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Greedy Approximation and Approximation Spaces in Quaternionic Hilbert Spaces

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Abstract. In this paper, we investigate greedy approximation associated with frames and Riesz bases in right quaternionic Hilbert spaces. We show that every Riesz basis in a right quaternionic Hilbert space is greedy, while every frame is quasi-greedy and almost greedy, with explicit constants depending only on the frame bounds. We then introduce approximation spaces associated with quaternionic frames and characterize them in terms of weak-$\ell_p(\mathbb{H})$ and Lorentz sequence spaces $\ell_{p,q}(\mathbb{H})$. For general frames, we establish sufficient conditions for membership in these approximation spaces, and for Riesz bases we obtain converse results, leading to complete characterizations. Our results extend several classical theorems from real and complex Hilbert spaces to the quaternionic setting.

1. Introduction and Preliminaries

Greedy approximation methods form a central topic in nonlinear approximation theory, harmonic analysis and computational mathematics. Given a redundant or non-orthogonal representation system in a Hilbert or Banach space, greedy algorithms provide adaptive procedures that construct approximations using only a small number of significant terms. Classical contributions in this area established the foundations of best m-term approximation and convergence properties of greedy methods; see [7–9, 11, 14–16].

Frame theory has emerged as a flexible framework for stable, redundant representations in Hilbert spaces. Frames generalize orthonormal

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bases and allow reconstruction even in the presence of redundancy. The classical theory of frames and Riesz bases is now well established; see [4, 5, 10].

The interaction between greedy approximation and frames has been well explored in the classical Hilbert space setting; see, for example, [8, 12, 14]. In particular, approximation spaces associated with frames and bases were introduced and characterized using sequence space techniques, leading to precise decay estimates for m-term approximation errors [3, 6, 7]. These results provide a bridge between abstract approximation theory and adaptive numerical methods.

An essential role in the characterization of approximation spaces is played by rearrangement-invariant sequence spaces such as weak-ℓ_p and Lorentz spaces. Norm equivalences and sharp constants for Lorentz sequence spaces were studied in detail in [2].

More recently, attention has turned to extending these classical theories to non-commutative settings. Quaternionic Hilbert spaces arise naturally in quantum mechanics and operator theory and their mathematical foundations have been developed in several works [1, 17]. While many structural properties of complex Hilbert spaces carry over to the quaternionic setting, the non-commutativity of the scalar field introduces additional technical challenges, particularly in operator theory and approximation problems.

The theory of frames in right quaternionic Hilbert spaces is still developing; [13] established several fundamental results, while their connection with greedy approximation and approximation spaces remains largely unexplored. The main objective of this paper is to address this gap by developing a theory of greedy approximation associated with frames and Riesz bases in right quaternionic Hilbert spaces. We focus on greedy approximants induced by frames through their canonical duals, thereby avoiding unnecessary technical complications while retaining the essential features of greedy approximation theory.

Our main results show that every Riesz basis is greedy, while every frame is quasi-greedy and almost greedy, with constants depending on the frame bounds. These results extend classical results to the quaternionic setting [9, 14, 15]. We further characterize approximation spaces associated with quaternionic frames using weak-$\ell_p(\mathbb{H})$ and Lorentz spaces $\ell_{p,q}(\mathbb{H})$, obtaining sufficient conditions for frames and complete characterizations for Riesz bases. Our analysis uses techniques from nonlinear approximation and Lorentz space theory [2, 3, 6, 7].

The paper is organized as follows. Section 2 develops greedy approximation results and Section 3 studies the associated approximation spaces and their sequence-space characterizations.

We recall the basic notions and results on right quaternionic Hilbert spaces and frame theory. Standard references for quaternionic Hilbert spaces and operators are [1, 17], while frame theory is discussed in [5, 10]; frames in the quaternionic setting were studied in [13]. Throughout the paper, $\mathbb{H}$ denotes the non-commutative field of quaternions, $\mathcal{H}_{\mathbb{H}}$ denotes a separable right quaternionic Hilbert space over $\mathbb{H}$ and $\mathbb{N}$ denotes the set of natural numbers. For $m \in \mathbb{N}$, we write $[m] = \{1, 2, \dots, m\}$. By a right linear operator on a right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ we mean a mapping $T : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ satisfying

$$T(uq_1 + vq_2) = T(u)q_1 + T(v)q_2, \quad u, v \in \mathcal{H}_{\mathbb{H}},\ q_1, q_2 \in \mathbb{H}.$$

Every bounded right linear operator $T : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ admits an adjoint T^* defined by

$$\langle Tu|\, v\rangle = \langle u|\, T^*v\rangle\,, \quad u, v \in \mathcal{H}_{\mathbb{H}}.$$

The notation $A \asymp B$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 A \le B \le c_2 A$.

Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_{\mathbb{H}}$. The family $\mathcal{U}$ is called a frame for $\mathcal{H}_{\mathbb{H}}$ if there exist constants $0 < r_1 \le r_2 < \infty$ such that

$$r_1\|u\|^2 \le \sum_{j\in\mathbb{N}} |\langle u_j|\, u\rangle|^2 \le r_2\|u\|^2, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

If only the upper inequality holds, $\mathcal{U}$ is called a Bessel sequence. If $r_1 = r_2$, the frame is called tight. Associated with a frame $\mathcal{U}$ are the synthesis operator $\mathcal{T}_{\mathcal{U}} : \ell^2(\mathbb{H}) \to \mathcal{H}_{\mathbb{H}}$ and the analysis operator $\mathcal{T}_{\mathcal{U}}^* : \mathcal{H}_{\mathbb{H}} \to \ell^2(\mathbb{H})$ defined by

$$\mathcal{T}_{\mathcal{U}}(\{q_j\}_{j\in\mathbb{N}}) = \sum_{j\in\mathbb{N}} u_j q_j, \qquad \mathcal{T}_{\mathcal{U}}^*(u) = \{\langle u_j|\, u\rangle\}_{j\in\mathbb{N}}\,.$$

The frame operator $\mathcal{S}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}\mathcal{T}_{\mathcal{U}}^*$ is a bounded, positive, self-adjoint and invertible right linear operator given by

$$\mathcal{S}_{\mathcal{U}}u = \sum_{j\in\mathbb{N}} u_j\, \langle u_j|\, u\rangle\,, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

The family $\left\{\mathcal{S}_{\mathcal{U}}^{-1}u_j\right\}_{j\in\mathbb{N}}$ is called the canonical dual frame of $\mathcal{U}$

A frame $\mathcal{U}$ is called a Riesz basis if there exist constants $0 < r_1 \le r_2 < \infty$ such that

$$r_1 \sum_{j\in\mathbb{N}} |q_j|^2 \le \left\| \sum_{j\in\mathbb{N}} u_j q_j \right\|^2 \le r_2 \sum_{j\in\mathbb{N}} |q_j|^2, \quad \{q_j\} \in \ell^2(\mathbb{H}).$$

Equivalently, a Riesz basis is the image of an orthonormal basis under a bounded bijective right linear operator; see [5, 10].

The following lemma plays a central role in the study of greedy approximation and approximation spaces.

For a finite subset $\sigma \subset \mathbb{N}$, we denote by

$$[\mathcal{U}]_\sigma = \text{rt-span}\{u_j : j \in \sigma\}$$

the right linear span of the corresponding frame elements. The family $\mathcal{U}_\sigma = \{u_j\}_{j\in\sigma}$ is a frame for $[\mathcal{U}]_\sigma$ with frame operator $\mathcal{S}_{\mathcal{U}_\sigma}$. Since $[\mathcal{U}]_\sigma$ is closed, there exists an orthogonal projection $\mathcal{P}_\sigma : \mathcal{H}_\mathbb{H} \to [\mathcal{U}]_\sigma$ given by

$$\mathcal{P}_\sigma(u) = \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}_\sigma}^{-1} u_j \middle| u \right\rangle, \quad u \in \mathcal{H}_\mathbb{H}.$$

For $m \in \mathbb{N}$ and $u \in \mathcal{H}_\mathbb{H}$, the m-term approximation error of u with respect to a frame $\mathcal{U}$ is denoted by $\sigma_m(u)$. The corresponding greedy approximant of order m is denoted by $G_m(u)$. All rearrangements of quaternionic coefficient sequences are performed via their moduli. For a sequence $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{H}$, the notation $\left\{q_j^*\right\}_{j\geq 1}$ denotes the non-increasing rearrangement of $\{|q_j|\}_{j\geq 1}$, in accordance with the classical theory of greedy approximation [7, 14].

2. Frames and Greedy Approximation in Quaternionic Hilbert Spaces

Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_\mathbb{H}$ and let $\sigma \subset \mathbb{N}$ with $|\sigma| = m \in \mathbb{N}$. We denote by $[\mathcal{U}]_\sigma := \text{rt-span}\{u_j : j \in \sigma\}$, the right closed linear span of $\{u_j\}_{j\in\sigma}$. Then $\mathcal{U}_\sigma = \{u_j\}_{j\in\sigma}$ is a frame for the subspace $[\mathcal{U}]_\sigma$. Let $\mathcal{S}_\sigma$ denote the frame operator associated with the frame $\mathcal{U}_\sigma$ on $[\mathcal{U}]_\sigma$. Since $[\mathcal{U}]_\sigma$ is a right closed subspace of $\mathcal{H}_\mathbb{H}$, there exists an orthogonal projection $\mathcal{P}_\sigma : \mathcal{H} \to [\mathcal{U}]_\sigma$. Consequently, for any $u \in \mathcal{H}_\mathbb{H}$,

$$\begin{aligned}
\mathcal{P}_\sigma u &= \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_\sigma^{-1} u_j \middle| \mathcal{P}_\sigma u \right\rangle \\
&= \sum_{j\in\sigma} u_j \left\langle \mathcal{P}_\sigma \mathcal{S}_\sigma^{-1} u_j \middle| u \right\rangle \\
&= \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_\sigma^{-1} u_j \middle| u \right\rangle.
\end{aligned}$$

To analyze m-term approximation errors associated with frames, we consider finite-dimensional subspaces generated by subsets of frame elements. For a frame $\mathcal{U} = \{u_n\}_{n\in\mathbb{N}}$ in a right quaternionic Hilbert space $\mathcal{H}_\mathbb{H}$, orthogonal projections onto such subspaces play a central role. The following lemma relates the m-term approximation error to these projections.

Lemma 2.1. *Let $\mathcal{U} = \{u_n\}_{n\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$. Let $\sigma \subset \mathbb{N}$ be a finite subset with $|\sigma| = m$ and let $\mathcal{P}_\sigma$ denote the orthogonal projection from $\mathcal{H}_{\mathbb{H}}$ onto $[\mathcal{U}]_\sigma := \text{rt-span}\{u_j : j \in \sigma\}$. Then, for any $u \in \mathcal{H}_{\mathbb{H}}$,*

$$\sigma_m(u) = \inf\big\{\|u - \mathcal{P}_\sigma u\| : \sigma \subset \mathbb{N},\ |\sigma| = m\big\}.$$

Proof. Let $u \in \mathcal{H}_{\mathbb{H}}$ and let $\sigma \subset \mathbb{N}$ with $|\sigma| = m$. Since $\mathcal{P}_\sigma u$ is the orthogonal projection of u onto the closed subspace $[\mathcal{U}]_\sigma$, we have

$$\|u - \mathcal{P}_\sigma u\| = \operatorname{dist}(u, [\mathcal{U}]_\sigma) = \inf\big\{\|u - w\| : w \in [\mathcal{U}]_\sigma\big\}.$$

Consequently, for every $w \in [\mathcal{U}]_\sigma$,

$$\|u - \mathcal{P}_\sigma u\| \le \|u - w\|.$$

Taking the infimum over all subsets $\sigma \subset \mathbb{N}$ with $|\sigma| = m$ yields the desired equality. □

Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a Riesz basis for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ and let $\mathcal{S}_\mathcal{U}$ denote the associated frame operator. For a finite subset $\sigma \subset \mathbb{N}$ with $|\sigma| = m$, define the operators $\mathcal{S}_{\mathcal{U}_\sigma} : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ and $\mathcal{Q}_{\mathcal{U}_\sigma} : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ by

$$\mathcal{S}_{\mathcal{U}_\sigma}(u) = \sum_{j\in\sigma} u_j \left\langle u_j \middle| u \right\rangle,$$

and

$$\mathcal{Q}_{\mathcal{U}_\sigma}(u) = \mathcal{S}_{\mathcal{U}_\sigma}\mathcal{S}_\mathcal{U}^{-1}(u) = \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_\mathcal{U}^{-1}u_j \middle| u \right\rangle, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

Let $\rho = \{j_n\}_{n=1}^\infty$ be a permutation of $\mathbb{N}$ such that

$$\left|\left\langle \mathcal{S}_\mathcal{U}^{-1}u_{j_n} \middle| u \right\rangle\right|^2 \ge \left|\left\langle \mathcal{S}_\mathcal{U}^{-1}u_{j_{n+1}} \middle| u \right\rangle\right|^2, \quad n \in \mathbb{N}.$$

Then the m-greedy approximant of $u \in \mathcal{H}_{\mathbb{H}}$ with respect to the Riesz basis $\mathcal{U}$ is defined by

$$G_m(u) = \sum_{n=1}^m u_{j_n} \left\langle \mathcal{S}_\mathcal{U}^{-1}u_{j_n} \middle| u \right\rangle.$$

In the quaternionic setting, the greedy selection is performed with respect to the modulus of the coefficients $\left\langle \mathcal{S}_\mathcal{U}^{-1}u_j \middle| u \right\rangle$, since quaternions do not admit a natural order. This ensures that the greedy algorithm is well defined and independent of the choice of representatives.

Next, we show that every Riesz basis in a right quaternionic Hilbert space is greedy.

Theorem 2.2. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a Riesz basis for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. Then, for any $m \in \mathbb{N}$ and all $u \in \mathcal{H}_{\mathbb{H}}$,*

$$\|u - G_m(u)\| \leq \sqrt{\frac{r_2}{r_1}}\sigma_m(u).$$

Proof. Since a Riesz basis is a frame with the same bounds, $\mathcal{U}$ is a frame for $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. By definition of the m-greedy approximant,

$$u - G_m(u) = \sum_{n>m} u_{j_n} \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle.$$

Therefore,

$$\begin{aligned}
\|u - G_m(u)\|^2 &= \sup_{\substack{v\in\mathcal{H}_{\mathbb{H}} \\ \|v\|=1}} \left| \sum_{n>m} \langle v| u_{j_n}\rangle \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle \right|^2 \\
&\leq \sum_{n>m} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle \right|^2 \sup_{\substack{v\in\mathcal{H}_{\mathbb{H}} \\ \|v\|=1}} \sum_{n>m} |\langle v| u_{j_n}\rangle|^2 \\
&\leq r_2 \sum_{n>m} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle \right|^2.
\end{aligned}$$

By the definition of the greedy ordering, for any subset $\sigma \subset \mathbb{N}$ with $|\sigma| = m$,

$$\sum_{n>m} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle \right|^2 \leq \sum_{j\in\mathbb{N}\setminus\sigma} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right|^2.$$

This gives,

$$\|u - G_m(u)\|^2 \leq r_2 \sum_{j\in\mathbb{N}\setminus\sigma} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right|^2.$$

Let $\mathcal{S}_\sigma$ denote the frame operator of the frame $\mathcal{U}_\sigma = \{u_j\}_{j\in\sigma}$ on $[\mathcal{U}]_\sigma$ and let $\mathcal{P}_\sigma$ be the orthogonal projection from $\mathcal{H}_{\mathbb{H}}$ onto $[\mathcal{U}]_\sigma$. Then

$$\mathcal{P}_\sigma(u) = \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_\sigma^{-1} u_j \middle| u \right\rangle.$$

Using the canonical dual frame inequalities, we obtain

$$\begin{aligned}
\frac{1}{r_1}\|u - \mathcal{P}_\sigma(u)\|^2 &\geq \sum_{j\in\mathbb{N}} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u - \mathcal{P}_\sigma(u) \right\rangle \right|^2 \\
&\geq \sum_{j\in\mathbb{N}\setminus\sigma} \left| \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right|^2.
\end{aligned}$$

Consequently, for any $\sigma \subset \mathbb{N}$ with $|\sigma| = m$,

$$\|u - G_m(u)\|^2 \leq \frac{r_2}{r_1}\|u - \mathcal{P}_\sigma(u)\|^2 = \frac{r_2}{r_1}\sigma_m^2(u). \qquad \square$$

Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ and let $\mathcal{S}_{\mathcal{U}}^{-1}\mathcal{U} = \left\{\mathcal{S}_{\mathcal{U}}^{-1}u_j\right\}_{j\in\mathbb{N}}$ denote its canonical dual frame. We define the m-term approximation manifolds associated with the frame $\mathcal{U}$ by

$$\sum_m \mathcal{U} := \left\{ \sum_{j\in\sigma} u_j q_j : \sigma \subset \mathbb{N},\ |\sigma| = m,\ q_j \in \mathbb{H} \right\},$$

$$\widetilde{\sum_m}\mathcal{U} := \left\{ \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_j \middle|\, u \right\rangle : \sigma \subset \mathbb{N},\ |\sigma| = m,\ u \in \mathcal{H}_{\mathbb{H}} \right\}.$$

For each $u \in \mathcal{H}_{\mathbb{H}}$, the corresponding m-term approximation errors are defined by

$$\sigma_m(u) := \inf\big\{\|u - v\| : v \in \Sigma_m(\mathcal{U})\big\},$$
$$\widetilde{\sigma}_m(u) := \inf\big\{\|u - v\| : v \in \widetilde{\Sigma}_m(\mathcal{U})\big\}.$$

For a finite subset $\sigma \subset \mathbb{N}$ with $|\sigma| = m$, define the operators $\mathcal{S}_{\mathcal{U}_\sigma}, \mathcal{Q}_{\mathcal{U}_\sigma} : \mathcal{H}_{\mathbb{H}} \to \mathcal{H}_{\mathbb{H}}$ by

$$\mathcal{S}_{\mathcal{U}_\sigma}(u) := \sum_{j\in\sigma} u_j \langle u_j |\, u\rangle,$$
$$\mathcal{Q}_{\mathcal{U}_\sigma}(u) := \mathcal{S}_{\mathcal{U}_\sigma}\big(\mathcal{S}_{\mathcal{U}}^{-1}u\big) = \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_j \middle|\, u \right\rangle, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

Let $\rho = \{j_n\}_{n\in\mathbb{N}}$ be a permutation of $\mathbb{N}$ such that

$$\left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n} \middle|\, u\right\rangle\right|^2 \geq \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_{n+1}} \middle|\, u\right\rangle\right|^2, \quad n \in \mathbb{N}.$$

For each $u \in \mathcal{H}_{\mathbb{H}}$, the m-greedy approximant of u with respect to the frame $\mathcal{U}$ is defined by

$$G_m(u) := \sum_{n=1}^{m} u_{j_n} \left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n} \middle|\, u \right\rangle.$$

In particular, there exists a subset $\sigma_0 \subset \mathbb{N}$ with $|\sigma_0| = m$ such that $G_m(u) = \mathcal{Q}_{\mathcal{U}_{\sigma_0}}(u)$.

Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ and let $\sigma \subset \mathbb{N}$ be a finite subset. Define the operators $\mathcal{T}_{\mathcal{U}_\sigma} : \ell^2(\sigma, \mathbb{H}) \to \mathcal{H}_{\mathbb{H}}$ and $\mathcal{T}_{\mathcal{U}_\sigma}^* : \mathcal{H}_{\mathbb{H}} \to \ell^2(\sigma, \mathbb{H})$ by

$$\mathcal{T}_{\mathcal{U}_\sigma}(\{q_j\}_{j\in\sigma}) := \sum_{j\in\sigma} u_j q_j, \quad \{q_j\}_{j\in\sigma} \in \ell^2(\sigma, \mathbb{H}),$$
$$\mathcal{T}_{\mathcal{U}_\sigma}^*(u) := \{\langle u_j |\, u\rangle\}_{j\in\sigma}, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

Observe that

$$\mathcal{S}_{\mathcal{U}_\sigma}(u) = \mathcal{T}_{\mathcal{U}_\sigma}\mathcal{T}^*_{\mathcal{U}_\sigma}(u) = \sum_{j\in\sigma} u_j\langle u_j|\, u\rangle, \quad u \in \mathcal{H}_{\mathbb{H}},$$

and that $\mathcal{T}^*_{\mathcal{U}_\sigma}$ is the adjoint of $\mathcal{T}_{\mathcal{U}_\sigma}$.

To estimate greedy and truncated frame approximations, it is necessary to control the norm of frame operators associated with finite subsets of frame elements. This can be achieved by viewing these operators through their corresponding analysis and synthesis mappings. The following lemma provides a uniform bound for such truncated frame operators.

Lemma 2.3. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2 and let $\sigma \subset \mathbb{N}$. Then*

$$\|\mathcal{S}_{\mathcal{U}_\sigma}(u)\| \le r_2\|u\|, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

Proof. Let $u \in \mathcal{H}_{\mathbb{H}}$. By the frame inequality, we have

$$\|\mathcal{T}^*_{\mathcal{U}_\sigma}(u)\|^2 = \sum_{j\in\sigma} |\langle u_j|\, u\rangle|^2 \le \sum_{j=1}^{\infty} |\langle u_j|\, u\rangle|^2 \le r_2\|u\|^2.$$

Hence,

$$\|\mathcal{T}^*_{\mathcal{U}_\sigma}(u)\| \le \sqrt{r_2}\|u\|, \quad u \in \mathcal{H}_{\mathbb{H}}.$$

Since $\|\mathcal{T}_{\mathcal{U}_\sigma}\| = \|\mathcal{T}^*_{\mathcal{U}_\sigma}\|$, it follows that

$$\|\mathcal{S}_{\mathcal{U}_\sigma}(u)\| = \|\mathcal{T}_{\mathcal{U}_\sigma}\mathcal{T}^*_{\mathcal{U}_\sigma}(u)\| \le \|\mathcal{T}_{\mathcal{U}_\sigma}\|\|\mathcal{T}^*_{\mathcal{U}_\sigma}\|\|u\| \le r_2\|u\|. \qquad \square$$

Next, we show that frames in right quaternionic Hilbert spaces satisfy the quasi-greedy condition.

Theorem 2.4. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. Then:*

(a) $\|G_m(u)\| \le \dfrac{r_2}{r_1}\|u\|$ *for all* $u \in \mathcal{H}_{\mathbb{H}}$.

(b) *For each* $u \in \mathcal{H}_{\mathbb{H}}$, $\|u - G_m(u)\| \to 0$ *as* $m \to \infty$.

Proof. (a) Let $G_m(u) = \mathcal{Q}_{\sigma_0}(u)$ for some subset $\sigma_0 \subset \mathbb{N}$ with $|\sigma_0| = m$. Then

$$\|G_m(u)\| = \|\mathcal{Q}_{\sigma_0}(u)\| = \left\|\mathcal{S}_{\mathcal{U}_{\sigma_0}}\mathcal{S}_{\mathcal{U}}^{-1}(u)\right\| \le \|\mathcal{S}_{\mathcal{U}_{\sigma_0}}\|\left\|\mathcal{S}_{\mathcal{U}}^{-1}\right\|\|u\|.$$

By Lemma 2.3, $\|\mathcal{S}_{\mathcal{U}_{\sigma_0}}\| \le r_2$ and since $\left\|\mathcal{S}_{\mathcal{U}}^{-1}\right\| \le r_1^{-1}$, we obtain

$$\|G_m(u)\| \le \frac{r_2}{r_1}\|u\|.$$

(b) By definition of the greedy approximant,

$$u - G_m(u) = \sum_{n>m} u_{j_n}\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n}|\, u\right\rangle.$$

Therefore,

$$\begin{aligned}\|u - G_m(u)\|^2 &= \sup_{\substack{v\in\mathcal{H}_{\mathbb{H}}\\ \|v\|=1}} \left|\sum_{n>m} \langle v|\, u_{j_n}\rangle \left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n}\right|\, u\right\rangle\right|^2 \\ &\leq \sum_{n>m} \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n}\right|\, u\right\rangle\right|^2 \sup_{\substack{v\in\mathcal{H}_{\mathbb{H}}\\ \|v\|=1}} \sum_{n>m} |\langle v|\, u_{j_n}\rangle|^2 \\ &\leq r_2 \sum_{n>m} \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n}\right|\, u\right\rangle\right|^2.\end{aligned}$$

By the greedy ordering, for any $\sigma \subset \mathbb{N}$ with $|\sigma| = m$,

$$\sum_{n>m} \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n}\right|\, u\right\rangle\right|^2 \leq \sum_{j\in\mathbb{N}\setminus\sigma} \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j}\right|\, u\right\rangle\right|^2.$$

Since $\left\{\mathcal{S}_{\mathcal{U}}^{-1}u_j\right\}_{j\in\mathbb{N}}$ is a Bessel sequence, the right-hand side converges to zero as $m \to \infty$. Consequently, $\|u - G_m(u)\| \longrightarrow 0 \quad \text{as } m \to \infty.$ □

Next, we state a lemma that will be used in the subsequent result.

Lemma 2.5. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with frame bounds r_1 and r_2 and let $\mathcal{T}_{\mathcal{U}} : \ell^2(\mathbb{H}) \to \mathcal{H}_{\mathbb{H}}$ denote its synthesis operator. Then*

$$\ell^2(\mathbb{H}) = \ker \mathcal{T}_{\mathcal{U}} \oplus \operatorname{Ran} \mathcal{T}_{\mathcal{U}}^*.$$

Moreover, for every sequence $\boldsymbol{q} = \{q_j\}_{j\in\mathbb{N}} \in \ker(\mathcal{T}_{\mathcal{U}})^{\perp} = \operatorname{Ran} \mathcal{T}_{\mathcal{U}}^*$, the following inequalities hold:

$$r_1 \sum_{j\in\mathbb{N}} |q_j|^2 \leq \left\|\sum_{j\in\mathbb{N}} u_j q_j\right\|^2 \leq r_2 \sum_{j\in\mathbb{N}} |q_j|^2.$$

Next, we show that frames in right quaternionic Hilbert spaces also satisfy the almost greedy condition.

Theorem 2.6. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. Then, for all $u \in \mathcal{H}_{\mathbb{H}}$ and $m \in \mathbb{N}$,*

$$\|u - G_m(u)\| \leq \sqrt{\frac{r_2}{r_1}}\widetilde{\sigma}_m(u).$$

Proof. As shown in Theorem 2.4, for any subset $\sigma \subset \mathbb{N}$ with $|\sigma| = m$ and any $u \in \mathcal{H}_{\mathbb{H}}$,

$$\|u - G_m(u)\|^2 \leq r_2 \sum_{j\in\mathbb{N}\setminus\sigma} \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j}\right|\, u\right\rangle\right|^2.$$

By Lemma 2.5, we have

$$r_1 \sum_{j\in\mathbb{N}} |q_j|^2 \le \left\| \sum_{j\in\mathbb{N}} u_j q_j \right\|^2 \le r_2 \sum_{j\in\mathbb{N}} |q_j|^2, \quad \{q_j\}_{j\in\mathbb{N}} \in \operatorname{Ran} \mathcal{T}_{\mathcal{U}}^*.$$

Moreover,

$$\left\{ \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right\}_{j\in\mathbb{N}} = \mathcal{T}_{\mathcal{U}}^* (\mathcal{S}_{\mathcal{U}}^{-1} u) \in \operatorname{Ran} \mathcal{T}_{\mathcal{U}}^*.$$

Therefore, for any $\sigma \subset \mathbb{N}$ with $|\sigma| = m$,

$$\|u - G_m(u)\|^2 \le \frac{r_2}{r_1} \left\| \sum_{j\in\mathbb{N}\setminus\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right\|^2.$$

Let

$$v = \sum_{j\in\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \in \widetilde{\sum_m} \mathcal{U}.$$

Then

$$\|u - v\|^2 = \left\| \sum_{j\in\mathbb{N}\setminus\sigma} u_j \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right\|^2.$$

Consequently,

$$\|u - G_m(u)\|^2 \le \frac{r_2}{r_1} \|u - v\|^2, \qquad v \in \widetilde{\sum_m} \mathcal{U}.$$

Taking the infimum over all such v yields

$$\|u - G_m(u)\| \le \sqrt{\frac{r_2}{r_1}}\, \widetilde{\sigma}_m(u). \qquad \square$$

3. Quaternionic Approximation Spaces

In this section, we introduce weak-$\ell_p(\mathbb{H})$ sequence spaces and Lorentz sequence spaces in the quaternionic setting. We also define the approximation spaces $\mathcal{A}^s(\mathcal{U})$ and $\mathcal{A}^s_q(\mathcal{U})$ associated with a frame $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ in a right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$.

Definition 3.1. Let $0 < p < \infty$. The weak-$\ell_p(\mathbb{H})$ sequence space, denoted by $w\ell_p(\mathbb{H})$, consists of all quaternionic sequences $\boldsymbol{q} = \{q_j\}_{j\in\mathbb{N}}$ such that

$$\|\boldsymbol{q}\|_{w\ell_p(\mathbb{H})} := \sup_{j\ge 1} \left(j^{1/p} q_j^* \right) < \infty,$$

where $\left\{ q_j^* \right\}_{j=1}^{\infty}$ denotes the non-increasing rearrangement of the sequence $\{|q_j|\}_{j=1}^{\infty}$.

Equivalently,

$$w\ell_p(\mathbb{H}) = \left\{\{q_j\}_{j\in\mathbb{N}} : \|\boldsymbol{q}\|_{w\ell_p(\mathbb{H})} < \infty\right\}$$

Definition 3.2. Let $0 < p < \infty$ and $1 \le q \le \infty$. The quaternionic Lorentz sequence space $\ell_{p,q}(\mathbb{H})$ consists of all quaternionic sequences $\boldsymbol{q} = \{q_j\}_{j\in\mathbb{N}}$ such that

$$\|\boldsymbol{q}\|_{\ell_{p,q}(\mathbb{H})} = \left(\sum_{j=1}^{\infty}(q_j^*)^q j^{\frac{q}{p}-1}\right)^{1/q} < \infty), \quad (1 \le q < \infty,$$

where $\left\{q_j^*\right\}_{j=1}^{\infty}$ denotes the non-increasing rearrangement of the sequence $\{|q_j|\}_{j=1}^{\infty}$. In the case $q = \infty$, the norm is defined by

$$\|\boldsymbol{q}\|_{\ell_{p,\infty}(\mathbb{H})} := \sup_{j\ge1} j^{1/p} q_j^*.$$

Remark 3.3. For $0 < p < \infty$ and $1 \le q < \infty$, the Lorentz quasi-norm admits the equivalent dyadic characterization

$$\|\boldsymbol{q}\|_{\ell_{p,q}(\mathbb{H})} = \left(\sum_{j=1}^{\infty}(q_j^*)^q j^{\frac{q}{p}-1}\right)^{1/q} \asymp \left(\sum_{k=0}^{\infty} 2^{k\frac{q}{p}}(q_{2^k}^*)^q\right)^{1/q},$$

where $\left\{q_j^*\right\}_{j\ge1}$ denotes the non-increasing rearrangement of $\{|q_j|\}_{j\ge1}$.

In the following, we define approximation spaces associated with frames in right quaternionic Hilbert spaces.

Definition 3.4. Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$. For $0 < s < \infty$, the approximation space $\mathcal{A}^s(\mathcal{U})$ consists of all $u \in \mathcal{H}_{\mathbb{H}}$ such that

$$\|u\|_{\mathcal{A}^s(\mathcal{U})} := \sup_{m\ge1} m^s \sigma_m(u) < \infty.$$

Definition 3.5. Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$. For $0 < s < \infty$ and $1 \le q \le \infty$, the approximation space $\mathcal{A}_q^s(\mathcal{U})$ consists of all $u \in \mathcal{H}_{\mathbb{H}}$ such that

$$\|u\|_{\mathcal{A}_q^s(\mathcal{U})} := \left(\sum_{m=1}^{\infty}(\sigma_m(u))^q\, m^{qs-1}\right)^{1/q} < \infty, \quad (1 \le q < \infty),$$

with the usual modification

$$\|u\|_{\mathcal{A}_\infty^s(\mathcal{U})} := \sup_{m\ge1} m^s \sigma_m(u).$$

Indeed, for $0 < s < \infty$ and $1 \le q < \infty$, the quasi-norm of the approximation space $\mathcal{A}_q^s(\mathcal{U})$ admits the equivalent dyadic characterization

$$\|u\|_{\mathcal{A}_q^s(\mathcal{U})} = \left(\sum_{m=1}^{\infty} (\sigma_m(u))^q \, m^{qs-1}\right)^{1/q} \asymp \left(\sum_{k=0}^{\infty} (\sigma_{2^k}(u))^q \, 2^{kqs}\right)^{1/q}.$$

We now address the following question: given $r > 0$, for which elements $u \in \mathcal{H}_{\mathbb{H}}$ does the m-term approximation error satisfy $\sigma_m(u) \le Cm^{-r}$ for all $m \in \mathbb{N}$ and some constant $C > 0$?

Theorem 3.6. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. If the canonical dual coefficients, $\left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_j \middle| u\right\rangle\right\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H})$, with $p < 2$ and $\frac{1}{p} = s + \frac{1}{2}$, then $u \in \mathcal{A}^s(\mathcal{U})$.*

Proof. Let $c := \left\|\left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_j \middle| u\right\rangle\right\}_{j\in\mathbb{N}}\right\|_{w\ell_p(\mathbb{H})}$. Let $\{j_n\}_{n\in\mathbb{N}}$ be a permutation of $\mathbb{N}$ such that

$$\left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n} \middle| u\right\rangle\right| \ge \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_{n+1}} \middle| u\right\rangle\right|, \quad n \in \mathbb{N},$$

and set $c_n^* := \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1}u_{j_n} \middle| u\right\rangle\right|$. By assumption, $n^{1/p}c_n^* \le c$ for all $n \in \mathbb{N}$. Then, by Theorem 2.4, for every $m \in \mathbb{N}$, we have

$$\|u - G_m(u)\|^2 \le r_2 \sum_{n=m+1}^{\infty} (c_n^*)^2.$$

Define the dyadic sums

$$F_k^2 := \sum_{n=2^k}^{2^{k+1}-1} (c_n^*)^2, \quad k = 0, 1, 2, \ldots.$$

Using $c_n^* \le cn^{-1/p}$ and $\frac{1}{p} = s + \frac{1}{2}$, we obtain

$$F_k^2 \le c^2 \sum_{n=2^k}^{2^{k+1}-1} n^{-2/p} \le c^2 2^{-2ks}, \quad k \ge 0.$$

Let $m \in \mathbb{N}$ and choose $l \in \mathbb{N}$ such that $2^l \le m < 2^{l+1}$. Then

$$\begin{aligned} m^{2s}\sigma_m^2(u) &\le m^{2s} \|u - G_m(u)\|^2 \\ &\le m^{2s} r_2 \sum_{n=m+1}^{\infty} (c_n^*)^2 \\ &\le r_2 2^{2s(l+1)} \sum_{n=2^l}^{\infty} (c_n^*)^2 \\ &= r_2 2^{2s(l+1)} \sum_{k=l}^{\infty} F_k^2 \end{aligned}$$

$$\leq r_2 2^{2s(l+1)} \sum_{k=l}^{\infty} c^2 2^{-2ks}$$
$$= r_2 c^2 \frac{2^{2s}}{1-2^{-2s}}.$$

Hence, $\sup_{m\geq 1} m^s \sigma_m(u) < \infty$. □

Next, we show that the converse of Theorem 3.6 holds for Riesz bases.

Theorem 3.7. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a Riesz basis for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. Then*

$$u \in \mathcal{A}^s(\mathcal{U}) \quad \text{if and only if} \quad \left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle\right\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H}),$$

where $\frac{1}{p} = s + \frac{1}{2}$ and $p < 2$.

Proof. If $\{\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j | u\rangle\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H})$, then the result follows directly from Theorem 3.6. Conversely, assume that $u \in \mathcal{A}^s(\mathcal{U})$. Let $\{j_n\}_{n\in\mathbb{N}}$ be a permutation of $\mathbb{N}$ such that

$$\left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle\right| \geq \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_{n+1}} \middle| u \right\rangle\right|, \quad n \in \mathbb{N},$$

and define

$$c_n^* := \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle\right|.$$

By the proof of Theorem 2.2, for every $m \in \mathbb{N}$ we have

$$\sum_{n=m+1}^{\infty} (c_n^*)^2 \leq \frac{1}{r_1} \sigma_m^2(u).$$

In particular,

$$(c_{2m}^*)^2 \leq \frac{1}{m} \sum_{n=m+1}^{2m} (c_n^*)^2 \leq \frac{1}{m} \sum_{n=m+1}^{\infty} (c_n^*)^2 \leq \frac{1}{r_1} m^{-1} \sigma_m^2(u), \quad m \geq 1.$$

A similar estimate holds for c_{2m+1}^*. Since $u \in \mathcal{A}^s(\mathcal{U})$, there exists a constant $C > 0$ such that

$$\sigma_m(u) \leq C m^{-s}, \quad m \geq 1.$$

Combining the above inequalities yields

$$c_m^* \leq C r_1^{-1/2} m^{-s-\frac{1}{2}}, \quad m \geq 1.$$

Therefore,

$$m^{1/p} c_m^* = m^{s+\frac{1}{2}} c_m^* \leq C r_1^{-1/2}, \quad m \geq 1,$$

which shows that $\left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle\right\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H})$. □

Next, we give a simple example to illustrate Theorem 3.7.

Example 3.8. Let $\mathcal{H}_{\mathbb{H}} = \ell^2(\mathbb{H})$ and let $\{e_j\}_{j\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell^2(\mathbb{H})$. Define $u_j = 1/3e_j, \quad j \in \mathbb{N}$. Then $\{u_j\}_{j\in\mathbb{N}}$ forms a Riesz basis for $\ell^2(\mathbb{H})$ with bounds $r_1 = r_2 = \frac{1}{9}$. The frame operator is

$$\mathcal{S}_{\mathcal{U}} u = \sum_{j\in\mathbb{N}} \frac{1}{9} e_j \langle e_j | u \rangle = \frac{1}{9} u, \qquad \mathcal{S}_{\mathcal{U}}^{-1} u = \sum_{j\in\mathbb{N}} 9 e_j \langle e_j | u \rangle = 9I.$$

Next, let $u = \sum\limits_{j\in\mathbb{N}} e_j q_j, q_j \in \mathbb{H}$ as $\mathcal{S}_{\mathcal{U}}^{-1} u_j = 9\left(\frac{1}{3} e_j\right) = 3e_j$. Then, we obtain

$$\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle = 3q_j, \quad j \in \mathbb{N}.$$

Therefore $\left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle\right\}_{j\in\mathbb{N}} = \{3q_j\}_{j\in\mathbb{N}}$, which implies

$$\{3q_j\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H}) \quad \iff \quad \{q_j\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H}).$$

Thus by Theorem 3.7,

$$u \in \mathcal{A}^s(\mathcal{U}) \quad \text{if and only if} \quad \{\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle\}_{j\in\mathbb{N}} \in w\ell_p(\mathbb{H}),$$

where $\frac{1}{p} = s + \frac{1}{2}$ and $0 < q < 2$.

We now give a characterization of the approximation space $\mathcal{A}_q^s(\mathcal{U})$ in terms of Lorentz sequence spaces.

Theorem 3.9. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a frame for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. If $\left\{\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle\right\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H})$, then $u \in \mathcal{A}_q^s(\mathcal{U})$, where*

$$\frac{1}{p} = s + \frac{1}{2}, \qquad 0 < q < 2.$$

Proof. Let $\{j_n\}_{n\in\mathbb{N}}$ be a permutation of $\mathbb{N}$ such that

$$\left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle\right| \geq \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_{n+1}} \middle| u \right\rangle\right|, \quad n \in \mathbb{N},$$

and define $c_n^* := \left|\left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_{j_n} \middle| u \right\rangle\right|$. By Theorem 2.2, for each $m \in \mathbb{N}$,

$$\sigma_m^2(u) \leq \|u - G_m(u)\|^2 \leq r_2 \sum_{n=m+1}^{\infty} (c_n^*)^2 .$$

In particular,

$$\sigma_{2^m}(u) \leq \sqrt{r_2} \left(\sum_{n=2^m}^{\infty} (c_n^*)^2 \right)^{1/2} .$$

Using a dyadic decomposition, we obtain

$$\sum_{n=2^m}^{\infty} (c_n^*)^2 \leq \sum_{k=m}^{\infty} 2^k \left(c_{2^k}^*\right)^2 ,$$

and since $0 < q < 2$, the embedding $\ell_q \hookrightarrow \ell_2$ yields

$$\left(\sum_{k=m}^{\infty} 2^k \left(c^*_{2^k}\right)^2\right)^{1/2} \leq \left(\sum_{k=m}^{\infty} 2^{kq/2} \left(c^*_{2^k}\right)^q\right)^{1/q}.$$

Therefore,

$$\sigma_{2^m}(u) \leq \sqrt{r_2} \left(\sum_{k=m}^{\infty} 2^{kq/2} \left(c^*_{2^k}\right)^q\right)^{1/q}.$$

Consequently,

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{msq} \sigma^q_{2^m}(u) &\leq (r_2)^{q/2} \sum_{m=1}^{\infty} 2^{msq} \sum_{k=m}^{\infty} 2^{kq/2} \left(c^*_{2^k}\right)^q \\
&\leq C(r_2)^{q/2} \sum_{k=1}^{\infty} 2^{k(s+1/2)q} \left(c^*_{2^k}\right)^q \\
&= C(r_2)^{q/2} \sum_{k=1}^{\infty} 2^{kq/p} \left(c^*_{2^k}\right)^q \\
&< \infty,
\end{aligned}$$

where $\frac{1}{p} = s + \frac{1}{2}$. Hence $u \in \mathcal{A}^s_q(\mathcal{U})$. □

Finally, we show that the converse of Theorem 3.9 holds for Riesz bases in right quaternionic Hilbert spaces.

Theorem 3.10. *Let $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ be a Riesz basis for the right quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$ with bounds r_1 and r_2. Then*

$$u \in \mathcal{A}^s_q(\mathcal{U}) \quad \textit{if and only if} \quad \left\{\left\langle \mathcal{S}^{-1}_{\mathcal{U}} u_j \middle| u \right\rangle\right\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H}),$$

where

$$\frac{1}{p} = s + \frac{1}{2}, \qquad 0 < q < 2.$$

Proof. If $\{\langle \mathcal{S}^{-1}_{\mathcal{U}} u_j | u\rangle\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H})$, then the result follows immediately from Theorem 3.9. Conversely, assume that $u \in \mathcal{A}^s_q(\mathcal{U})$. Since $\mathcal{U}$ is a Riesz basis, Theorem 3.7 yields

$$c^*_j := \left|\left\langle \mathcal{S}^{-1}_{\mathcal{U}} u_{j_n} \middle| u \right\rangle\right| \leq \frac{1}{\sqrt{r_1}} j^{-1/2} \sigma_j(u), \quad j \in \mathbb{N}.$$

Therefore,

$$\sum_{j=1}^{\infty} j^{\frac{q}{p}-1} \left(c^*_j\right)^q \leq \frac{1}{(r_1)^{q/2}} \sum_{j=1}^{\infty} j^{\frac{q}{p}-1} j^{-q/2} \sigma^q_j(u)$$

$$= \frac{1}{(r_1)^{q/2}} \sum_{j=1}^{\infty} j^{qs-1} \sigma_j^q(u)$$
$$< \infty,$$

where $\frac{1}{p} = s + \frac{1}{2}$. Hence $\left\{ \left\langle \mathcal{S}_{\mathcal{U}}^{-1} u_j \middle| u \right\rangle \right\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H})$. □

To illustrate the abstract characterization results obtained above, we conclude this section with simple examples in a standard right quaternionic Hilbert space. The examples demonstrate how the approximation spaces associated with a Riesz basis reduce to classical Lorentz sequence spaces.

Example 3.11. Let $\mathcal{H}_{\mathbb{H}} = \ell^2(\mathbb{H})$ and let $\lambda_j = 1 + \frac{1}{1+j}, j \in \mathbb{N}$ be a bounded sequence satisfying $1 < \lambda_j \leq \frac{3}{2}$ for all $j \in \mathbb{N}$. Define $u_j = (1 + \frac{1}{1+j})e_j$, where $\{e_j\}_{j\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell^2(\mathbb{H})$. Then $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}_{\mathbb{H}}$ with bounds 1 and $\frac{9}{4}$. The frame operator for $u = \sum_{j\in\mathbb{N}} e_j q_j, q_j \in \mathbb{H}$ is

$$\mathcal{S}_{\mathcal{U}} u = \sum_{j\in\mathbb{N}} \left(\frac{j+2}{j+1}\right)^2 e_j \langle e_j | u \rangle = \left(\frac{j+2}{j+1}\right)^2 e_j q_j$$
$$\mathcal{S}_{\mathcal{U}}^{-1} u = \left(\frac{j+1}{j+2}\right)^{-2} e_j q_j,$$

Hence, by Theorem 3.10,

$$u \in \mathcal{A}_q^s(\mathcal{U}) \quad \text{if and only if} \quad \left\{ \left(\frac{j+1}{j+2}\right) q_j \right\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H}),$$

where $\frac{1}{p} = s + \frac{1}{2}$ and $0 < q < 2$. As $\frac{2}{3} < \left(\frac{j+1}{j+2}\right) q_j < 1$, we get

$$\left\{ \left(\frac{j+1}{j+2}\right) q_j \right\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H}) \quad \iff \quad \{q_j\} \in \ell_{p,q}(\mathbb{H}).$$

Thus,

$$\mathcal{A}_q^s(\mathcal{U}) = \left\{ \sum_{j\in\mathbb{N}} e_j q_j : \quad q_j \in \ell_{p,q}(\mathbb{H}) \right\}.$$

Example 3.12. Let $\mathcal{H}_{\mathbb{H}} = \ell^2(\mathbb{H})$ and let $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ be a bounded sequence satisfying $0 < a \leq \lambda_j \leq b < \infty$ for all $j \in \mathbb{N}$. Define $u_j = \lambda_j e_j$, where $\{e_j\}_{j\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell^2(\mathbb{H})$. Then $\mathcal{U} = \{u_j\}_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}_{\mathbb{H}}$ with bounds a^2 and b^2. The

corresponding frame operator satisfies

$$\mathcal{S}_{\mathcal{U}}u = \sum_{j\in\mathbb{N}} \lambda_j^2 e_j \langle e_j | u\rangle, \qquad \mathcal{S}_{\mathcal{U}}^{-1}u = \sum_{j\in\mathbb{N}} \lambda_j^{-2} e_j \langle e_j | u\rangle.$$

For $u = \sum_j e_j q_j$, we obtain

$$\langle \mathcal{S}_{\mathcal{U}}^{-1}u_j | u\rangle = \lambda_j^{-1} q_j, \quad j \in \mathbb{N}.$$

Hence, by Theorem 3.10,

$$u \in \mathcal{A}_q^s(\mathcal{U}) \quad \text{if and only if} \quad \{\lambda_j^{-1}q_j\}_{j\in\mathbb{N}} \in \ell_{p,q}(\mathbb{H}),$$

where $\frac{1}{p} = s + \frac{1}{2}$ and $0 < q < 2$.

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