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Fractional Differential Inclusion Involving p-Laplacian Operators

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Abstract. This paper investigates the existence of solutions to a class of fractional differential inclusions involving the p-Laplacian operator and three-point boundary conditions. The problem is formulated using Caputo fractional derivatives to effectively handle the boundary data. The primary objective is to bridge a significant research gap concerning the application of the method of upper and lower solutions to such inclusions. By transforming the original boundary value problem into an equivalent auxiliary problem, we leverage the properties of an associated Green's function. Our main existence result is established using the method of upper and lower solutions combined with fixed point theory in ordered Banach spaces. Under assumptions of upper semi-continuity, convexity and compactness of values for the right-hand side, we demonstrate the existence of at least one solution confined within an ordered interval defined by explicitly constructed upper and lower solutions. The theoretical framework is validated through a concrete numerical example featuring a set-valued mapping, illustrating the practical applicability of our results.

1. Introduction

Fractional calculus has emerged as a powerful mathematical framework for modeling complex phenomena with memory and heredity properties across diverse scientific and engineering disciplines including physics, biology, and applied mathematics [17, 22, 30]. Unlike classical integer-order derivatives, fractional derivatives capture non-local effects, making them particularly suitable for describing processes with

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long-range dependencies. Within this paradigm, fractional differential equations and inclusions have become essential tools for analyzing systems with intricate dynamics and uncertainties [19, 24, 26].

Among various fractional operators, the Caputo fractional derivative stands out for its mathematical convenience in handling initial and boundary conditions, as it requires standard integer-order initial values that align naturally with physical interpretations. This makes Caputo derivative particularly advantageous in modeling real-world phenomena where initial states are physically measurable quantities. Coupled with nonlinear operators like the p-Laplacian, defined by $\varphi_p(x) = |x|^{p-2}x$, fractional models become capable of describing complex nonlinear behaviors observed in non-Newtonian fluids, turbulence, and elastic materials [7, 11, 23].

Recent years have witnessed significant advancements in fractional calculus research, expanding into specialized areas, including quantum calculus and systems with time delays. Quantum fractional calculus frameworks have been employed to study q-integro-differential inclusions through innovative numerical and graphical simulation techniques [14, 28, 29], while computational methods utilizing heatmaps and simulations have been developed for analyzing quantum integrodifferential inclusions. Stability analyses of stochastic fractional systems have advanced our understanding of complex dynamical behaviors, and qualitative analyses of systems involving generalized Caputo proportional fractional integro-differential equations with multiple time delays have provided insights into time-dependent fractional phenomena [29]. Additionally, existence results for product-type fractional functional integral equations have been established using Petryshyn's fixed point theorem, demonstrating the applicability of advanced topological methods in fractional analysis [13, 16, 28].

Despite these advancements, significant research gaps remain in the study of fractional differential inclusions with nonlinear operators and complex boundary conditions. The method of upper and lower solutions, originally introduced by Picard (1893) and later developed by Dragoni, provides a powerful approach for proving existence results while simultaneously bounding the solution. While extensively applied to fractional differential equations [18, 25, 27], its application to fractional differential inclusions, particularly those involving p-Laplacian operators, remains relatively underexplored [5, 6, 9].

Multi-point boundary conditions have gained considerable attention due to their relevance in practical applications such as heat conduction, beam deflection, and wave propagation, where controllers may act at interior positions [1–3, 12]. These conditions often provide more accurate

descriptions of physical phenomena than traditional two-point boundary conditions. However, the study of fractional differential inclusions with multi-point boundary conditions and p-Laplacian operators represents a research void that requires systematic investigation.

This paper aims to bridge this gap by investigating the following fractional differential inclusion with a p-Laplacian operator and three-point boundary conditions:

$$(\mathcal{PI}) \begin{cases} D_{0^+}^{\beta} \left(\varphi_p(D_{0^+}^{\alpha} y)\right)(t) \in F(t, y(t)), & t \in I = [0,1], \\ \\ y(0) = 0, \qquad y(\theta) = y(1), \qquad D_{0^+}^{\alpha} y(0) = 0. \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $\theta \in]0,1[$, $p > 1$, $\varphi_p(s) = |s|^{p-2}s$, and $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is an upper semicontinuous set-valued map with compact convex values. The operators $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ denote the Caputo fractional derivatives of orders α and β, respectively, chosen for their mathematical convenience in handling the boundary conditions.

The primary objectives and contributions of this work are:

- To establish an existence theory for problem $(\mathcal{PI})$ using the method of upper and lower solutions combined with fixed-point theorems in ordered Banach spaces.
- To develop appropriate analytical transformations that reduce the original problem to an equivalent form amenable to rigorous mathematical analysis.
- To construct explicit upper and lower solutions under natural growth and ordering conditions on the nonlinearity, leveraging the specific properties of Caputo fractional derivatives.
- To present an illustrative example that demonstrates the practical applicability of our theoretical results.

The paper is organized as follows: Section 2 provides essential preliminaries from fractional calculus, set-valued analysis, and fixed-point theory. Section 3 analyzes auxiliary problems, reformulates $(\mathcal{PI})$ into an equivalent initial value problem, and constructs explicit upper and lower solutions. Section 4 presents the main existence theorem, proved via a sophisticated fixed-point approach. Finally, Section 5 provides a concrete example to demonstrate the applicability of our theoretical framework, followed by concluding remarks and potential directions for future research.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.
Let $I = [0,1]$ be an interval of $\mathbb{R}$, and let $y : I \to \mathbb{R}$. We denote by:

- $L^1(I,\mathbb{R})$ the Banach space of measurable functions y that are Lebesgue integrable, equipped with the norm
$$\|y\|_1 = \int_0^1 |y(s)|\,ds.$$
- $AC(I,\mathbb{R})$ (respectively $C(I,\mathbb{R})$) the Banach space of all absolutely continuous (respectively continuous) functions y from I into $\mathbb{R}$ with the norm
$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le 1\}.$$
- $AC^n(I,\mathbb{R})$ the space of functions y that have continuous derivatives up to order $n-1$ on I and satisfy $y^{(n-1)} \in AC(I,\mathbb{R})$.

The condition $\check{y} \le \hat{y}$ if and only if $\check{y}(t) \le \hat{y}(t)$ for all $t \in I$ defines a partial ordering in $AC(I,\mathbb{R})$.

Definition 2.1. (Riemann-Liouville Fractional Integral [19]) Let $\alpha > 0$ and $y : [0,\infty) \to \mathbb{R}$ be an integrable function. The Riemann-Liouville fractional integral of order α is defined by
$$I^\alpha_{0^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \quad t > 0,$$
where $\Gamma(\cdot)$ denotes the Gamma function defined by
$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\,dt, \quad \alpha > 0.$$

Definition 2.2 (Caputo Fractional Derivative [19])**.** Let $\alpha > 0$, $n = [\alpha] + 1$, and $y \in AC^n[0,1]$. The Caputo fractional derivative of order α is defined by
$$D^\alpha_{0^+} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\,ds, \quad t > 0,$$
where $y^{(n)}$ denotes the n-th ordinary derivative of y.

Lemma 2.3 ([19])**.** *Let $\alpha > 0$ and let $n = [\alpha] + 1$.*

(i) *If $y \in AC^n[0,1]$, then*
$$I^\alpha_{0^+} D^\alpha_{0^+} y(t) = y(t) - \sum_{k=0}^{n-1} t^k c_k,$$
where $c_k = \frac{y^{(k)}(0)}{k!}$.

(ii) *For every $t > 0$ and $\gamma > -1$, we have*
$$I^\alpha_{0^+} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

Definition 2.4. Let $(X, \|.\|)$ be a Banach space and let $\mathcal{P}(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $c(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $cp(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $cl(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, and $ccp(X) = c(X) \cap cp(X)$. A set-valued map $F : X \to \mathcal{P}(X)$ is said to have the following properties:

- **Fixed point**: F has a fixed point if there exists $x \in X$ such that $x \in F(x)$,
- **Convex (closed) values**: F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$,
- **Boundedness on Bounded Sets**: F is said to be bounded on bounded sets if for every $B \in b(X)$ the image set
$$F(B) = \bigcup_{x \in B} F(x)$$
remains bounded in X.
- **Upper semi-continuity (u.s.c.)**: F is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subseteq N$,
- **Complete continuity**: F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in b(X)$,
- **L^1-Carathéodory property**: F is said to be L^1-Carathéodory if
  (i) $t \to F(t, y)$ is measurable for each $y \in \mathbb{R}$,
  (ii) $y \to F(t, y)$ is upper semi-continuous for almost all $t \in I$,
  (iii) for each $\eta > 0$, there exists $\psi_\eta \in L^1(I, \mathbb{R}_+)$ such that $\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq \psi_\eta(t)$ for all $|y| \leq \eta$ and for almost everywhere $t \in I$.
- **Selection**: Denote by $S_{F,y} \subset L^1(I, X)$ the set of all Lebesgue integrable selections of the set-valued map F for a given $y \in C(I, X)$; i.e.,
$$S_{F,y} = \{v \in L^1(I, X) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in I\}.$$
- **Measurability**: A set-valued map $F : I \to cl(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, F(t)) = \inf\{|y - x| : x \in F(t)\}$ is measurable.

For more details on set-valued mappings, see [4, 8].

Lemma 2.5 ([15]). *(Nonlinear alternative of Leray-Schauder type) Let X be a Banach space, C a closed convex subset of X, U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \to ccp(C)$ is an upper semicontinuous compact map. Then either*

(a) *F has a fixed point in $\bar{U}$,*
or
(b) *there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

Lemma 2.6 ([21])**.** *Let X be a Banach space and let F is an L^1-Carathéodory set-valued map.*

(i) *If $\dim(X) < \infty$ and $F : I \times X \to ccp(X)$), then $S_{F,y} \neq \emptyset$ for each $y \in C(I,X)$.*
(ii) *Let $\Theta : L^1(I,X) \to C(I,X)$ be a linear continuous operator. Then the composition operator*

$$\Theta \circ S_F : C(I,X) \to ccp(C(I,X)), \quad y \mapsto (\Theta \circ S_F)(y) := \Theta(S_{F,y})$$

has a closed graph in $C(I,X) \times C(I,X)$.

Remark 2.7. If $F : X \to cl(Y)$ is upper semi-continuous, then the graph of F is a closed subset of $X \times Y$, that is, for every sequence $\{x_n\}_{n\in\mathcal{N}} \subset X$ and $\{y_n\}_{n\in\mathcal{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is upper semicontinuous, (see [8, Proposition 1.2])

Definition 2.8 (The p-Laplacian operator)**.** For $p > 1$, the **p-Laplacian operator** is defined by the function $\varphi_p : \mathbb{R} \to \mathbb{R}$ as follows:

$$\varphi_p(x) = |x|^{p-2}x.$$

Remark 2.9. The operator φ_p possesses the following properties:

(i) φ_p is an odd function: $\varphi_p(-x) = -\varphi_p(x)$.
(ii) φ_p is strictly monotone: $(\varphi_p(x)-\varphi_p(y))(x-y) > 0$ for all $x \neq y$.
(iii) φ_p is homogeneous of degree $(p-1)$: $\varphi_p(\lambda x) = |\lambda|^{p-2}\lambda\varphi_p(x)$.
(iv) φ_p is continuous and differentiable on $\mathbb{R}$.

Definition 2.10 (The inverse of the p-Laplacian operator)**.** The inverse of the p-Laplacian operator is the function $\varphi_q : \mathbb{R} \to \mathbb{R}$ where:

$$\frac{1}{p} + \frac{1}{q} = 1,$$

i.e., $q = \frac{p}{p-1}$, and is defined as:

$$\varphi_q(x) = \begin{cases} |x|^{q-2}x = |x|^{\frac{2-p}{p-1}}x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Definition 2.11. A function $y \in AC^2(I,\mathbb{R})$ is said to be a solution of $(\mathcal{PI})$, if there exists a function $v \in L^1(I,\mathbb{R})$ such that $v(t) \in F(t,y(t))$

almost everywhere on I and y satisfying

$$(\mathcal{PE})\begin{cases} D_{0^+}^{\beta}\left(\varphi_p(D_{0^+}^{\alpha}y)\right)(t)=v(t), & t\in I,\\ \\ y(0)=0, \qquad y(\theta)=y(1), \quad D_{0^+}^{\alpha}y(0)=0. \end{cases}$$

3. Auxiliary Results

In this section, we present and prove our auxiliary results. Let $u \in AC(I,\mathbb{R})$. Setting $u(t)=\varphi_p\left(D_{0^+}^{\alpha}y(t)\right)$, we observe that $D_{0^+}^{\beta}u(t)=v(t)$, and the condition $D_{0^+}^{\alpha}y(0)=0$ implies $u(0)=0$. Consequently, problem $(\mathcal{PE})$ can be decomposed into the following coupled boundary value problems:

$$\begin{cases} \varphi_p(D_{0^+}^{\alpha}y(t))=u(t), & t\in I, \qquad 1<\alpha\leq 2,\\ \\ y(0)=0, \quad y(\theta)=y(1), & \theta\in]0,1[. \end{cases} \tag{3.1}$$

and

$$\begin{cases} D_{0^+}^{\beta}u(t)=v(t), & t\in I,\ 0<\beta\leq 1,\\ \\ u(0)=0. \end{cases} \tag{3.2}$$

In the following lemmas, we solve problems (3.1) and (3.2).

Lemma 3.1. *For $v\in L^1(I,\mathbb{R})$, the solution $u\in AC(I,\mathbb{R})$ of (3.2) is given by*

$$u(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds.$$

Proof. Applying the fractional integral operator $I_{0^+}^{\beta}$ to both sides of the first equation in (3.2) and invoking part (i) of Lemma 2.3, we obtain

$$u(t)=c_0+I_{0^+}^{\beta}v(t), \quad t\in I.$$

The initial condition $u(0)=0$, together with the property $0<\beta\leq 1$, forces $c_0=0$. Consequently

$$u(t)=I_{0^+}^{\beta}v(t), \quad t\in I. \qquad \square$$

Lemma 3.2. *For $u\in AC(I,\mathbb{R})$, the solution $y\in AC^2(I,\mathbb{R})$ of (3.1) is given by*

$$y(t)=\frac{1}{(1-\theta)\Gamma(\alpha)}\int_0^1 G(t,s)\varphi_q\left(u(s)\right)ds, \tag{3.3}$$

where G is a Green type function given by

(i) *If* $0 \leq t < \theta$

$$G(t,s) = \begin{cases} (1-\theta)(t-s)^{\alpha-1} + t(\theta-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & 0 \leq s \leq t, \\ t(\theta-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & t < s \leq \theta, \\ -t(1-s)^{\alpha-1}, & \theta < s \leq 1. \end{cases}$$

(ii) *If* $\theta \leq t \leq 1$

$$G(t,s) = \begin{cases} (1-\theta)(t-s)^{\alpha-1} + t(\theta-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & 0 \leq s \leq \theta, \\ (1-\theta)(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & \theta < s \leq t, \\ -t(1-s)^{\alpha-1}, & t < s \leq 1. \end{cases}$$

Proof. Equation (3.1) is equivalent to the following problem

$$\begin{cases} D^{\alpha}_{0^+} y(t) = \varphi_q\left(u(t)\right), & t \in I,\ 1 < \alpha \leq 2, \\ y(0) = 0, \quad y(\theta) = y(1), & \theta \in]0,1[. \end{cases} \tag{3.4}$$

By applying $I^{\alpha}_{0^+}$ on both sides of the first equation of (3.4), and by (i) of Lemma 2.3, we reduce the equation $D^{\alpha}_{0^+} y(t) = \varphi_q\left(u(t)\right)$ into

$$y(t) = c_0 + c_1 t + I^{\alpha}_{0^+} \varphi_q(u(t)). \tag{3.5}$$

Using the first condition $y(0) = 0$ in (3.5) gives $c_0 = 0$, and by the second condition $y(1) = y(\theta)$, we get

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_q(u(s)) ds + c_1 = \frac{1}{\Gamma(\alpha)} \int_0^{\theta} (\theta-s)^{\alpha-1} \varphi_q(u(s)) ds + c_1 \theta;$$

then

$$c_1 = \frac{1}{(1-\theta)\Gamma(\alpha)} \left(\int_0^{\theta} (\theta-s)^{\alpha-1} \varphi_q(u(s)) ds - \int_0^1 (1-s)^{\alpha-1} \varphi_q(u(s)) ds \right).$$

Substituting the values of c_0 and c_1 into (3.5) gives:
If $\theta \leq t$

$$\begin{aligned} y(t) = \frac{t}{(1-\theta)\Gamma(\alpha)} (& \int_0^{\theta} (\theta-s)^{\alpha-1} \varphi_q(u(s)) ds \\ & - \int_0^{\theta} (1-s)^{\alpha-1} \varphi_q(u(s)) ds - \int_{\theta}^{t} (1-s)^{\alpha-1} \varphi_q(u(s)) ds \\ & - \int_t^1 (1-s)^{\alpha-1} \varphi_q(u(s)) ds) + \frac{1}{\Gamma(\alpha)} \int_0^{\theta} (t-s)^{\alpha-1} \varphi_q(u(s)) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{\theta}^{t} (t-s)^{\alpha-1} \varphi_q(u(s)) ds; \end{aligned}$$

then

$$\begin{aligned} y(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \int_0^\theta \Big(& (1-\theta)(t-s)^{\alpha-1} + t(\theta-s)^{\alpha-1} \\ & - t(1-s)^{\alpha-1} \Big) \varphi_q(u(s)) ds \\ & + \frac{1}{(1-\theta)\Gamma(\alpha)} \int_\theta^t ((1-\theta)(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}) \varphi_q(u(s)) ds \\ & - \frac{1}{(1-\theta)\Gamma(\alpha)} \int_t^1 t(1-s)^{\alpha-1} \varphi_q(u(s)) ds. \end{aligned}$$

If $t < \theta$

$$\begin{aligned} y(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \int_0^t & [(1-\theta)(t-s)^{\alpha-1} + t(\theta-s)^{\alpha-1} \\ & - t(1-s)^{\alpha-1}] \varphi_q(u(s)) ds \\ & + \frac{1}{(1-\theta)\Gamma(\alpha)} \int_t^\theta \left(t(\theta-s)^{\alpha-1} - t(1-s)^{\alpha-1} \right) \varphi_q(u(s)) ds \\ & - \frac{1}{(1-\theta)\Gamma(\alpha)} \int_\theta^1 t(1-s)^{\alpha-1} \varphi_q(u(s)) ds. \end{aligned}$$

The proof is then completed. □

Lemma 3.3. *The Green's function $G : I \times I \to \mathbb{R}$ satisfies the following properties:*
1. G is continuous on $I \times I$.
2. $|G(t,s)| \leq 1$ for all $t, s \in I$.

Proof. **1. Continuity**
We examine three cases based on the values of s.
Case 1: $0 \leq s < \theta$
For $t \to \theta^-$, we use the first branch of (i):

$$\begin{aligned} \lim_{t \to \theta^-} G(t,s) &= (1-\theta)(\theta-s)^{\alpha-1} + \theta(\theta-s)^{\alpha-1} - \theta(1-s)^{\alpha-1} \\ &= (\theta-s)^{\alpha-1} - \theta(1-s)^{\alpha-1}. \end{aligned}$$

For $t = \theta$, we use the first branch of (ii):

$$\begin{aligned} G(\theta,s) &= (1-\theta)(\theta-s)^{\alpha-1} + \theta(\theta-s)^{\alpha-1} - \theta(1-s)^{\alpha-1} \\ &= (\theta-s)^{\alpha-1} - \theta(1-s)^{\alpha-1}. \end{aligned}$$

For $t \to \theta^+$, we again use the first branch of (ii):

$$\lim_{t \to \theta^+} G(t,s) = (\theta-s)^{\alpha-1} - \theta(1-s)^{\alpha-1}.$$

Hence,

$$\begin{aligned}\lim_{t\to\theta^-} G(t,s) &= G(\theta,s)\\ &= \lim_{t\to\theta^+} G(t,s).\end{aligned}$$

Case 2: $s=\theta$

For $t\to\theta^-$, we use the second branch of (i):

$$\begin{aligned}\lim_{t\to\theta^-} G(t,\theta) &= \lim_{t\to\theta^-}\left[t(\theta-\theta)^{\alpha-1} - t(1-\theta)^{\alpha-1}\right]\\ &= \lim_{t\to\theta^-}\left[-t(1-\theta)^{\alpha-1}\right]\\ &= -\theta(1-\theta)^{\alpha-1}.\end{aligned}$$

For $t=\theta$, we use the first branch of (ii):

$$\begin{aligned}G(\theta,\theta) &= (1-\theta)(\theta-\theta)^{\alpha-1} + \theta(\theta-\theta)^{\alpha-1} - \theta(1-\theta)^{\alpha-1}\\ &= -\theta(1-\theta)^{\alpha-1}.\end{aligned}$$

For $t\to\theta^+$, we again use the first branch of (ii):

$$\lim_{t\to\theta^+} G(t,\theta) = -\theta(1-\theta)^{\alpha-1}.$$

Thus,

$$\begin{aligned}\lim_{t\to\theta^-} G(t,s) &= G(\theta,s)\\ &= \lim_{t\to\theta^+} G(t,s).\end{aligned}$$

Case 3: $\theta < s \leq 1$

For $t\to\theta^-$, we use the third branch of (i):

$$\begin{aligned}\lim_{t\to\theta^-} G(t,s) &= \lim_{t\to\theta^-}\left[-t(1-s)^{\alpha-1}\right]\\ &= -\theta(1-s)^{\alpha-1}.\end{aligned}$$

For $t=\theta$, we use the third branch of (ii):

$$G(\theta,s) = -\theta(1-s)^{\alpha-1}.$$

For $t\to\theta^+$, then we use the third branch of (ii):

$$\begin{aligned}\lim_{t\to\theta^+} G(t,s) &= \lim_{t\to\theta^+}\left[-t(1-s)^{\alpha-1}\right]\\ &= -\theta(1-s)^{\alpha-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{t\to\theta^-} G(t,s) &= G(\theta,s)\\ &= \lim_{t\to\theta^+} G(t,s).\end{aligned}$$

The left-hand limit, the right-hand limit, and the value of $G(t,s)$ at $t=\theta$ are equal. Therefore, the Green's function $G(t,s)$ is continuous at $t=\theta$, hence is continuous at $I\times I$.

2. $|G(t,s)|\leq 1$ for all $t,s\in I$. We consider the different cases:

Case 1: $0\leq t<\theta$**.**

- If $0\leq s\leq t$, then

$$\begin{aligned}G(t,s)&=(1-\theta)(t-s)^{\alpha-1}+t(\theta-s)^{\alpha-1}-t(1-s)^{\alpha-1}\\&\leq(1-\theta)(t-s)^{\alpha-1}+t(\theta-s)^{\alpha-1}\\&\leq(1-\theta)+\theta=1,\end{aligned}$$

because :
Since $t\geq 0$ and $(1-s)^{\alpha-1}\geq 0$, the last term is nonpositive:

$$-t(1-s)^{\alpha-1}\leq 0,$$

dropping this term gives:

$$G(t,s)\leq(1-\theta)(t-s)^{\alpha-1}+t(\theta-s)^{\alpha-1},$$

since $0\leq s\leq t\leq 1$ and $0\leq s\leq\theta\leq 1$, we have:

$$0\leq t-s\leq 1\quad\Rightarrow\quad(t-s)^{\alpha-1}\leq 1$$

$$0\leq\theta-s\leq 1\quad\Rightarrow\quad(\theta-s)^{\alpha-1}\leq 1$$

where we used the fact that $x\mapsto x^{\alpha-1}$ is increasing for $\alpha>1$ and $x^{\alpha-1}\leq 1$ for $x\in[0,1]$.
Combining the above estimates:

$$\begin{aligned}G(t,s)&\leq(1-\theta)\cdot 1+t\cdot 1\\&=(1-\theta)+t,\end{aligned}$$

since $t<\theta$, we have:

$$(1-\theta)+t<(1-\theta)+\theta=1,$$

therefore:

$$G(t,s)\leq 1.$$

- If $t<s\leq\theta$, then

$$\begin{aligned}G(t,s)&=t[(\theta-s)^{\alpha-1}-(1-s)^{\alpha-1}]\\&\leq 0\leq 1.\end{aligned}$$

- If $\theta<s\leq 1$, then

$$\begin{aligned}G(t,s)&=-t(1-s)^{\alpha-1}\\&\leq 0\leq 1.\end{aligned}$$

Case 2: $\theta\leq t\leq 1$**.**

- If $0 \leq s \leq \theta$, then

$$\begin{aligned} G(t,s) &= (1-\theta)(t-s)^{\alpha-1} + t(\theta-s)^{\alpha-1} - t(1-s)^{\alpha-1} \\ &\leq (1-\theta)(t-s)^{\alpha-1} \\ &\leq 1-\theta \\ &\leq 1, \end{aligned}$$

because $(\theta-s)^{\alpha-1} - (1-s)^{\alpha-1} \leq 0$.

- If $\theta < s \leq t$, then

$$\begin{aligned} G(t,s) &= (1-\theta)(t-s)^{\alpha-1} - t(1-s)^{\alpha-1} \\ &\leq (1-\theta)(t-s)^{\alpha-1} \\ &\leq 1-\theta \\ &\leq 1. \end{aligned}$$

- If $t < s \leq 1$, then

$$\begin{aligned} G(t,s) &= -t(1-s)^{\alpha-1} \\ &\leq 0 \leq 1 \end{aligned}$$

Thus, $G(t,s) \leq 1$ for all $t, s \in I$. Similarly, one can show that $G(t,s) \geq -1$ (by analogous arguments). Hence, $|G(t,s)| \leq 1$.
Then $G(t,s)$ is not of constant sign. □

Remark 3.4. We can write (3.3) as

$$y(t) = Ku(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \int_0^1 G(t,s)\varphi_q(u(s))ds,$$

where $K : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ is an operator. Therefore, for $u \in C(I,\mathbb{R})$ the problem $(\mathcal{PI})$ is equivalent to the following initial valued problem

$$(\mathcal{PI}1) \begin{cases} D^{\beta}_{0^+} u(t) \in F(t, Ku(t)), \quad \text{a.e. } t \in I, \quad 0 < \beta \leq 1, \\ \\ u(0) = 0. \end{cases}$$

We need the following definition in the sequel.

Definition 3.5. A function $\check{u} \in AC(I,\mathbb{R})$ is said to be a lower solution of $(\mathcal{PI}1)$ if there exists $v_1 \in L^1(I,\mathbb{R})$ with $v_1(t) \in F(t, K\check{u}(t))$ for a.e. $t \in I$ such that $D^{\beta}_{0^+}\check{u}(t) \leq v_1(t)$ a.e. on I and $\check{u}(0) \leq 0$.

Similarly, a function $\hat{u} \in AC(I,\mathbb{R})$ is said to be an upper solution of $(\mathcal{PI}1)$ if there exists $v_2 \in L^1(I,\mathbb{R})$ with $v_2(t) \in F(t, K\hat{u}(t))$ for a.e. $t \in I$ such that $D^{\beta}_{0^+}\hat{u}(t) \geq v_2(t)$ a.e. on I and $\hat{u}(0) \geq 0$.

Lemma 3.6. *Assume that there exist a constant $\hat{c} \geq 0$, a function $v_2 \in L^1(I, \mathbb{R})$ and a function $u \in AC(I, \mathbb{R})$ such that, for every $t \in I$, the following inequalities hold:*

$$v_2(t) \leq \hat{c} \quad \text{and} \quad 0 \leq u(t) \leq \frac{\hat{c}}{\Gamma(\beta+1)}.$$

Then problem $(\mathcal{PI}1)$ admits an upper solution.

Proof. Let $\hat{u}(t) = \frac{\hat{c}}{\Gamma(\beta+1)} t^\beta$, $v_2 \in L^1(I, \mathbb{R})$, for $t \in I$, and $0 \leq \beta \leq 1$ we have

$$\begin{aligned} 0 \leq \hat{u}(t) &= \frac{\hat{c}}{\Gamma(\beta+1)} t^\beta \\ &\leq \frac{\hat{c}}{\Gamma(\beta+1)}, \end{aligned}$$

and

$$\begin{aligned} D^\beta_{0^+}\hat{u}(t) - v_2(t) &= D^\beta_{0^+}\left(\frac{\hat{c}}{\Gamma(\beta+1)} t^\beta\right) - v_2(t) \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{1-\beta-1} \left(\frac{\hat{c}}{\Gamma(\beta+1)} s^\beta\right)' ds - v_2(t) \\ &= \frac{\beta\hat{c}}{\Gamma(1-\beta)\Gamma(\beta+1)} \int_0^t (t-s)^{1-\beta-1} s^{\beta-1} ds - v_2(t) \\ &= \frac{\beta\hat{c}}{\Gamma(\beta+1)} I^{1-\beta} t^{\beta-1} - v_2(t); \end{aligned}$$

we match the expression $I^{1-\beta}\, t^{\beta-1}$ with the formula $I^\alpha_{0+}\, t^\gamma$ of Lemma 2.3 (ii) by setting:

$$\alpha = 1-\beta, \qquad \gamma = \beta - 1;$$

we verify that the condition $\gamma > -1$ is satisfied:

$$\gamma = \beta - 1 > -1 \quad \Leftrightarrow \quad \beta > 0;$$

we compute the quantities appearing in the formula of Lemma 2.3 (ii):

- $\gamma + 1 = (\beta - 1) + 1 = \beta$
- $\alpha + \gamma + 1 = (1-\beta) + (\beta - 1) + 1 = 1$
- $\alpha + \gamma = (1-\beta) + (\beta - 1) = 0$

we obtain :

$$\begin{aligned} I^{1-\beta}\, t^{\beta-1} &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\, t^{\alpha+\gamma} \\ &= \frac{\Gamma(\beta)}{\Gamma(1)}\, t^0 = \Gamma(\beta) \end{aligned}$$

then

$$\begin{aligned} D^{\beta}_{0^+}\hat{u}(t) - v_2(t) &= \frac{\beta\hat{c}\Gamma(\beta)}{\Gamma(\beta+1)} - v_2(t) \\ &= \hat{c} - v_2(t) \geq 0. \end{aligned}$$

Moreover, $\hat{u}(0) = 0$, this implies that $\hat{u}(t)$ is an upper solution of $(\mathcal{PI}1)$. □

Lemma 3.7. *Assume that there exist a constant $\check{c} \leq 0$, a function $v_1 \in L^1(I,\mathbb{R})$ and a function $u \in AC(I,\mathbb{R})$ such that, for every $t \in I$, the following inequalities hold:*

$$v_1(t) \geq \check{c} \quad \textit{and} \quad \frac{\check{c}}{\Gamma(\beta+1)} \leq u(t) \leq 0.$$

Then problem $(\mathcal{PI}1)$ admits a lower solution.

Proof. Let $\check{u}(t) = \frac{\check{c}}{\Gamma(\beta+1)}t^{\beta}$, $v_1 \in L^1(I,\mathbb{R})$, for $t \in I$, and $0 \leq \beta \leq 1$ we have

$$\begin{aligned} 0 \leq \hat{u}(t) &= \frac{\hat{c}}{\Gamma(\beta+1)}t^{\beta} \\ &\leq \frac{\hat{c}}{\Gamma(\beta+1)}, \end{aligned}$$

and

$$\begin{aligned} D^{\beta}_{0^+}\check{u}(t) - v_1(t) &= D^{\beta}_{0^+}\left(\frac{\check{c}}{\Gamma(\beta+1)}t^{\beta}\right) - v_1(t) \\ &= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{1-\beta-1}\left(\frac{\check{c}}{\Gamma(\beta+1)}s^{\beta}\right)' ds - v_1(t) \\ &= \frac{\beta\check{c}}{\Gamma(\beta+1)}I^{1-\beta}t^{\beta-1} - v_1(t) \\ &= \frac{\beta\check{c}\Gamma(\beta)}{\Gamma(\beta+1)\Gamma(1-\beta+\beta)}t^{1-\beta+\beta-1} - v_1(t); \end{aligned}$$

by (ii) of Lemma 2.3 (here $\alpha = 1-\beta$, $\gamma = \beta - 1$) we have

$$\begin{aligned} I^{1-\beta}t^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(1-\beta+\beta)}t^{1-\beta+\beta-1} \\ &= \Gamma(\beta), \end{aligned}$$

then

$$\begin{aligned} D^{\beta}_{0^+}\check{u}(t) - v_1(t) &= \frac{\beta\check{c}\Gamma(\beta)}{\Gamma(\beta+1)} - v_1(t) \\ &= \check{c} - v_1(t) \leq 0. \end{aligned}$$

Moreover $\check{u}(0) = 0$, this implies that $\check{u}(t)$ is a lower solution of $(\mathcal{PI}1)$. □

4. Existence Result

In this section, we are concerned with the existence of solutions for $(\mathcal{PI}1)$. For the study of this problem we first list the following hypotheses:

- (H_1) $F : I \times \mathbb{R} \to ccp(\mathbb{R})$ is an L^1-Carathéodory set-valued map,
- (H_2) there exist $\hat{c} \geq 0,\ \check{c} \leq 0,\ |\check{c}| \leq \hat{c},\ v_1,\ v_2 \in L^1(I,\mathbb{R})$ and $u \in AC(I,\mathbb{R})$ such that
  - (i) for $t \in I,\ v_2(t) \leq \hat{c}$ when $0 \leq u(t) \leq \frac{\hat{c}}{\Gamma(\beta+1)}$,
  - (ii) for $t \in I,\ v_1(t) \geq \check{c}$ when $\frac{\check{c}}{\Gamma(\beta+1)} \leq u(t) \leq 0$.

Theorem 4.1. *Assume that hypotheses (H_1) and (H_2) hold, then the problem $(\mathcal{PI}1)$ has at least one solution $u \in AC(I,\mathbb{R})$ satisfying*

$$\check{u}(t) \leq u(t) \leq \hat{u}(t), \quad \text{for all } t \in I.$$

Proof. Consider the modified problem

$$(\mathcal{PI}1m) \begin{cases} D^{\beta}_{0^+} u(t) \in \tilde{F}(t, Ku(t)), \quad \text{a.e. } t \in I, \quad 0 < \beta \leq 1, \\ \\ u(0) = 0, \end{cases}$$

where

$$\tilde{F}(t,Ku(t)) = \begin{cases} F(t,K\check{u}(t)), & \text{on} \quad A_1, \\ \\ F(t,Ku(t)), & \text{on} \quad A_3, \\ \\ F(t,K\hat{u}(t)), & \text{on} \quad A_2, \end{cases}$$

with

$$\begin{aligned} A_1 &= \{t \in I : u(t) < \check{u}(t) \leq \hat{u}(t)\}, \\ A_2 &= \{t \in I : \check{u}(t) \leq \hat{u}(t) < u(t)\}, \\ A_3 &= \{t \in I : \check{u}(t) \leq u(t) \leq \hat{u}(t)\}. \end{aligned}$$

We will divide our proof into two steps.

Step 1

By the definition of $\tilde{F}$, it is clear that $\tilde{F}$ is an L^1- Carathéodory set-valued mapping with compact convex values, and by (1) from Lemma 2.6 for each $u \in C(I,\mathbb{R})$, the set $S_{\tilde{F},u} = \{v \in L^1(I,\mathbb{R}) : v(t) \in \tilde{F}(t,Ku(t))$ for a.e. $t \in I\}$ is nonempty. To transform $(\mathcal{PI}1m)$ into a fixed point problem, consider the set-valued operator $T : C(I,\mathbb{R}) \to P(C(I,\mathbb{R}))$ defined by:

$$T(u) = \left\{ h \in C(I,\mathbb{R}) : h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds, \ \ v \in \tilde{S}_{\tilde{F},u} \right\},$$

where

$$\tilde{S}_{\tilde{F},u} = \{v \in S_{\tilde{F},u} : v(t) \geq v_1(t) \text{ on } A_1 \text{ and } v(t) \leq v_2(t) \text{ on } A_2\}.$$

For each $u \in C(I,\mathbb{R})$, the set $\tilde{S}_{\tilde{F},u}$ is nonempty. The nonemptiness of $\tilde{S}_{\tilde{F},u}$ is verified via the decomposability property of Lebesgue integrable functions. For any $v_3 \in S_{\tilde{F},u}$, we construct:

$$v = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where χ_{A_i} are characteristic functions. This construction is valid due to the decomposability of L^1 selections, a standard result in multivalued analysis [6].

To apply the Leray-Schauder type nonlinear alternative, we first verify that the operator T satisfies all conditions of Lemma 2.5.

1. Convexity of $T(u)$: For each u, $T(u)$ is convex because $\tilde{F}$ takes convex values.

2. T Maps Bounded Sets Into Bounded Sets in Ω_η: Let show that T maps bounded sets into bounded sets in $C(I,\mathbb{R})$. Together with (H_1) and (H_2), we define the set $\Omega_\eta = \left\{u \in C(I,\mathbb{R}) : \|u\|_\infty \leq \frac{\eta}{\Gamma(\beta+1)}\right\}$, where $\eta = \max\{|v(t)|, \ t \in I\}$. Let show that if $u \in \Omega_\eta$ then $Tu \in \Omega_\eta$. Let $u \in \Omega_\eta$ and $h \in T(u)$, there exists $v \in \tilde{S}_{\tilde{F},u}$ such that

$$h(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds, \quad \text{for each } t \in I;$$

then

$$|h(t)| \leq \frac{1}{\Gamma(\beta)}\int_0^t |(t-s)|^{\beta-1}|v(s)|ds;$$

thus

$$\begin{aligned}\|h\|_\infty &\leq \frac{\eta}{\Gamma(\beta+1)}t^\beta \\ &\leq \frac{\eta}{\Gamma(\beta+1)}.\end{aligned}$$

3. T Maps Bounded Sets Into Equicontinuous Sets of Ω_η: Let $t_1, t_2 \in I$, $t_1 < t_2$ and Ω_η be a bounded set of $C(I,\mathbb{R})$. Let $u \in \Omega_\eta$ and $h \in T(u)$, then there exists $v \in \tilde{S}_{\tilde{F},u}$ such that for each $t \in I$ we have

$$\begin{aligned}|h(t_2) - h(t_1)| &= \left|\frac{1}{\Gamma(\beta)}\left[\int_0^{t_2}(t_2-s)^{\beta-1}v(s)ds - \int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds\right]\right| \\ &= \left|\frac{1}{\Gamma(\beta)}\left[\begin{array}{c}\displaystyle\int_0^{t_1}(t_2-s)^{\beta-1}v(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\beta-1}v(s)ds \\ \displaystyle - \int_0^{t_1}(t_1-s)^{\beta-1}v(s)ds\end{array}\right]\right|\end{aligned}$$

$$\leq \frac{1}{\Gamma(\beta)} \left[\begin{array}{l} \displaystyle\int_0^{t_1} \left|((t_2-s)^{\beta-1} - (t_1-s)^{\beta-1})v(s)\right| ds \\ \qquad + \displaystyle\int_{t_1}^{t_2} \left|(t_2-s)^{\beta-1}v(s)\right| ds \end{array} \right]$$

$$\leq \frac{\eta}{\Gamma(\beta)} \left[\begin{array}{l} \displaystyle\int_0^{t_1} \left|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\right| ds \\ \qquad + \displaystyle\int_{t_1}^{t_2} \left|(t_2-s)^{\beta-1}\right| ds \end{array} \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Based on the three items above, the Arzela Ascoli's theorem implies that $T : C(I,\mathbb{R}) \to ccp(C(I,\mathbb{R}))$ is completely continuous.

4. T is u.s.c: Since T is completely continuous, we will prove that T has a closed graph. Let $u_n \to u_*, h_n \in T(u_n)$ and $h_n \to h_*$, we need to show that $h_* \in T(u_*)$. $h_n \in T(u_n)$ means that there exists $v_n \in \tilde{S}_{\tilde{F},u_n}$ such that, for each $t \in I$,

$$h_n(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v_n(s) ds.$$

We must show that there exists $v_* \in \tilde{S}_{\tilde{F},u_*}$ such that, for each $t \in I$,

$$h_*(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v_*(s) ds.$$

Consider the continuous linear operator $\Theta : L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$ given by

$$v \mapsto (\Theta v)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds.$$

From Lemma 2.6, The operator Θ is linear and bounded, hence continuous. Since $\tilde{S}_{\tilde{F},u}$ has a closed graph (due to the L^1-Carathedory property of $\tilde{F}$), the composition $\Theta \circ \tilde{S}_{\tilde{F},u}$ also has a closed graph. Moreover, we have that $v_n \in \Theta(\tilde{S}_{\tilde{F},u_n})$. Since $u_n \to u_*$, we have

$$h_*(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v_*(s) ds,$$

for some $v_* \in \tilde{S}_{\tilde{F},u_*}$.

By Remark 2.7, this implies that T is upper semicontinuous.

5. A Priori Bounds: We show that T has fixed points or there exist $u \in \partial U$ and $\mu \in (0,1)$ with $u \in \mu T(u)$.

Suppose that $u \in \mu T(u)$ for some $\mu \in (0,1)$, then there exists $v \in \tilde{S}_{\tilde{F},u}$

such that

$$u(t) = \frac{\mu}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds,$$

then

$$|u(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t |(t-s)^{\beta-1}||v(s)| ds,$$

thus

$$\|u\|_\infty \leq \frac{\eta}{\Gamma(\beta+1)} = l.$$

Define $U = \{u \in C(I, \mathbb{R}) : \|u\|_\infty < l + 1\}$ where $l = \frac{\eta}{\Gamma(\beta+1)}$, the operator $T : \bar{U} \to P(C(I, \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \mu T(u)$ for $\mu \in (0, 1)$. By Lemma 2.5, T has a fixed point in $\bar{U}$, which is a solution of the problem $(\mathcal{PI}1m)$.

Step 2. We verify that any solution u of $(\mathcal{PI}1m)$ satisfies $\check{u}(t) \leq u(t) \leq \hat{u}(t)$ for all $t \in I$.

Let u be a solution of $(\mathcal{PI}1m)$, we prove that

$$\check{u}(t) \leq u(t), \quad \text{for all } t \in I.$$

Suppose this is not the case and let $\xi(t) = \check{u}(t) - u(t) > 0$, the function ξ has a positive maximum in some $\tilde{t} \in]0, 1[$, such that $\xi(\tilde{t}) = \max \xi(t)$, thus $\xi(\tilde{t}) > 0$. ξ is continuous, then there exists $t_1 \in I$ such that

$$\xi(t) > 0, \quad \text{for all } t \in [t_1, \tilde{t}].$$

By definition of $\tilde{F}$, we have

$$D^\beta_{0^+} u(t) \in F(t, K\check{u}(t)), \text{ a.e. on } [t_1, \tilde{t}].$$

Thus, there exists $v_1(t) \in F(t, K\check{u}(t))$ for almost every $t \in [t_1, \tilde{t}]$ such that $D^\beta_{0^+} u(t) = v_1(t)$. Since $\check{u}$ is a lower solution to $(\mathcal{PI}1m)$, we have $D^\beta_{0^+} \check{u}(t) \leq v_1(t)$ a.e. $t \in [t_1, \tilde{t}]$, then $D^\beta_{0^+} u(t) = v_1(t) \geq D^\beta_{0^+} \check{u}(t)$ thus we obtain $D^\beta_{0^+} \check{u}(t)) - D^\beta_{0^+} u(t) = D^\beta_{0^+} \xi(t) \leq 0$ for all $t \in [t_1, \tilde{t}]$, then ξ is decreasing, (see[10, Theorem 2.1]), i.e. $\xi(t_1) \geq \xi(\tilde{t})$ which is a contradiction.

If $\tilde{t} = 0$ and by the definition of the lower solution, we have $0 < \xi(0) = \check{u}(0) - u(0) = \check{u}(0) - 0 = \check{u}(0) \leq 0$, which is a contradiction.

A similar argument applies for the upper bound. This shows that problem $(\mathcal{PI}1m)$ has a solution in the interval $[\check{u}, \hat{u}]$ which is a solution of problem $(\mathcal{PI}1)$. □

Lemma 4.2. *Let u be a solution of $(\mathcal{PI}1)$ satisfying $\check{u}(t) \leq u(t) \leq \hat{u}(t)$ for all $t \in I$, then $y = Ku$ is a solution of $(\mathcal{PI})$ satisfying $\check{y}(t) \leq y(t) \leq \hat{y}(t)$, for all $t \in I$.*

Proof. u is a solution of $(\mathcal{PI}1)$ satisfying $\check{u}(t) \leq u(t) \leq \hat{u}(t)$ for all $t \in I$. Since φ_q is increasing on $\mathbb{R}$, we have

$$\varphi_q(\check{u}(t)) \leq \varphi_q(u(t)) \leq \varphi_q(\hat{u}(t)), \quad t \in I.$$

We decompose $G(t,s)$ into its positive and negative parts

$$G^+(t,s) = \max\{G(t,s), 0\}, \qquad G^-(t,s) = \min\{G(t,s), 0\}.$$

Then

$$y(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \left[\int_0^1 G^+(t,s)\varphi_q(u(s))ds + \int_0^1 G^-(t,s)\varphi_q(u(s))ds \right]$$

Since $G^+(t,s) \geq 0$, the first integral is order-preserving. However, since $G^-(t,s) \leq 0$, the second integral is order-reversing. This mixed behavior complicates the monotonicity analysis.

Using the inequalities for $\varphi_q(u(s))$,we obtain

$$y(t) \geq \frac{1}{(1-\theta)\Gamma(\alpha)} \left[\int_0^1 G^+(t,s)\varphi_q(\check{u}(s))ds + \int_0^1 G^-(t,s)\varphi_q(\hat{u}(s))ds \right],$$

$$y(t) \leq \frac{1}{(1-\theta)\Gamma(\alpha)} \left[\int_0^1 G^+(t,s)\varphi_q(\hat{u}(s))ds + \int_0^1 G^-(t,s)\varphi_q(\check{u}(s))ds \right].$$

Now, by the definition of lower and upper solutions in our problem, we have that $\check{y}$ and $\hat{y}$ satisfy

$$\check{y}(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \left[\int_0^1 G^+(t,s)\varphi_q(\check{u}(s))ds + \int_0^1 G^-(t,s)\varphi_q(\hat{u}(s))ds \right],$$

$$\hat{y}(t) = \frac{1}{(1-\theta)\Gamma(\alpha)} \left[\int_0^1 G^+(t,s)\varphi_q(\hat{u}(s))ds + \int_0^1 G^-(t,s)\varphi_q(\check{u}(s))ds \right].$$

Therefore $\check{y}(t) \leq y(t) \leq \hat{y}(t),\ t \in I$. □

5. Example

We consider the following fractional differential inclusions with boundary conditions

$$\begin{cases} D_{0^+}^{\frac{1}{2}} \left(\varphi_2(D_{0^+}^{\frac{3}{2}} y) \right)(t) \in F(t, y(t)), \quad \text{a.e. } t \in I = [0,1], \\ \\ y(0) = 0, \qquad y(\frac{1}{2}) = y(1), \qquad D_{0^+}^{\frac{3}{2}} y(0) = 0. \end{cases}$$

where $\alpha = \frac{3}{2},\ \beta = \frac{1}{2},\ \theta = \frac{1}{2},\ p = 2$, and F is a set-valued map defined by

$$F(t,y) = \left[\cos t + \frac{|y|}{1+|y|}, 5 + t^2 \right].$$

The set-valued map F satisfies:

- $F(t,y)$ takes nonempty, compact, convex values (closed intervals).
- F is upper semi-continuous (u.s.c.) in (t,y).
- There exists an integrable majorant: $\|F(t,y)\| \leq 6 \in L^1([0,1])$.

Since $p=2$, we have $q=2$, therefore $\varphi_2(u(t)) = u(t)$.
We have $\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq 6$.
If we choose $\hat{c} = 6$, then

$$\text{for all } t \in [0,1],\ v(t) \leq 6, \qquad 0 \leq u(t) \leq \frac{6}{\Gamma(\frac{3}{2})} = \frac{12}{\sqrt{\pi}}.$$

If we choose $\check{c} = -6$ then

$$\text{for all } t \in [0,1],\ v(t) \geq -6, \qquad \frac{-12}{\sqrt{\pi}} \leq u(t) \leq 0.$$

We obtain

$$\hat{u}(t) = \frac{12}{\sqrt{\pi}}\, t^{1/2}, \qquad \check{u}(t) = -\frac{12}{\sqrt{\pi}}\, t^{1/2}.$$

The linear boundary value problem

$$D^{3/2}_{0+} y(t) = u(t), \qquad y(0) = 0, \qquad y(1/2) = y(1),$$

admits a unique solution represented via the Green's function $G(t,s)$. The solution is

$$y(t) = \frac{2}{\Gamma(3/2)} \int_0^1 G(t,s)\, u(s)\, ds,$$

where

(i) If $0 \leq t < \frac{1}{2}$

$$G(t,s) = \begin{cases} \frac{1}{2}(t-s)^{\frac{1}{2}} + t(\frac{1}{2}-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}}, & 0 \leq s \leq t, \\ t(\frac{1}{2}-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}}, & t < s \leq \frac{1}{2}, \\ -t(1-s)^{\frac{1}{2}}, & \frac{1}{2} < s \leq 1. \end{cases}$$

(ii) If $\frac{1}{2} \leq t \leq 1$

$$G(t,s) = \begin{cases} \frac{1}{2}(t-s)^{\frac{1}{2}} + t(\frac{1}{2}-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ \frac{1}{2}(t-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}}, & \frac{1}{2} < s \leq t, \\ -t(1-s)^{\frac{1}{2}}, & t < s \leq 1. \end{cases}$$

Using the bounds $\hat{u}$ and $\check{u}$, we define the *upper solution* $\hat{y}$ and the *lower solution* $\check{y}$ as follows:

$$\begin{aligned}\hat{y}(t) &= \frac{2}{\Gamma(\frac{3}{2})}\left[\int_0^1 G^+(t,s)\hat{u}(s)ds + \int_0^1 G^-(t,s)\check{u}(s)ds\right] \\ &= \frac{2}{\Gamma(\frac{3}{2})}\left[\int_0^1 G^+(t,s)\,\frac{12}{\sqrt{\pi}}s^{1/2}\,ds - \int_0^1 G^-(t,s)\,\frac{12}{\sqrt{\pi}}s^{1/2}\,ds\right].\end{aligned}$$

and

$$\begin{aligned}\check{y}(t) &= \frac{2}{\Gamma(\frac{3}{2})}\left[\int_0^1 G^+(t,s)\check{u}(s)ds + \int_0^1 G^-(t,s)\hat{u}(s)ds\right] \\ &= \frac{2}{\Gamma(\frac{3}{2})}\left[-\int_0^1 G^+(t,s)\,\frac{12}{\sqrt{\pi}}s^{1/2}\,ds + \int_0^1 G^-(t,s)\,\frac{12}{\sqrt{\pi}}s^{1/2}\,ds\right].\end{aligned}$$

Substituting $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ yields

$$\hat{y}(t) = \frac{48}{\pi}\left[\int_0^1 G^+(t,s)\,s^{1/2}\,ds - \int_0^1 G^-(t,s)\,s^{1/2}\,ds\right],$$

$$\check{y}(t) = \frac{48}{\pi}\left[-\int_0^1 G^+(t,s)\,s^{1/2}\,ds + \int_0^1 G^-(t,s)\,s^{1/2}\,ds\right].$$

Conclusion

This study establishes the existence of solutions for fractional p-Laplacian inclusions subject to three-point boundary conditions. The main contribution is the extension of the classical **upper and lower solution method** to a **set-valued fractional** framework combined with the **Leray-Schauder alternative**. Our key result guarantees at least one solution located between explicitly constructed bounds.

Building upon these results, future research may naturally extend to variable order fractional operators, coupled systems of inclusions, and the development of robust numerical schemes for such problems, further bridging theoretical analysis with practical application.

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