Sahand Communications in Mathematical Analysis (SCMA) Vol. ··· No. ··· (20??), ···-···
http://scma.maragheh.ac.ir
DOI: ...

Fixed Point Theory for a Family of Mappings with Application in Nonlinear Integral Equations

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Abstract. In this paper, we introduce a novel three step iterative algorithm and present a weak convergence result. We prove that the sequence generated by our algorithm converges weakly to a common fixed point of the family of mappings satisfying condition (E). We also present an application of our result for finding the solution of a family of nonlinear integral equations. Lastly, we present a couple of non-trivial examples which satisfy the condition (E) but fails to satisfy nonexpansiveness and present the convergence behaviour of our algorithm for different choices of initial guesses and constants.

1. Introduction

Suppose $\mathcal{H}$ is a Banach space along with norm $\|\cdot\|$, $\mathcal{C} \neq \emptyset$ is a convex and closed subset of $\mathcal{H}$. Suppose S is a mapping defined on $\mathcal{C}$. Then, any point $p \in \mathcal{C}$ such that

$$S(p) = p,$$

is said to be a fixed point of S and the set of fixed points of S. We denote the set of fixed point of S by $F(S) = \{p \in \mathcal{C} : S(p) = p\}$. The objective of finding a point $p \in \mathcal{C}$ such that $S(p) = p$ is known as fixed point problem.

A mapping S defined on $\mathcal{C}$ to itself is said to be a contraction mapping if it satisfies the condition:

$$\|S(x) - S(y)\| \leq \mu \|x - y\|, \quad \forall \mu \in [0, 1]. \tag{1.1}$$

2020 *Mathematics Subject Classification.* 47H10, 47H09, 54A20.
Key words and phrases. Fixed point iteration, Condition (E) mapping, Weak convergence
Received: 2025-12-09, Accepted: 2026-09-14.
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In (1.1), if we take μ as 1 then S is said to be a nonexpansive mapping. A number of extensions and generalizations of nonexpansive mappings and their results have been introduced by many authors. Some of the notable extensions and generalizations of nonexpansive mappings can be found in [1, 3, 4, 17].

In 1967, Diaz [3] generalized the concept of nonexpansive mappings and introduced quasi nonexpansive mappings as follows:

$$\|S(x) - p\| \leq \|x - p\|, \quad \forall x \in \mathcal{C} \text{ and } p \in F(S).$$

In 2008, Suzuki [17] introduced a new class of nonexpansive mappings, called mappings satisfying condition (C), which is weaker than nonexpansive mappings but stronger than quasi nonexpansive mappings, and defined as follows if

$$\frac{1}{2}\|x - S(x)\| \leq \|x - y\| \quad \Rightarrow \quad \|S(x) - S(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{C}.$$

Recently in 2011, Fuster et.al. [4] proposed a new class of nonexpansive mappings which are weaker than nonexpansive but stronger than condition (C), called mapping satisfying condition (E_μ) and defined as: The mapping $S : \mathcal{C} \to \mathcal{C}$ is said to satisfy conditions (E_μ) on $\mathcal{C}$, if $\exists$ $\mu \geq 1$ such that $\forall x, y \in \mathcal{C}$,

$$\|x - S(y)\| \leq \mu \|x - S(x)\| + \|x - y\|.$$

We say that S satisfies condition (E) on $\mathcal{C}$ whenever S satisfies (E_μ) for some $\mu \geq 1$.

We know that the fixed point of a nonexpansive mapping may not exist, but we can approximate it using some techniques. A well known way to find a fixed point of a nonexpansive mapping S is to use contraction to approximate it [2]. More precisely, fix a point $x \in \mathcal{C}$ and define the mapping $S_\nu : \mathcal{C} \to \mathcal{C}$ as

$$S_\nu(x) = \nu x + (1 - \nu) S(x), \quad \forall\ x \in \mathcal{C},$$

and any $\nu \in (0, 1)$. One can easily see S_ν is a contraction mapping on $\mathcal{C}$ and the classical Banach contraction principle assures that S_ν has a unique fixed point $x_\nu \in \mathcal{C}$, i.e. $x_\nu = \nu x_\nu + (1 - \nu) S(x_\nu)$.

In this direction, a well known iteration approach for solving fixed point problem is the picard iteration process which is defined as

$$u_{n+1} = S(u_n), \quad n \geq 1, \tag{1.2}$$

where the initial value u_1 is selected arbitrarily. The Picard iteration may not converge to a fixed point of the nonexpansive mapping. Hence many authors have proposed various iterative techniques to approximate fixed points of nonexpansive type mappings and their generalizations.

In 1953, Mann [7] established a new iteration which is determined as follows

$$u_{n+1} = (1-\theta_n)\, u_n + \theta_n S(u_n), \quad n \in \mathbb{N}, \tag{1.3}$$

where $\{\theta_n\} \in [0,1]$ and the initial value u_1 is selected arbitrary.

In 1974, Ishikawa [6] generalized the above one step algorithm to a two step algorithm to approximate fixed point of nonexpansive mapping as follows

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = (1-\alpha_n)\, u_n + \alpha_n S(u_n), \\ u_{n+1} = (1-\theta_n)\, u_n + \theta_n S(w_n), \end{cases} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\theta_n\} \in [0,1]$.

In 2000, Noor [8] developed a three step iterative process to approximate solution in Hilbert space as follows

$$\begin{cases} u_1 \in \mathcal{C}, \\ v_n = (1-\delta_n)\, u_n + \delta_n S(u_n), \\ w_n = (1-\alpha_n)\, u_n + \alpha_n S(v_n), \\ u_{n+1} = (1-\theta_n)\, u_n + \theta_n S(w_n), \end{cases} \quad n \in \mathbb{N},$$

where $\{\theta_n\}, \{\alpha_n\}, \{\delta_n\} \in [0,1]$.

In 2016, Thakur et. al. [19] proposed a new iterative process to approximate fixed point for the mapping satisfying condition (C) as follows

$$\begin{cases} u_1 \in \mathcal{C}, \\ v_n = (1-\alpha_n)\, u_n + \alpha_n S(u_n), \\ w_n = S((1-\theta_n)\, u_n + \theta_n v_n), \\ u_{n+1} = S(w_n), \end{cases} \quad n \in \mathbb{N},$$

where the mapping S satisfies condition (C) and $\{\theta_n\}$, $\{\alpha_n\}$ are coming from $(0,1)$.

Recently, in 2024, Paimsang et al. [11] introduced a new iterative process, called the CT-iteration, for approximating fixed points of mappings satisfying Condition (E). The iterative scheme is defined as follows

$$\begin{cases} u_1 \in \mathcal{C}, \\ v_n = (1-\alpha_n)\, u_n + \alpha_n S(u_n), \\ w_n = (1-\delta_n-\gamma_n)\, u_n + \delta_n S(u_n) + \gamma_n v_n, \\ u_{n+1} = (1-\theta_n-\sigma_n)\, v_n + \theta_n S(v_n) + \sigma_n S(w_n), \end{cases} \quad n \in \mathbb{N},$$

where the sequences $\{\alpha_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$, $\{\sigma_n\}$, $\{\delta_n+\gamma_n\}$, and $\{\theta_n+\sigma_n\}$ are coming from $(0,1)$. They established weak convergence results under suitable assumptions on the control parameters.

In recent years, inertial techniques have attracted significant attention in fixed point theory because of their ability to accelerate the convergence of classical iterative algorithms. The concept of inertial extrapolation originated from Polyak's heavy-ball method [22], in which the next iterate is generated by incorporating information from the previous step through the extrapolation formula

$$u_{n+1} = u_n + \theta_n \left(u_n - u_{n-1}\right),$$

where $\{\theta_n\}$ is an inertial parameter. Motivated by the remarkable success of this approach in optimization, many researchers have incorporated inertial terms into fixed point iterative schemes to enhance their convergence behaviour while preserving the underlying convergence properties. In this direction, very recently, Sunthrayuth et al. [21] proposed a Krasnosel'skii-Mann type iterative algorithm (namely DIKM) with double inertial steps for approximating common fixed points of an infinite family of nonexpansive mappings in real Hilbert spaces. They established a weak convergence theorem under suitable assumptions on the control parameters. The proposed iterative process (DIKM) is defined as follows

$$\begin{cases} u_0, u_1 \in \mathcal{H} \\ v_n = u_n + \beta_n \left(u_n - u_{n-1}\right), \\ w_n = u_n + \alpha_n \left(u_n - u_{n-1}\right), \\ u_{n+1} = \left(1-\theta_n\right) w_n + \alpha_n S\left(v_n\right), \end{cases} \qquad n \in \mathbb{N},$$

where $0 \leq \beta_n \leq \beta_{n+1} \leq 1$, $0 \leq \alpha_n \leq \alpha_{n+1} \leq \alpha < \frac{\epsilon-\sqrt{2\epsilon}}{\epsilon}$, $\epsilon \in (2,\infty)$ and $0 < \theta \leq \theta_n \leq \theta_{n+1} \leq \frac{1}{1+\epsilon}$ with $\theta_n > \alpha_n, \forall n \in \mathbb{N}$.

In recent years, many authors have introduced and explored various iterative processes for nonexpansive mappings and their generalization such as iterative algorithm for finding a common fixed point of nonexpansive mappings, where the sequence generated by the algorithm converges to the unique solution of variational inequality problem [5, 15], fixed point results for a class of generalized nonexpansive type mappings and associated Krasnosel'skiĭ type mappings and Mann type iteration procedure for finding a common fixed point of a nonexpansive type semigroup [12, 13], iterative algorithm for finding a common solution to the variational inclusion problem and set of common fixed points of nonexpansive mappings [14], Mann–Dotson's algorithm to approximate common fixed points of a countable family of k-strict pseudocontractive non self mappings [16]. Recently, Tadesse et al. presented fixed point results for

Abbas-Nazir three step iterative algorithm for two mappings satisfying the generalized Suzuki type nonexpansive mapping and proved strong and weak convergence results [18].

Motivated by above, we introduce a new three step algorithm for a family of mappings satisfying condition (E). We represent a weak convergence result and prove that the algorithm converges weakly to a common fixed point. We also present an application of mapping satisfying condition (E) to finding the solution of a nonlinear integral equations. Lastly, We present a couple of non-trivial examples and present the convergence behaviour of our algorithm to verify our result.

2. Preliminaries

In this section, we provide key Lemmas, and results which are useful in proving our result.

Proposition 2.1 ([4]). *Suppose $\mathcal{C} \neq \emptyset$ is a subset of $\mathcal{H}$. Suppose that the mapping $S : \mathcal{C} \to \mathcal{H}$ satisfies condition (E) with set of fixed point $F(S) \neq \emptyset$ then, S is a quasi nonexpansive mapping.*

Lemma 2.2 ([10]). *Suppose $\{u_n\}$, $\{w_n\}$ and $\{\delta_n\}$ are nonnegative sequences of real numbers satisfying following:*

$$u_{n+1} \leq (1 + \delta_n) u_n + w_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} w_n < \infty$ then $\lim_{n\to\infty} u_n = 0$.

Lemma 2.3 ([10]). *Consider $u_{-1} \geq 0$, $u_0 \geq 0$, and $\{u_n\}$, $\{a_n\}$, $\{b_n\}$ are sequences of nonnegative real numbers which satisfy:*

$$u_{n+1} \leq (1 + a_n) u_n + (a_n + b_n) u_{n-1} + b_n u_{n-2}, \quad \forall n \geq 1.$$

Then

$$u_{n+1} \leq K \prod_{i=i}^{n} (1 + 2a_i + 2b_i), \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then u_n is bounded.

Lemma 2.4 ([20]). *Suppose $r > 0$ is a fixed real number. Then a real Banach space $\mathcal{H}$ is said to be uniformly convex if and only $\exists$ a strictly increasing continuous function $\Psi : [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$ such that*

$$\|\alpha x + (1 - \alpha) y\|^2 \leq \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha (1 - \alpha) \Psi (\|x - y\|),$$

$\forall x, y \in B_r = \{x \in \mathcal{H} : \|x\| \leq r\}$, $\alpha \in [0, 1]$.

Lemma 2.5 ([9]). *A Banach space $\mathcal{H}$ is said to satisfy the Opial condition, if for every weakly convergent sequence $\{u_n\}$ with weak limit $u^\dagger \in \mathcal{H}$, holds following condition,*

$$\liminf_{n\to\infty} \left\|u_n - u^\dagger\right\| < \liminf_{n\to\infty} \|u_n - u^*\|,$$

$\forall u^* \in \mathcal{H}$ and $u^\dagger \neq u^*$.

Theorem 2.6 ([4]). *Suppose $\mathcal{C} \neq \emptyset$ is a subset of $\mathcal{H}$ and S is a mapping satisfying condition (E) defined on $\mathcal{C}$. Then $\mathcal{C}$ admits approximate fixed point sequence if and only if S has a fixed point in $\mathcal{C}$.*

Lemma 2.7 ([4, 23]). *Let $\mathcal{H}$ be a Banach space having the Opial property, and let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$. Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying Condition (E). If a sequence $\{u_n\} \subset \mathcal{C}$ satisfies*

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n\to\infty} \|S(u_n) - u_n\| = 0,$$

then $S(u) = u$.

3. Main Results

In this section, we introduce a novel three step iterative algorithm with following assumptions:

(A1) $\mathcal{H}$ is a real uniformly convex Banach space which satisfies Opial's property and $\mathcal{C}$ is a nonempty subset of $\mathcal{H}$.

(A2) $\mathcal{S}_n : \mathcal{C} \to \mathcal{C}$, $n \in \mathbb{N}$, be a infinite family of mappings satisfying condition (E).

(A3) $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset, n \in \mathbb{N}$.

(A4) The sequences $\{\beta_n\}, \{\alpha_n\}, \{\theta_n\}, \{\gamma_n\}, \{\sigma_n\}$ satisfy the following conditions:

(I) $\{\beta_n\} \in (0,1)$ with $\sum_{n=1}^{\infty} \beta_n < \infty$,

(II) $\{\alpha_n\}, \{\theta_n\} \in (0,1)$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} \theta_n \leq \limsup_{n\to\infty} \theta_n < 1$,

(III) $\{\gamma_n\}, \{\sigma_n\} \geq 0$ with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Algorithm 3.1. First we introduce the following algorithm

$$\begin{cases} u_{-1}, u_0, u_1 \in \mathcal{C}, \\ v_n = u_n + \beta_n (u_n - u_{n-1}), \\ w_n = (1-\alpha_n) v_n \\ \qquad +\alpha_n S_n [u_n + \gamma_n (u_n - u_{n-1}) + \sigma_n (u_{n-1} - u_{n-2})], \\ u_{n+1} = (1-\theta_n) w_n + \theta_n S_n (w_n), \end{cases} \qquad n \in \mathbb{N}.$$

Theorem 3.2. *Suppose that assumptions* (A1)–(A4) *hold. Let* $\{u_n\}$ *be the sequence generated by Algorithm 3.1. Then the following results hold:*

(i) $\lim_{n\to\infty} \|u_n - p\|$ *exists,* $\forall p \in \bigcap_{n=1}^{\infty} F(S_n), n \in \mathbb{N}$.

(ii) $\{u_n\}$ *converges weakly to a point in* $\bigcap_{n=1}^{\infty} F(S_n), n \in \mathbb{N}$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} F(S_n)$. Then,

$$\begin{aligned} \|u_{n+1} - p\| &= \|(1-\theta_n) w_n + \theta_n S_n(w_n) - p\| \\ &\leq (1-\theta_n) \|w_n - p\| + \theta_n \|S_n(w_n) - p\| \\ &\leq \|w_n - p\|, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|w_n - p\| &= \|(1-\alpha_n) v_n + \alpha_n S_n(u_n + \gamma_n(u_n - u_{n-1}) \\ &\quad + \sigma_n(u_{n-1} - u_{n-2})) - p\| \\ &\leq (1-\alpha_n) \|v_n - p\| + \alpha_n \|S_n(u_n + \gamma_n(u_n - u_{n-1}) \\ &\quad + \sigma_n(u_{n-1} - u_{n-2})) - p\| \\ &\leq (1-\alpha_n) \|v_n - p\| + \alpha_n \|u_n - p\| + \alpha_n\gamma_n \|u_n - u_{n-1}\| \\ &\quad + \alpha_n\sigma_n \|u_{n-1} - u_{n-2}\|. \end{aligned} \tag{3.2}$$

Since,

$$\begin{aligned} \|v_n - p\| &= \|u_n + \beta_n(u_n - u_{n-1}) - p\| \\ &\leq \|u_n - p\| + \beta_n \|u_n - u_{n-1}\|. \end{aligned} \tag{3.3}$$

Then, from (3.1), (3.2) and (3.3)

$$\begin{aligned} \|u_{n+1} - p\| &\leq (1-\alpha_n)\{\|u_n - p\| + \beta_n \|u_n - u_{n-1}\|\} + \alpha_n \|u_n - p\| \\ &\quad + \alpha_n\gamma_n \|u_n - u_{n-1}\| + \alpha_n\sigma_n \|u_{n-1} - u_{n-2}\| \\ &= \|u_n - p\| + \beta_n(1-\alpha_n) + \alpha_n\gamma_n)) \|u_n - u_{n-1}\| \\ &\quad + \alpha_n\sigma_n \|u_{n-1} - u_{n-2}\|. \end{aligned} \tag{3.4}$$

From (3.2) and (3.3), we get

$$\begin{aligned} \|u_{n+1} - p\| &\leq (1-\alpha_n)\{\|u_n - p\| + \beta_n \|u_n - u_{n-1}\|\} + \alpha_n \|u_n - p\| \\ &\quad + \alpha_n\gamma_n \|u_n - u_{n-1}\| + \alpha_n\sigma_n \|u_{n-1} - u_{n-2}\| \\ &\leq (1-\alpha_n) \|u_n - p\| + \beta_n(1-\alpha_n) \|u_n - p\| \\ &\quad + \beta_n(1-\alpha_n) \|u_{n-1} - p\| + \alpha_n \|u_n - p\| + \alpha_n\gamma_n \|u_n - p\| \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+ \alpha_n\gamma_n \|u_{n-1} - p\| + \alpha_n\sigma_n \|u_{n-1} - p\| + \alpha_n\sigma_n \|u_{n-2} - p\| \\
&= \{(1-\alpha_n) + \beta_n (1-\alpha_n) + \alpha_n + \alpha_n\gamma_n\} \|u_n - p\| \\
&\quad + \{\beta_n (1-\alpha_n) + \alpha_n\gamma_n + \alpha_n\sigma_n\} \|u_{n-1} - p\| \\
&\quad + \alpha_n\sigma_n \|u_{n-2} - p\| \\
&= \{1 + \beta_n (1-\alpha_n) + \alpha_n\gamma_n\} \|u_n - p\| \\
&\quad + \{\beta_n (1-\alpha_n) + \alpha_n\gamma_n + \alpha_n\sigma_n\} \|u_{n-1} - p\| \\
&\quad + \alpha_n\sigma_n \|u_{n-2} - p\| \\
&= (1 + a_n) \|u_n - p\| + (a_n + b_n) \|u_{n-1} - p\| + b_n \|u_{n-2} - p\|,
\end{aligned}
$$

where, $a_n = \beta_n (1-\alpha_n) + \alpha_n\gamma_n$ and $b_n = \alpha_n\sigma_n$.

Therefore, by Lemma 2.3, we obtain

$$
\|u_{n+1} - p\| \leq K \prod_{i=1}^{n} (1 + 2a_i + 2b_i). \tag{3.6}
$$

where $K = \max\{\|u_{-1} - p\|, \|u_0 - p\|, \|u_1 - p\|\}$.

Since, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, it follows that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Hence, by Lemma 2.3, the sequence $\{\|u_n - p\|\}$ is bounded.

From (3.4), we have

$$
\begin{aligned}
\|u_{n+1} - p\| \leq \|u_n - p\| &+ (\beta_n (1-\alpha_n) + \alpha_n\gamma_n) \|u_n - u_{n-1}\| \\
&+ \alpha_n\sigma_n \|u_{n-1} - u_{n-2}\|.
\end{aligned}
$$

Since $\|u_n - p\|$ is bounded, there exists a constant $M > 0$ such that

$$
\|u_n - p\| \leq M, \quad \forall n \geq 0.
$$

Consequently,

$$
\|u_n - u_{n-1}\| \leq \|u_n - p\| + \|u_{n-1} - p\| \leq 2M,
$$

and

$$
\|u_{n-1} - u_{n-2}\| \leq \|u_{n-1} - p\| + \|u_{n-2} - p\| \leq 2M.
$$

Therefore

$$
\begin{aligned}
\|u_{n+1} - p\| &\leq \|u_n - p\| + 2M (\beta_n (1-\alpha_n) + \alpha_n\gamma_n + \alpha_n\sigma_n) \\
&= \|u_n - p\| + t_n,
\end{aligned} \tag{3.7}
$$

where $t_n = 2M (\beta_n (1-\alpha_n) + \alpha_n\gamma_n + \alpha_n\sigma_n)$.

Since, $\sum_{n=1}^{\infty}\beta_n<\infty$, $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\sigma_n<\infty$ and $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1$, it follows that $\sum_{n=1}^{\infty}t_n<\infty$. By applying Lemma 2.2 to (3.7) , we conclude that $\lim_{n\to\infty}\|u_n-p\|$ exists.

Now, we will show that sequence $\{u_n\}$ converges weakly to some point in $\bigcap_{n=1}^{\infty}F(S_n)$.

Using, Lemma 2.4 $\exists$ a strictly increasing and continuous function $\Psi:[0,\infty)\to[0,\infty)$ with $\Psi(0)=0$ such that

(3.8)

$$\begin{aligned}
\|w_n-p\|^2 &= \|(1-\alpha_n)v_n+\alpha_nS_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-p\|^2\\
&\leq(1-\alpha_n)\|v_n-p\|^2+\alpha_n\|S_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-p\|^2\\
&\quad-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|)\\
&\leq(1-\alpha_n)\|v_n-p\|^2+\alpha_n\|(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-p\|^2\\
&\quad-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|)\\
&\leq(1-\alpha_n)\|v_n-p\|^2+\alpha_n(\|u_n-p\|+\gamma_n\|u_n-u_{n-1}\|\\
&\quad+\sigma_n\|u_{n-1}-u_{n-2}\|)^2\\
&\quad-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|)\\
&=(1-\alpha_n)\|v_n-p\|^2+\alpha_n\|u_n-p\|^2+\alpha_n\gamma_n^2\|u_n-u_{n-1}\|^2\\
&\quad+\alpha_n\sigma_n^2\|u_{n-1}-u_{n-2}\|^2+2\alpha_n\gamma_n\|u_n-p\|\|u_n-u_{n-1}\|\\
&\quad+2\alpha_n\sigma_n^2\|u_n-p\|\|u_{n-1}-u_{n-2}\|\\
&\quad+2\alpha_n\gamma_n\sigma_n\|u_n-u_{n-1}\|\|u_{n-1}-u_{n-2}\|\\
&\quad-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1})\\
&\quad+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|)\\
&\leq(1-\alpha_n)\{\|u_n-p\|+\beta_n\|u_n-u_{n-1}\|\}^2+\alpha_n\|u_n-p\|^2\\
&\quad+\alpha_n\gamma_n^2\|u_n-u_{n-1}\|^2+\alpha_n\sigma_n^2\|u_{n-1}-u_{n-2}\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_n\gamma_n \|u_n-p\| \|u_n-u_{n-1}\| \\
&+2\alpha_n\sigma_n^2 \|u_n-p\| \|u_{n-1}-u_{n-2}\| \\
&+2\alpha_n\gamma_n\sigma_n \|u_n-u_{n-1}\| \|u_{n-1}-u_{n-2}\| \\
&-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1}) \\
&+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|) \\
=\ &\|u_n-p\|^2+\beta_n^2(1-\alpha_n)\|u_n-u_{n-1}\|^2 \\
&+2\beta_n(1-\alpha_n)\|u_n-p\| \|u_n-u_{n-1}\| \\
&+\alpha_n\gamma_n^2\|u_n-u_{n-1}\|^2+\alpha_n\sigma_n^2\|u_{n-1}-u_{n-2}\|^2 \\
&+2\alpha_n\gamma_n\|u_n-p\| \|u_n-u_{n-1}\| \\
&+2\alpha_n\sigma_n^2\|u_n-p\| \|u_{n-1}-u_{n-2}\| \\
&+2\alpha_n\gamma_n\sigma_n\|u_n-u_{n-1}\| \|u_{n-1}-u_{n-2}\| \\
&-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1}) \\
&+\sigma_n(u_{n-1}-u_{n-2}))-v_n\|).
\end{aligned}
$$

Since,

$$
\begin{aligned}
\|u_{n+1}-p\|^2 &= \|(1-\theta_n)w_n+\theta_nS_n(w_n)-p\|^2 \\
&\le (1-\theta_n)\|w_n-p\|^2+\theta_n\|S_n(w_n)-p\|^2 \\
&\quad -(1-\theta_n)\theta_n\Psi(\|S_n(w_n)-w_n\|) \\
&\le (1-\theta_n)\|w_n-p\|^2+\theta_n\|w_n-p\|^2 \\
&\quad -(1-\theta_n)\theta_n\Psi(\|S_n(w_n)-w_n\|) \\
&\le \|w_n-p\|^2-(1-\theta_n)\theta_n\Psi(\|S_n(w_n)-w_n\|).
\end{aligned} \tag{3.9}
$$

By using (3.8), we get

(3.10)

$$
\begin{aligned}
\|u_{n+1}-p\|^2 &\le \|u_n-p\|^2+\beta_n^2(1-\alpha_n)\|u_n-u_{n-1}\|^2 \\
&\quad +2\beta_n^2(1-\alpha_n)\|u_n-p\| \|u_n-u_{n-1}\|+\alpha_n\gamma_n^2\|u_n-u_{n-1}\|^2 \\
&\quad +\alpha_n\sigma_n^2\|u_{n-1}-u_{n-2}\|^2+2\alpha_n\gamma_n\|u_n-p\| \|u_n-u_{n-1}\| \\
&\quad +2\alpha_n\sigma_n^2\|u_n-p\| \|u_{n-1}-u_{n-2}\| \\
&\quad +2\alpha_n\gamma_n\sigma_n\|u_n-u_{n-1}\| \|u_{n-1}-u_{n-2}\| \\
&\quad -(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1}) \\
&\quad +\sigma_n(u_{n-1}-u_{n-2}))-v_n\|) \\
&\quad -(1-\theta_n)\theta_n\Psi(\|S_n(w_n)-w_n\|) \\
&\le \|u_n-p\|^2+\varepsilon_n-(1-\alpha_n)\alpha_n\Psi(\|S_n(u_n+\gamma_n(u_n-u_{n-1})
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma_n\left(u_{n-1}-u_{n-2}\right))-v_n\|) \\
&-\left(1-\theta_n\right)\theta_n\Psi\left(\|S_n\left(w_n\right)-w_n\|\right),
\end{aligned}
$$

where,

$$
\begin{aligned}
\varepsilon_n = {} & \beta_n^2\left(1-\alpha_n\right)\|u_n-u_{n-1}\|^2+2\beta_n^2\left(1-\alpha_n\right)\|u_n-p\|\,\|u_n-u_{n-1}\| \\
& +\alpha_n\gamma_n^2\|u_n-u_{n-1}\|^2+\alpha_n\sigma_n^2\|u_{n-1}-u_{n-2}\|^2 \\
& +2\alpha_n\gamma_n\|u_n-p\|\,\|u_n-u_{n-1}\|+2\alpha_n\sigma_n^2\|u_n-p\|\,\|u_{n-1}-u_{n-2}\| \\
& +2\alpha_n\gamma_n\sigma_n\|u_n-u_{n-1}\|\,\|u_{n-1}-u_{n-2}\|.
\end{aligned}
$$

Since $\|u_n-p\|$ is bounded and

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\beta_n<\infty, \\
&\sum_{n=1}^{\infty}\gamma_n<\infty, \\
&\sum_{n=1}^{\infty}\sigma_n<\infty, \\
&0<\liminf_{n\to\infty}\theta_n\le\limsup_{n\to\infty}\theta_n<1, \\
&0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1,
\end{aligned}
$$

it follows that

$$
\sum_{n=1}^{\infty}\varepsilon_n<\infty.
$$

Therefore, by (3.10), we obtain

$$
\begin{aligned}
&\left(1-\alpha_n\right)\alpha_n\Psi\left(\|S_n\left(u_n+\gamma_n\left(u_n-u_{n-1}\right)\right.\right.\quad\left.\left.+\sigma_n\left(u_{n-1}-u_{n-2}\right)\right)-v_n\|\right) \\
&\quad+\left(1-\theta_n\right)\theta_n\Psi\left(\|S_n\left(w_n\right)-w_n\|\right) \\
&\le\|u_n-p\|^2+\varepsilon_n-\|u_{n+1}-p\|^2.
\end{aligned}
$$

Since $0<\liminf_{n\to\infty}\theta_n\le\limsup_{n\to\infty}\theta_n<1$, and $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$, we get

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1-\alpha_n\right)\alpha_n\Psi\left(\|S_n\left(u_n+\gamma_n\left(u_n-u_{n-1}\right)+\sigma_n\left(u_{n-1}-u_{n-2}\right)\right)-v_n\|\right) \\
&\quad+\lim_{n\to\infty}\left(1-\theta_n\right)\theta_n\Psi\left(\|S_n\left(w_n\right)-w_n\|\right) \\
&\le 0,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lim_{n\to\infty}\Psi\left(\|S_n\left(w_n\right)-w_n\|\right)=0, \\
&\lim_{n\to\infty}\Psi\left(\|S_n\left(u_n+\gamma_n\left(u_n-u_{n-1}\right)+\sigma_n\left(u_{n-1}-u_{n-2}\right)\right)-v_n\|\right)=0.
\end{aligned}
$$

Hence, by the properties of Ψ, we get

$$\lim_{n\to\infty} \|S_n(w_n) - w_n\| = 0,$$

and

$$\lim_{n\to\infty} \|S_n(u_n + \gamma_n(u_n - u_{n-1}) + \sigma_n(u_{n-1} - u_{n-2})) - v_n\| = 0. \tag{3.11}$$

Since,

$$\begin{aligned} \|w_n - v_n\| &= \|(1-\alpha_n)v_n - \alpha_n S_n(u_n + \gamma(u_n - u_{n-1}) \\ &\quad + \sigma(u_{n-1} - u_{n-2})) - v_n\| \\ &\leq \alpha_n \|S_n(u_n + \gamma(u_n - u_{n-1}) + \sigma(u_{n-1} - u_{n-2})) - v_n\|. \end{aligned} \tag{3.12}$$

Then from (3.11) and (3.12), we get

$$\lim_{n\to\infty} \|w_n - v_n\| = 0.$$

Moreover, using the definition of $\{v_n\}$, we have

$$\begin{aligned} \lim_{n\to\infty} \|v_n - u_n\| &= \lim_{n\to\infty} \|u_n + \beta_n(u_n - u_{n-1}) - u_n\| \\ &= \lim_{n\to\infty} \beta_n \|u_n - u_{n-1}\| \\ &= 0. \end{aligned} \tag{3.13}$$

Since,

$$\begin{aligned} \|w_n - u_n\| &= \|w_n - v_n + v_n - u_n\| \\ &\leq \|w_n - v_n\| + \|v_n - u_n\|. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|w_n - u_n\| = 0.$$

Now, assume that q is a weak cluster point of $\{w_n\}$, then $\exists$ a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which converges weakly to a point q. Moreover, we have $\lim_{n\to\infty} \|S_n(w_n) - w_n\| = 0$. By Lemma 2.7, $q \in F(S_n)$, for each $n \in \mathbb{N}$. Hence,

$$q \in \bigcap_{n=1}^{\infty} F(S_n), \quad n \in \mathbb{N}.$$

To construct that the sequence $\{w_n\}$ converges weakly to the point in $\bigcap_{n=1}^{\infty} F(S_n)$, it sufficient to show that the weak limit set consists of a single point. Consider that q and q^* are two weak limit points of $\{w_n\}$ and let $\{w_{n_i}\}$, $\{w_{n_j}\}$ be two sub sequences of $\{w_n\}$ converging weakly to the points q and q^*, respectively. Assume $q \neq q^*$, then by Opial's condition:

$$\lim_{n\to\infty} \|w_n - q\| = \lim_{i\to\infty} \|w_{n_i} - q\|$$

$$
\begin{aligned}
&< \lim_{i\to\infty} \|w_{n_i} - q^*\| \\
&= \lim_{j\to\infty} \left\|w_{n_j} - q^*\right\| \\
&< \lim_{j\to\infty} \left\|w_{n_j} - q\right\| \\
&= \lim_{n\to\infty} \|w_n - q\| .
\end{aligned}
$$

Which is a contradiction of our assumption, $q \neq q^*$. So, we can conclude that the sequence $\{w_n\}$ converges weakly to a point in $\bigcap_{n=1}^{\infty} F(S_n)$. Moreover, since $\lim_{n\to\infty} \|w_n - u_n\| = 0$, hence the sequence $\{u_n\}$ generated by algorithm 3.1 also converges weakly to a point in $\bigcap_{n=1}^{\infty} F(S_n)$, $n \in \mathbb{N}$. □

Algorithm 3.3. We introduce the following algorithm.

$$
\begin{cases}
u_{-1}, u_0, u_1 \in \mathcal{C}, \\
v_n = u_n + \beta_n (u_n - u_{n-1}), \\
w_n = (1-\alpha_n) v_n + \alpha_n S [u_n + \gamma_n (u_n - u_{n-1}) + \sigma_n (u_{n-1} - u_{n-2})], \\
u_{n+1} = (1-\theta_n) w_n + \theta_n S (w_n).
\end{cases}
$$

Where $\{\theta_n\}$, $\{\beta_n\}$, $\{\alpha_n\} \in (0,1)$, $\{\gamma_n\} \geq 0$ and $\{\sigma_n\} \geq 0$.

Corollary 3.4. *Suppose that $\{u_n\}$ is a sequence generated by Algorithm 3.3, and let S be a mapping satisfying Condition (E) defined on a nonempty closed convex subset of a uniformly convex Banach space. Assume that $F(S) \neq \emptyset$ and that the sequences $\{\beta_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, and $\{\theta_n\}$ satisfy Assumption $(A4)$. Then, the sequence $\{u_n\}$ converges weakly to a point in $F(S)$.*

4. Applications

In this section, we provide the application of our obtained results.

Suppose $\mathcal{C}$ is a nonempty compact subset of $C[0,1]$, $I = [0,1]$ and define the nonlinear functional integral equation

$$
\begin{aligned}
\Phi(u) = \digamma(u) &+ \lambda \int_0^u K(u,v)\, G(v, \Phi(v))\, ds \\
&+ \eta \int_0^1 K^*(u,v)\, G^*(v, \Phi(v))\, ds,
\end{aligned} \tag{4.1}
$$

$\forall u, v \in I$, λ, $\eta \in \mathbb{R}^+$ and $\digamma : I \to \mathbb{R}^+$, $G, G^* : I \times \mathcal{C} \to \mathcal{C}$, $K, K^* : I \times I \to \mathbb{R}^+$. The following conditions satisfies:

(C_1) The function $\digamma : I \to \mathbb{R}^+$ is continuous.

(C_2) The function $K, K^* : I \times I \to \mathbb{R}^+$ are continuous such that $\forall\,(u,v) \in I \times I$ with

$$\int_0^u K\,(u,v)\,ds \leq K_1, \qquad \int_0^1 K^*\,(u,v)\,ds \leq K_2.$$

(C_3) The function $G, G^* : I \times \mathcal{C} \to \mathcal{C}$ are two continuous functions such that $\forall u \in I$ and $\Phi_1, \Phi_2 \in \mathcal{C}$,

$$|G\,(u,\Phi_1) - G\,(u,\Phi_2)| \leq L_1\,|\Phi_1\,(u) - \Phi_2\,(u)|\,,$$
$$|G^*\,(u,\Phi_1) - G^*\,(u,\Phi_2)| \leq L_2\,|\Phi_1\,(u) - \Phi_2\,(u)|\,.$$

(C_4) $\lambda L_1 K_1 + \eta L_2 K_2 \leq 1$.

Theorem 4.1. *Suppose $\mathcal{C}$ is a compact subset of $C\,[0,1]$ and supremum norm is defined as*

$$\|\Phi_1 - \Phi_2\| = \sup_{u \in \mathcal{C}} \|\Phi_1(u) - \Phi_2(u)\|\,.$$

Define the mapping $S : \mathcal{C} \to \mathcal{C}$ as

$$S\,(\Phi\,(u)) = \digamma\,(u) + \lambda \int_0^u K\,(u,v)\,G\,(v,\Phi\,(v))\,ds + \eta \int_0^1 K^*\,(u,v)\,G^*\,(v,\Phi\,(v))\,ds, \tag{4.2}$$

which satisfies assumptions (C_1) to (C_4). Then S admits approximate fixed point sequence if and only if (4.1) has a solution in $\mathcal{C}$.

Proof. Suppose that $\Phi_1,\ \Phi_2 \in \mathcal{C}$ then,

$$\begin{aligned}
|\Phi_1\,(u) - S\,(\Phi_2\,(u))| = \Bigg|\Phi_1\,(u) - \Bigg(&\digamma\,(u) + \lambda \int_0^u K\,(u,v)\,G\,(v,\Phi_1\,(v))\,ds \\
&+\eta \int_0^1 K^*\,(u,v)\,G^*\,(v,\Phi_1\,(v))\,ds\Bigg) \\
&+ \Bigg(\digamma\,(u) + \lambda \int_0^u K\,(u,v)\,G\,(v,\Phi_1\,(v))\,ds \\
&+\eta \int_0^1 K^*\,(u,v)\,G^*\,(v,\Phi_1\,(v))\,ds\Bigg) \\
&-\digamma\,(u) - \lambda \int_0^u K\,(u,v)\,G\,(v,\Phi_2\,(v))\,ds \\
&-\eta \int_0^1 K^*\,(u,v)\,G^*\,(v,\Phi_2\,(v))\,ds\Bigg| \\
\leq\ &|\Phi_1\,(u) - S\,(\Phi_1\,(u))| \\
&+ \lambda \int_0^u K(u,v)\,|G\,(v,\Phi_1\,(v)) - G\,(v,\Phi_2\,(v))|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\eta\int_0^1 K^*(u,v)\left|G^*(v,\Phi_1(v))-G^*(v,\Phi_2(v))\right|ds\\
&\leq\left|\Phi_1(u)-S(\Phi_1(u))\right|\\
&+\lambda\int_0^u K(u,v)L_1\left|\Phi_1(v)-\Phi_2(v)\right|ds\\
&+\eta\int_0^1 K^*(u,v)L_2\left|\Phi_1(v)-\Phi_2(v)\right|ds.
\end{aligned}
$$

Taking supremum of both sides, we get

$$\|\Phi_1-S(\Phi_2)\|\leq\|\Phi_1-S(\Phi_1)\|+(\lambda K_1L_1+\eta K_2L_2)\|\Phi_1-\Phi_2\|.$$

Since, $\lambda K_1L_1+\eta K_2L_2\leq 1$ this implies that

$$\|\Phi_1-S(\Phi_2)\|\leq\|\Phi_1-S(\Phi_1)\|+\|\Phi_1-\Phi_2\|.$$

Hence, the mapping S satisfying condition (E) on $\mathcal{C}$. It satisfies all the conditions of Theorem 2.6 so (4.1) has a solution in $\mathcal{C}$. □

Example 4.2. To illustrate the applicability of Theorem 4.1, consider the nonlinear functional integral equation

(4.3)

$$\Phi(u)=u+\cos u+\frac{1}{8}\int_0^u(u+v+1)\sin(\Phi(v))\,dv+\frac{1}{10}\int_0^1\frac{e^{-uv}\Phi(v)}{1+\Phi(v)^2}dv,$$

where $u\in I=[0,1]$. Here,

$$F(u)=u+\cos u,\qquad K(u,v)=u+v+1,\qquad K^*(u,v)=e^{-uv}$$

and

$$G(v,\Phi(v))=\sin(\Phi(v)),\qquad G^*(v,\Phi(v))=\frac{\Phi(v)}{1+\Phi(v)^2},$$

with

$$\lambda=\frac{1}{8},\qquad\eta=\frac{1}{10}.$$

We verify assumptions (C_1)–(C_4).

(i) Since $F(u)=u+\cos u$ is continuous on $I=[0,1]$, assumption (C_1) is satisfied.

(ii) The kernel functions $K(u,v)=u+v+1$ and $K^*(u,v)=e^{-uv}$ are continuous on $I\times I$. Moreover,

$$\int_0^u K(u,v)\,dv=\int_0^u(u+v+1)\,dv=\frac{3}{2}u^2+u\leq\frac{5}{2},$$

for all $u\in I$. Hence, we choose

$$K_1=\frac{5}{2}.$$

Also,

$$\int_0^1 K^*(u,v)dv = \int_0^1 e^{-uv}dv$$
$$= \begin{cases} \dfrac{1-e^{-u}}{u}, & u > 0, \\ 1, & u = 0, \end{cases}$$

which implies

$$\int_0^1 K^*(u,v)\,dv \leq 1.$$

Thus, we take $K_2 = 1$. Hence, assumption (C_2) holds.

(iii) Since $|\sin x - \sin y| \leq |x-y|$, the function G is Lipschitz continuous with Lipschitz constant $L_1 = 1$. Further, define

$$f(x) = \frac{x}{1+x^2}.$$

Then

$$f'(x) = \frac{1-x^2}{(1+x^2)^2},$$

and

$$\left|f'(x)\right| \leq 1, \quad \forall x \in \mathbb{R}.$$

Hence,

$$\left|\frac{x}{1+x^2} - \frac{y}{1+y^2}\right| \leq |x-y|,$$

which shows that $L_2 = 1$. Therefore, assumption (C_3) is satisfied.

(iv) Finally,

$$\lambda L_1 K_1 + \eta L_2 K_2 = \frac{1}{8}\left(\frac{5}{2}\right) + \frac{1}{10}(1) = \frac{5}{16} + \frac{1}{10} = \frac{33}{80} < 1.$$

Thus, $\lambda L_1 K_1 + \eta L_2 K_2 \leq 1$, and assumption (C_4) is satisfied.

Therefore, all the hypotheses of Theorem 4.1 are fulfilled. Consequently, the operator

$$S(\Phi(u)) = u + \cos u + \frac{1}{8}\int_0^u (u+v+1)\sin(\Phi(v))\,dv$$
$$+ \frac{1}{10}\int_0^1 \frac{e^{-uv}\Phi(v)}{1+\Phi(v)^2}dv,$$

satisfies Condition (E). Hence, by Theorem 4.1, the nonlinear functional integral equation (4.3) a solution in $\mathcal{C}$. This example validates the applicability of Theorem 4.1.

5. Example

In this section, we provide a couple of nontrivial examples of mappings that satisfy condition (E) but not nonexpansive.

Example 5.1. Let $\mathcal{H} = \mathcal{R}^2$ be a Banach space with $\|\cdot\|_2$ and $\mathcal{K}^2 = [0,\infty) \times [0,\infty)$ be a subset of $\mathcal{R}^2$. Suppose $S_n : \mathcal{K}^2 \to \mathcal{K}^2$, $n \in \mathbb{N}$ is a family of mappings defined as:

$$S_n(u,v) = \begin{cases} (\max\{n,u\},v), & \text{if } u \neq 0, \\ (0,0), & \text{if } u = 0. \end{cases}$$

Then,

Case (a): $u_1 \geq n$

(5.1)
$$\begin{aligned} \|S_n(u_1,u_2) - (v_1,v_2)\| &= \|(\max\{n,u_1\},u_2) - (v_1,v_2)\| \\ &= \|(u_1,u_2) - (v_1,v_2)\| \\ &= \|(u_1 - v_1, u_2 - v_2)\| \\ &\leq \|S_n(v_1,v_2) - (v_1,v_2)\| + \|(u_1 - v_1, u_2 - v_2)\|. \end{aligned}$$

Hence S_n satisfies condition (E), if $u_1 \geq n$, $\forall u_2 \in \mathcal{K}$ and $(v_1,v_2) \in \mathcal{K}^2$.
Case (b): $u_1 < n$

$$\begin{aligned} \|S_n(u_1,u_2) - (v_1,v_2)\| &= \|(\max\{n,u_1\},u_2) - (v_1,v_2)\| \\ &= \|(n,u_2) - (v_1,v_2)\| \\ &= \|(n - v_1, u_2 - v_2)\| \\ &= \sqrt{(n-v_1)^2 + (u_2 - v_2)^2}. \end{aligned} \tag{5.2}$$

Sub case (i): $v_1 \leq n$

$$\begin{aligned} &\|S_n(v_1,v_2) - (v_1,v_2)\| + \|(u_1,u_2) - (v_1,v_2)\| \\ &= \|(\max\{n,v_1\},v_2) - (v_1,v_2)\| + \|(u_1,u_2) - (v_1,v_2)\| \\ &= \|(n,v_2) - (v_1,v_2)\| + \|(u_1 - v_1, u_2 - v_2)\| \\ &= \|(n - v_1, v_2 - v_2)\| + \|(u_1 - v_1, u_2 - v_2)\| \\ &= \|(n - v_1, 0)\| + \|(u_1 - v_1, u_2 - v_2)\| \\ &= \sqrt{(n-v_1)^2} + \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}. \end{aligned} \tag{5.3}$$

Now,

$$\begin{aligned} (n-v_1)^2 + (u_2 - v_2)^2 \leq{}& (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ &+ (n-v_1)^2 + (n - v_1)\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \end{aligned}$$

$$= \left((n - v_1) + \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \right)^2,$$

i.e.

$$\sqrt{(n - v_1)^2 + (u_2 - v_2)^2} \leq (n - v_1) + \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}.$$

From (5.2) and (5.3),

$$\|S_n(u_1, u_2) - (v_1, v_2)\| \leq \|S_n(v_1, v_2) - (v_1, v_2)\| + \|(u_1, u_2) - (v_1, v_2)\|.$$

Sub case (ii): $v_1 > n$

$$\begin{aligned} \|S_n(v_1, v_2) - (v_1, v_2)\| + \|(u_1, u_2) - (v_1, v_2)\| &= \|(\max\{n, v_1\}, v_2) - (v_1, v_2)\| + \|(u_1, u_2) - (v_1, v_2)\| \\ &= \|(v_1, v_2) - (v_1, v_2)\| + \|(u_1 - v_1, u_2 - v_2)\| \\ &= \|(v_1 - v_1, v_2 - v_2)\| + \|(u_1 - v_1, u_2 - v_2)\| \\ &= \|(u_1 - v_1, u_2 - v_2)\| \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}. \end{aligned} \tag{5.4}$$

Since, $u_1 < n < v_1$,

$$\begin{aligned} (n - v_1)^2 &< (u_1 - v_1)^2 \\ (n - v_1)^2 + (u_2 - v_2)^2 &< (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ \sqrt{(n - v_1)^2 + (u_2 - v_2)^2} &< \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}. \end{aligned}$$

From (5.2) and (5.4), we get

$$\|S_n(u_1, u_2) - (v_1, v_2)\| < \|S_n(v_1, v_2) - (v_1, v_2)\| + \|(u_1, u_2) - (v_1, v_2)\|.$$

The family of mappings S_n satisfies condition (E) in all cases with $\bigcap_{n=1}^{\infty} F(S_n) = (0, 0)$.

If we take $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with $u_1 < n$, $v_1 = 0$ and $u_2 = v_2$ then

$$\begin{aligned} \|S_n(u) - S_n(v)\| &= \|S_n(u_1, u_2) - S_n(0, v_2)\| \\ &= \|(n, u_2) - (0, 0)\| \\ &= \|(n, u_2)\| \\ &= \sqrt{n^2 + u_2^2}. \end{aligned} \tag{5.5}$$

Again,

$$\|u - v\| = \|(u_1, u_2) - (v_1, v_2)\| \tag{5.6}$$

$$\begin{aligned} &= \|(u_1, u_2) - (0, u_2)\| \\ &= \|(u_1, 0)\| \\ &= u_1. \end{aligned}$$

Since,

$$\begin{aligned} n &> u_1 \\ \sqrt{n^2 + u_2^2} &> \sqrt{u_1^2}. \end{aligned}$$

From (5.5) and (5.6), we get

$$\|S_n(u) - S_n(v)\| > \|u - v\|.$$

This inequality shows that the family of mappings S_n is not nonexpansive.

Remark 5.2. Observe that the family of mappings S_n is discontinuous at every point of the form $(0, v)$, $v \in [0, \infty)$. Indeed,

$$\lim_{u \to 0^+} S_n(u, v) = (n, v) \neq (0, 0) = S_n(0, v).$$

This discontinuity does not affect the validity of Example 5.1, since condition (E) does not require continuity. The example is included only to illustrate that a mapping satisfying condition (E) need not be nonexpansive.

Example 5.3. Let $\mathcal{H} = \mathbb{R}$ be the Banach space with the usual norm, and let $\mathcal{K} = [0, 1] \subset \mathcal{H}$. Define the mapping $S : \mathcal{K} \to \mathcal{K}$ by

$$S(u) = \begin{cases} \frac{u}{3}, & u \in \left[0, \frac{2}{3}\right], \\ \frac{u}{6} + \frac{1}{5}, & u \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

then,

Case (a): $u, v \in \left[0, \frac{2}{3}\right)$. Then

$$|S(u) - S(v)| = \frac{1}{3}|u - v| < |u - v|$$

and we have,

$$|u - S(v)| \leq |u - S(u)| + |u - v|.$$

Case (b): $u, v \in \left[\frac{2}{3}, 1\right]$. Then

$$|S(u) - S(v)| = \frac{1}{6}|u - v| < |u - v|$$

$$|u - S(v)| \leq |u - S(v)| + |u - v|.$$

Case (c): $u \in \left[0, \frac{2}{3}\right)$, $v \in \left[\frac{2}{3}, 1\right]$. Then we have,

$$|v - S(u)| = \left|v - \frac{u}{3}\right|$$

$$\begin{aligned}
&\leq \left|5u - \frac{6}{5}\right| + |u - v| \\
&= 6\left|v - \frac{v}{6} - \frac{1}{5}\right| + |u - v| \\
&= 6\,|v - S(v)| + |u - v|\,.
\end{aligned}$$

and

$$\begin{aligned}
|u - S(v)| &= \left|u - \frac{v}{6} - \frac{1}{5}\right| \\
&\leq |u| + \left|\frac{v}{6} + \frac{1}{5}\right| \\
&\leq |u| + \left|\frac{1}{6} + \frac{1}{5}\right| \\
&\leq |u| + \frac{11}{30}.
\end{aligned}$$

Sub case (i): If $u \in \left[0, \frac{9}{30}\right)$,

$$\begin{aligned}
|u - S(v)| &\leq |u| + \frac{11}{30} \\
&\leq 4\,|u| + \frac{11}{30} \\
&\leq 6\,|u - S(u)| + |u - v|\,.
\end{aligned}$$

Sub case (ii): If $u \in \left[\frac{9}{30}, \frac{2}{3}\right)$,

$$\begin{aligned}
&6\,|u - S(u)| = 4\,|u| \geq \frac{36}{30}, \\
&|u - S(v)| \leq \frac{2}{3} + \frac{11}{30} = \frac{31}{30}, \\
&|u - S(v)| \leq \frac{31}{30} \leq \frac{36}{30} \leq 6\,|u - S(u)| + |u - v|\,.
\end{aligned}$$

Therefore in all the cases mapping S satisfies condition (E).

On the other hand, for $u = \frac{3}{5}$, $v = \frac{2}{3}$ we have $S(u) = \frac{1}{5}$, $S(v) = \frac{31}{100}$ and

$$|S(u) - S(v)| = 0.11 > 0.06 = |u - v|\,.$$

Hence S is not a nonexpansive mapping.

Now we present the convergence behaviour for different initial guesses and different values of $\{\beta_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\theta_n\}$. Lastly, we present a comparative analysis of Algorithm 3.3 against the CT iteration, DIKM iteration, Noor iteration and Thakur iteration.

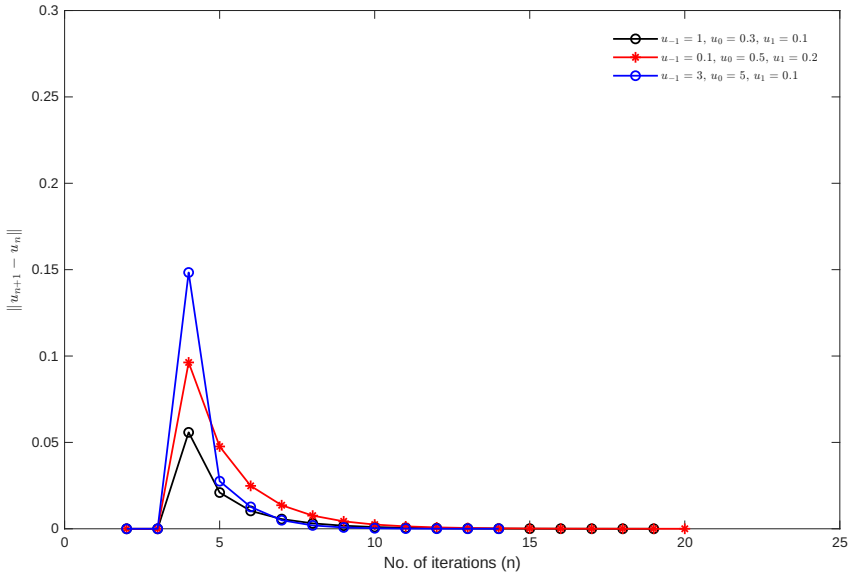

FIGURE 1. Convergence behaviour for fixed choice of $\beta_n = \frac{1}{(n+1)^2 \log(n+1)}$, $\alpha_n = \frac{1}{2}$, $\sigma_n = \frac{2}{(2n+1)^2}$, $\gamma_n = \frac{1}{n^{\frac{3}{2}}}$ and $\theta_n = \frac{1}{5}$ with different choices of initial guesses for u_{-1}, u_0 and u_1.

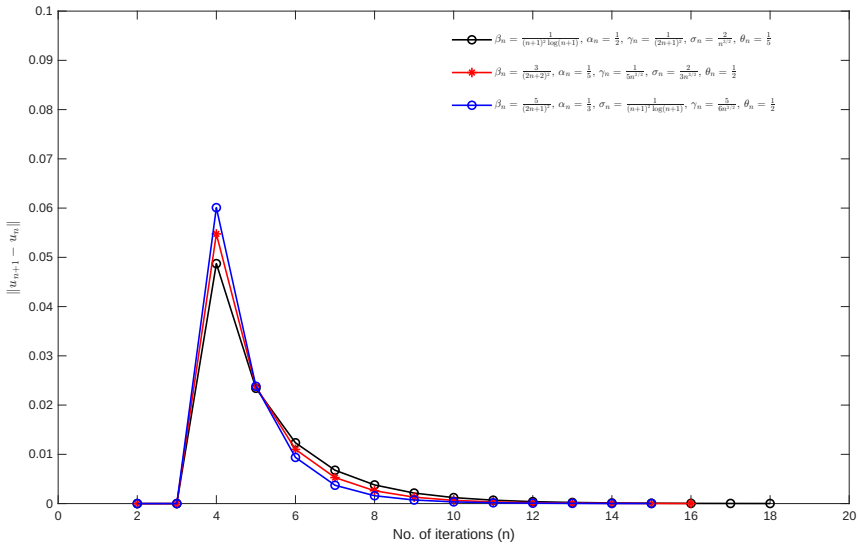

FIGURE 2. Convergence behaviour for different choices of $\{\beta_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\theta_n\}$ with initial guesses $u_{-1} = 0.5$,$u_0 = 0.2$ and $u_2 = 0.1$.

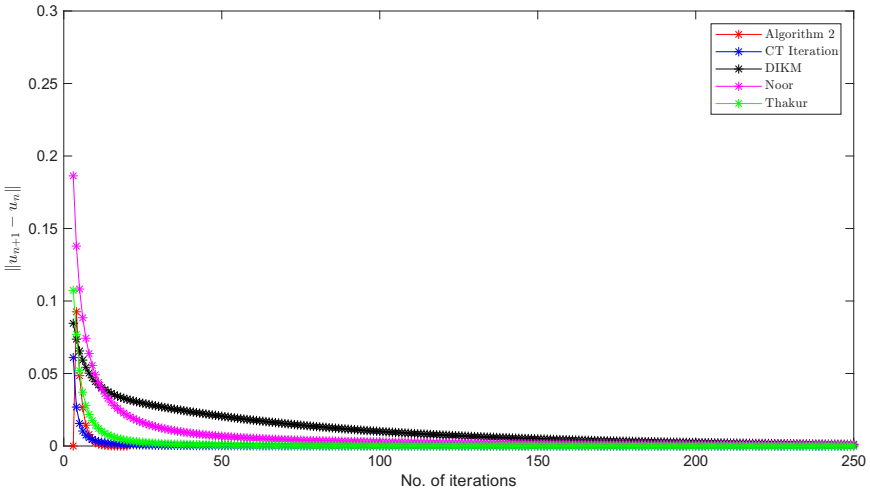

FIGURE 3. Comparison of Algorithm 3.3, the CT Iteration, the DIKM Iteration, Noor's Iteration and Thakur's Iteration.

6. Conclusions

In this paper, we introduced a new three-step iterative algorithm for approximating common fixed points of a infinite family of mappings satisfying Condition (E). We proved that the proposed algorithm converges weakly to a common fixed point under suitable assumptions. We also presented an application of mapping satisfying condition (E) to solve a nonlinear integral equation by converting it into a fixed point problem. Finally, we presented two nontrivial numerical examples and compared the proposed algorithm with the CT Iteration, DIKM Iteration, Noor's Iteration and Thakur's Iteration. The numerical results show that the proposed algorithm is effective and has good convergence behaviour.

Acknowledgment. We are very much thankful to the reviewer for their constructive comments and suggestions which have been useful for the improvement of this paper.

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