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# Tensor Product in Locally Convex Cones

Saeed Vazifeh[1] and Asghar Ranjbari[2*]

Abstract. An interesting generalization of locally convex spaces is locally convex cones, in which one can study special mathematical properties in these structures. A motivation for the proposal of locally convex cones arises from the study of set-valued functions, which calls for a domain that extends beyond vector spaces. The collection of all nonempty convex subsets of a vector space allows for the addition and multiplication of its elements by nonnegative reals, but not for additive inverses. It does not satisfy the cancellation law and, therefore, cannot be embedded into a vector space. In this paper, we introduce tensor product in locally convex cones. We show that under some conditions, tensor product of two dual pairs is also a dual pair. Also, we prove that tensor product of $b$-bornological cones is $b$-bornological. We verify precompactness and separatedness in tensor product of locally convex cones.

## 1. Introduction

The general theory of locally convex cones, as developed by K. Keimel and W. Roth in [3], deals with an order or a convex quasiuniform structure on a cone. We review the fundamental definitions. For more details, we refer [3–5].

A set $\mathcal{C}$ is called a cone if it is endowed with an addition and a scalar multiplication for non-negative real numbers. As vector spaces, the addition is associative and commutative and there is a neutral element $0 \in \mathcal{C}$. The scalar multiplication has the usual associative and distributive properties. We have $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in \mathcal{C}$. The cancelation law, that is, $a + x = b + x$ implies $a = b$, does not hold

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* Corresponding author.

generally in cones. It holds if and only if the cone $\mathcal{C}$ is embedded into a real vector space.

A Cone $\mathcal{C}$ is called a preordered cone (ordered cone for simplicity) if there is a preorder, i.e., a reflexive and transitive relation $\leq$ on $\mathcal{C}$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha \cdot a \leq \alpha \cdot c$ for all $a, b, c \in \mathcal{C}$ and $\alpha \in \mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$, i.e. $\leq$ is compatible with the addition and the scalar multiplication. Let $\mathcal{C}$ and $\mathcal{C}'$ be cones. A mapping $t : \mathcal{C} \to \mathcal{C}'$ is called a linear operator if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ hold for $a, b \in \mathcal{C}$ and $\alpha \geq 0$. The family of all linear operators from $\mathcal{C}$ to $\mathcal{C}'$ is denoted by $\mathcal{L}(\mathcal{C}, \mathcal{C}')$. A linear operator $\mu : \mathcal{C} \to \overline{\mathbb{R}}$ is called a linear functional on $\mathcal{C}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

**Definition 1.1.** Let $\mathcal{V}$ be a subset of the preordered cone $\mathcal{C}$. We say that $\mathcal{V}$ is an (abstract) 0-neighborhood system, if

- $\forall\ v \in \mathcal{V}$, $0 < v$;
- $\forall u, v \in \mathcal{V}$, $\exists\ w \in \mathcal{V}$ such that $w \leq u$ and $w \leq v$;
- $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

Let $a \in \mathcal{C}$ and $v \in \mathcal{V}$ be arbitrary. We define a neighborhood of $a$ in the upper, resp. lower topologies on $\mathcal{C}$ as $v(a) = \{b \in \mathcal{C} : b \leq a + v\}$, resp. $(a)v = \{b \in \mathcal{C} : a \leq b + v\}$. The common refinement of these topologies is called symmetric topology. The neighborhoods of the symmetric topology are as $v(a) \cap (a)v$ and denoted by $v(a)v$ for $a \in \mathcal{C}$ and $v \in \mathcal{V}$.

**Definition 1.2.** The pair $(\mathcal{C}, \mathcal{V})$ is called a full locally convex cone, if each element of $\mathcal{C}$ is bounded below, i.e. for each $a \in \mathcal{C}$ and each $v \in \mathcal{V}$ there exists $\rho > 0$ such that $0 \leq a + \rho v$.

Every subcone of $\mathcal{C}$, that is not necessarily containing $\mathcal{V}$, is called a locally convex cone.

An element $a$ of a locally convex cone $(\mathcal{C}, \mathcal{V})$ is called bounded above if for each $v \in \mathcal{V}$ there is a $\rho > 0$ such that $a \leq \rho v$. The element $a$ is bounded if it is bounded below and bounded above.

Let $\mathcal{C}$ be a cone. Let $U, V \subseteq \mathcal{C}^2 = \mathcal{C} \times \mathcal{C}$. We denote

$$U \circ V = \left\{(a, b) \in p^2 : \exists c \in \mathcal{C}, \text{s.t.} (a, c) \in U \text{ and } (c, b) \in V\right\},$$

$$U^{-1} = \{(b, a) : (a, b) \in U\},$$

$$\triangle = \{(a, a) : a \in \mathcal{C}\}.$$

**Definition 1.3.** A collection $\mathcal{U}$ of convex subsets $U \subset \mathcal{C}^2 = \mathcal{C} \times \mathcal{C}$ is called a convex quasiuniform structure, if the following properties hold:

($U1$) $\triangle \subset U$ for each $U \in \mathcal{U}$;
($U2$) $\forall\ U, V \in \mathcal{U}, \exists W \in \mathcal{U}$ such that $W \subset U \cap V$;
($U3$) $\lambda U \circ \mu U \subset (\lambda + \mu)U, \forall U \in \mathcal{U}$ and $\forall \lambda, \mu > 0$;

$(U4)$ $\lambda U \in \mathcal{U}, \forall U \in \mathcal{U}$ and $\forall \lambda > 0$.

**Definition 1.4.** Let $\mathcal{U}$ and $\mathcal{W}$ be two convex quasiuniform structures on $\mathcal{C}$. We say that $\mathcal{U}$ is finer than $\mathcal{W}$ whenever for each $W \in \mathcal{W}$ there exists $U \in \mathcal{U}$ such that $U \subset W$.

The convex quasiuniform structures $\mathcal{U}$ and $\mathcal{W}$ on $\mathcal{C}$ are equivalent if they are finer than each other.

We associate a preorder to each quasiuniform structure $\mathcal{U}$ on $\mathcal{C}$: For $a, b \in \mathcal{C}$ it was defined

$$a \leq b \text{ if and only if } (a, b) \in U, \quad \forall U \in \mathcal{U}.$$

Also, we consider two topologies: the neighborhood bases for an element $a$ in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{C} : (b, a) \in U\}, \text{ resp.}(a)U = \{b \in \mathcal{C} : (a, b) \in U\}, \quad U \in \mathcal{U}.$$

The common refinement of the upper and lower topologies is indeed, the topology associated with the uniform structure $\mathcal{U}_s = \left\{U \cap U^{-1} : U \in \mathcal{U}\right\}$.

The notions of a convex quasiuniform structure $\mathcal{U}$ and an (abstract) 0-neighborhood system $\mathcal{V}$ on a cone $\mathcal{C}$ are equivalent:

Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone and $v \in \mathcal{V}$. We set

$$\tilde{v} = \{(a, b) \in \mathcal{C} \times \mathcal{C} : a \leq b + v\}.$$

The family $\widetilde{\mathcal{V}} = \{\tilde{v} : v \in \mathcal{V}\}$ is a convex quasiuniform structure on $\mathcal{C}$ and $\widetilde{\mathcal{V}}$ induces a (global) preorder on $\mathcal{C}$ and the same lower, upper and symmetric topologies. On the other hand: if $\mathcal{C}$ is a cone and $\mathcal{U}$ is a convex quasiuniform structure on $\mathcal{C}$, then there exists a preorder and an (abstract) 0-neighborhood system $\mathcal{V}$ such that $\mathcal{U}$ is equivalent to the convex quasiuniform structure $\widetilde{\mathcal{V}}$ (see [3], I.5.5).

If $(\mathcal{C}, \mathcal{V})$ is a locally convex cone, then by the definition each element of $\mathcal{C}$ is bounded below, i.e.

$$\text{for all } a \in \mathcal{C} \text{ and } \tilde{v} \in \widetilde{\mathcal{V}} \text{ there is some } \rho > 0 \text{ such that } (0, a) \in \rho\tilde{v}.$$

Conversely, if $\mathcal{U}$ is a convex quasiuniform structure on $\mathcal{C}$ which satisfies also

$$(U5) \text{ for all } a \in \mathcal{C} \text{ and } U \in \mathcal{U}, (0, a) \in \rho U \text{ for some } \rho > 0,$$

then with the preorder generated by $\mathcal{U}$, the cone $\mathcal{C}$ is locally convex.

Let $\mathcal{C}$ and $\mathcal{C}'$ be locally convex cones with convex quasiuniform structures $\mathcal{U}$ and $\mathcal{V}$ respectively and let $t : \mathcal{C} \to \mathcal{C}'$ be a linear mapping. We say that $t$ is uniformly continuous (u-continuous) if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $T(U) \subset V$, where $T = t \times t$, i.e., $(a, b) \in U$ implies $(t(a), t(b)) \in V$. Now, let $\mathcal{V}$ and $\mathcal{W}$ be (abstract) 0-neighborhood systems on $\mathcal{C}$ and $\mathcal{C}'$, respectively. The linear mapping $t$ is u-continuous

if and only if for each $w \in \mathcal{W}$ one can find a $v \in \mathcal{V}$ such that $(a, b) \in \tilde{v}$ implies $(t(a), t(b)) \in \tilde{w}$, or equivalently, $t(a) \leq t(b) + w$ whenever $a \leq b + v$. If $t$ is uniformly continuous, then it is continuous with respect to the lower, upper and symmetric topologies on $\mathcal{C}$ and $\mathcal{C}'$.

The cone $\overline{\mathbb{R}}$ endowed with the (abstract) 0-neighborhood system $\boldsymbol{\epsilon} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, is a full locally convex cone.

A dual pair $(\mathcal{C}, \mathcal{D})$ consists of two cones $\mathcal{C}$ and $\mathcal{D}$ with a bilinear mapping

$$(a, x) \to \langle a, x \rangle : \mathcal{C} \times \mathcal{D} \to \overline{\mathbb{R}}$$

(see [3], II, 3.1). We say that $\mathcal{C}$ separates the points of $\mathcal{D}$, if for $p_1, p_2 \in \mathcal{C}$, with $p_1 \neq p_2$, one can find an element $q \in \mathcal{D}$ such that $\langle p_1, q \rangle \neq \langle p_2, q \rangle$. We say that $\mathcal{C}$ weakly separates the points of $\mathcal{D}$, if for every nonzero element $a \in \mathcal{C}$ one can find an element $b \in \mathcal{D}$ such that $\langle a, b \rangle \neq 0$. In vector spaces, these two definitions are equivalent. The dual pair $(\mathcal{C}, \mathcal{D})$ is (weakly) separating if $\mathcal{C}$ and $\mathcal{D}$ (weakly) separate the points of each other.

Let $(\mathcal{C}, \mathcal{C}')$ and $(\mathcal{D}, \mathcal{D}')$, be dual pairs and $t : \mathcal{C} \to \mathcal{D}$ be a linear mapping. Then $\langle t(x), y' \rangle$ is a bilinear functional in the two variables $x \in \mathcal{C}$ and $y' \in \mathcal{D}'$. Denote by $t'(y')$ the linear functional on $\mathcal{C}$ which results from this bilinear functional by fixing $y' \in \mathcal{D}'$, so that $t'$ is defined by the identity

$$\left\langle x, t'\left(y'\right)\right\rangle = \left\langle t(x), y'\right\rangle,$$

valid for all $x \in \mathcal{C}$ and $y' \in \mathcal{D}'$. Then for every $y' \in \mathcal{D}'$, $t'(y') \in \mathcal{L}(\mathcal{C}, \bar{\mathbb{R}})$, where $\mathcal{L}(\mathcal{C}, \bar{\mathbb{R}})$ is the cone of all linear functionals from $\mathcal{C}$ to $\bar{\mathbb{R}}$, i.e. the algebraic dual of $\mathcal{C}$ and $t'$ is a linear mapping of $\mathcal{D}'$ into $\mathcal{L}(\mathcal{C}, \bar{\mathbb{R}})$. We say $t'$ is the transpose of the linear mapping $t$.

## 2. Bilinear Mappings and Tensor Product of Cones

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two cones and

$$\mathcal{T}^* = \left\{h : \mathcal{C} \times \mathcal{D} \to \bar{\mathbb{R}} : h \text{ is bilinear}\right\}.$$

It is easy to see that $\mathcal{T}^*$ is a cone.

Let $\mathcal{T}^{**} = \{\mu : \mathcal{T}^* \to \bar{\mathbb{R}} : \mu \text{ is linear}\}$ be the (algebraic) dual of $\mathcal{T}^*$. There is a natural mapping $\phi : \mathcal{C} \times \mathcal{D} \to \mathcal{T}^{**}$ such that $\phi(x, y) \in \mathcal{T}^{**}$, obtained by putting:

$$(\phi(x, y))(h) = h(x, y),$$

for $(x, y) \in \mathcal{C} \times \mathcal{D}$ and $h \in \mathcal{T}^*$. The mapping $\phi$ is bilinear.

If $x \neq 0$ and $y \neq 0$, then $\phi(x, y) \neq 0$. Indeed, if $x \in \mathcal{C}$ and $x \neq 0$, then take $x^* \in \mathcal{C}^*$ and $y^* \in \mathcal{D}^*$ such that $\langle x, x^* \rangle \neq 0$ and $\langle y, y^* \rangle \neq 0$. Put $h(x, y) = \langle x, x^* \rangle \langle y, y^* \rangle$. We have $h \in \mathcal{T}^*$ and $h(x, y) \neq 0$. The image $\phi(\mathcal{C}, \mathcal{D})$ of $\mathcal{C} \times \mathcal{D}$ is not in general a subcone of $\mathcal{T}^{**}$, though it is clear

that if $w \in \phi(\mathcal{C}, \mathcal{D})$ and $\alpha \geq 0$, then $\alpha w \in \phi(\mathcal{C}, \mathcal{D})$. The subcone of $\mathcal{T}^{**}$ spanned by $\phi(\mathcal{C}, \mathcal{D})$ is called the tensor product of $\mathcal{C}$ and $\mathcal{D}$ and denoted by $\mathcal{C} \otimes \mathcal{D}$. We denote $\phi(x, y)$ by $x \otimes y$. Every element of $\mathcal{C} \otimes \mathcal{D}$ can be written as a finite sum of the form $\sum_i x_i \otimes y_i$, but, in general, this representation is not unique.

We note that $(\mathcal{T}^*, \mathcal{C} \otimes \mathcal{D})$ is a weak separating dual pair. Indeed, for $h \in \mathcal{T}^*$, $h \neq 0$ means that there are elements $x \in \mathcal{C}, y \in \mathcal{D}$ with $h(x, y) \neq 0$ and this is equivalent to $\langle h, x \otimes y \rangle \neq 0$. Moreover, $\mathcal{T}^*$ is the (algebraic) dual of $\mathcal{C} \otimes \mathcal{D}$, since to any linear functional $f$ on $\mathcal{C} \otimes \mathcal{D}$ corresponds the element $f \circ \phi$ of $\mathcal{T}^*$. Thus the tensor product $\mathcal{T} = \mathcal{C} \otimes \mathcal{D}$ has the property that its algebraic dual $\mathcal{T}^*$ is the set of all bilinear functionals on $\mathcal{C} \times \mathcal{D}$. In the following proposition, we extend this result to bilinear mappings.

**Proposition 2.1.** *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{R}$ be three cones and let $\phi$ be the canonical mapping of $\mathcal{C} \times \mathcal{D}$ into $\mathcal{C} \otimes \mathcal{D}$. Then to every linear mapping $f$ from $\mathcal{C} \otimes \mathcal{D}$ into $\mathcal{R}$ corresponds a bilinear mapping $f \circ \phi$ of $\mathcal{C} \times \mathcal{D}$ into $\mathcal{R}$ and this correspondence is an isomorphism of the cone of linear mappings of $\mathcal{C} \otimes \mathcal{D}$ into $\mathcal{R}$ onto the cone of bilinear mappings of $\mathcal{C} \times \mathcal{D}$ into $\mathcal{R}$.*

*Proof.* We define the mapping $t$ from $\mathcal{L}(\mathcal{C} \otimes \mathcal{D}, \mathcal{R})$ (the cone of all linear mappings from $\mathcal{C} \otimes \mathcal{D}$ into $\mathcal{R}$) into the cone of bilinear mappings from $\mathcal{C} \times \mathcal{D}$ into $\mathcal{R}$ by $t(f) = f \circ \phi$. It is clear that $t$ is linear. Let $t(f) = t(g)$. Then $f \circ \phi = g \circ \phi$, i.e. $f(\phi(x, y)) = g(\phi(x, y))$ for all $(x, y) \in \mathcal{C} \times \mathcal{D}$ and then $f = g$. This yields that $t$ is one to one. Now, we prove that $t$ is onto. Let $h : \mathcal{C} \times \mathcal{D} \to \mathcal{R}$ be a bilinear mapping. For every $z^* \in \mathcal{R}^*$ (algebraic dual of $\mathcal{R}$), we define a bilinear functional $f^*(z^*)$ on $\mathcal{C} \times \mathcal{D}$, by putting

$$(f^*(z^*))(x, y) = \langle h(x, y), z^* \rangle ,$$

for all $(x, y) \in \mathcal{C} \times \mathcal{D}$. Let $f$ be the transpose of $f^*$, restricted on $\mathcal{C} \otimes \mathcal{D}$. We have

$$\langle f(\phi(x, y)), z^* \rangle = \langle \phi(x, y), f^*(z^*) \rangle = f^*(z^*)(x, y) = \langle h(x, y), z^* \rangle ,$$

for all $(x, y) \in \mathcal{C} \times \mathcal{D}$ and all $z^* \in \mathcal{R}^*$. Then $f$ maps $\mathcal{C} \otimes \mathcal{D}$ into $\mathcal{R}$ and $h = f \circ \phi = t(f)$. This yields that $t$ is onto. □

**Lemma 2.2.** *Let $f_i : \mathcal{C} \to \bar{\mathbb{R}}$ be linear functionals on the cone $\mathcal{C}$ such that*

$$f_i(x)y \in \mathcal{C}, \quad \forall x, y \in \mathcal{C}, \tag{2.1}$$

*for all* $i = 0, 1, 2, \ldots, n$. *Then either* $f_0 = \sum_{i=1}^{n} \lambda_i f_i$ *for real numbers* $\lambda_i$ *or there is an element* $a \in \mathcal{C}$ *with* $f_1(a) = f_2(a) = \cdots = f_n(a) = 0$ *and* $f_0(a) = 1$.

*Proof.* If $f_1, f_2, \ldots, f_n$ are linearly dependent, the result holds. Then we suppose that $f_1, f_2, \ldots, f_n$ are linearly independent. If $n = 0$ there is nothing to prove. Suppose that the result is true for the integer $n - 1$. Then for each $i \in \{1, \ldots, n\}$, $f_i$ is not a linear combination of $f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$. Thus by the induction hypothesis, there are $a_i \in \mathcal{C}$ such that

$$f_j(a_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $x \in \mathcal{C}$ be arbitrary. Fix $i \in \{1, 2, \ldots, n\}$. By (2.1), $f_i(x)a_i$ is meaningful and $f_i(f_i(x)a_i) = f_i(x)$ and $f_j(f_i(x)a_i) = 0$ for $j \neq i$. Then

$$x \in \sum_{i=1}^{n} f_i(x)a_i + \bigcap_{i=1}^{n} f_i^{-1}(0),$$

where $f_i^{-1}(0) = \{x : f_i(x) = 0\}$. Indeed, $f_i\left(\sum_{i=1}^{n} f_i(x)a_i\right) = f_i(x)$ and then $f_i(x) \in f_i(x) + f_i\left(\bigcap_{i=1}^{n} f_i^{-1}(0)\right)$ (since $0 \in f_i\left(\bigcap_{i=1}^{n} f_i^{-1}(0)\right)$). Then either there is $a \in \bigcap_{i=1}^{n} f_i^{-1}(0)$ such that $f_0(a) = 1$ and $f_i(a) = 0$, for all $i = 1, \ldots, n$ or $f_0(y) = 0$ for all $y \in \bigcap_{i=1}^{n} f_i^{-1}(0)$. In this case, for all $x \in \mathcal{C}$, we have $f_0(x) = \sum_{i=1}^{n} f_i(x)f_0(a_i)$. Set $\lambda_i = f_0(a_i)$. We have $f_0 = \sum_{i=1}^{n} \lambda_i f_i$. □

**Remark 2.3.** If the condition (2.1) is not satisfied, it is not necessary that Lemma 2.2 holds. For example, put $\mathcal{C} = \mathbb{R}_+ = [0, \infty)$ which is a cone (as a subcone of $\bar{\mathbb{R}}$) and $f_0 : \mathbb{R}_+ \to \bar{\mathbb{R}}$ by $f_0(0) = 0$ and $f_0(a) = \infty$, for all $a \in (0, \infty)$. Also set $f_i : \mathbb{R}_+ \to \bar{\mathbb{R}}$ defined by $f_i(x) = ix$ for $= i = 1, 2, \ldots, n$. The mappings $f_0, f_1, f_2, \ldots, f_n$ are linear, but Lemma 2.2 does not hold for them. Note that $f_0$ from $\bar{\mathbb{R}}$ into $\bar{\mathbb{R}}$ is not linear.

The concepts of a base and dimension for cones are defined in [1]. We briefly review them. Suppose $\mathcal{C}$ is a cone and $A \subset \mathcal{C}$. We set

$$span(A) = \left\{ \sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, a_1, \ldots, a_n \in A, \alpha_1, \ldots, \alpha_n \geq 0 \right\}.$$

If $A = ptyset$, then we set $span(A) = \{0\}$. The subset $A$ of $\mathcal{C}$ is called linearly independent whenever $span(A) \neq span(E)$ for each $E \subsetneqq A$. Obviously, for every $x \neq 0$, $\{x\}$ is linearly independent.

**Definition 2.4** ([1])**.** Let $\mathcal{C}$ be a cone. A subset $A$ of $\mathcal{C} \setminus \{0\}$ is called a base for $\mathcal{C}$ whenever

(1) for every $a \in \mathcal{C}$ there are $a_1, \ldots a_n \in A, \alpha_1, \ldots, \alpha_n \geq 0$ such that $a = \sum_{i=1}^{n} \alpha_i a_i$, in the other words $\mathcal{C} = span(A)$,
(2) $A$ is linearly independent.

We note that a cone dose not necessarily have a base.

The cone $\mathcal{C}$ is called an $n$-dimensional cone whenever $\mathcal{C}$ has a finite base $A = \{a_1, a_2 \ldots a_n\}$, $n \in \mathbb{N}$ and for every base $B$ for $\mathcal{C}$, $Card(B) \geq n$ (see [1]).

**Proposition 2.5.** *If $\mathcal{C}$ and $\mathcal{D}$ are $m$ and $n$-dimensional cones, respectively, then $\mathcal{C} \otimes \mathcal{D}$ is an $mn$-dimensional cone.*

*Proof.* Let $A = \{p_1, p_2, \ldots, p_m\}$ be a base for $\mathcal{C}$ and $B = \{q_1, q_2, \ldots, q_n\}$ be a base for $\mathcal{D}$. It is easy to see that the set $E = \{p_i \otimes q_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ is a base for $\mathcal{C} \otimes \mathcal{D}$. Let $F = \{a_r \otimes b_s : a_r \in \mathcal{C} \text{ and } b_s \in \mathcal{D}\}$ be another base for $\mathcal{C} \otimes \mathcal{D}$. Then $A' = \{a_r : a_r \otimes b_s \in F\}$ is a base for $\mathcal{C}$, with $Card(A') \geq m$ and $B' = \{b_s : a_r \otimes b_s \in F\}$ is a base for $\mathcal{D}$ with $Card(B') \geq n$. These yield that $Card(F) \geq mn$. □

**Theorem 2.6.** *Let $(\mathcal{C}, \mathcal{C}')$ and $(\mathcal{D}, \mathcal{D}')$ be weak separating dual pairs, such that $\langle x, x' \rangle\, w \in \mathcal{C}$, $\langle x, x' \rangle\, w' \in \mathcal{C}'$, $\langle y, y' \rangle\, z \in \mathcal{D}$, and $\langle y, y' \rangle\, z' \in \mathcal{D}'$ for all $x, w \in \mathcal{C}$, $x', w' \in \mathcal{C}'$, $y, z \in \mathcal{D}$ and $y', z' \in \mathcal{D}'$. Then $(\mathcal{C} \otimes \mathcal{D}, \mathcal{C}' \otimes \mathcal{D}')$ is also a weak separating dual pair, with the bilinear functional*

$$\langle a, a' \rangle = \sum_{i,j} \langle x_i, x'_j \rangle \langle y_i, y'_j \rangle,$$

*where $a = \sum_i x_i \otimes y_i \in \mathcal{C} \otimes \mathcal{C}'$ and $a' = \sum_j x'_j \otimes y'_j \in \mathcal{D} \otimes \mathcal{D}'$.*

*Proof.* Let $a = \sum_i x_i \otimes y_i \in \mathcal{C} \otimes \mathcal{C}'$ be arbitrary with $a \neq 0$. Suppose that $\{e_1, e_2, \ldots, e_m\}$ is a base in the finite dimensional subcone of $\mathcal{C}$ spanned

by the $x_i$ and $\{f_1, f_2, \ldots, f_n\}$ is a base in the subcone of $\mathcal{D}$ spanned by $y_i$. We can express $a$ in the form

$$a = \sum_{r=1}^{m} \sum_{s=1}^{n} \lambda_{rs} e_r \otimes f_s.$$

Since $a \neq 0$, at least one $\lambda_{rs}$ is different from zero; suppose $\lambda_{11} \neq 0$. By Lemma 2.2, there is $x' \in \mathcal{C}'$ with $\langle e_1, x' \rangle = 1$ and $\langle e_r, x' \rangle = 0$ for $r = 2, 3, \ldots, m$ and there exists $y' \in \mathcal{D}'$ with $\langle f_1, y' \rangle = 1$ and $\langle f_s, y' \rangle = 0$ for $s = 2, 3, \ldots, n$. We have

$$\langle a, x' \otimes y' \rangle = \lambda_{11} \neq 0.$$

We conclude that $\langle a, a' \rangle = 0$ for all $a' \in \mathcal{C}' \otimes \mathcal{D}'$ implies $a = 0$. Similarly $\langle a, a' \rangle = 0$ for all $a \in \mathcal{C} \otimes \mathcal{D}$ implies $a' = 0$. Then $(\mathcal{C} \otimes \mathcal{D}, \mathcal{C}' \otimes \mathcal{D}')$ is a weak separating dual pair. □

## 3. Tensor Product of Locally Convex Cones

In this section, we equip tensor products of cones with a preorder or a convex quasiuniform structure.

Let $\mathcal{C}$ and $\mathcal{D}$ be preordered cones. For two elements $\sum_{i=1}^{n} a_i \otimes b_i$ and $\sum_{j=1}^{m} c_j \otimes d_j$ of $\mathcal{C} \otimes \mathcal{D}$, we consider following preorder:

$$\sum_{i=1}^{n} a_i \otimes b_i \preceq \sum_{j=1}^{m} c_j \otimes d_j \quad \Leftrightarrow \quad n = m, a_i \leq c_i \text{ and } b_i \leq d_i. \tag{3.1}$$

**Remark 3.1.** By (3.1), it is enough we consider $n = m = 1$, i.e. $a \otimes b \preceq c \otimes d$ if and only if $a \leq c$ and $b \leq d$, for all $a \otimes b, c \otimes d \in \mathcal{C} \otimes \mathcal{D}$.

Suppose that $\mathcal{V}$ and $\mathcal{W}$ are (abstract) 0-neighborhood systems for preordered cones $\mathcal{C}$ and $\mathcal{D}$, respectively, such that $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ are locally convex cones. The set $\mathcal{V} \otimes \mathcal{W} = \{v \otimes w : v \in \mathcal{V} \text{ and } w \in \mathcal{W}\}$ with the following property is an (abstract) 0-neighborhood system for $\mathcal{C} \otimes \mathcal{D}$:

$$a \otimes b \preceq c \otimes d + v \otimes w \quad \Leftrightarrow \quad a \leq c + v \text{ and } b \leq d + w. \tag{3.2}$$

It is easy to see that $\mathcal{V} \otimes \mathcal{W}$ is an (abstract) 0-neighborhood system for $\mathcal{C} \otimes \mathcal{D}$ and $(\mathcal{C} \otimes \mathcal{D}, \mathcal{V} \otimes \mathcal{W})$ is a locally convex cone.

In the following theorem, we use convex quasiuniform structure corresponding to (abstract) 0-neighborhood system. As it is mentioned in the introduction, they are equivalent.

Let $\mathcal{C}$ be a cone. A subset $B$ of $\mathcal{C}^2$ is called uniformly convex whenever it has the properties $(U_1)$ and $(U_3)$. For $F \subseteq \mathcal{C}^2$, let $uch(F)$ be the smallest uniformly convex subset of $\mathcal{C}^2$ which contains $F$.

**Theorem 3.2.** *Let $(\mathcal{C},\mathcal{U})$ and $(\mathcal{D},\mathcal{V})$ be two locally convex cones, where $\mathcal{U}$ and $\mathcal{V}$ are convex quasiuniform structures on $\mathcal{C}$ and $\mathcal{D}$ respectively. Also, let $\phi$ be the canonical mapping of $\mathcal{C}\times\mathcal{D}$ into $\mathcal{C}\otimes\mathcal{D}$. Then there is a finest convex quasiuniform structure on $\mathcal{C}\otimes\mathcal{D}$ under which $\phi$ is u-continuous. A linear mapping $f$ of $\mathcal{C}\otimes\mathcal{D}$ under this convex quasiuniform structure into a third locally convex cone $(\mathcal{R},\mathcal{W})$ is u-continuous if and only if the bilinear mapping $f\circ\phi$ of $(\mathcal{C}\times\mathcal{D},\mathcal{U}\times\mathcal{V})$ into $(\mathcal{R},\mathcal{W})$ is u-continuous.*

*Proof.* Let $W$ be an element of any convex quasiuniform structure on $\mathcal{C}\otimes\mathcal{D}$ under which $\phi$ is u-continuous. There are $U\in\mathcal{U}$ and $V\in\mathcal{V}$ such that $U\times V\subset\Phi^{-1}(W)$ and then $uch(\Phi(U\times V))\subset W$, where $\Phi=\phi\times\phi$. The collection $\mathcal{X}=\{uch(\Phi(U\times V)):U\in\mathcal{U},V\in\mathcal{V}\}$ is a convex quasiuniform structure on $\mathcal{C}\otimes\mathcal{D}$; in fact, it is the finest convex quasiuniform structure on $\mathcal{C}\otimes\mathcal{D}$ making $\phi$ u-continuous. Now, let $f:(\mathcal{C}\otimes\mathcal{D},\mathcal{X})\to(\mathcal{R},\mathcal{W})$ be a linear mapping. If $f$ is u-continuous, then $f\circ\phi$ is u-continuous, since $\phi$ is u-continuous. Now, let $f\circ\phi$ be u-continuous and put $F=f\times f$. Let $R\in\mathcal{R}$. There is $U\times V\in\mathcal{U}\times\mathcal{V}$ such that $F\circ\Phi(U\times V)\subset R$, i.e. $\Phi(U\times V)\subset F^{-1}(R)$. We have $uch(\Phi(U\times V))\subset uch\left(F^{-1}(R)\right)=F^{-1}(R)$. This shows that $f$ is u-continuous. □

The convex quasiuniform structure $\mathcal{X}$ constructed in Theorem 3.2 is called tensor product convex quasiuniform structure and the equivalent (abstract) 0-neighborhood system is called the tensor product (abstract) 0-neighborhood system. The cone $\mathcal{C}\otimes\mathcal{D}$ with this convex quasiuniform structure (or (abstract) 0-neighborhood system) is called locally convex tensor product of locally convex cones $\mathcal{C}$ and $\mathcal{D}$.

Let $(\mathcal{C},\mathcal{V})$ be a locally convex cone. The closure of $a\in\mathcal{C}$ is defined to be the set

$$\bar{a}=\bigcap_{v\in\mathcal{V}}v(a),$$

(see [3], chapter I). We say that the locally convex cone $(\mathcal{C},\mathcal{V})$ is separated if $\bar{a}=\bar{b}$ implies $a=b$ for $a,b\in\mathcal{C}$ (see [3], I, 3.8).

**Proposition 3.3.** *If $(\mathcal{C},\mathcal{V})$ and $(\mathcal{D},\mathcal{W})$ are separated locally convex cone, then $(\mathcal{C}\otimes\mathcal{D},\mathcal{V}\otimes\mathcal{W})$ is also separated.*

*Proof.* Let $\overline{a\otimes b}=\overline{c\otimes d}$. Then

$$a\otimes b\preceq c\otimes d+v\otimes w,\qquad c\otimes d\preceq a\otimes b+v\otimes w,$$

for all $v\otimes w\in\mathcal{V}\otimes\mathcal{W}$. By considering the definition of the preorder in tensor product, we have

$$a\leq c+v,\qquad c\leq a+v,\qquad b\leq d+w,\qquad d\leq b+w.$$

We conclude that $\bar{a} = \bar{c}$ and $\bar{b} = \bar{d}$. We have $a = c$ and $b = d$, since $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ are separated. Then $a \otimes b = c \otimes d$. This yields that $\mathcal{C} \otimes \mathcal{D}$ is separated. □

We say that the locally convex cone $(\mathcal{C}, \mathcal{V})$ is a *uc*-cone whenever $\mathcal{V} = \{\alpha v : \alpha > 0\}$ for some $v \in \mathcal{V}$ (see [2]).

**Proposition 3.4.** *If $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ are two uc-cone, then $(\mathcal{C}\otimes\mathcal{D}, \mathcal{V}\otimes\mathcal{W})$ is a uc-cone.*

*Proof.* Let $\mathcal{V} = \{\alpha v : \alpha > 0\}$ and $\mathcal{W} = \{\beta w : \beta > 0\}$. The cone $\mathcal{C} \otimes \mathcal{D}$ with the abstract 0-neighborhood system $\mathcal{V} \otimes \mathcal{W} = \{\gamma(v \otimes w) : \gamma > 0\}$ is a locally convex cone and the proof is complete. □

Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone. We say that $F \subseteq \mathcal{C}^2$ is uniformly bounded ($u$-bounded) if there exists a positive real number $\rho$ such that $F \subset \tilde{\rho v}$ for each $v \in \mathcal{V}$, where $\tilde{v} = \{(a, b) \in \mathcal{C}^2 : a \leq b + v\}$. A subset $A$ of $\mathcal{C}$ is called bounded below (above) whenever $\{0\} \times A$ (res. $A \times \{0\}$) is $u$-bounded, that is, for every $v \in \mathcal{V}$ there is a positive real number $\rho$ such that $0 \leq a + \rho v$ ($a \leq \rho v$) (see [2]).

Bornological and $b$-bornological locally convex cones have been studied in [2]. Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be two locally convex cones. We say that the linear mapping $t : \mathcal{C} \to \mathcal{D}$ is $u$-bounded if $(t \times t)(F)$ is $u$-bounded in $\mathcal{D}^2$ for each $u$-bounded subset $F$ of $\mathcal{C}^2$. We shall say $(\mathcal{C}, \mathcal{V})$ is a bornological cone if every $u$-bounded linear operator from $(\mathcal{C}, \mathcal{V})$ into any locally convex cone is $u$-continuous (see [2]).

Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be locally convex cones. The linear operator $t : \mathcal{C} \to \mathcal{D}$ is called bounded below whenever $t$ maps bounded below subsets of $\mathcal{C}$ into bounded below subsets of $\mathcal{D}$. The locally convex cone $(\mathcal{C}, \mathcal{V})$ is called b-bornological whenever every bounded below linear operator from $(\mathcal{C}, \mathcal{V})$ into other locally convex cone is u-continuous (see [2]).

**Theorem 3.5.** *Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be locally convex cones. If $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ are b-bornological, then $(\mathcal{C}\otimes\mathcal{D}, \mathcal{V}\otimes\mathcal{W})$ is also b-bornological.*

*Proof.* Let $(\mathcal{R}, \mathcal{Z})$ be a locally convex cone. Let $T : (\mathcal{C} \otimes \mathcal{D}, \mathcal{V} \otimes \mathcal{W}) \to (\mathcal{R}, \mathcal{Z})$ be a bounded below linear operator. Let $y \in \mathcal{D}$ be fixed and define $T_y : \mathcal{C} \to \mathcal{R}$, by $T_y(x) = T(x \otimes y)$, for all $x \in \mathcal{C}$. We show that the mapping $T_y$ is bounded below. Let $A \subset \mathcal{C}$ is a bounded below subset, then $A \otimes \{y\} \subset \mathcal{C} \otimes \mathcal{D}$ is bounded below. Indeed, for given $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$, there exists $\lambda_1 > 0$ such that $0 \leq a + \lambda_1 v$ for all $a \in A$, since $A$ is bounded below. On the other hand there is $\lambda_2 > 0$ such that $0 \leq y + \lambda_2 w$, since every element of a locally convex cone is bounded below. Set $\lambda = \max\{\lambda_1, \lambda_2\}$. We will have $0 \leq a + \lambda v$ and $0 \leq y + \lambda w$ and so $0 \preceq a \otimes y + \lambda(v \otimes w)$ for all $a \in A$, by (3.2). This yields that $A \otimes \{y\}$ is bounded below. The set $T(A \otimes \{y\})$ is bounded below

in $(\mathcal{R}, \mathcal{Z})$, since $T$ is a bounded below operator. We conclude that $T_y$ is $u$-continuous, since $(\mathcal{C}, \mathcal{V})$ is $b$-bornological. Similarly, for each fixed $x \in \mathcal{C}$, the mapping $T_x : \mathcal{D} \to \mathcal{R}$, by $T_x(y) = T(x \otimes y)$, for all $y \in \mathcal{D}$, is u-continuous. Then, For every $z \in \mathcal{Z}$, there are $v \in \mathcal{V}$ and $w \in \mathcal{W}$ such that $T_y(a) \leq T_y(b) + z$ and $T_x(c) \leq T_x(d) + z$, whenever $a \leq b + v$ and $c \leq d + w$, respectively. We conclude that $T(a \otimes b) \leq T(c \otimes d) + z$, whenever $a \otimes b \preceq c \otimes d + v \otimes w$. Hence $T$ is $u$-continuous. □

**Remark 3.6.** We do not know if tensor product of bornological cones is bornological or no.

A subset A of a locally convex cone $(\mathcal{C}, \mathcal{V})$ is called precompact with respect to the symmetric topology if for every $v \in \mathcal{V}$, there are $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} v(a_i)v$. One can define similarly precompactness with respect to the upper and lower topologies (see [3]).

**Proposition 3.7.** *Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be two locally convex cones and $A \subseteq \mathcal{C}$ and $B \subseteq \mathcal{D}$ be precompact with respect to the symmetric topology. Then the subset $A \otimes B$ of $\mathcal{C} \otimes \mathcal{D}$ is precompact with respect to the symmetric topology.*

*Proof.* Let the subsets $A \subseteq \mathcal{C}$ and $B \subseteq \mathcal{D}$ be precompact. Let $v \otimes w \in \mathcal{V} \otimes \mathcal{W}$ be arbitrary. There are finite subsets $\{x_1, x_2, \ldots, x_n\} \subseteq A$ and $\{y_1, y_2, \ldots, y_m\} \subseteq B$ such that $A \subseteq \bigcup_{i=1}^{n} v(x_i)v$ and $B \subseteq \bigcup_{j=1}^{m} w(y_j)w$. Let $a \in A$ and $b \in B$. We have $a \leq x_i + v$, $x_i \leq a + v$, $b \leq y_j + w$ and $y_j \leq b + w$ for some $1 \leq i \leq n$ and $1 \leq j \leq m$. Then $a \otimes b \preceq x_i \otimes y_j + v \otimes w$ and $x_i \otimes y_j \preceq a \otimes b + v \otimes w$ and so $a \otimes b \in (v \otimes w)(x_i \otimes y_j)(v \otimes w)$. This yields that $A \otimes B$ is precompact in $\mathcal{C} \otimes \mathcal{D}$ with respect to the symmetric topology. □

**Remark 3.8.**
(a) Proposition 3.7 also holds for precompactness with respect to the upper and lower topologies.
(b) Similar to Proposition 3.7, one can show that the upper and lower boundedness are also preserved by tensor product.

**Example 3.9.** As we mentioned in the introduction, $(\bar{\mathbb{R}}, \mathcal{V})$ is a locally convex cone, where $\mathcal{V} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$. Tensor product $\bar{\mathbb{R}} \otimes \bar{\mathbb{R}}$ is a locally convex cone with (abstract) 0-neighborhood system $\mathcal{V} \otimes \mathcal{V} = \{\epsilon \otimes \epsilon \in \mathbb{R} \otimes \mathbb{R} : \epsilon > 0\} = \{\epsilon(1 \otimes 1) : \epsilon > 0\}$ and that is a $uc$-cone. The set $\mathcal{A} = \{(+\infty) \otimes (+\infty)\}$ is a precompact and bounded below subset of $\bar{\mathbb{R}} \otimes \bar{\mathbb{R}}$, but not bounded above.

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[1] Department of Pure Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Tabriz, Tabriz, Iran.
*Email address*: s.vazifeh@tabrizu.ac.ir

[2] Department of Pure Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Tabriz, Tabriz, Iran.
*Email address*: ranjbari@tabrizu.ac.ir; ranjbari@gmail.com