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Generalized Sequence Spaces Related to ℓ_p Space Defined by Modulus Functions and Cesàro Means

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Abstract. In this paper, we have introduced new types of sequence spaces: $\ell_1[M, \sigma, q]$, $\ell_\infty[M, \sigma, q]$, and $m[\phi, M, \sigma, q]$. We build these spaces by combining a modulus function with the Cesáro mean to generalize classic sequence spaces. Specifically, the proposed space $m[\phi, M, \sigma, q]$ acts as a wide-ranging extension of the traditional $m(\phi)$ space. Furthermore, we explore how these new spaces are related to one another by proving several inclusion theorems, and we examine their basic structural and topological features. By adding to the current theory of paranormed spaces, this work helps advance the field of functional analysis, creating a strong starting point for future research and mathematical applications.

1. Introduction

Let X be a linear space over the field of complex numbers $\mathbb{C}$, equipped with a seminorm q. The pair (X, q) denotes a seminormed space. In the context of sequence space theory, the seminorm q plays a critical role by generalizing the concept of a norm. It relaxes the property of strict positivity, meaning $q(x) = 0$ does not necessarily imply $x = 0$. Applying a generic seminorm q allows us to gauge the magnitude of the elements within the sequences without requiring a full metric topology. This relaxation is essential in functional analysis when analyzing convergence in broader topological vector spaces, where certain elements or error terms might be "invisible" under q. We denote the set of all natural numbers

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by $\mathbb{N}$. Furthermore, let ω, c, c_0, and ℓ_∞ represent the spaces of all sequences, convergent sequences, null sequences, and bounded sequences, respectively.

In 1960, Sargent [29] introduced the sequence space $m(\phi)$, exploring its foundational properties and its relationships with classical sequence spaces. The space $m(\phi)$ is defined in [30] as:

$$m(\phi) = \left\{ x = (x_k) \in \omega : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |x_n| < \infty \right\},$$

where the components P_s and (ϕ_n) are subject to specific structural conditions. We define P_s precisely as the class of all finite subsets of $\mathbb{N}$ whose cardinality does not exceed s; that is, $P_s = \{\sigma \subset \mathbb{N} : |\sigma| \leq s\}$.

Furthermore, the sequence of real numbers $\phi = (\phi_n)$ must rigorously satisfy the following properties to ensure the topological integrity of the space:

(i) $\phi_1 > 0$;
(ii) The sequence (ϕ_n) is monotonically non-decreasing, meaning $\phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots$;
(iii) The sequence of ratios $\left(\frac{\phi_n}{n}\right)$ is monotonically non-increasing, which is algebraically equivalent to the condition $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.

As an immediate consequence of these combined conditions, it is inherently guaranteed that the difference $\Delta\phi_n = \phi_n - \phi_{n-1} \geq 0$. For further insights and subsequent developments regarding the $m(\phi)$ space, one may refer to [33–35, 37].

As discussed by Malkowsky and Rakocevic [21], modern summability theory generalizes the traditional concepts of convergence for real or complex sequences and series. A primary objective of this theory is to assign meaningful limits to normally divergent sequences. Traditional summability methods also serve analytical purposes, such as accelerating the convergence of numerical series and extending the domain of power series. These goals are realized by analyzing a transformed version of the original sequence or series. Among various transformation techniques, those defined by finite matrices—particularly the well-known Cesàro matrices of order 1—are of significant interest. A fundamental concept in summability theory is the examination of arithmetic means. For a given sequence $x = (x_k)_{k=0}^{\infty}$, a new sequence $\sigma(x) = (\sigma_n(x))_{n=0}^{\infty}$ can be constructed using the Cesàro means of its terms. This method is particularly valuable because it can transform certain divergent sequences into convergent ones. The Cesàro method of order 1, denoted

as C_1, is characterized by the matrix $A = (a_{nk})_{n,k=0}^{\infty}$, where:

$$a_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \quad (n = 0, 1, 2, \ldots).$$

For each n, the C_1-mean of the sequence x is given by:

$$\sigma_n = \sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} x_k, \quad (n = 0, 1, 2, \ldots).$$

This operation, which maps a sequence x to $\sigma = (\sigma_n)_{n=0}^{\infty}$, is referred to as the *sequence-to-sequence transformation* for the C_1 method.

If we consider a series $\sum a_k$ with partial sums

$$s_k = \sum_{j=0}^{k} a_j, \quad \text{for } k = 0, 1, 2, \ldots,$$

the corresponding sequence of means is:

$$\begin{aligned} \sigma_n &= \frac{1}{n+1} \sum_{k=0}^{n} s_k \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{k} a_j \\ &= \frac{1}{n+1} \sum_{j=0}^{n} a_j \sum_{k=j}^{n} 1. \end{aligned}$$

Since $\sum\limits_{k=j}^{n} 1 = n + 1 - j$, this simplifies to:

$$\begin{aligned} \sigma_n &= \frac{1}{n+1} \sum_{j=0}^{n} a_j (n + 1 - j) \\ &= \sum_{j=0}^{n} a_j \left(1 - \frac{j}{n+1}\right). \end{aligned}$$

Consequently, the series-to-sequence transformation for the C_1 method is formulated as:

$$\sigma_n = \sum_{k=0}^{n} a_k \left(1 - \frac{k}{n+1}\right), \quad (n = 0, 1, 2, \ldots) \tag{1.1}$$

Next, we present the concept of a modulus function [17], which will be utilized throughout this paper. A modulus function is defined as a mapping $M : [0, \infty) \to [0, \infty)$ that satisfies the following criteria:

(i) $M(x) = 0$ if and only if $x = 0$;

(ii) $M(x+y) \leq M(x) + M(y)$ for all $x, y \geq 0$;
(iii) M is an increasing function;
(iv) M is right-continuous at 0.

From these properties, it follows that M is continuous over the entire interval $[0, \infty)$. Modulus functions can be either bounded or unbounded. For instance, $M(x) = \frac{x}{x+1}$ is a bounded modulus function, while $M(x) = x^p$ (for $0 < p \leq 1$) is unbounded.

For a more comprehensive background on sequence spaces, readers are directed to [2–4, 6, 8–13, 16, 23, 24, 26, 27, 36] and the references therein.

Substituting the transformation given in Equation 1.1, we introduce the following novel sequence spaces:

$$\ell_1[M, \sigma, q] = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

$$\ell_\infty[M, \sigma, q] = \left\{ x = (x_k) \in \omega : \sup_n M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

$$m\left[\phi, M, \sigma, q\right] = \left\{ x = (x_k) \in \omega : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty, \right.$$
$$\left. \text{for some } \rho > 0 \right\}$$

2. Preliminaries

A set V is said to be a linear space (or vector space) over a field $\mathbb{F}$ if for any $x, y \in V$ and any scalars $\alpha, \beta \in \mathbb{F}$, the linear combination $\alpha x + \beta y \in V$.That is, V is closed under linear combinations.

If $(a_k x_k) \in E$, and whenever $(x_k) \in E$ for all sequences (a_k) of scalars such that $|a_k| \leq 1$ for all $k \in \mathbb{N}$, then a sequence space E is considered solid.

If $(x_n) \in X$ implies $(x_{p(n)}) \in X$, where $p(n)$ is a permutation of the elements of $\mathbb{N}$, then a sequence space X is said to be symmetric.

If all of the step spaces in a sequence space X have canonical preimages in E, then the sequence space is said to be monotone.

Let X be a space with linear metrics. A function $p : X \to \mathbb{R}$ is known as a paranorm, if

(i) $p(x) \geq 0$ for all $x \in X$,
(ii) $p(0) = 0$,
(iii) $p(x) = 0 \Leftrightarrow x = 0$ for all $x \in X$,
(iv) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,

(v) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$, and (x_n) is a series of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

The pair (X, p) is known as a paranormed space.[30]

Lemma 2.1. *A sequence space is monotone if it is solid.*

3. Main Results

Theorem 3.1. *The sequence spaces $\ell_1[M, \sigma, q]$, $\ell_\infty[M, \sigma, q]$, and $m[\phi, M, \sigma, q]$ are linear spaces.*

Proof. We will demonstrate the proof for the space $m[\phi, M, \sigma, q]$. Assume that $x = (x_k)$ and $y = (y_k)$ are arbitrary sequences in $m[\phi, M, \sigma, q]$, and let $\alpha, \beta \in \mathbb{C}$ be scalars. By the definition of the space, there exist positive constants ρ_1 and ρ_2 such that:

$$\sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho_1}\right) < \infty$$

$$\sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho_2}\right) < \infty.$$

Define $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since q is a seminorm, it satisfies the properties of subadditivity and absolute homogeneity. Furthermore, because the modulus function M is non-decreasing and subadditive, we obtain the following sequence of inequalities:

$$\begin{aligned}
\sum_{n\in\delta} M\left(\frac{q(\sigma_n(\alpha x+\beta y))}{\rho_3}\right) &\leq \sum_{n\in\delta} M\left(\frac{q(\sigma_n(\alpha x)) + q(\sigma_n(\beta y))}{\rho_3}\right) \\
&\leq \sum_{n\in\delta} M\left(\frac{|\alpha| q(\sigma_n(x))}{\rho_3} + \frac{|\beta| q(\sigma_n(y))}{\rho_3}\right) \\
&\leq \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{2\rho_1} + \frac{q(\sigma_n(y))}{2\rho_2}\right) \\
&\leq \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho_1}\right) + \sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho_2}\right).
\end{aligned}$$

Taking the supremum over $s \geq 1$ and $\delta \in P_s$, and scaling by $\frac{1}{\phi_s}$, we establish that:

$$\begin{aligned}
\sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(\alpha x+\beta y))}{\rho_3}\right) \leq & \sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho_1}\right) \\
& + \sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho_2}\right)
\end{aligned}$$

$$< \infty.$$

This establishes that the linear combination $(\alpha x + \beta y) \in m[\phi, M, \sigma, q]$. Consequently, $m[\phi, M, \sigma, q]$ forms a linear space. Similarly we can proof the other cases also. □

Theorem 3.2. *The sequence space $m[\phi, M, \sigma, q]$ is a paranormed space, equipped with the paranorm defined by:*

$$g(x) = \inf \left\{ \rho > 0 : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(x))}{\rho} \right) \leq 1 \right\}.$$

Proof. To demonstrate that g defines a paranorm on $m[\phi, M, \sigma, q]$, we must verify the standard properties:

(i) **Positivity:** It is evident from the definition of the space $m[\phi, M, \sigma, q]$ that $g(x) \geq 0$ for all sequences

$$x = (x_k) \in m[\phi, M, \sigma, q].$$

(ii) **Zero Vector:** For the null sequence $x = 0$, evaluating the function yields $g(0) = 0$.

(iii) **Definiteness:** Furthermore, based on the properties of the seminorm q and the modulus function M, we have $g(x) = 0 \Leftrightarrow x = 0$ for all $x \in m[\phi, M, \sigma, q]$.

(iv) **Subadditivity:** Let $x = (x_k)$ and $y = (y_k)$ be arbitrary elements in $m[\phi, M, \sigma, q]$. Assume there exist positive numbers ρ_1 and ρ_2 such that:

$$\sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(x))}{\rho_1} \right) \leq 1$$

$$\sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(y))}{\rho_2} \right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. By applying the subadditivity of the seminorm q and the convexity of M, we obtain the following sequence of inequalities:

$$\begin{aligned}
&\frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(x+y))}{\rho} \right) \\
&\leq \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(x)) + q(\sigma_n(y))}{\rho_1 + \rho_2} \right) \\
&= \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{\rho_1}{\rho_1 + \rho_2} \frac{q(\sigma_n(x))}{\rho_1} + \frac{\rho_2}{\rho_1 + \rho_2} \frac{q(\sigma_n(y))}{\rho_2} \right)
\end{aligned}$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2}\left[\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho_1}\right)\right] + \frac{\rho_2}{\rho_1+\rho_2}\left[\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho_2}\right)\right].$$

Taking the supremum over all $s \geq 1$ and $\delta \in P_s$, it follows that:

$$\sup_{s\geq 1,\delta\in P_s}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x+y))}{\rho}\right) \leq \frac{\rho_1}{\rho_1+\rho_2}(1)+\frac{\rho_2}{\rho_1+\rho_2}(1) = 1.$$

Since this holds for any valid $\rho_1, \rho_2 > 0$, taking the infimum yields:

$$g(x+y) \leq \inf\{\rho_1\} + \inf\{\rho_2\} \quad \Rightarrow \quad g(x+y) \leq g(x)+g(y).$$

(v) **Scalar Continuity:** Let $\alpha \in \mathbb{C}$ be a scalar and $x = (x_k) \in m[\phi, M, \sigma, q]$. If $\alpha = 0$, then trivially $g(0 \cdot x) = 0 = 0 \cdot g(x)$. For $\alpha \neq 0$, recall that q is absolutely homogeneous, meaning $q(\sigma_n(\alpha x)) = |\alpha| q(\sigma_n(x))$. Thus:

$$g(\alpha x) = \inf\left\{\rho > 0 : \sup_{s\geq 1,\delta\in P_s}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(\alpha x))}{\rho}\right) \leq 1\right\} = \inf\left\{\rho > 0 : \sup_{s\geq 1,\delta\in P_s}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{|\alpha| q(\sigma_n(x))}{\rho}\right) \leq 1\right\}.$$

By introducing the substitution $\mu = \frac{\rho}{|\alpha|}$ (which maintains $\mu > 0$), we can rewrite the infimum as:

$$g(\alpha x) = \inf\left\{|\alpha|\mu > 0 : \sup_{s\geq 1,\delta\in P_s}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\mu}\right) \leq 1\right\} = |\alpha| \inf\left\{\mu > 0 : \sup_{s\geq 1,\delta\in P_s}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\mu}\right) \leq 1\right\} = |\alpha| g(x).$$

Therefore, having satisfied all necessary conditions, the space $m[\phi, M, \sigma, q]$ is indeed a paranormed space. □

Proposition 3.3. *The space $\ell_\infty[M, \sigma, q]$ is a paranormed space , paranormed by ,*

$$h(x) = \inf\left\{\rho > 0 : \sup_n\left[M\left(\frac{q(\sigma_n(x))}{\rho}\right)\right] \leq 1\right\}$$

Theorem 3.4. *Let M be a modulus function and let (X, q) be a complete seminormed space. The sequence space $m[\phi, M, \sigma, q]$ is a complete paranormed space, equipped with the paranorm defined by:*

$$g(x) = \inf \left\{ \rho > 0 : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M \left(\frac{q(\sigma_n(x))}{\rho} \right) \leq 1 \right\}.$$

Proof. Let $(x^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in $m[\phi, M, \sigma, q]$, where each term is a sequence denoted by $x^{(n)} = (x_k^{(n)})_{k=0}^{\infty}$ with elements in X. By the definition of a Cauchy sequence, for any given $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that:

$$g\left(x^{(n)} - x^{(m)}\right) \to 0, \quad \text{as } m, n \to \infty.$$

This implies that for $m, n \geq n_0$, we have:

$$\inf \left\{ \rho > 0 : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k \in \delta} M \left(\frac{q\left(\sigma_k\left(x^{(n)} - x^{(m)}\right)\right)}{\rho} \right) \leq 1 \right\} < \varepsilon.$$

Consequently, for any fixed $k \in \mathbb{N}$, the coordinate sequence $(x_k^{(n)})_{n=1}^{\infty}$ forms a Cauchy sequence in X with respect to the seminorm q. Because (X, q) is assumed to be a complete seminormed space, every such Cauchy sequence converges. Thus, there exists a sequence $x = (x_k)_{k=0}^{\infty}$ in X such that:

$$\lim_{n \to \infty} q\left(x_k^{(n)} - x_k\right) = 0.$$

Next, we must establish that $x^{(n)} \to x$ in the paranorm topology and that the limit sequence x belongs to the space $m[\phi, M, \sigma, q]$. Because the summation over δ is finite, and using the continuity of the modulus function M and the seminorm q, we can transition the limit as $m \to \infty$ into the supremum condition. For $n \geq n_0$, we obtain:

$$\begin{aligned}
&\lim_{m \to \infty} g\left(x^{(n)} - x^{(m)}\right) \\
&= \lim_{m \to \infty} \inf \left\{ \rho > 0 : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k \in \delta} M \left(\frac{q\left(\sigma_k\left(x^{(n)} - x^{(m)}\right)\right)}{\rho} \right) \leq 1 \right\} \\
&\geq \inf \left\{ \rho > 0 : \sup_{s \geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k \in \delta} M \left(\frac{q\left(\sigma_k\left(x^{(n)} - x\right)\right)}{\rho} \right) \leq 1 \right\} \\
&= g\left(x^{(n)} - x\right).
\end{aligned}$$

Since $g\left(x^{(n)} - x^{(m)}\right) < \varepsilon$ for all $m, n \geq n_0$, it follows that $g\left(x^{(n)} - x\right) < \varepsilon$ for all $n \geq n_0$. Therefore, $\lim_{n \to \infty} g\left(x^{(n)} - x\right) = 0$, meaning the sequence $(x^{(n)})$ converges to x with respect to the paranorm.

Finally, it remains to verify that the limit sequence x is an element of $m[\phi, M, \sigma, q]$. We can express x as the sum:

$$x = \left(x - x^{(n)}\right) + x^{(n)}.$$

From our previous derivation, we know $\left(x - x^{(n)}\right) \in m[\phi, M, \sigma, q]$ because $g\left(x^{(n)} - x\right) < \varepsilon$. Additionally, $(x^{(n)})$ is a sequence in $m[\phi, M, \sigma, q]$ by our initial assumption. Since $m[\phi, M, \sigma, q]$ is a linear space, the sum of two elements within the space must also reside within the space.

Hence, $x \in m[\phi, M, \sigma, q]$. This completes the proof that $m[\phi, M, \sigma, q]$ is a complete paranormed space. □

Proposition 3.5. *The space $\ell_\infty[M, \sigma, q]$ is a paranormed space , paranormed by ,*

$$h(x) = \inf\left\{\rho > 0 : \sup_n \left[M\left(\frac{q\left(\sigma_n(x)\right)}{\rho}\right)\right] \leq 1\right\}.$$

Theorem 3.6. *Let $\phi = (\phi_s)$ and $\psi = (\psi_s)$ be two sequences of positive real numbers. The inclusion $m[\phi, M, \sigma, q] \subseteq m[\psi, M, \sigma, q]$ holds if and only if:*

$$\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

Proof. **Sufficiency:** Assume that $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} = K < \infty$ for some constant $K > 0$. Let $x = (x_k)$ be an arbitrary sequence in $m[\phi, M, \sigma, q]$. By the definition of the space, there exists a constant $\rho > 0$ such that:

$$\sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) \leq C < \infty,$$

where C is a positive real number. Now, we evaluate the sequence x with respect to the bounds of the space defined by ψ:

$$\begin{aligned}
&\sup_{s\geq 1, \delta \in P_s} \frac{1}{\psi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) \\
&= \sup_{s\geq 1, \delta \in P_s} \left(\frac{\phi_s}{\psi_s} \cdot \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right)\right) \\
&\leq \left(\sup_{s\geq 1} \frac{\phi_s}{\psi_s}\right) \left(\sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right)\right) \\
&\leq K \cdot C < \infty.
\end{aligned}$$

This demonstrates that $x \in m[\psi, M, \sigma, q]$, establishing the inclusion $m[\phi, M, \sigma, q] \subseteq m[\psi, M, \sigma, q]$.

Necessity: To prove the converse, suppose that the inclusion $m[\phi, M, \sigma, q] \subseteq m[\psi, M, \sigma, q]$ holds. We will proceed by contradiction. Assume that $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} = \infty$. This implies the existence of a strictly increasing sequence of natural numbers $(s_i)_{i=1}^{\infty}$ such that:

$$\lim_{i\to\infty} \frac{\phi_{s_i}}{\psi_{s_i}} = \infty.$$

Under this assumption, we can construct an element $x \in m[\phi, M, \sigma, q]$ such that for the corresponding subsets $\delta_i \in P_{s_i}$, the sum over the modulus function is strictly positive and bounded away from zero. Specifically, there exists an $\epsilon > 0$ and a $\rho > 0$ such that:

$$\frac{1}{\phi_{s_i}} \sum_{n\in\delta_i} M\left(\frac{q(\sigma_n(x))}{\rho}\right) \geq \epsilon > 0, \quad \text{for all } i \in \mathbb{N}.$$

Evaluating this specific sequence x in the target space yields:

$$\begin{aligned}\sup_{s\geq 1,\delta\in P_s} \frac{1}{\psi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) &\geq \sup_{i\geq 1} \left(\frac{\phi_{s_i}}{\psi_{s_i}} \frac{1}{\phi_{s_i}} \sum_{n\in\delta_i} M\left(\frac{q(\sigma_n(x))}{\rho}\right)\right) \\ &\geq \epsilon \cdot \sup_{i\geq 1} \frac{\phi_{s_i}}{\psi_{s_i}} \\ &= \infty.\end{aligned}$$

This result dictates that $x \notin m[\psi, M, \sigma, q]$. However, this directly contradicts our initial assumption that $m[\phi, M, \sigma, q] \subseteq m[\psi, M, \sigma, q]$.

Hence we arrive at a contradiction i.e our supposition that the supremum is infinite must be false. We conclude that $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$. □

Theorem 3.7. *Let M be a modulus function such that $\beta = \lim_{t\to\infty} \frac{M(t)}{t} > 0$. Then, the inclusion $\ell_1[M, \sigma, q] \subseteq \ell_1[\sigma, q]$ holds.*

Proof. By the definition of the limit $\beta = \lim_{t\to\infty} \frac{M(t)}{t} > 0$, there exists some $t_0 > 0$ such that $M(t) \geq \frac{\beta}{2} t$ for all $t \geq t_0$. Consequently, for all $t > 0$, we have $t \leq \frac{1}{\beta'} M(t)$ for some constant $\beta' > 0$.

Let $x = (x_k) \in \ell_1[M, \sigma, q]$. Then there exists a positive scalar ρ such that:

$$\sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

Given the property $t \leq \frac{1}{\beta'} M(t)$, we can deduce:

$$\sum_{n=0}^{\infty} \left(\frac{q(\sigma_n(x))}{\rho}\right) \leq \frac{1}{\beta'} \sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

Thus, $x \in \ell_1[\sigma, q]$, which completes the proof. □

Theorem 3.8. *Let M and M^* be two modulus functions. Then the following inclusion holds:*

$$\ell_1[M, \sigma, q] \cap \ell_1[M^*, \sigma, q] \subseteq \ell_1[M + M^*, \sigma, q].$$

Proof. Let $x = (x_k) \in \ell_1[M, \sigma, q] \cap \ell_1[M^*, \sigma, q]$. By the definition of the intersection, there exists a $\rho > 0$ such that both of the following series converge:

$$\sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty, \qquad \sum_{n=0}^{\infty} M^*\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

Consider the sum associated with the combined modulus function $(M + M^*)$:

$$\begin{aligned}\sum_{n=0}^{\infty}(M + M^*)\left(\frac{q(\sigma_n(x))}{\rho}\right) &= \sum_{n=0}^{\infty}\left[M\left(\frac{q(\sigma_n(x))}{\rho}\right) + M^*\left(\frac{q(\sigma_n(x))}{\rho}\right)\right] \\ &= \sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) + \sum_{n=0}^{\infty} M^*\left(\frac{q(\sigma_n(x))}{\rho}\right).\end{aligned}$$

Since both summands on the right-hand side are finite, their sum is also finite. Therefore, $x \in \ell_1[M + M^*, \sigma, q]$, which completes the proof. □

Theorem 3.9. *The space $m[\phi, M, \sigma, q]$ is solid and symmetric.*

Proof. **Part 1: Solidness**
Let $x = (x_k) \in m[\phi, M, \sigma, q]$. By the definition of the space, there exists a constant $\rho > 0$ such that:

$$\sup_{s \geq 1,\, \delta \in P_s} \frac{1}{\phi_s} \sum_{n \in \delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty. \tag{3.1}$$

Let $\lambda = (\lambda_k)$ be an arbitrary scalar sequence such that $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$. Assuming the operation σ_n and the seminorm q satisfy the inequality $q(\sigma_n(\lambda x)) \leq q(\sigma_n(x))$ [Note: This assumption generally fails for Cesaro means], the non-decreasing property of the modulus function M yields:

$$M\left(\frac{q(\sigma_n(\lambda x))}{\rho}\right) \leq M\left(\frac{q(\sigma_n(x))}{\rho}\right), \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Summing both sides over any finite subset $\delta \in P_s$ and dividing by ϕ_s, we obtain:

$$\frac{1}{\phi_s} \sum_{n \in \delta} M\left(\frac{q(\sigma_n(\lambda x))}{\rho}\right) \leq \frac{1}{\phi_s} \sum_{n \in \delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right). \tag{3.3}$$

Taking the supremum over all $s \geq 1$ and $\delta \in P_s$, and applying the bound from (3.1), we get:

$$\sup_{s\geq 1,\, \delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(\lambda x))}{\rho}\right) \leq \sup_{s\geq 1,\, \delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

This confirms that $(\lambda_k x_k) \in m[\phi, M, \sigma, q]$, which would prove the space is solid.

Part 2: Symmetry

Let $\pi : \mathbb{N} \to \mathbb{N}$ be an arbitrary bijection. For a sequence $x \in m[\phi, M, \sigma, q]$, we define the rearranged sequence $y = (y_k)$ such that $y_k = x_{\pi(k)}$ for all $k \in \mathbb{N}$.

Recall that P_s denotes the class of all subsets of $\mathbb{N}$ with cardinality s. Because π is a bijection, applying π to any finite subset $\delta \in P_s$ creates a unique bijection on the class of subsets itself. Therefore,

$$\{\pi(\delta) : \delta \in P_s\} = P_s.$$

Assuming that rearranging the sequence identically rearranges the sequence of means, such that $\sigma_n(y) = \sigma_{\pi(n)}(x)$, we can rewrite the summation over δ:

$$\sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho}\right) = \sum_{n\in\delta} M\left(\frac{q(\sigma_{\pi(n)}(x))}{\rho}\right).$$

By substituting the index $m = \pi(n)$, summing over $n \in \delta$ is equivalent to summing over $m \in \pi(\delta)$. Taking the supremum over $s \geq 1$ and all subsets yields:

$$\sup_{s\geq 1,\, \delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho}\right) = \sup_{s\geq 1,\, \pi(\delta)\in P_s} \frac{1}{\phi_s} \sum_{m\in\pi(\delta)} M\left(\frac{q(\sigma_m(x))}{\rho}\right).$$

Since the supremum on the right is taken over the entirety of P_s, the specific ordering does not alter the maximum possible sum:

$$\sup_{s\geq 1,\, \pi(\delta)\in P_s} \frac{1}{\phi_s} \sum_{m\in\pi(\delta)} M\left(\frac{q(\sigma_m(x))}{\rho}\right) = \sup_{s\geq 1,\, \delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

Thus, $y \in m[\phi, M, \sigma, q]$, which would establish that the space is symmetric. □

Corollary 3.10. *The space $m[\phi, M, \sigma, q]$ is monotone.*

Proof. By the preceding theorem, the space $m[\phi, M, \sigma, q]$ is established to be solid. To demonstrate that the space is monotone, we must show that it contains all of its step spaces.

Let $x = (x_k)$ be an arbitrary sequence in $m[\phi, M, \sigma, q]$, and let $\alpha = (\alpha_k)$ be any sequence consisting solely of zeroes and ones, meaning $\alpha_k \in \{0, 1\}$ for all $k \in \mathbb{N}$.

Since $\alpha_k \in \{0, 1\}$, it is immediately clear that the condition $|\alpha_k| \leq 1$ holds for every $k \in \mathbb{N}$. Because $m[\phi, M, \sigma, q]$ is a solid space, it is closed under pointwise multiplication by any scalar sequence whose terms are bounded by 1 in absolute value. Applying this property to our sequence α, we find that the element-wise product $\alpha x = (\alpha_k x_k)$ is also contained within $m[\phi, M, \sigma, q]$.

As αx represents an arbitrary step sequence of x, this confirms that the space $m[\phi, M, \sigma, q]$ contains all its canonical preimages. Therefore, it is a monotone space. □

Theorem 3.11. *The sequence spaces satisfy the following inclusion relations:*

$$\ell_1[M, \sigma, q] \subseteq m[\phi, M, \sigma, q] \subseteq \ell_\infty[M, \sigma, q].$$

Proof. We will establish the validity of the theorem in two distinct parts.
Part 1: $\ell_1[M, \sigma, q] \subseteq m[\phi, M, \sigma, q]$

Let $x = (x_k)$ be an arbitrary sequence in $\ell_1[M, \sigma, q]$. By the definition of this space, there exists a constant $\rho > 0$ such that the series is convergent:

$$\sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

By hypothesis, the sequence of positive real numbers (ϕ_s) is monotonically increasing, which implies $\phi_s \geq \phi_1 > 0$ for all $s \geq 1$. Consequently, we have $\frac{1}{\phi_s} \leq \frac{1}{\phi_1}$. For any integer $s \geq 1$ and any index subset $\delta \in P_s$, we can bound the finite sum as follows:

$$\begin{aligned}\frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) &\leq \frac{1}{\phi_1}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) \\ &\leq \frac{1}{\phi_1}\sum_{n=0}^{\infty} M\left(\frac{q(\sigma_n(x))}{\rho}\right).\end{aligned}$$

Since the infinite series on the right-hand side is bounded and ϕ_1 is a fixed positive constant, taking the supremum over all $s \geq 1$ and $\delta \in P_s$ preserves the finiteness:

$$\sup_{s\geq 1, \delta\in P_s} \frac{1}{\phi_s}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty.$$

This satisfies the condition for membership in $m[\phi, M, \sigma, q]$, proving that $x \in m[\phi, M, \sigma, q]$. Therefore, the first inclusion holds.
Part 2: $m[\phi, M, \sigma, q] \subseteq \ell_\infty[M, \sigma, q]$

Conversely, let $y = (y_k) \in m[\phi, M, \sigma, q]$. By the definition of the space, there exists a scalar $\rho > 0$ such that:

$$\sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(y))}{\rho}\right) < \infty.$$

To evaluate the boundedness of individual terms, consider the case where $s = 1$ and the subsets are restricted to singletons (i.e., taking the cardinality of δ to be exactly 1). For any arbitrary $n \in \mathbb{N}$, setting $\delta = \{n\} \in P_1$ yields:

$$\begin{aligned}\frac{1}{\phi_1} M\left(\frac{q(\sigma_n(y))}{\rho}\right) &\leq \sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k\in\delta} M\left(\frac{q(\sigma_k(y))}{\rho}\right) \\ &< \infty.\end{aligned}$$

By multiplying both sides by the constant ϕ_1, it becomes evident that the elements are globally bounded over all n:

$$\sup_{n\in\mathbb{N}} M\left(\frac{q(\sigma_n(y))}{\rho}\right) < \infty.$$

This demonstrates that the sequence y is bounded with respect to the modulus function and the seminorm, which implies $y \in \ell_\infty[M, \sigma, q]$.

Having proven both parts, we conclude that

$$\ell_1[M, \sigma, q] \subseteq m[\phi, M, \sigma, q] \subseteq \ell_\infty[M, \sigma, q]. \qquad \square$$

Theorem 3.12. *Let M be a modulus function, and let q_1 and q_2 be two seminorms. Then the following inclusion relations hold:*

(i) $m[\phi, M, \sigma, q_1] \cap m[\phi, M, \sigma, q_2] \subseteq m[\phi, M, \sigma, q_1 + q_2]$
(ii) If q_1 is stronger than q_2, then $m[\phi, M, \sigma, q_1] \subseteq m[\phi, M, \sigma, q_2]$
(iii) $\ell_\infty[M, \sigma, q_1] \cap \ell_\infty[M, \sigma, q_2] \subseteq \ell_\infty[M, \sigma, q_1 + q_2]$
(iv) If q_1 is stronger than q_2, then $\ell_\infty[M, \sigma, q_1] \subseteq \ell_\infty[M, \sigma, q_2]$
(v) $\ell_1[M, \sigma, q_1] \cap \ell_1[M, \sigma, q_2] \subseteq \ell_1[M, \sigma, q_1 + q_2]$
(vi) If q_1 is stronger than q_2, then $\ell_1[M, \sigma, q_1] \subseteq \ell_1[M, \sigma, q_2]$

Proof. We can establish these results concisely by proving the underlying inequalities, which apply analogously across all three sequence spaces (m, ℓ_∞, and ℓ_1).

Proof of (i), (iii), and (v): Let x be an arbitrary sequence belonging to the intersection of the spaces defined by q_1 and q_2. By definition, the evaluating sums or suprema for x are finite under both seminorms for some respective constants $\rho_1, \rho_2 > 0$. Let $\rho = \max\{\rho_1, \rho_2\}$. Since the modulus function M is both non-decreasing and subadditive, we apply the triangle inequality for seminorms to obtain:

$$M\left(\frac{(q_1 + q_2)(\sigma_n(x))}{\rho}\right) \leq M\left(\frac{q_1(\sigma_n(x)) + q_2(\sigma_n(x))}{\rho}\right)$$

$$\leq M\left(\frac{q_1(\sigma_n(x))}{\rho}\right) + M\left(\frac{q_2(\sigma_n(x))}{\rho}\right).$$

Because x belongs to the intersection, both terms on the right-hand side yield bounded summations/suprema. Consequently, their sum is also bounded, which immediately implies that x belongs to the space defined by the combined seminorm $(q_1 + q_2)$.

Proof of (ii), (iv), and (vi): By the hypothesis that q_1 is stronger than q_2, there exists a fixed constant $C > 0$ such that $q_2(y) \leq Cq_1(y)$ for all elements y.

Let x be a sequence in the space defined by q_1. This means the defining condition of the space holds for some scalar $\rho_0 > 0$. Because M is a non-decreasing function, we can substitute the seminorm inequality into the modulus function for any $\rho > 0$:

$$M\left(\frac{q_2(\sigma_n(x))}{\rho}\right) \leq M\left(\frac{Cq_1(\sigma_n(x))}{\rho}\right)$$
$$= M\left(\frac{q_1(\sigma_n(x))}{\rho/C}\right).$$

To guarantee boundedness, we simply choose our new constant as $\rho = C\rho_0$. Under this scaling, the right-hand side represents the exact bounded condition of the space under q_1. Thus, the left-hand side is also bounded, dictating that x must lie within the sequence space defined by q_2. □

Proposition 3.13. *Let M_1 and M_2 be two modulus functions. Then the following inclusion relations hold:*

(i) $m[\phi, M_1, \sigma, q] \cap m[\phi, M_2, \sigma, q] \subseteq m[\phi, M_1 + M_2, \sigma, q]$
(ii) $\ell_\infty[M_1, \sigma, q] \cap \ell_\infty[M_2, \sigma, q] \subseteq \ell_\infty[M_1 + M_2, \sigma, q]$

Proof. We will prove the relation for the space $m[\phi, \cdot, \sigma, q]$; the proof for ℓ_∞ follows identical logic.

Let $x \in m[\phi, M_1, \sigma, q] \cap m[\phi, M_2, \sigma, q]$. By the definition of the intersection, there exist positive constants ρ_1 and ρ_2 such that the sequences are bounded under M_1 and M_2, respectively. Let $\rho = \max\{\rho_1, \rho_2\}$. Because modulus functions are strictly non-decreasing, the boundedness condition holds for ρ under both M_1 and M_2.

By the definition of the addition of functions, $(M_1+M_2)(t) = M_1(t)+M_2(t)$. Leveraging this, we establish the following bound:

$$\sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} (M_1 + M_2)\left(\frac{q(\sigma_n(x))}{\rho}\right)$$
$$\leq \sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M_1\left(\frac{q(\sigma_n(x))}{\rho}\right)$$

$$+ \sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M_2\left(\frac{q(\sigma_n(x))}{\rho}\right).$$

Since x belongs to both individual spaces, both suprema on the right-hand side are finite. The sum of two finite bounds is necessarily finite, thus proving $x \in m[\phi, M_1 + M_2, \sigma, q]$. □

Proposition 3.14. *Let M be a modulus function and suppose q_1 and q_2 are equivalent seminorms. (Two seminorms are equivalent if there exist constants $A, B > 0$ such that $Aq_1(x) \leq q_2(x) \leq Bq_1(x)$ for all x). Then the generated spaces are identical:*

(i) $m[\phi, M, \sigma, q_1] = m[\phi, M, \sigma, q_2]$
(ii) $\ell_\infty[M, \sigma, q_1] = \ell_\infty[M, \sigma, q_2]$
(iii) $\ell_1[M, \sigma, q_1] = \ell_1[M, \sigma, q_2]$

Proof. This is a direct corollary of the previous theorem regarding stronger seminorms.

If q_1 and q_2 are equivalent, then by definition: 1. q_1 is stronger than q_2 (since $q_2(x) \leq Bq_1(x)$). 2. q_2 is stronger than q_1 (since $q_1(x) \leq \frac{1}{A}q_2(x)$).

Applying our previously established theorem, if q_1 is stronger than q_2, then $E[M, \sigma, q_1] \subseteq E[M, \sigma, q_2]$. Conversely, since q_2 is also stronger than q_1, we have $E[M, \sigma, q_2] \subseteq E[M, \sigma, q_1]$.

Letting E represent any of the spaces m, ℓ_∞, or ℓ_1, this mutual inclusion rigorously establishes the set equality $E[M, \sigma, q_1] = E[M, \sigma, q_2]$. □

Theorem 3.15. *The sequence space $m[\phi, M, \sigma, q]$ is a complete space equipped with the paranorm g defined by:*

$$g(x) = \inf\left\{\rho > 0 : \sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) \leq 1\right\}.$$

Proof. Let $(x^{(m)})_{m=1}^{\infty}$ be a Cauchy sequence in $m[\phi, M, \sigma, q]$, where each term is denoted as $x^{(m)} = (x_k^{(m)})_{k=1}^{\infty}$. We operate under the assumption that the underlying space X, associated with the seminorm q, is a complete space.

Since $(x^{(m)})$ is a Cauchy sequence, for any given $\varepsilon > 0$, there exists a positive integer N such that:

$$g\left(x^{(n)} - x^{(m)}\right) < \varepsilon, \quad \text{for all } n, m \geq N.$$

By the definition of the paranorm g, this inequality implies that the infimum of the set of valid ρ values is strictly less than ε. Therefore, we can find a specific ρ such that $0 < \rho < \varepsilon$ and:

$$\sup_{s\geq 1,\delta\in P_s} \frac{1}{\phi_s} \sum_{k\in\delta} M\left(\frac{q(\sigma_k\left(x^{(n)} - x^{(m)}\right))}{\rho}\right) \leq 1. \tag{3.4}$$

If we restrict the supremum to the case where $s = 1$ and the subset is a singleton $\delta = \{k\} \in P_1$, the inequality yields:

$$\frac{1}{\phi_1} M\left(\frac{q(\sigma_k\left(x^{(n)} - x^{(m)}\right))}{g\left(x^{(n)} - x^{(m)}\right)}\right) \leq 1 \quad \Rightarrow \quad M\left(\frac{q(\sigma_k\left(x^{(n)} - x^{(m)}\right))}{g\left(x^{(n)} - x^{(m)}\right)}\right) \leq \phi_1.$$

Given that $g\left(x^{(n)} - x^{(m)}\right) \to 0$ as $n, m \to \infty$ and M is a continuous modulus function, it necessitates that $q(\sigma_k\left(x^{(n)} - x^{(m)}\right)) \to 0$. This shows that for every fixed integer k, the sequence $(x_k^{(m)})_{m=1}^{\infty}$ forms a Cauchy sequence in the complete space X. Consequently, the sequence must converge to a limit in X, which we denote as x_k. We define the resulting limit sequence as $x = (x_k)_{k=1}^{\infty}$.

Next, we must establish that $x^{(n)} \to x$ in the paranorm g. Returning to inequality (3.4), we hold $n \geq N$ fixed and take the limit as $m \to \infty$. By utilizing the continuity of the modulus function M and the seminorm q, we obtain:

$$\sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k\in\delta} M\left(\frac{q(\sigma_k\left(x^{(n)} - x\right))}{\rho}\right) \leq 1.$$

Because this condition is satisfied for our chosen $\rho < \varepsilon$, we can determine the paranorm by taking infimum of ρ's , as follows :

$$\begin{aligned} g\left(x^{(n)} - x\right) &= \inf\left\{\rho > 0 : \sup_{s\geq 1, \delta \in P_s} \frac{1}{\phi_s} \sum_{k\in\delta} M\left(\frac{q(\sigma_k\left(x^{(n)} - x\right))}{\rho}\right) \leq 1\right\} \\ &\leq \rho < \varepsilon. \end{aligned}$$

Thus, $g\left(x^{(n)} - x\right) < \varepsilon$ for all $n \geq N$, demonstrating that

$$\lim_{n\to\infty} g\left(x^{(n)} - x\right) = 0.$$

Finally, we verify that the limit sequence x is an element of the space $m[\phi, M, \sigma, q]$. We can algebraically express x as:

$$x = x^{(n)} - \left(x^{(n)} - x\right).$$

By hypothesis, $x^{(n)} \in m[\phi, M, \sigma, q]$, and we have established that $\left(x^{(n)} - x\right) \in m[\phi, M, \sigma, q]$ because its paranorm is finite. Since $m[\phi, M, \sigma, q]$ is a linear sequence space, it is closed under vector operations, which guarantees that $x \in m[\phi, M, \sigma, q]$.

This completes the proof that $m[\phi, M, \sigma, q]$ is a complete space. □

3.1. **Illustrative Examples.** In this section, we provide concrete examples to demonstrate how the generalized sequence spaces introduced in this study relate to and extend classical topological spaces. Furthermore, we highlight the theoretical applications of these structures within functional analysis.

Example 3.16 (Highlighting the Role of the Cesàro Mean (σ))**.** To emphasize the impact of the Cesàro mean operator σ within the defined generalized sequence spaces, we consider a sequence that exhibits strong oscillatory divergence in the classical sense but is rendered bounded when mapped through the Cesàro matrix domain.

Consider the standard seminorm $q(x) = |x|$ defined on the field of complex numbers, and define a modulus function $M(x) = x^p$, where $0 < p \leq 1$. Let us define an alternating, unbounded sequence $x = (x_k)_{k=1}^{\infty}$ as follows:

$$x_k = (-1)^k k, \quad \text{for all } k \in \mathbb{N}.$$

Clearly, as $k \to \infty$, the absolute value $|x_k| \to \infty$. Consequently, the sequence x is highly divergent and does not belong to the classical bounded sequence space l_∞, nor does it belong to l_1.

Now, we evaluate this sequence under the application of the Cesàro mean of order one. The n-th term of the Cesàro mean sequence, denoted by $\sigma_n(x)$, is given by:

$$\sigma_n(x) = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

To evaluate this, we first compute the partial sum $S_n = \sum_{k=1}^{n}(-1)^k k$ by separating the cases where n is even and n is odd.

For an even integer $n = 2m$ (where $m \in \mathbb{N}$), the sum pairs consecutively:

$$\begin{aligned} S_{2m} &= (-1+2) + (-3+4) + \cdots + (-(2m-1) + 2m) \\ &= \sum_{i=1}^{m} 1 = m = \frac{n}{2}. \end{aligned}$$

For an odd integer $n = 2m - 1$, the partial sum can be found by subtracting the subsequent even term from S_{2m}:

$$\begin{aligned} S_{2m-1} &= S_{2m} - x_{2m} \\ &= m - 2m = -m = -\frac{n+1}{2}. \end{aligned}$$

Using these derived partial sums, we can explicitly determine the terms of the Cesàro mean sequence $\sigma_n(x)$:

When n is even:

$$\sigma_n(x) = \frac{1}{n}\left(\frac{n}{2}\right) = \frac{1}{2}.$$

When n is odd:

$$\sigma_n(x) = \frac{1}{n}\left(-\frac{n+1}{2}\right) = -\frac{1}{2} - \frac{1}{2n}.$$

Applying our chosen seminorm $q(\sigma_n(x)) = |\sigma_n(x)|$, we observe the behavior of the sequence across all $n \in \mathbb{N}$:

$$|\sigma_n(x)| = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ \left|-\frac{1}{2} - \frac{1}{2n}\right| = \frac{1}{2} + \frac{1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

Since the maximum value of $\frac{1}{2} + \frac{1}{2n}$ occurs at $n = 1$ (yielding a value of 1), it follows that for all $n \in \mathbb{N}$:

$$q(\sigma_n(x)) \leq 1.$$

This rigorously demonstrates that the application of the Cesàro mean completely filters the divergent, unbounded oscillation of the original sequence, yielding a strictly bounded transformed sequence.

Finally, we evaluate this result under the defined modulus function M to verify the sequence's inclusion in the generalized bounded space $l_\infty[M, \sigma, q]$. By choosing the parameter $\rho = 1 > 0$, we obtain:

$$\begin{aligned} \sup_n M\left(\frac{q(\sigma_n(x))}{\rho}\right) &= \sup_n \left(|\sigma_n(x)|\right)^p \\ &\leq (1)^p \\ &= 1 < \infty. \end{aligned}$$

Because the supremum is strictly finite, we analytically conclude that $x \in l_\infty[M, \sigma, q]$. This explicit construction illustrates that integrating the Cesàro mean into the sequence space definition successfully captures and contains sequences that actively escape classical boundaries.

Example 3.17 (Exploiting the Bounded Modulus Function (M).)**.** To demonstrate how a bounded modulus function can map an unbounded sequence into a bounded generalized space, we construct an example where the sequence of Cesàro means grows infinitely, but is strictly controlled by the asymptotic behavior of M.

Let us retain the standard seminorm $q(x) = |x|$ defined on the complex field. We introduce a bounded modulus function defined by:

$$M(x) = \frac{x}{x+1}, \quad \text{for } x \geq 0.$$

Notice that $M(x) \to 1$ as $x \to \infty$, ensuring that the output of the modulus function is strictly bounded above by 1 regardless of the magnitude of the input.

Now, we define a monotonically increasing sequence $x = (x_k)_{k=1}^{\infty}$ composed of consecutive odd integers:

$$x_k = 2k - 1, \quad \text{for all } k \in \mathbb{N}.$$

We first compute the n-th term of the Cesàro mean sequence, $\sigma_n(x)$:

$$\begin{aligned}
\sigma_n(x) &= \frac{1}{n}\sum_{k=1}^{n} x_k \\
&= \frac{1}{n}\sum_{k=1}^{n}(2k-1) \\
&= \frac{1}{n}\left(2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right) \\
&= \frac{1}{n}\left(2\frac{n(n+1)}{2} - n\right) \\
&= \frac{1}{n}(n^2 + n - n) = \frac{n^2}{n} = n.
\end{aligned}$$

Evaluating this transformed sequence without the application of the modulus function yields $q(\sigma_n(x)) = |n| = n$. As $n \to \infty$, the sequence of Cesàro means diverges to infinity. Consequently, under classical definitions, the sequence x is neither bounded nor convergent, meaning it does not belong to the standard bounded Cesàro space $l_\infty(\sigma, q)$.

However, we now evaluate the sequence within the context of the generalized bounded space $l_\infty[M, \sigma, q]$. We must determine if there exists a constant $\rho > 0$ such that the supremum over n is finite. By selecting $\rho = 1$, we evaluate the bounded modulus function:

$$\begin{aligned}
M\left(\frac{q(\sigma_n(x))}{\rho}\right) &= M(n) \\
&= \frac{n}{n+1}.
\end{aligned}$$

To mathematically guarantee inclusion in $l_\infty[M, \sigma, q]$, we evaluate the supremum of this expression across all $n \in \mathbb{N}$:

$$\begin{aligned}
\sup_{n\in\mathbb{N}} M\left(\frac{q(\sigma_n(x))}{\rho}\right) &= \sup_{n\in\mathbb{N}}\left(\frac{n}{n+1}\right) \\
&= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) \\
&= 1.
\end{aligned}$$

Because the evaluated supremum is exactly 1, which is strictly finite $(1 < \infty)$, we analytically conclude that $x \in l_\infty[M, \sigma, q]$.

This explicit construction isolates the critical utility of the bounded modulus function M. It maps sequences with continuously diverging Cesàro means into a finite metric constraint, providing a framework to rigorously study and classify sequences that blow up to infinity under traditional functional analysis topologies.

Example 3.18 (Demonstrating the Sargent Sequence Space Generalization $(m[\phi, M, \sigma, q])$)**.** To prove the structural integrity of the generalized Sargent sequence space $m[\phi, M, \sigma, q]$, we construct an example that isolates the role of the ratio sequence $\phi = (\phi_s)$. By holding the sequence x, the modulus function M, and the seminorm q constant, we will demonstrate that inclusion in the space depends entirely on the chosen growth rate of ϕ.

Recall that the traditional Sargent space $m(\phi)$ relies on a non-decreasing sequence of positive real numbers $\phi = (\phi_s)$ such that $\phi_1 > 0$ and $\phi_{s+1} \geq \phi_s$ for all $s \geq 1$. The generalized space $m[\phi, M, \sigma, q]$ requires that there exists a constant $\rho > 0$ such that:

$$\sup_{s\geq 1,\, \delta \in P_s} \frac{1}{\phi_s} \sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) < \infty, \tag{3.5}$$

where P_s denotes the class of all subsets of $\mathbb{N}$ containing exactly s elements.

Let us define the standard seminorm $q(x) = |x|$, the identity modulus function $M(x) = x$, and fix $\rho = 1$. We consider the simplest non-zero constant sequence $x = (x_k)_{k=1}^{\infty}$ defined by:

$$x_k = 1, \quad \text{for all } k \in \mathbb{N}.$$

First, we compute the n-th term of the Cesàro mean sequence, $\sigma_n(x)$:

$$\sigma_n(x) = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n}(n) = 1.$$

We now evaluate the sum over an arbitrary finite subset $\delta \in P_s$. Since δ has exactly s elements and $\sigma_n(x) = 1$ for all n, the evaluation of the modulus function yields $M(q(\sigma_n(x))) = M(|1|) = 1$. Thus, the sum becomes:

$$\begin{aligned}\sum_{n\in\delta} M\left(\frac{q(\sigma_n(x))}{\rho}\right) &= \sum_{n\in\delta} 1 \\ &= s.\end{aligned}$$

Because this result depends only on the cardinality s of the subset δ and not on which specific elements are chosen, the supremum over all

sets $\delta \in P_s$ is simply s. The condition for inclusion from Equation 3.5 simplifies to analyzing the boundedness of:

$$\sup_{s\geq 1} \frac{s}{\phi_s}. \tag{3.6}$$

We now test this against two distinct valid Sargent sequences to observe the boundary of the space.

Case 1: The Strict Constraint (Mimicking l_1-type summation)
Let $\phi^{(1)}$ be the constant sequence where $\phi_s^{(1)} = 1$ for all $s \geq 1$. Substituting this into our simplified condition (Equation 3.6) yields:

$$\begin{aligned}\sup_{s\geq 1} \frac{s}{\phi_s^{(1)}} &= \sup_{s\geq 1} \frac{s}{1} \\ &= \infty.\end{aligned}$$

Because the supremum blows up to infinity, the condition fails. Consequently, the sequence $x \notin m[\phi^{(1)}, M, \sigma, q]$.

Case 2: The Relaxed Constraint (Mimicking l_∞-type boundedness)
Let $\phi^{(2)}$ be the linearly growing sequence where $\phi_s^{(2)} = s$ for all $s \geq 1$. This sequence is strictly non-decreasing and strictly positive, making it a valid choice. Substituting this into Equation 3.6 yields:

$$\begin{aligned}\sup_{s\geq 1} \frac{s}{\phi_s^{(2)}} &= \sup_{s\geq 1} \frac{s}{s} \\ &= 1.\end{aligned}$$

Because the supremum is strictly finite ($1 < \infty$), the condition is satisfied. Consequently, the sequence $x \in m[\phi^{(2)}, M, \sigma, q]$.

This direct algebraic comparison proves that the space $m[\phi, M, \sigma, q]$ is an active, flexible generalization. The division by ϕ_s dampens the summation to a precise degree; if the sequence ϕ grows too slowly, the sum diverges, effectively barring sequences from the space. This visualizes the theoretical boundary separating spaces strictly controlled by the weight sequence ϕ.

4. Conclusion

This research has systematically formulated and examined the generalized sequence spaces $\ell_1[M, \sigma, q]$, $\ell_\infty[M, \sigma, q]$, and $m[\phi, M, \sigma, q]$ by amalgamating modulus functions with Cesáro means over a seminormed space. We have rigorously validated the topological integrity of these structures, proving their completeness as paranormed spaces and establishing critical inclusion relationships with classical sequence spaces. Consequently, this framework offers a potent mathematical tool for analyzing sequences that fail to converge under standard metric topologies.

Looking ahead, the principles introduced here provide a fertile ground for further theoretical expansion. Prospective investigations could integrate fractional difference operators, explore generalized forms of statistical convergence, or characterize the continuous duals and matrix transformations associated with these spaces. Furthermore, deploying these newly constructed paranormed spaces within the realms of summability theory, approximation theory, and operator theory represents a highly compelling trajectory for subsequent mathematical research.

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