple calculations, we get

$$I_1 = \Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)^{\frac{1}{2}} \times \frac{1}{\left[{}_{\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)}\mathrm{I}^{\alpha}_{*} \Upsilon\left(\mathfrak{v}_o\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o - \mathfrak{u}_o)^\alpha}}},$$

$$I_2 = \Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)^{\frac{1}{2}} \times \frac{1}{\left[{}_{*}\mathrm{I}^{\alpha}_{\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)} \Upsilon\left(\mathfrak{u}_o\right) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o - \mathfrak{u}_o)^\alpha}}}.$$

Combining the values of I_1 and I_2, we get the desired result □

Remark 4.5. By putting $\alpha = 1$ in Lemma 4.4, we achieve the subsequent result for multiplicative convex function in [3].

$$\frac{\Upsilon\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)}{\left(\int_{\mathfrak{u}_o}^{\mathfrak{v}_o} (\Upsilon(\zeta))^{\mathrm{d}\zeta}\right)^{\frac{1}{\mathfrak{v}_o - \mathfrak{u}_o}}} = \left[\int_0^1 \left(\Upsilon^* \left((1-\lambda)\,\mathfrak{u}_o + \lambda \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right)^{\lambda} \right)^{\mathrm{d}\lambda} \right]^{\frac{\mathfrak{v}_o - \mathfrak{u}_o}{4}} \tag{4.5}$$

$$\times \left[\int_0^1 \left(\Upsilon^* \left((1-\lambda) \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} + \lambda \mathfrak{v}_o \right)^{(\lambda-1)} \right) \mathrm{d}\lambda \right]^{\frac{\mathfrak{v}_o - \mathfrak{u}_o}{4}}.$$

Theorem 4.6. *Considering the same conditions of Lemma 4.4, if $|\Upsilon^*|$ is MTEC function, then*

$$\left| \frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\left[{}_*\mathrm{I}^{\alpha}_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)} \Upsilon(\mathfrak{u}_o) \cdot_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)} \mathrm{I}^{\alpha}_{*} \Upsilon(\mathfrak{v}_o) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}} \right|$$
$$\leq \left[(\Upsilon^*(\mathfrak{u}_o))^{\mathfrak{C}_1} \cdot \left(\Upsilon^* \left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right) \right)^{\mathfrak{C}_2} \right.$$
$$\left. \times \left(\Upsilon^* \left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right) \right)^{\mathfrak{C}_3} \cdot (\Upsilon^*(\mathfrak{v}_o))^{\mathfrak{C}_4} \right]^{\frac{\mathfrak{v}_o - \mathfrak{u}_o}{4}}.$$

Where

$$\mathfrak{C}_1 = \int_0^1 \lambda^{\alpha} \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \mathrm{d}\lambda$$
$$\mathfrak{C}_2 = \int_0^1 \lambda^{\alpha} \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \mathrm{d}\lambda$$
$$\mathfrak{C}_3 = \int_0^1 (1-\lambda)^{\alpha} \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}} \mathrm{d}\lambda$$
$$\mathfrak{C}_4 = \int_0^1 (1-\lambda)^{\alpha} \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}} \mathrm{d}\lambda,$$

holds.

Proof. From the equality (4.4), we have

$$\left| \frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\left[{}_*\mathrm{I}^{\alpha}_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)} \Upsilon(\mathfrak{u}_o) \cdot_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)} \mathrm{I}^{\alpha}_{*} \Upsilon(\mathfrak{v}_o) \right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}} \right|$$
$$= \exp\left\{ \frac{\mathfrak{v}_o - \mathfrak{u}_o}{4} \left[\int_0^1 \lambda^{\alpha} \left| \ln \Upsilon^* \left((1-\lambda)\mathfrak{u}_o + \lambda \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} \right) \right| \mathrm{d}\lambda \right.\right.$$
$$\left.\left. + \int_0^1 (\lambda-1)^{\alpha} \left| \ln \Upsilon^* \left((1-\lambda) \frac{\mathfrak{u}_o + \mathfrak{v}_o}{2} + \lambda \mathfrak{v}_o \right) \right| \mathrm{d}\lambda \right] \right\}$$

$$\leq \exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left[\int_0^1 \lambda^\alpha\left(\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\ln|\Upsilon^*(\mathfrak{u}_o)|\right.\right.\right.$$
$$+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|\Bigg)\mathrm{d}\lambda+\int_0^1(1-\lambda)^\alpha\left(\frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\right.$$
$$\left.\left.\left.\times\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|+\frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\ln|\Upsilon^*(\mathfrak{v}_o)|\right)\mathrm{d}\lambda\right]\right\},$$

It implies that

$$=\exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left[\mathfrak{C}_1\ln|\Upsilon^*(\mathfrak{u}_o)|+\mathfrak{C}_2\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|\right.\right.$$
$$\left.\left.+\mathfrak{C}_3\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|+\mathfrak{C}_4\ln|\Upsilon^*(\mathfrak{v}_o)|\right]\right\}$$
$$\leq\left[(\Upsilon^*(\mathfrak{u}_o))^{\mathfrak{C}_1}\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)^{\mathfrak{C}_2}\left(\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right)^{\mathfrak{C}_3}(\Upsilon^*(\mathfrak{v}_o))^{\mathfrak{C}_4}\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}}.$$

This completes the proof. □

Theorem 4.7. *Considering the same conditions of Lemma 4.4, if* $|\Upsilon^*|^q$ *for* $q>1$ *is MTEC function, then*

$$\left|\frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\left[{}_*\mathrm{I}^\alpha_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\Upsilon(\mathfrak{u}_o)\cdot_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\mathrm{I}^\alpha_*\Upsilon(\mathfrak{v}_o)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o-\mathfrak{u}_o)^\alpha}}}\right|$$
$$\leq\left[\left|\Upsilon^*(\mathfrak{u}_o)\left(\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right)^2\Upsilon^*(\mathfrak{v}_o)\right|^q\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4q}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, *holds.*

Proof. From the equality (4.4), we have

$$\left|\frac{\Upsilon\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}{\left[{}_*\mathrm{I}^\alpha_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\Upsilon(\mathfrak{u}_o)\cdot_{\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)}\mathrm{I}^\alpha_*\Upsilon(\mathfrak{v}_o)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{v}_o-\mathfrak{u}_o)^\alpha}}}\right|$$
$$=\exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left[\left(\int_0^1\lambda^{p\alpha}d\lambda\right)^{\frac{1}{p}}\right.\right.$$

$$
\begin{aligned}
&\times \left(\int_0^1 \left|\ln \Upsilon^* \left((1-\lambda)\,\mathfrak{u}_o + \lambda \frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q \mathrm{d}\lambda\right)^{\frac{1}{q}} \\
&+ \int_0^1 ((\lambda-1)^{p\alpha})^{\frac{1}{p}} \left(\int_0^1 \left|\ln \Upsilon^* \left((1-\lambda)\,\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2} + \lambda \mathfrak{v}_o\right)\right|^q \mathrm{d}\lambda\right)^{\frac{1}{q}}\Bigg]\Bigg\} \\
&\leq \exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left[\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left(\int_0^1 \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\mathrm{d}\lambda \ln|\Upsilon^*(\mathfrak{u}_o)|^q\right.\right. \\
&+ \int_0^1 \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\mathrm{d}\lambda \ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q\Bigg)^{\frac{1}{q}} \\
&+ \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left(\int_0^1 \frac{\sin\left(\frac{\pi\lambda}{2}\right)}{\exp\{1-\lambda\}}\mathrm{d}\lambda \ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q\right. \\
&\left.\left.+ \int_0^1 \frac{\cos\left(\frac{\pi\lambda}{2}\right)}{\exp\{\lambda\}}\mathrm{d}\lambda \ln|\Upsilon^*(\mathfrak{v}_o)|^q\right)^{\frac{1}{q}}\right]\Bigg\} \\
&= \exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left[\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\right.\right. \\
&\times \left(\left(\frac{2}{\pi}\right)\ln|\Upsilon^*(\mathfrak{u}_o)|^q + \left(\frac{2}{\pi}\right)\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&\left.+ \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \left(\left(\frac{2}{\pi}\right)\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q + \left(\frac{2}{\pi}\right)\ln|\Upsilon^*(\mathfrak{v}_o)|^q\right)^{\frac{1}{q}}\right]\Bigg\} \\
&= \exp\left\{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\left(\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)\ln|\Upsilon^*(\mathfrak{u}_o)|^q\right.\right. \\
&+ \left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q\Bigg)^{\frac{1}{q}} \\
&+ \left(\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)\ln\left|\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right|^q\right. \\
&\left.\left.+ \left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)\ln|\Upsilon^*(\mathfrak{v}_o)|^q\right)^{\frac{1}{q}}\right]\Bigg\} \\
&\leq \left[\left|\Upsilon^*(\mathfrak{u}_o)\left(\Upsilon^*\left(\frac{\mathfrak{u}_o+\mathfrak{v}_o}{2}\right)\right)^2\Upsilon^*(\mathfrak{v}_o)\right|^q\right]^{\frac{\mathfrak{v}_o-\mathfrak{u}_o}{4q}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}}.
\end{aligned}
$$

This completes the proof. □

5. Applications

This section elaborates the applications of our results in terms of special functions such that q-digamma function and modified Bessel function, due to which we get the new recurrence relations.

q-Digamma Function. The detail concept of q-digamma mapping $\delta_q(\vartheta)$ is available in [19] and defined as follows:

$$\delta_q(\vartheta) = -\ln(q-1) + \ln q \left(\vartheta - \frac{1}{2} - \sum_{j=1}^{\infty} \frac{q^{-j\vartheta}}{1-q^{-j\vartheta}} \right),$$

with $q > 1$ and $\vartheta > 0$.

Let $\delta : I \to \mathbb{R}$ be a positive and completely monotone function. Then, the q-digamma function can be applied to the class of $MTEC$ functions.

Proposition 5.1. *For $0 < \mathfrak{u}_o < \mathfrak{v}_o$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we achive*

$$\left| \sqrt{\delta_q'(\mathfrak{u}_o)\delta_q'(\mathfrak{v}_o)} \cdot \left[{}_{\mathfrak{u}_o}\mathrm{I}_*^{\alpha}\delta_q'(\mathfrak{v}_o) \cdot_* \mathrm{I}_{\mathfrak{v}_o}^{\alpha}\delta_q'(\mathfrak{u}_o) \right]^{\frac{\Gamma(\alpha+1)}{2(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}} \right|$$
$$\leq \left[|\delta_q{}''(\mathfrak{u}_o) \cdot \delta_q{}''(\mathfrak{v}_o)|^q \right]^{\frac{(\mathfrak{v}_o-\mathfrak{u}_o)}{2q}\left(\frac{2}{p\alpha+1}\left(1-\frac{1}{2^{p\alpha}}\right)\right)^{\frac{1}{p}}\left(\frac{2\pi+4\exp\{1\}}{\exp\{1\}(\pi^2+4)}\right)^{\frac{1}{q}}},$$

Proof. It is straightforward to know that for any $\epsilon > 0$, the mapping $\vartheta \longmapsto \delta' q(\vartheta)$ is completely monotonic on $(0, \infty)$. That is to say, on the interval $(0, \infty)$, $|\Upsilon'(\vartheta)|^q = |\delta_q''(\vartheta)|^q$ is a non-negative $MTEC$ mapping, which ensures that $|\delta_q''(\vartheta)|$ is also a $MTEC$ mapping. Applying $|\Upsilon(\vartheta)| = |\delta_q'(\vartheta)|$ for $\vartheta > 0$ to Theorem 4.3, one obtains the desired outcome. This terminates the proof. □

Modified Bessel Function. The detail theory of type-1 modified Bessel function $\mathfrak{J}_p$ is available in [29]:

$$\mathfrak{J}_p(\zeta) = \sum_{n=0}^{\infty} \frac{\left(\frac{n}{2}\right)^{p+2n}}{n!\Gamma(p+n+1)}, \quad \zeta \in \mathbb{R}.$$

Next, we introduce a new function $\Upsilon_p : [0, \infty[\to [0, \infty[$ for $p \in \mathbb{Z}^+$ such that $p \geq 1$, and defined by

$$\Upsilon_p(\zeta) = \zeta^p \mathfrak{J}_p(\zeta), \qquad \Upsilon'_p(\zeta) = \zeta^p \mathfrak{J}_{p-1}(\zeta).$$

The function $\Upsilon'_p(\zeta)$ is a TEC for $\zeta > 1$. Clearly, $\Upsilon' p(\zeta)$ can be represented as the logarithm of $\exp \Upsilon' p(\zeta)$, which is a function with log $MTEC$. This shows that $\exp \Upsilon'_p(\zeta)$ is an $MTEC$ for $p \geq 1$.

Proposition 5.2. *Let $p \geq 1$, $\mathfrak{u}_o, \mathfrak{v}_o \in [0, \infty[$ such that $\mathfrak{u}_o < \mathfrak{v}_o$, then the inequality holds:*

$$\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right)^p \mathfrak{J}_p\left(\frac{\mathfrak{u}_o + \mathfrak{v}_o}{2}\right) \tag{5.1}$$

$$\leq \left[\exp\left\{(\mathfrak{u}_o\mathfrak{v}_o)^p\right\}({}_{\mathfrak{u}_o}\mathrm{I}^{\alpha}_{*}\mathfrak{J}_p(\mathfrak{v}_o) \cdot_* \mathrm{I}^{\alpha}_{\mathfrak{v}_o}\mathfrak{J}_p(\mathfrak{u}_o))\right]^{\frac{\Gamma(\alpha+1)}{\sqrt{2\exp\{1\}}(\mathfrak{v}_o-\mathfrak{u}_o)^{\alpha}}}$$

$$\leq \left[\exp\left\{(\mathfrak{u}_o\mathfrak{v}_o)^p\right\}(\mathfrak{J}_p(\mathfrak{u}_o) \cdot \mathfrak{J}_p(\mathfrak{v}_o))\right]^{\frac{\alpha \mathrm{Z}}{\sqrt{2\exp\{1\}}}}.$$

Proof. The $MTEC$ function $\exp\{\Upsilon_p(\zeta)\}$, is applied in Theorem 3.1 for $\alpha > 0$ to achieve the stated result, where

$$\Upsilon_p(\zeta) = \zeta^p \mathfrak{J}_p(\zeta). \qquad \square$$

6. Conclusion

In this paper, we introduced a new class of $MTEC$ functions within the framework of multiplicative calculus and investigated their fundamental properties as well as their relationships with several existing notions of convexity. Based on this new structure, we established novel HH type inequalities via the multiplicative RL fractional integral operator. By considering products and quotients of $MTEC$ functions, a variety of new fractional HH inequalities were derived, along with refined midpoint and trapezoidal type inequalities that provide sharper bounds under the proposed convexity assumptions. The validity and effectiveness of the obtained results were supported through graphical illustrations, confirming their sharpness. Finally, several applications to special functions were presented, demonstrating the applicability and strength of the proposed framework in multiplicative fractional analysis.

The results obtained in this study open several promising directions for future research in fractional analysis and integral inequalities within multiplicative settings.

Problem 1: HH-type inequalities for multiplicative trigonometric exponential convex functions via multiplicative Atangana-Baleanu fractional integrals.

Problem 2: Fractional version of HH-type inequalities for multiplicative trigonometric exponential convex functions via multiplicative Katugampola fractional integrals.

Problem 3: Fractional estimates of HH-type for multiplicative trigonometric exponential convex functions via multiplicative k-RL fractional integrals.

Problem 4: Fractional estimates of HH-type for multiplicative trigonometric exponential convex functions via multiplicative Hadamard fractional integrals.

Problem 5: HH-type inequalities for (P, m)-multiplicative trigonometric exponential convex functions.
Problem 6: HH-type inequalities for h-multiplicative trigonometric exponential convex functions.

Another potential avenue is to investigate stochastic, fuzzy, or quantum variants of these inequalities, which may uncover deeper insights into uncertainty modeling and quantum information theory. Moreover, extending the established results to multi-dimensional or operator-valued settings could open up further applications in functional analysis, optimization, and applied mathematical modeling.

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