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Soft Compact Extension for Soft Topological Spaces

Umamaheshwari Oyyavandan[1], Jeyabalan Ramasamy[2*] and Ramkumar Solai[3]

Abstract. This paper discusses the extension of the tube lemma to soft topological spaces and examines the nature and behaviour of arbitrary soft products of soft compact soft topological spaces. In particular, we prove that the arbitrary soft product of soft compact soft topological spaces is soft compact under the soft product topology. Furthermore, we construct a soft compact extension for soft topological spaces.

1. Introduction

Because of uncertainties in engineering, economics, environmental science, sociology, medical science etc. classical mathematical tools are not always successful when solving complicated problems in these domains. Rough sets theory, fuzzy sets theory, probability theory, and some other mathematical tools are useful approaches to describe uncertainty. All of these theories have its inherent difficulties. A soft set is a function from a paramer set E to the power set of initial universe set X. Molodtsov[15] in 1999 introduced the soft set theory, for modelling vagueness and uncertainity. Maji and etc. [14] applied the soft set theory to solve a decision-making problem in 2002. Same authors [13] in 2003 discussed several basic properties of soft set theory. soft sets were compared with related fuzzy sets and rough sets by Aktas and etc. [1] in 2007. Babitha and etc. [4] defined and studied the properties of equivalence relations and partitions on soft sets in 2010. Shabir and etc. [21] in 2011, introduced the definitions of soft open sets, soft closed sets, soft neighbourhood of a soft point, soft interior points, soft closure and

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* Corresponding author.

soft separation axioms and investigated their basic properties. Further investigations on the properties of soft interior and soft closure was done by Hussain and etc [9]. Kharral and etc. [12] defined the soft mappings and derived the properties of images of soft sets and inverse images of soft sets. Soft continuity of soft mappings was discussed by Aygunoglu and etc. [3] in 2012. Also they introduced soft product topology through the soft projection mappings and studied the properties of them. Das and etc. [5] studied some of the basic properties of the soft real set and soft real numbers. Zorlutuna and etc. [27] provided the relation between fuzzy topology and soft topology. Notion of fixed point for a soft function was defined by Wardowski [26] in 2013 and proved a new fixed-point result for a soft function in a soft compact soft Hausdorff soft topological space. Varol and etc. [25] introduced the concepts soft convergence of sequences, soft homeomorphism and investigate the relations between these concepts. Soft neighborhood properties were studied in a soft topological space by Nazmul and etc. [17]. Georgiou [6] discussed the properties concerning the soft continuity, the soft open maps, soft closed maps and the soft homeomorphism in 2014. soft compact set was defined on soft topological space by Ozturk and etc. [19] and discussed the properties of them. Soft separation axioms for different types of soft points were studied by several authors [7], [10] and [24]. Tietze extension theorem and Urysohn lemma for soft topological spaces were derived by Mondal and etc.[16] in 2015. In 2016, Atmaca [2] derived the one point soft compactification for soft topological spaces. Relation between all those soft separation axioms was discussed by Singh [22] in 2017. In the same year, Aras and etc. [8] studied about Soft continuity through soft points. Nordo [18] in 2020, presented the definition of family of soft functions which soft separates soft points from soft closed sets and derived the soft Embedding Lemma for soft topological spaces. Ramkumar and etc. [20] extended soft separation axioms to the soft product soft topological spaces in 2020. Second sectionconsists basic definitions and preliminaries. Third section of this article deals with tube lemma for soft topological spaces. In the fourth section, we derive the Tychonoff theorem for soft topological spaces which states that the arbitrary soft product of soft compact soft topological space is soft compact soft topological space. Fifth section deals with the properties of soft completely regular spaces. At the end, we discuss the Stone Cech compactification for soft topological spaces. Throughout this article, (X, τ, E_1) will denote soft topological space, where X and E denote an initial universe set and a set of parameters. All notations and symbols not explicitly defined here are adopted from the book chapter [18]

2. Finite Soft Product of soft compact soft topological spaces

Definition 2.1 ([15])**.** If $(F,\ E)$ is a soft set over X. Then $(F,\ E)$ is a soft point if

$$F(e)=\begin{cases}x, & \text{if } e=e_i,\\ \emptyset, & \text{if } e\neq e_i.\end{cases}$$

We denote this soft point $(F,\ E)$ as (x_{e_i},E).

Definition 2.2 ([15])**.** If $(x_e,\ E)$ is a soft point and if $(F,\ E)$ is a soft set such that $x_e(e_i)\subseteq F(e_i)$, for all $e_i\in E$, then we say that $(x_e,\ E)$ is soft belong to $(F,\ E)$ which is denoted as $(x_e,\ E)\,\tilde{\in}\,(F,\ E)$.

Definition 2.3 ([15])**.** Let $(F,\ E)$ and $(G,\ E)$ be two soft sets over X. Then $(F,\ E)$ is a soft subset of $(G,\ E)$ if $F(e)\subseteq G(e)$, for all $e\in E$. We denote this soft subset as $(F,\ E)\,\tilde{\subseteq}\,(G,\ E)$.

Definition 2.4 ([15])**.** Let $\{(F_i,\ E):i\in I\}\,\tilde{\subseteq}SS(X)_E$, where $SS(X)_E$ is the collection of all soft sets over X and I is an arbitrary index set. Then

(i) the soft union of $\{(F_i,\ E):i\in I\}$ is the soft set $(F,\ E)$, where F is the mapping defined as $F(e)=\cup\{F_i(e):i\in I\}$, for every $e\in E$ and is denoted as $(F,\ E)=\underset{i\in I}{\tilde{\cup}}(F_i,\ E)$.

(ii) the soft intersection of $\{(F_i,\ E):i\in I\}$ is the soft set $(F,\ E)$, where F is the mapping defined as $F(e)=\cap\{F_i(e):i\in I\}$, for every $e\in E$ and is denoted as $(F,\ E)=\underset{i\in I}{\tilde{\cap}}(F_i,\ E)$.

Definition 2.5 ([27])**.** Let $x_0\in X$ and $(T,E)\,\tilde{\in}\tau$. We write $x_0\tilde{\in}(T,E)$, if for every $e\in E$, $x_0\in T(e)$.

Theorem 2.6. *Let $x_0\in X$. If $x_0\in(F,E)$, for all non empty soft open sets (F,E) of (X,τ,E), then there is a soft homeomorphism from $SS(Y)_E$ to $SS(x_0\times Y)_{\psi(E)}$.*

Proof. Define $\phi(y)=x_0\times y$ and $\psi(e)=e\times e$ for all $y\in Y$ and $e\in E$ so that $\phi_\psi:SS(Y)_E\to SS(x_0\times Y)_{E\times E}$ is a soft function. Here $(x_0\times Y,\tau\times\sigma|_{x_0\times Y},E\times E)$ is considered as a soft subspace of $(X\times Y,\tau\times\sigma,E\times E)$.

Let (x_{ei},E), $(y_{ej},E)\,\tilde{\in}SP(Y)_E$ and $(x_{ei},E)\,\tilde{\neq}\,(y_{ej},E)$, for some $ei,ej\in E$. Then either $x\neq y$ or $ei\neq ej$, so that either $\phi(x)\neq\phi(y)$ or $\psi(ei)\neq\psi(ej)$. Let $\phi_\psi(x_{ei},E)\,\tilde{=}\,(F,E\times E)$, then

$$\begin{aligned}F(e\times e)&=\phi\left(x_{ei}\left(\psi^{-1}(e\times e)\right)\right)\\&=\phi(x_{ei}(e))\end{aligned}$$

$$= \begin{cases} \phi(x), & \text{if } e = ei, \\ \emptyset, & \text{if } e \neq ei, \end{cases}$$

$$= \begin{cases} x_0 \times x, & \text{if } e = ei, \\ \emptyset, & \text{if } e \neq ei. \end{cases}$$

Similarly, $\phi_\psi(y_{ej}, E) \tilde{=} (G, E \times E)$, then

$$\begin{aligned} G(e \times e) &= \phi\left(y_{ej}\left(\psi^{-1}(e \times e)\right)\right) \\ &= \phi(y_{ej}(e)) \\ &= \begin{cases} \phi(y), & \text{if } e = ej, \\ \emptyset, & \text{if } e \neq ej, \end{cases} \\ &= \begin{cases} x_0 \times y, & \text{if } e = ej, \\ \emptyset, & \text{if } e \neq ej. \end{cases} \end{aligned}$$

Clearly $\phi(x)$ and $\phi(y)$ are singleton sets. So that the soft points $\phi_\psi(x_{ei}, E)$ and $\phi_\psi(y_{ej}, E)$ are distinct soft points in

$$(x_0 \times Y, \tau \times \sigma|_{x_0 \times Y}, E \times E).$$

Suppose $(x_0 \times y_{ei \times ei}, E \times E)$ is a soft point in $(x_0 \times Y, \tau \times \sigma|_{x_0 \times Y}, E \times E)$. Then $(\phi_\psi)^{-1}(x_0 \times y_{ei \times ei}, E \times E) \tilde{=} (G, E)$, where

$$\begin{aligned} G(e) &= \phi^{-1}(x_0 \times y_{ei \times ei}(\psi(e))) \\ &= \phi^{-1}(x_0 \times y_{ei \times ei}(e \times e)) \\ &= \begin{cases} \phi^{-1}(x_0 \times y), & \text{if } (e \times e) = (ei \times ej), \\ \emptyset, & \text{if } (e \times e) \neq (ei \times ej), \end{cases} \\ &= \begin{cases} y, & \text{if } (e \times e) = (ei \times ej), \\ \emptyset, & \text{if } (e \times e) \neq (ei \times ej). \end{cases} \end{aligned}$$

That is, for any soft point $(x_0 \times y_{ei \times ei}, E \times E)$ in $x_0 \times Y$, there is a $(G, E) = (y_{e_i}, E) \tilde{\in} SP(Y)$ such that $\phi_\psi((y_{e_i}, E)) \tilde{=} (x_0 \times y_{ei \times ei}, E \times E)$, where $e = e_i = e_j$. Thus ϕ_ψ is a soft bijective function from $SS(Y)_E$ to $SS(x_0 \times Y)_{\psi(E)}$.

If $(G, E) \tilde{\in} \sigma$ such that $G(e) = V_e (\subseteq Y)$, then $\phi_\psi(G, E) \tilde{=} (H, \psi(E))$, where $H(e \times e) = \phi\left(G\left(\psi^{-1}(e \times e)\right)\right) = \phi(G(e)) = x_0 \times V_e$ for $e \in E$. We know that $x_0 \times V_e = (X \times V_e) \cap (x_0 \times Y)$. That is $(H, \psi(E)) \tilde{=} [(X, E) \times (G, E)] \tilde{\cap} (x_0 \times Y)$, where $(X, E) \tilde{\in} \tau$. So that $\phi_\psi(G, E)$ is soft open function.

If $(H, \psi(E)) \tilde{\in} \tau \times \sigma_{|x_0 \times Y}$ and $H(e \times e) = x_0 \times v_e$ in $(x_0 \times Y, \tau \times \sigma_{x_0 \times Y}$, $\psi(E))$, then $H(e \times e) = [u_e \times v_e] \cap (x_0 \times Y)$ for $e \in E$, where $(T, E) \tilde{\in} \tau$

and $(S,E)\,\tilde{\in}\sigma$ and $T(e)=u_e$ and $S(e)=v_e$. $(\phi_\psi)^{-1}(H,\psi(E))\,\tilde{=}\left(\phi^{-1}_{\psi^{-1}}\right)(H,\psi(E))\,\tilde{=}\,(F,E)$ where $F(e)=\phi^{-1}(H(\psi(e)))=\phi^{-1}(H(e\times e))=\phi^{-1}(x_0\times v_e)=v_e$, because by hypothesis, $x_0\in(F,E)$, for all non empty soft open sets (F,E). ϕ_ψ is a soft homeomorphism from $SS(Y)_E$ to $SS(x_0\times Y)_{\psi(E\times E)}$. □

In the above theorem 2.6, we can not omit the condition "$x_0\in(F,E)$, for all non empty soft open sets (F,E)", If we omit the above condition, soft continuity of ϕ_ψ would fail. Following example shows it.

Example 2.7. Let $x_0\in X$ be fixed. Consider (X,τ,E) and (Y,σ,E), where $E=\{e_1,e_2,e_3\}$, $\tau=\{(\emptyset,E),(X,E),(V,E)\}$, $\sigma=\{(\emptyset,E),(Y,E),(U,E)\}$,

$$(V,E)=\begin{cases}X, & \text{if } e=e_1,\\ \emptyset, & \text{if } e=e_2,\\ \emptyset, & \text{if } e=e_3,\end{cases}$$

and

$$(U,E)=\begin{cases}U_{e_1}, & \text{if } e=e_1,\\ U_{e_2}, & \text{if } e=e_2,\\ \emptyset, & \text{if } e=e_3.\end{cases}$$

Note that $U_{e_1},U_{e_2}\subseteq Y$. Define a soft function $\phi_\psi: SS(Y)_E\to SS(x_0\times Y)_{E\times E}$ induced by $\phi: Y\to x_0\times Y$ and $\psi: E\to E\times E$, where $\phi(y)=x_0\times y$ and $\psi(e)=e\times e$.

Take $(V,E)\,\tilde{\in}\tau$ and $(U,E)\,\tilde{\in}\sigma$. Then their soft product is $(V,E)\times(U,E)\,\tilde{=}\,(F,E)$ where

$$F(e_i\times e_j)=\begin{cases}X\times U_{e_1}, & \text{if } e_i\times e_j=e_1\times e_1,\\ X\times U_{e_2}, & \text{if } e_i\times e_j=e_1\times e_2\\ \emptyset, & \text{if } e_i\times e_j=e_1\times e_3,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_1,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_2\ \ \tilde{\in}\tau\times\sigma \text{ in } X\times Y,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_3,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_1,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_2,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_3.\end{cases}$$

Then $(F,E)\tilde{\cap}(x\times Y, E\times E)\tilde{=}(G,E)$ where

$$G(e_i\times e_j)=\begin{cases} x\times U_{e_1} & \text{if } e_i\times e_j=e_1\times e_1\\ x\times U_{e_2}, & \text{if } e_i\times e_j=e_1\times e_2,\\ \emptyset, & \text{if } e_i\times e_j=e_1\times e_3,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_1,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_2,\\ \emptyset, & \text{if } e_i\times e_j=e_2\times e_3,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_1,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_2,\\ \emptyset, & \text{if } e_i\times e_j=e_3\times e_3, \end{cases}$$

is a soft open set in $x\times Y$. Now $(\phi_\psi)^{-1}(G,E)\tilde{=}(H,E)$ where

$$H(e)=\begin{cases} U_{e_1}, & \text{if } e=e_1,\\ \emptyset, & \text{if } e=e_2\ \tilde{\notin}\sigma \text{ in } (Y,E),\\ \emptyset, & \text{if } e=e_3. \end{cases}$$

That inverse image of soft open set (G,E) is (H,E) which is not a soft open set, so that ϕ_ψ is not a soft continuous map.

Theorem 2.8 (Tube Lemma for Soft Topological Space)**.** *Let $(X\times Y,\tau\times\sigma,E\times E)$ be the soft product of (X,τ,E) and a soft compact space (Y,σ,E). Let $x_0\in(F,E)$ for all $(F,E)\in\tau$ except $(\emptyset,E)$ and $(N,E\times E)$ be any soft open set in $X\times Y$ containing $(x_0\times Y,E\times E)$, then there exists a $(G,E)\,\tilde{\in}\tau$ containing (x_0,E) such that*

$$(x_0\times Y,E\times E)\,\tilde{\subseteq}\,(G,E)\times(Y,E)\,\tilde{\subseteq}\,(N,E\times E).$$

Proof. Let $(X\times Y,\tau\times\sigma,E\times E)$ be the soft product space and $(N,E\times E)\tilde{\in}\tau\times\sigma$ in $X\times Y$ containing $(x_0\times Y,E\times E)$. For each $(x_0\times y_{e_i\times e_j},E\times E)\tilde{\in}(x_0\times Y,E\times E)$, we can find a $\left(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E\right)\tilde{\in}\tau\times\sigma$ such that

$$\left(x_0\times y_{e_i\times e_j},E\times E\right)\tilde{\in}\left(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E\right)\tilde{\subseteq}(N,E\times E).$$

Now the collection

$$\left\{\left(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E\right):\left(x_0\times y_{e_i\times e_j},E\times E\right)\tilde{\in}(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E)\right\}$$

is a soft open cover for $(x_0\times Y,E\times E)$.

$$\Big\{(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E)\tilde{\cap}(x_0\times Y,E\times E):\left(x_0\times y_{e_i\times e_j},E\times E\right)\\ \tilde{\in}\left(U_{x_{0e_i}}\times V_{y_{e_j}},E\times E\right)\tilde{\in}\tau\Big\}$$

is the soft open cover for $(x_0 \times Y, \psi(E))$. Since the soft compact space (Y, σ, E) is soft homeomorphic with $(x_0 \times Y, \psi(E))$, $(x_0 \times Y, \psi(E))$ is a soft compact space. So that we can find finite number of $e_i'^s$ and $e_j'^s$ such that

$$\tilde{\cup}\Big\{ \Big(U_{x_{0e_i}} \times V_{y_{e_j}}, E \times E\Big) \tilde{\cap} (x_0 \times Y, E \times E) : \big(x_0 \times y_{e_i \times e_j}, E \times E\big) \tilde{\in} \Big(U_{x_{0e_i}} \times V_{y_{e_j}}, E \times E\Big) \Big\} \tilde{\supseteq} (x_0 \times Y, \psi(E)),$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Let

$$(F, E) \doteq \tilde{\bigcap_{i=1}^{n}} \Big(U_{x_{0e_i}}, E\Big).$$

Then $(F, E) \tilde{\in} \tau$ and $(x_0, E) \tilde{\in} (F, E)$.

$$(x_0 \times Y, E \times E) \tilde{\subseteq} (F, E) \times (Y, E) \doteq (F \times Y, E \times E) \tilde{\subseteq} (N, E \times E).$$

That is, $x_0 \in F(e_i)$ for all $e_i \in E$.

Let $(x_0 \times y_{e_k \times e_l}, E \times E) \tilde{\in} (x_0 \times Y, E \times E)$ where $(y_{e_l}, E) \tilde{\in} (Y, E)$. There exists some e_j and for all e_i,

$$\begin{aligned}(x_0 \times y_{e_k \times e_l}, E \times E) \tilde{\in} & \Big[\Big(U_{x_{0e_i}}, E\Big) \times (v_{y_{e_j}}, E)\Big] \tilde{\cap} (x_0 \times Y, \psi(E)) \\ & \doteq \Big(U_{x_{0e_i}} \times V_{y_{e_j}},\ E \times E\Big) \tilde{\cap}\ (x_0 \times Y,\ \psi(E)),\end{aligned}$$

where $(x_0, E) \tilde{\in} \Big(U_{x_{0e_i}}, E\Big)$ for all $i = 1, 2, \ldots, n$. Then $(x_0, E) \tilde{\in} (F, E)$, so that $(x_0 \times y_{e_k \times e_l}, E \times E) \tilde{\in} (F, E) \times (Y, E)$. Also each $(U_{x_{0e_i}} \times V_{y_{e_j}}, E \times E) \tilde{\subseteq} (N, E \times E)$, we have $(F, E) \times (Y, E) \tilde{\subseteq} (N, E \times E)$. □

Theorem 2.9. *Finite soft product of soft compact soft topological spaces is a soft compact soft topological space under soft product topology.*

Proof. Consider a soft open cover $\{(U_\alpha, E \times E)\}_{\alpha \in I}$ for $(X \times Y, \tau \times \sigma, E \times E)$. Fix $\big(x_{0_{e_i}}, E\big) \tilde{\in} (X, E)$. For $y \in Y$, if

$$\Big(x_0 \times y_{e_i \times e_j}, E \times E\Big) \tilde{\in} (X \times Y, E \times E),$$

we can find some $\alpha_k \in I$ such that

$$\Big(x_0 \times y_{e_i \times e_j}, E \times E\Big) \tilde{\in} (U_{\alpha_k}, E \times E).$$

There exists $\big(V_{e_j}, E\big) \tilde{\in} \tau$ containing $\big(x_{0_{e_i}}, E\big)$ and $\big(W_{e_j}, E\big) \tilde{\in} \sigma$ containing $\big(y_{e_j}, E\big)$ such that

$$\Big(x_0 \times y_{e_i \times e_j}, E \times E\Big) \tilde{\in} \big(V_{e_j} \times W_{e_j}, E \times E\big) \tilde{\subseteq} (U_{\alpha_k}, E \times E).$$

Let

$$\mathcal{W} = \Big\{ \big(W_{e_j}, E\big) : \Big(x_0 \times y_{e_i \times e_j}, E \times E\Big)\ \tilde{\in} (V_{e_j} \times W_{e_j}, E \times E)$$

$$\tilde{\subseteq} \ (U_{\alpha_k}, E \times E) \ \text{ for some } \alpha_k \in I \text{ and } y \in Y \Big\}.$$

Then $\mathcal{W}$ is a soft open cover for (Y, E). Since (Y, σ, E) is a soft compact space, we can find $e_{j1}, e_{j2}, \ldots, e_{jn} \in E$ such that

$$(Y, E) \tilde{=} \overset{n}{\underset{l=1}{\tilde{\cup}}} (W_{e_{jl}}, E).$$

Take

$$\left(V_{(x_0,e_i)}, E\right) \tilde{=} \overset{n}{\underset{l=1}{\tilde{\cap}}} \left(V_{e_{jl}}, E\right).$$

Since each $\left(V_{(x_0,e_i)}, E\right) \tilde{\in} \tau$. $\mathcal{V} = \{(V_{(x,e_i)}, E) : (x_{e_i}, E) \tilde{\in} (X, E)\}$ is a soft open cover for (X, E). Then we can find $i1, i2, \ldots, im$ and

$$(X, E) \tilde{=} \overset{m}{\underset{k=1}{\tilde{\cup}}} \left(V_{(x,e_ik)}, E\right).$$

Consider the collection $\mathcal{H} = \{\left(V_{(x,e_ik)}, E\right) \times \left(W_{e_{jl}}, E\right) : k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n\}$ in $(X \times Y, \tau \times \sigma, E \times E)$. Let

$$\Big(x \times y_{e_i \times e_j}, E \times E\Big) \tilde{\in} SP\left(X \times Y\right)_{E \times E}.$$

Since $\overset{m}{\underset{k=1}{\tilde{\cup}}} \left(V_{(x,e_ik)}, E\right) \tilde{=} (X, E)$, there exists i_k for some $k \in \{1, 2, \ldots, m\}$ and $(x_{e_i}, E) \tilde{\subseteq} \left(V_{(x,e_ik)}, E\right)$. Similarly there exists jl for some $l \in \{1, 2, \ldots, n\}$ and $\left(y_{e_j}, E\right) \tilde{\subseteq} \left(W_{e_{jl}}, E\right)$, because $(Y, E) \tilde{=} \overset{n}{\underset{l=1}{\tilde{\cup}}} \left(W_{e_{jl}}, E\right)$. Then $(x_{e_i}, E) \times \left(y_{e_j}, E\right) \tilde{\in} \left(V_{(x,e_ik)}, E\right) \times \left(W_{e_{jl}}, E\right)$ for some $l \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. That is

$$\Big(x \times y_{e_i \times e_j}, E \times E\Big) \tilde{\in} (V_{(x,e_ik)} \times W_{e_{jl}}, E \times E),$$

for some $l \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. This means

$$(X \times Y, E \times E) \tilde{=} \overset{}{\underset{k,l}{\tilde{\cup}}} \left(V_{(x,e_ik)} \times W_{e_{jl}}, E \times E\right).$$

Let $\left(V_{(x,e_ik)}, E\right) \times \left(W_{e_jl}, E\right) \tilde{\in} \mathcal{H}$, there exist $i1, i2, \ldots, im$ and

$$\left(V_{(x,e_ik)}, E\right) \tilde{=} \overset{m}{\underset{k=1}{\tilde{\cap}}} \left(V_{e_j}, E\right).$$

Also we can find $\left(W_{e_j}, E\right)$ and

$$\left(V_{(x,e_ik)}, E\right) \times \left(W_{e_jl}, E\right) \tilde{\subseteq} (V_{e_j} \times W_{e_j}, E \times E) \tilde{\subseteq} \left(U_{\alpha_k}, E \times E\right),$$

for some $\alpha_k \in I$. □

3. Tychonoff Theorem for Soft Topological Spaces

Lemma 3.1. *Let $\mathscr{A} \subseteq SS(X)_E$ having finite soft intersection property. Then $\mathscr{A}$ has a maximal element.*

Proof. Let $\mathscr{F} = \{\mathscr{B} \subseteq SS(X)_E : \mathscr{B} \supseteq \mathscr{A}$ and $\mathscr{B}$ has finite soft intersection property$\}$. Clearly $\mathscr{F}$ is partial ordered by the soft inclusion relation "$\tilde{\subseteq}$". Let $\mathscr{G} \subset \mathscr{F}$ be a simply ordered set and $\mathscr{D} = \underset{\mathscr{C} \in \mathscr{G}}{\cup} \mathscr{C}$. If $\mathscr{C} \in \mathscr{G}$, then $\mathscr{C} \in \mathscr{F}$ which means $\mathscr{C} \supseteq \mathscr{A}$ and $\mathscr{C}$ has finite soft intersection property. So $\mathscr{D} \supseteq \mathscr{A}$. Let $(G_1, E), (G_2, E), \ldots, (G_n, E)$ be any finite number of members of $\mathscr{D}$, then there exists some $\mathscr{C}_i \in \mathscr{G}$ such that $(G_i, E) \in \mathscr{C}_i$. For each $i \in \{1, 2, \ldots, n\}$, $\mathscr{C}_i \subseteq SS(X)_E$ with finite soft intersection property such that $\mathscr{C}_i \supseteq \mathscr{A}$. Also $\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$ are taken from a simply ordered set $\mathscr{G}$. So there exists some $k \in \{1, 2, \ldots, n\}$ such that $\mathscr{C}_i \tilde{\subseteq} \mathscr{C}_k$ for all $i \in \{1, 2, \ldots, n\}$ and $\overset{n}{\underset{i=1}{\tilde{\cup}}} \mathscr{C}_i = \mathscr{C}_k = \mathscr{D}$. This means that $(G_1, E), (G_2, E), \ldots, (G_n, E)$ are the elements of $\mathscr{C}_k$. Since $\mathscr{C}_k$ has finite soft intersection property, $(G_1, E) \tilde{\cap} (G_2, E) \tilde{\cap} \cdots \tilde{\cap} (G_n, E) \neq (\emptyset, E)$. So $\mathscr{D}$ has finite soft intersection property. Thus, every simply ordered subset $\mathscr{G}$ of the partially ordered set $\mathscr{F}$ has an upper bound $\mathscr{D}$. By Zorn's lemma, we can find a maximal element $\mathscr{H}$ which contains $\mathscr{A}$ and having finite soft intersection property. □

Lemma 3.2. *Let $\mathscr{A} \subseteq SS(X)_E$ and $\mathscr{A}$ having finite soft intersection property. If $\mathscr{F}$ is a maximal element with the property of containing $\mathscr{A}$ and having finite soft intersection property, then*

(i) $\overset{n}{\underset{i=1}{\tilde{\cap}}} (A_i, E) \tilde{\in} \mathscr{F}$, *for each* $(A_i, E) \tilde{\in} \mathscr{F}$, *where* $i \in \{1, 2. \ldots, n\}$,

(ii) *If* $(A, E) \tilde{\cap} (T, E) \tilde{\neq} \phi$ *for every* $(T, E) \tilde{\in} \mathscr{F}$, *then* $(A, E) \tilde{\in} \mathscr{F}$.

Proof. (i). Let $(A, E) \tilde{=} \overset{n}{\underset{i=1}{\tilde{\cap}}} (A_i, E)$ where $(A_i, E) \tilde{\in} \mathscr{F}$ for all i. Let $\mathscr{B} = \mathscr{F} \cup \{(A, E)\}$. For $j = 1, 2, \ldots, m$, choose $(B_j, E) \in \mathscr{B}$.

If $(B_j, E) \tilde{\neq} (A, E)$ for all j, then $\overset{m}{\underset{j=1}{\tilde{\cap}}} (B_j, E) \tilde{\neq} (\emptyset, E)$ because $\mathscr{F}$ has finite soft intersection property.

If $(B_j, E) \tilde{=} (A, E)$ for some $j (= k \; say)$, then

$$\begin{aligned} \overset{m}{\underset{j=1}{\tilde{\cap}}} (B_j, E) &\tilde{=} \overset{m}{\underset{j=1 \& j \neq k}{\tilde{\cap}}} (B_j, E) \tilde{\cap} (A, E) \\ &\tilde{=} \overset{m}{\underset{j=1 \; \& \; j \neq k}{\tilde{\cap}}} (B_j, E) \tilde{\cap} \overset{n}{\underset{i=1}{\tilde{\cap}}} (A_i, E). \end{aligned}$$

Since m and n are finite and $\mathscr{F}$ has finite soft intersection property, we have

$$\underset{j=1j\neq k}{\overset{m}{\tilde{\cap}}}(B_j,E)\,\tilde{\cap}\,\overset{n}{\underset{i=1}{\tilde{\cap}}}(A_i,E)\,\tilde{\neq}\,(\emptyset,E)\,.$$

In either cases, $(A,E)\in\mathscr{F}$, because $\mathscr{F}$ is maximal element.

(ii). Let $(A,E)\,\tilde{\cap}\,(T,E)\,\tilde{\neq}\phi$ for every $(T,E)\,\tilde{\in}\mathscr{F}$. Let $\mathscr{B}=\mathscr{F}\cup\{(A,E)\}$. For $j=1,2,\ldots,m$, choose $(C_j,E)\in\mathscr{B}$.

If $(C_j,E)\,\tilde{\neq}\,(A,E)$ for all j, then $\overset{m}{\underset{j=1}{\tilde{\cap}}}(C_j,E)\,\tilde{\neq}\,(\emptyset,E)$ because $\mathscr{F}$ has finite soft intersection property.

If $(C_j,E)\,\tilde{=}(A,E)$ for some $j(=k\ say)$, then

$$\overset{m}{\underset{j=1}{\tilde{\cap}}}(C_j,E)\,\tilde{=}\,\underset{j=1\ \&\ j\neq k}{\overset{m}{\tilde{\cap}}}(C_j,E)\,\tilde{\cap}\,(A,E)\,.$$

By part (i), $\overset{m}{\underset{j=1j\neq k}{\tilde{\cap}}}(C_j,E)\in\mathscr{F}$. Since (A,E) soft intersects every member of $\mathscr{F}$,

$$\overset{m}{\underset{j=1j\neq k}{\tilde{\cap}}}(C_j,E)\,\tilde{\cap}\,(A,E)\,\tilde{\neq}\,(\emptyset,E)\,.$$

This proves that $(F,E)\in\mathscr{F}$. □

Theorem 3.3 (Tychonoff theorem for soft topological spaces)**.** *If $\{(X_\alpha,\tau_\alpha,E):\alpha\in J\}$ is a collection of soft compact spaces. then*

$$\left(\prod X_\alpha,\prod\tau_\alpha,\prod E\right),$$

is a soft compact space under the soft product topology.

Proof. Let

$$\mathscr{A}=\left\{\left(U_\gamma,\prod E\right)\in SS(\prod X_\alpha)_{[\prod E]}:\gamma\in J\right\},$$

be a collection of soft sets having finite soft intersection property. Then the collection

$$\mathscr{B}=\left\{\overline{(U_\gamma,E)}:(U_\gamma,E)\in\mathscr{A}\right\},$$

is a family of soft closed subsets of

$$\left(\prod X_\alpha,\prod\tau_\alpha,\prod E\right),$$

with finite soft intersection property. By the lemma 3.1, we can find $\mathscr{C}$ of soft subsets of

$$\left(\prod X_\alpha,\prod\tau_\alpha,\prod E\right),$$

such that $\mathscr{C}$ is maximal with respect to $\mathscr{C}\supseteq\mathscr{A}$ and having finite soft intersection property.

Fix $\beta \in J$, consider the soft projection map

$$(\rho_p)_\beta =: SS\left(\prod X_\alpha\right)_{\prod E} \to SS(X_\beta)_E,$$

induced by the mappings $\rho_\beta : \prod X_\alpha \to X_\beta$ and $p_\beta : \prod E \to E$. Take

$$\mathscr{D} = \left\{(\rho_p)_\beta\left(C, \prod E\right) : \left(C, \prod E\right) \in \mathscr{C}\right\}.$$

Note that $\mathscr{D}$ is a collection of soft subsets of a soft compact space (X_β, τ_β, E). Suppose there exist $(C_1, \prod E), (C_2, \prod E), \ldots, (C_n, \prod E)$ such that

$$\begin{aligned}&(\rho_p)_\beta\left[\left(C_1, \prod E\right) \tilde{\cap} (\rho_p)_\beta\left(C_2, \prod E\right) \tilde{\cap} \ldots \tilde{\cap} (\rho_p)_\beta\left(C_n, \prod E\right)\right]\\ &\quad = (\emptyset, E).\end{aligned}$$

$$\begin{aligned}&\left(\rho_\beta(C_1), p_\beta\left(\prod E\right)\right) \tilde{\cap} \left(\rho_\beta(C_2), p_\beta\left(\prod E\right)\right) \tilde{\cap} \ldots\\ &\quad \tilde{\cap} \left(\rho_\beta(C_n), p_\beta\left(\prod E\right)\right)\\ &= (\emptyset, E).\end{aligned}$$

$$(\rho_\beta(C_1), E) \tilde{\cap} (\rho_\beta(C_2), E) \tilde{\cap} \ldots \tilde{\cap} (\rho_\beta(C_n), E) = (\emptyset, E).$$

$${(\rho_p)_\beta}^{-1}\left[(\rho_\beta(C_1), E) \tilde{\cap} (\rho_\beta(C_2), E) \tilde{\cap} \ldots \tilde{\cap} (\rho_\beta(C_n), E)\right] = {(\rho_p)_\beta}^{-1}(\emptyset, E).$$

$$\begin{aligned}&\left({\rho_\beta}^{-1}(\rho_\beta(C_1)), {p_\beta}^{-1}(E)\right) \tilde{\cap} \left({\rho_\beta}^{-1}(\rho_\beta(C_2)), {p_\beta}^{-1}(E)\right) \tilde{\cap} \ldots\\ &\quad \tilde{\cap} \left({\rho_\beta}^{-1}(\rho_\beta(C_n)), {p_\beta}^{-1}(E)\right)\\ &= \left({\rho_\beta}^{-1}(\emptyset), {p_\beta}^{-1}(E)\right).\end{aligned}$$

So

$$\left(C_1, \prod E\right), \left(C_2, \prod E\right), \ldots, \left(C_n, \prod E\right) = \left(\emptyset, \prod E\right).$$

Thus $\mathscr{D}$ has finite soft intersection property so that

$$\tilde{\mathscr{D}} = \left\{\overline{(\rho_p)_\beta\left(C, \prod E\right)} : \left(C, \prod E\right) \in \mathscr{C}\right\},$$

is a collection of soft closed subsets of (X_β, τ_β, E) with finite soft intersection property. Since (X_β, τ_β, E) is soft compact space,

$$\tilde{\bigcap_{(D,E)\in\mathscr{D}}} \overline{(D, E)} = \tilde{\bigcap_{(C,\prod E)\in\mathscr{C}}} \overline{(\rho_p)_\beta\left(C, \prod E\right)} \tilde{\neq} \left(\emptyset, \prod E\right).$$

Take

$$\left(x_{\beta_{e_i}}, E\right) \tilde{\in} \tilde{\bigcap}_{(C,\prod E)\in\mathscr{C}} \overline{(\rho_p)_\beta \left(C, \prod E\right)}.$$

Choose a soft point

$$\left(x_{\prod e_i}, \prod E\right) \tilde{\in} SS(\prod X_\alpha)_{\prod E},$$

such that

$$(\rho_p)_\beta \left(x_{\prod e_i},\ \prod E\right) \tilde{=}\ \left(x_{\beta_{e_i}}, E\right),$$

for each $\beta \in I$.

If $\delta \in J$ and if (V_δ, E) is a soft open set containing the soft point $\left(x_{\delta_{e_j}}, E\right)$ in $(X_\delta, \tau_\delta, E)$, then $(\rho_p)_\delta^{-1}(V_\delta, E)$ is a subbasic soft open set containing the point $\left(x_{\prod e_i}, \prod E\right)$ in $(\prod X_\alpha, \prod \tau_\alpha, \prod E)$. Since

$$\left(x_{\delta_{e_j}}, E\right) \tilde{\in} \tilde{\tilde{\bigcap}}_{(C,\prod E)\tilde{\in}\mathscr{C}} \overline{(\rho_p)_\delta \left(C, \prod E\right)}. (V_\delta, E) \tilde{\cap} (\rho_p)_\delta \left(C, \prod E\right) \tilde{\neq} (\emptyset, E).$$

Take a soft point

$$\left(y_{\delta_{e_k}}, E\right) \tilde{\in} (V_\delta, E) \tilde{\cap} (\rho_p)_\delta \left(C, \prod E\right).$$

Let $\left(y_{\prod e_k}, \prod E\right)$ be a soft point in $(\prod X_\alpha, \prod \tau_\alpha, \prod E)$ such that

$$(\rho_p)_\delta \left(y_{\prod \delta_k},\ \prod E\right) \tilde{=}\ \left(y_{\delta_{e_k}},\ E\right).$$

Now

$$(\rho_p)_\delta^{-1} \left(y_{\delta_{e_k}},\ E\right) \tilde{\in} (\rho_p)_\delta^{-1}\ \ (V_\delta,\ E) \tilde{\cap} (\rho_p)_\delta^{-1} \left((\rho_p)_\delta \left(C,\ \prod E\right)\right).$$

This means that

$$\left(y_{\prod e_k}, \prod E\right) \tilde{\in} (\rho_p)_\delta^{-1} \left(V_\delta, \prod E\right) \tilde{\cap} \left(C, \prod E\right).$$

Thus every subbasic soft open set of $(\rho_p)_\delta^{-1}(V_\delta, E)$ soft intersects $(C, \prod E)$, for all $(C, \prod E) \in \mathscr{C}$. By the previous lemma 3.2 $(\rho_p)_\delta^{-1}(V_\delta, E) \in \mathscr{C}$. Then every basic soft open set soft intersects every member of $\mathscr{C}$. Thus $\left(x_{\prod e_i},\ \prod E\right) \tilde{\in} \overline{(C,\ \prod E)}$, for all $(C,\ \prod E) \in\ \mathscr{C}$.

$$\left(x_{\prod\ e_i},\ \prod\ E\right) \tilde{\in} \tilde{\bigcap}_{(C,\ \prod\ E)\ \in\ \mathscr{C}} \overline{\left(C,\ \prod\ E\right)},$$

which means

$$\tilde{\bigcap}_{(C,\prod E)\in\mathscr{C}} \overline{\left(C, \prod E\right)} \tilde{\neq} \left(\emptyset, \prod E\right).$$

Thus the soft product space $(\prod X_\alpha, \prod \tau_\alpha, \prod E)$ is soft compact under the soft product topology. □

4. Soft Compact Extension of Soft Topological Spaces

Definition 4.1 ([23])**.** Let $(S,E)\,(\tilde{\in}SS(X)_E)$ is called as soft dense in X if $\overline{(S,E)}\tilde{=}\,(X,E)$

Definition 4.2 ([2])**.** A soft compact soft Hausdorff space (X_1,σ,E) is a soft compact extension (or) soft compactification of (X,τ,E) if (X_1,σ,E) contains (X,τ,E) as soft dense subspace.

Definition 4.3 ([16])**.** The fixed soft set on the set of all real numbers R is (F,E) such that $F\,(e)=U$, for all $e\in E$, where $U\subseteq R$. Take $\mathcal{B}=\{(G,E):(G,E)\,\tilde{\in}SS(R)_E \textit{ and } G(e)=(a,b)\,,\textit{for all } e\in E\ \&\ a,b\in R\}$. If $(x,E)\,\tilde{\in}\,(R,E)$, there exists an open interval (a,b) containing x such that $(x,E)\,\tilde{\in}\,((a,b)\,,E)$, where $(a,b)\,(e)=(a,b)$, for all $e\in E$. If $(x,E)\,\tilde{\in}\,((a,b)\,,E)\,\tilde{\cap}\,((c,d),E)$, then $(x,E)\,\tilde{\in}\,(((a,b)\,\tilde{\cap}\,(c,d))\,,E)$ which is also a member of $\mathcal{B}$. Thus $\mathcal{B}$ is a soft base for some soft topology τ on R. Then τ in R is called fixed soft topology. Note that if $(F,E)\in\tau$, then $F\,(e)=U$, for all $e\in E$, where U is either an open interval or union of open intervals.

Definition 4.4 ([5])**.** The soft arithmetic product on (R,τ,E) is defined as $.\psi : SS(R)_E\times SS(R)_E\to SS(R)_E$. If (S,E) and (T,E) are soft sets, then their soft arithmetic product is $.\psi\,((S,E)\,,(T,E))=(S.T,E)$, where

$$S.T\,(e)=S\,(e)\,.T\,(e)$$
$$=\begin{cases}\{x.y : x\in S(e) \text{ and } y\in T(e)\}, & \text{if } S(e)\neq\emptyset \text{ and } T(e)\neq\emptyset,\\ \emptyset, & vS(e)=\emptyset \text{ or } T(e)=\emptyset.\end{cases}$$

Similarly soft arithmetic sum is defined by replacing . by + in the above definition.

Lemma 4.5. *The soft arithmetic product defined on $SS(R)_E$ is soft continuous in R with fixed soft topology.*

Proof. Soft arithmetic product is defined as $.\psi : SS(R)_E\times SS(R)_E\to SS(R)_E$. Let (F,E) be any soft open set containing (x_0y_0,E). We can find the $\epsilon(>0)$ open interval $(x_0y_0-\epsilon,x_0y_0+\epsilon)$ such that

$$(x_0y_0,E)\,\tilde{\in}\,((x_0y_0-\epsilon,x_0y_0+\epsilon)\,,E)\,\tilde{\subseteq}\,(F,E)\,.$$

We have to find the δ_1 open interval for x_0 and δ_2 open interval for y_0 such that the $.\psi$ image of product of the δ_1 open interval for x_0 and δ_2 open interval for y_0 is soft contained in (F,E).

$$|xy-x_0y_0| = |xy-xy_0+xy_0-x_0y_0| \tag{4.1}$$
$$\leq |xy-xy_0|+|xy_0-x_0y_0|$$

$$= |x|\,|y - y_0| + |y_0|\,|x - x_0|\,.$$

The equation (4.1) is less than ϵ only if

$$|x - x_0| < \frac{\epsilon}{2\,|y_0|},$$
$$|y - y_0| < \frac{\epsilon}{2\,|x|},$$
$$x_0 - \frac{\epsilon}{2\,|y_0|} < x < x_0 + \frac{\epsilon}{2\,|y_0|},$$
$$y_0 - \frac{\epsilon}{2\,|x|} < y < y_0 + \frac{\epsilon}{2\,|x|}.$$

That is xy is in the ϵ-open set of x_0y_0, only if x is in the δ_1-open set of x_0 and δ_2-open set of y_0, where $\delta_1 = \frac{\epsilon}{2|y_0|}$ and $\delta_2 = \frac{\epsilon}{2|x|}$.

Let

$$(H,\ E) \tilde{=} \left(\left(x_0 - \frac{\epsilon}{2\,|y_0|},\ x_0 + \frac{\epsilon}{2\,|y_0|}\right),\ E\right) \tilde{\in} SS(R)_E,$$

and

$$(K,\ E) \tilde{=} \left(\left(y_0 - \frac{\epsilon}{2\,|x|}, y_0 + \frac{\epsilon}{2\,|x|}\right), E\right) \tilde{\in} SS(R)_E,$$

where

$$H(e) = \left(x_0 - \frac{\epsilon}{2\,|y_0|}, x_0 + \frac{\epsilon}{2\,|y_0|}\right),$$

and

$$K\,(e) = \left(y_0 - \frac{\epsilon}{2\,|x|}, y_0 + \frac{\epsilon}{2\,|x|}\right),$$

$$\cdot_\psi\,((H, E) \times (K, E)) \tilde{=} (H.K, E)\,\tilde{\subseteq}\,(F, E) \qquad \square$$

Theorem 4.6. *The soft arithmetic sum defined on $SS(R)_E$ is soft continuous in R with fixed soft topology.*

Definition 4.7. Let (x_e, E) be a soft point not contained in the soft closed set (F, E). Then (X, τ, E) is said to be soft completely regular if there exists a soft continuous function $\phi_\psi : SS(X)_E \to SS([0,1])_E$ induced by the functions $\phi : X \to [0,1]$ and $\psi : E \to E$ such that $\phi_\psi\,(x_e, E) \tilde{=} (1, E)$ and $\phi_\psi\,(F, E) \tilde{=} (0, E)$.

Example 4.8. All soft normal spaces and soft discrete soft topological spaces are soft completely regular.

Theorem 4.9. *A subspace of a soft completely regular space is soft completely regular.*

Proof. Let $\left(Y, \tau_{|_Y}, E\right)$ be a soft subspace of a soft completely regular space (X, τ, E). Let $(C, E) \tilde{\in} \tau^c_{|_Y}, E)$ and $(x_e, E) \tilde{\in} SS(Y)_E$ such that $(x_e, E) \tilde{\notin} (C, E)$. There exists $(F, E) \tilde{\in} \tau^c$ such that $(C, E) \tilde{=} (F, E) \tilde{\cap} (Y, E)$ and $(x_e, E) \tilde{\notin} (F, E)$. Since (X, τ, E) is soft completely regular, we can find a soft continuous function $\phi_\psi : SS(X)_E \to SS([0,1])_E$ induced by $\phi : X \to [0,1]$ and $\psi : E \to E$ such that $\phi_\psi (x_e, E) \tilde{=} (1, E)$ and $\phi_\psi (F, E) \tilde{=} (0, E)$. The restriction map $\phi_{\psi_{|_Y}} : SS(Y)_E \to SS([0,1])_E$ induced by $\phi_{|_Y} : Y \to [0,1]$ and $\psi : E \to E$ is soft continuous such that $\phi_{\psi_{|_Y}} (x_e, E) \tilde{=} (1, E)$ and $\phi_{\psi_{|_Y}} (F, E) \tilde{=} \phi_{\psi_{|_Y}} (C, E) \tilde{=} (0, E)$. □

Theorem 4.10. *Soft product of soft completely regular spaces is soft completely regular.*

Proof. Let $(\prod X_\alpha, \prod \tau_\alpha, \prod E)$ be the soft product of soft completely regular spaces $(X_\alpha, \tau_\alpha, E)$, where $\alpha \in I$. Let

$$\left(F, \prod E\right) \tilde{\in} \left(\prod \tau_\alpha\right)^c, \qquad \left(x_{\beta_{e_\beta}}, \prod E\right) \tilde{\in} \prod X_\alpha,$$

such that

$$\left(x_\beta, \prod E\right) \tilde{\cap} \left(F, \prod E\right) \tilde{=} \left(\emptyset, \prod E\right),$$

where $x_\beta = (x_{\beta_i})_{\beta \in I}$ and $e_\beta = (e_{\beta_i})_{\beta \in I}$. Consider the soft projection map

$$\pi_{\rho_\alpha} : SS\left(\prod X_\alpha\right)_{\prod E} \to SS\left(X_\alpha\right)_E,$$

induced by the mapping $\pi_\alpha : \prod X_\alpha \to X_\alpha$ and $\rho_\alpha : \prod E \to E$. Since

$$\left(x_{\beta_{e_\beta}}, \prod E\right) \tilde{\in} \left(F, \prod E\right)^c,$$

we can find a $(U_{\alpha_i}, E) \tilde{\in} \tau_{\alpha_i}$, for $i = 1, 2, \ldots, n$ such that

$$\left(x_\beta, \prod E\right) \tilde{\in} \tilde{\bigcap_{i=1}^{n}} \pi^{-1}_{\rho_{\alpha_i}} (U_{\alpha_i}, E) \tilde{\subseteq} \left(F, \prod E\right)^c.$$

Clearly,

$$\tilde{\bigcap_{i=1}^{n}} \pi^{-1}_{\rho_{\alpha_i}} (U_{\alpha_i}, E) \tilde{\cap} \left(F, \prod E\right) \tilde{=} \left(\emptyset, \prod E\right).$$

Now

$$\begin{aligned} \pi_{\rho_{\alpha_i}} \left(x_\beta, \prod E\right) &\tilde{=} \pi_{\alpha_i \rho_{\alpha_i}} \left(x_\beta, \prod E\right) \\ &\tilde{=} \left(\pi_{\alpha_i} (x_\beta), \rho_{\alpha_i} \left(\prod E\right)\right) \\ &\tilde{=} (x_{\alpha_i}, E) \tilde{\in} (U_{\alpha_i}, E) \in \tau_{\alpha_i}. \end{aligned}$$

Since $(X_{\alpha_i}, \tau_{\alpha_i}, E)$ is soft completely regular, we can find soft continuous mappings $\phi_{\psi_{\alpha_i}} : SS(X_{\alpha_i})_E \to SS([0,1])_E$ induced by the mappings $\phi_{\alpha_i} : X_{\alpha_i} \to [0,1]$ and $\psi_{\alpha_i} : E \to E$ such that

$$\phi_{\psi_{\alpha_i}}(x_{\alpha_i}, E) \tilde{=} (1, E), \qquad \phi_{\psi_{\alpha_i}}((U_{\alpha_i}, E)^c) \tilde{=} (0, E),$$

for $i = 1, 2, \ldots, n$. Since

$$\phi_{\psi_{\alpha_i}} : SS(X_{\alpha_i})_E \to SS([0,1])_E, \qquad \pi_{\rho_\alpha} : SS\left(\prod X_\alpha\right)_{\prod E} \to SS(X_\alpha)_E,$$

are soft continuous functions, their soft composition $\phi_{\psi_{\alpha_i}} \circ \pi_{\rho_\alpha}$ is also a soft continuous function. Let $\phi_\psi = \phi_{\psi_{\alpha_1}} \circ \pi_{\rho_{\alpha_1}} . \phi_{\psi_{\alpha_2}} \circ \pi_{\rho_{\alpha_2}} \ldots \phi_{\psi_{\alpha_n}} \circ \pi_{\rho_{\alpha_n}}$. By previous lemma 4.5, ϕ_ψ is a soft continuous function. Now

$$\begin{aligned}
\phi_\psi(x_\beta, \prod E) \tilde{=} & \phi_{\psi_{\alpha_1}} \circ \pi_{\rho_{\alpha_1}}\left(x_\beta, \prod E\right) . \phi_{\psi_{\alpha_2}} \circ \pi_{\rho_{\alpha_2}}\left(x_\beta, \prod E\right) \ldots \\
& \phi_{\psi_{\alpha_n}} \circ \pi_{\rho_{\alpha_n}}\left(x_\beta, \prod E\right) \\
\tilde{=} & \phi_{\alpha_1 \psi_{\alpha_1}}\left(\pi_{\alpha_1 \rho_{\alpha_1}}\left(x_\beta, \prod E\right)\right) . \phi_{\alpha_2 \psi_{\alpha_2}}\left(\pi_{\alpha_2 \rho_{\alpha_2}}\left(x_\beta, \prod E\right)\right) \\
& \ldots \phi_{\alpha_n \psi_{\alpha_n}}\left(\pi_{\alpha_n \rho_{\alpha_n}}\left(x_\beta, \prod E\right)\right) \\
\tilde{=} & \phi_{\alpha_1 \psi_{\alpha_1}}\left(\pi_{\alpha_1}(x_\beta), \rho_{\alpha_1}\left(\prod E\right)\right) \\
& . \phi_{\alpha_2 \psi_{\alpha_2}}\left(\pi_{\alpha_2}(x_\beta), \rho_{\alpha_2}\left(\prod E\right)\right) \\
& \ldots \phi_{\alpha_n \psi_{\alpha_n}}\left(\pi_{\alpha_n}(x_\beta), \rho_{\alpha_n}\left(\prod E\right)\right) \\
\tilde{=} & \phi_{\alpha_1 \psi_{\alpha_1}}(x_{\alpha_1}, E) .
\end{aligned}$$

$\phi_{\alpha_2 \psi_{\alpha_2}}(x_{\alpha_2}, E) \ldots \phi_{\alpha_n \psi_{\alpha_n}}(x_{\alpha_n}, E) = (1, E) . (1, E) \ldots (1, E) \tilde{=} (1, E)$ and if

$$\left(y_\beta, \prod E\right) \tilde{\notin} \tilde{\bigcap_{i=1}^{n}} \pi^{-1}_{\rho_{\alpha_i}}(U_{\alpha_i}, E) \tilde{\subseteq} \left(F, \prod E\right),$$

then

$$\begin{aligned}
\phi_\psi\left(y_\beta, \prod E\right) \tilde{=} & \phi_{\psi_{\alpha_1}} \circ \pi_{\rho_{\alpha_1}}\left(y_\beta, \prod E\right) . \phi_{\psi_{\alpha_2}} \circ \pi_{\rho_{\alpha_2}}\left(y_\beta, \prod E\right) \ldots \phi_{\psi_{\alpha_n}} \circ \\
& \pi_{\rho_{\alpha_n}}\left(y_\beta, \prod E\right) \\
\tilde{=} & \phi_{\alpha_1 \psi_{\alpha_1}}\left(\pi_{\alpha_1 \rho_{\alpha_1}}\left(y_\beta, \prod E\right)\right) . \phi_{\alpha_2 \psi_{\alpha_2}}\left(\pi_{\alpha_2 \rho_{\alpha_2}}\left(y_\beta, \prod E\right)\right) \\
& \ldots \phi_{\alpha_n \psi_{\alpha_n}}\left(\pi_{\alpha_n \rho_{\alpha_n}}\left(y_\beta, \prod E\right)\right) \\
\tilde{=} & \phi_{\alpha_1 \psi_{\alpha_1}}\left(\pi_{\alpha_1}(y_\beta), \rho_{\alpha_1}\left(\prod E\right)\right) \\
& . \phi_{\alpha_2 \psi_{\alpha_2}}\left(\pi_{\alpha_2}(y_\beta), \rho_{\alpha_2}\left(\prod E\right)\right)
\end{aligned}$$

$$\begin{aligned}&\ldots\phi_{\alpha_n\psi_{\alpha_n}}\left(\pi_{\alpha_n}\left(y_\beta\right),\rho_{\alpha_n}\left(\prod E\right)\right)\\ \tilde{=}&\phi_{\alpha_1\psi_{\alpha_1}}\left(y_{\alpha_1},E\right).\phi_{\alpha_2\psi_{\alpha_2}}\left(y_{\alpha_2},E\right)\ldots\phi_{\alpha_n\psi_{\alpha_n}}\left(y_{\alpha_n},E\right)\\ \tilde{=}&\left(0,E\right).\left(0,E\right)\ldots\left(0,E\right)\tilde{=}\left(0,E\right).\end{aligned}$$

Thus $(\prod X_\alpha, \prod \tau_\alpha, \prod E)$ is a soft completely regular space. □

Definition 4.11. Let $(S,E)\in SS(R)_E$. Then (S,E) is soft bounded if we can find $N\in R$ such that $s\leq N$, for all $s\in\{|x|\in S(e_i):$ for all $e_i\in E$

Example 4.12. Consider $E=\{e_1,e_2,e_3\}$. Let $(F,E)\tilde{\in}SS(\mathbb{R})_E$ such that

$$F\left(e\right)=\begin{cases}\left(0,1\right), & \text{if } e=e_1,\\ \left(-1,1\right), & \text{if } e=e_2,\\ \left\{-1,1\right\}, & \text{if } e=e_3.\end{cases}$$

We can choose $M=1\in\mathbb{R}$ such that $t\leq M$ for all $t\in\{x\in F\left(e_i\right):$ for all $e_i\in E\}$

Example 4.13. Let $E=\mathbb{N}$ and a a soft set (G,E) on $(\mathbb{R},E)$ such that

$$G\left(e\right)=\begin{cases}\left(-1,1\right), & \text{if } e=e_1,\\ \left(-2,2\right), & \text{if } e=e_2,\\ \left(-3,3\right), & \text{if } e=e_3,\\ \quad\vdots & \\ \left(-n,n\right), & \text{if } e=e_n.\\ \quad\vdots & \end{cases}$$

We can not find $M\in\mathbb{R}$ such that $t\leq M$ for all $t\in\{x\in F(e_i):$ for all $e_i\in E\}$

Definition 4.14. A soft function $\zeta_\eta : SS(X)_E\to SS(\mathbb{R})_E$ is soft bounded if we can find $N\in\mathbb{R}$ such that $s\leq N$ for all $s\in\{|\zeta(x)|:x\in X\}$.

Example 4.15. Let $E=\{e_1,e_2,e_3\}$. Let $\zeta_\eta : SS(\mathbb{R})_E\to SS(\mathbb{R})_E$, where $\zeta:\mathbb{R}\to\mathbb{R}$ and $\eta:E\to E$ defined by $\zeta(x)=x$ and $\eta(e)=e$. We can not find $N\in\mathbb{R}$ such that $s\leq N$ for all $s\in\{|\phi(x)|:x\in X\}$. So that ζ_η is not a soft bounded function. Consider another soft function $\zeta_\eta : SS(\mathbb{R})_E\to SS(\mathbb{R})_E$ induced by $\phi:\mathbb{R}\to\mathbb{R}$ and $\psi:E\to E$ by $\phi\left(x\right)=\frac{1}{x}$ and $\psi\left(e\right)=e$. We can choose any $N\geq 1\in\mathbb{R}$ such that $s\leq M$ for all $s\in\{|\phi(x)|:x\in X\}$. So that ζ_η is a soft bounded function.

Theorem 4.16. *If a soft function $\zeta_\eta : SS(X)_E\to SS(\mathbb{R})_E$ is soft bounded, then $\zeta_\eta\left(T,E\right)$ is soft bounded for all soft set $(T,E)\tilde{\in}SS(X)_E$.*

Theorem 4.17. *If $(T,E)\tilde{\in}SS(X)_E$ is soft bounded, then for each $e_i \in E$, $T(e_i)$ is soft bounded.*

Theorem 4.18. *Let (Z,σ,E_2) be a soft compact soft Hausdorff space. If $\phi_\psi : SS(X)_{E_1} \to SS(Z)_{E_2}$ is a soft embedding, then there is a soft compactification (Y,Σ,E_1) of (X,τ,E_1).*

Proof. Let $\phi_\psi : SS(X)_{E_1} \to SS(Z)_{E_2}$ be a soft embedding. Then (X,E_1) is soft homeomorphic with $\phi_\psi(X,E_1)$. Since (Z,σ,E_2) be a soft compact soft Hausdorff space, $\overline{\phi_\psi(X,E_1)}$ is soft closed and so it is a soft compact space. $\overline{\phi_\psi(X,E_1)}$ is a soft compactification of $\phi_\psi(X,E_1)$.

Let $(T,\psi(E_1))\tilde{=}\overline{\phi_\psi(X,E_1)}$ and $(A,\psi(E_1))\tilde{=}\overline{\phi_\psi(X,E_1)}-\phi_\psi(X,E_1)$. Let (B,E_1) be any soft set such that $\gamma_\delta : SS(B)_{E_1} \to SS(A)_{\psi(E_1)}$ is soft bijection induced by the functions $\gamma : B \to A$ and $\delta : E_1 \to E_2$, where γ is a bijective function and $\delta(e)=\psi(e)$. Take $(Y,E_1)\tilde{=}(X,E_1)\tilde{\cup}(B,E_1)$. Define a soft function $\zeta_\eta : SS(Y)_{E_1} \to SS(T)_{\psi_{(E_1)}}$ induced by the functions

$$\zeta(y)=\begin{cases}\phi(y), & \text{if } y\in X,\\ \gamma(y), & \text{if } y\in B,\end{cases}$$

and $\eta(e)=\psi(e)$, for all $e\in E_1$. Induce soft topology on (Y,E_1) by (U,E_1) is soft open in (Y,E_1) if and only if $\zeta_\eta(U,E_1)$ is soft open in $(T,\psi(E_1))$

Let $(x_{e_i},\ E_1)$, $\left(y_{e_j},E_1\right)\tilde{\in}SP(Y)_{E_1}$ such that $(x_{e_i},E_1)\ \tilde{\neq}\ \left(y_{e_j},E_1\right)$. Then either $x\neq y$ or $e_i\neq e_j$. If $e_i\neq e_j$, then $\psi(e_i)\neq\psi(e_j)$ so that

$$\left(\zeta(x)_{\eta(e_i)},\psi(E_1)\right)\tilde{\neq}\left(\zeta(y)_{\eta(e_j)},\psi(E_1)\right).$$

If $x\neq y$, and if $x,y\in X$, then $\phi(x)\neq\phi(y)$ which implies

$$\left(\zeta(x)_{\eta(e_i)},\psi(E_1)\right)\tilde{\neq}\left(\zeta(y)_{\eta(e_j)},\psi(E_1)\right).$$

If $x\neq y$, and if $x,y\in B$, then $\gamma(x)\neq\gamma(y)$ which implies

$$\left(\zeta(x)_{\eta(e_i)},\psi(E_1)\right)\tilde{\neq}\left(\zeta(y)_{\eta(e_j)},\psi(E_1)\right).$$

If $x\neq y$, and if $x\in X$ and $y\in B$ ($x\in B$ and $y\in X$), then

$$\phi(x)\neq\gamma(y)\,(\gamma(x)\neq\phi(y)),$$

implies

$$\left(\zeta(x)_{\eta(e_i)},\psi(E_1)\right)\tilde{\neq}\left(\zeta(y)_{\eta(e_j)},\psi(E_1)\right).$$

If $\left(y_{e_j},\psi(E_1)\right)\tilde{\in}SP(T,\psi(E_1))$, there exists $e_i\in E_1$ such that $\psi(e_i)=e_j$. Also we can find exactly one possibility of $x\in X$ or $x\in B$ such that $\phi(x)=y$ or $\gamma(x)=y$, because ϕ_ψ and γ_δ are soft bijective soft functions. In either cases $x\in Y$ and $e_i\in E_1$ such that $\zeta_\eta(x_{e_i},E_1)\tilde{=}\left(y_{e_j},\psi(E_1)\right)$.

By the definition of soft topology induced on (Y, E_1), ζ_η is a soft open and soft continuous function. Thus $\zeta_\eta : SS(Y)_{E_1} \to SS(T)_{\psi(E_1)}$ is a soft homeomorphism.

If $(U, E_1) \tilde{\in} \tau$, then $\phi_\psi(U, E_1)$ is soft open in $\phi_\psi(X, E_1)$. Since $\overline{\phi_\psi(X, E_1)}$ is a soft compactification of $\phi_\psi(U, E_1)$, there exists a soft open set $(H, \psi(E_1)) \tilde{\in} (T, \psi(E_1))$ such that $\phi_\psi(U, E_1) \tilde{=} (H, \psi(E_1)) \tilde{\cap} \phi_\psi(X, E_1)$. Since ζ_η is a soft homeomorphism, $\zeta_\eta^{-1}(H, \psi(E_1))$ is a soft open set and $(U, E_1) \tilde{=} \zeta_\eta^{-1}(H, \psi(E_1)) \tilde{\cap} (X, E_1)$. Thus (X, E_1) is a soft subspace of (Y, E_1) so that (Y, E_1) is a soft compactification of (X, E_1) such that ζ_η is a soft continuous function from $SS(Y)_{E_1}$ to $SS(T)_{\psi(E_1)}$ and $\zeta_\eta|_{(X,E_1)} = \phi_\psi$. □

Theorem 4.19. *Let (Z, σ, E_2) be a soft compact soft Hausdorff space. If $\phi_\psi : SS(X)_{E_1} \to SS(Z)_{E_2}$ is a soft embedding, then there is a unique soft compactification (Y, Σ, E_1) of (X, τ, E_1).*

Proof. By previous theorem 4.18, there is a soft compactification (Y, E_1) of (X, E_1). Let (Y_1, τ_1, E_1) and (Y_2, τ_2, E_1) be two soft compactification of (X, τ, E_1). By previous theorem 4.18, we can find is a soft homeomorphisms ζ_{η_1} from $SS(Y_1)_{E_1}$ to $SS(\overline{\phi_\psi(X, E_1)})_{\psi(E_1)}$ and ζ_{η_2} from $SS(Y_2)_{E_1}$ to $SS(\overline{\phi_\psi(X, E_1)})_{\psi(E_1)}$ such that $\zeta_{\eta_1}|_{(X,E_1)} = \zeta_{\eta_2}|_{(X,E_1)} = \phi_\psi$. Now $\zeta_{\eta_2}^{-1} \circ \zeta_{\eta_1}$ is a soft homeomorphism from $SS(Y_1)_{E_1}$ to $SS(Y_2)_{E_1}$. Thus the soft compactifications (Y_1, τ_1, E_1) and (Y_2, τ_2, E_1) are same. □

Theorem 4.20. *Let $(A, \tau|_A, E_1)$ be a soft subspace of (X, τ, E_1) and (Z, σ, E_2) be a soft Hausdorff space. Let $\phi_\psi : SS(A)_{E_1} \to SS(Z)_{E_2}$ be a soft continuous function. If there is a soft continuous function $\zeta_\eta : SS(\overline{A})_{E_1} \to SS(Z)_{E_2}$ such that $\zeta_\eta|_{(A,E_1)} = \phi_\psi$, then it is unique.*

Proof. Let $\zeta_{\eta_1} : SS(\overline{A})_{E_1} \to SS(Z)_{E_2}$ and $\zeta_{\eta_2} : SS(\overline{A})_{E_1} \to SS(Z)_{E_2}$ be two soft continuous functions such that $\zeta_{\eta_1}|_{(A,E_1)} = \zeta_{\eta_2}|_{(A,E_1)} = \phi_\psi$. Note that $SS(\overline{A})_{E_1}$ is the set of soft sets in $\left(\overline{A}, E_1\right)$. Let (x_{e_i}, E_1) be a soft point in (A, E_1) such that $\zeta_{\eta_1}(x_{e_i}, E_1) \tilde{\neq} \zeta_{\eta_2}(x_{e_i}, E_1)$. Since (Z, σ, E_2) is a soft Hausdorff space, we can find two disjoint soft open sets (V_1, E_2) and (V_2, E_2) of $\zeta_{\eta_1}(x_{e_i}, E_1)$ and $\zeta_{\eta_2}(x_{e_i}, E_1)$. Since ζ_{η_1} and ζ_{η_2} are soft continuous functions, there exist soft open sets (U_1, E_1) and (U_2, E_1) of (x_{e_i}, E_1) such that $\zeta_{\eta_1}(U_1, E_1) \tilde{\subseteq} (V_1, E_2)$ and $\zeta_{\eta_2}(U_2, E_1) \tilde{\subseteq} (V_2, E_2)$. Let $(U, E_1) \tilde{=} (U_1, E_1) \tilde{\cap} (U_2, E_1)$. Since (U, E_1) is soft open in $\overline{(A, E_1)}$ and (A, E_1) is soft dense in $\overline{(A, E_1)}$, $(U, E_1) \tilde{\cap} (A, E_1) \tilde{\neq} (\emptyset, E_1)$. Let $\left(y_{e_j}, E_1\right) \tilde{\in} (U, E_1) \tilde{\cap} (A, E_1)$. Since $\left(y_{e_j}, E_1\right) \tilde{\in} (A, E_1)$ and

$$\zeta_{\eta_1}|_{(A,E_1)} = \zeta_{\eta_2}|_{(A,E_1)} = \phi_\psi,$$

then

$$\zeta_{\eta_1}\left(y_{e_j}, E_1\right) \tilde{=} \zeta_{\eta_2}\left(y_{e_j}, E_1\right)$$

$$\tilde{=}\phi_\psi\left(y_{e_j}, E_1\right)\tilde{\in}\zeta_{\eta_1}\left(U, E_1\right)\tilde{\cap}\zeta_{\eta_2}(U, E_1)\tilde{\subseteq}\left(V_1, E_2\right)\tilde{\cap}\left(V_2, E_2\right),$$

which is a contradiction to (V_1, E_2) and (V_2, E_2) are disjoint soft open sets. □

Theorem 4.21. *Let (X, τ, E) be a soft completely regular space. Then there exists a soft compactification (Y, Σ, E) of (X, τ, E) having the property "If $\phi_\psi : SS(X)_E \to SS(\mathbb{R})_E$ is a soft bounded soft continuous function, there exists a soft bounded soft continuous function $\zeta_\eta : SS(Y)_E \to SS(\mathbb{R})_E$ such that $\zeta_\eta|_{(X,E)} = \phi_\psi$"*

Proof. Let $\left\{ \left(\phi_\psi\right)_\alpha : SS(X)_E \to SS(\mathbb{R})_E \ : \ \alpha \in I \right\}$ be the collection of all soft bounded soft continuous functions from $SS(X)_E$ to $SS(\mathbb{R})_E$. Then for each $\alpha \in I$, there exists $M_\alpha \in \mathbb{R}$ such that $t \leq M_\alpha$ for all $t \in \{ |\phi_\alpha(x)| : x \in x\}$. Choose $I_\alpha = [\ inf\ \phi_\alpha(x),\ sup\ \phi_\alpha(x)\ ]$. Define $\phi_\psi : SS(X)_E \to SS(\prod_{\alpha\in I} I_\alpha)_{\prod E}$ such that $\phi(x) = (\phi_\alpha(x))_{\alpha\in I}$ and $\psi(e) = (\psi_\alpha(e))_{\alpha\in I}$. By the Tychonoff theorem for soft topological spaces, $(\prod I_\alpha, \prod E)$ is a soft compact space. Since (X, τ, E) is a soft completely regular space,the collection $\left\{\left(\phi_\psi\right)_\alpha : SS(X)_E \to SS(\mathbb{R})_E : \alpha \in I\right\}$ seperates soft points and soft points from soft closed sets. Therefore by soft embedding lemma, $\phi_\psi : SS(X)_E \to SS(\prod_{\alpha\in I} I_\alpha)_{\prod E}$ is a soft embedding, where $\phi = \Delta_{\alpha\in I}\,\phi_\alpha : X \to \prod I_\alpha$ and $\psi = \Delta_{\alpha\in I}\,\psi_\alpha : E \to \prod E$. Then by the theorem 4.18, we can construct a soft compactification (Z, E) of (X, E). Also by theorem 4.18, there is a soft embedding $\zeta_\eta : SS(Z)_E \to SS(\prod I_\alpha)_{\prod E}$ such that $\zeta_\eta|_{(X,E)} = \phi_\psi$. If γ_δ is a soft bounded soft continuous function from $SS(X)_E$ to $SS(R)_E$, then there exists some $\beta \in I$ such that $\gamma_\delta = \left(\phi_\psi\right)_\beta$. If we consider a soft projection function $(\pi_\rho)_\beta : SS(\prod I_\alpha)_{\prod E} \to SS(I_\beta)_E$, then $(\pi_\rho)_\beta \circ \zeta_\eta : SS(Z)_E \to SS(I_\beta)_E$ is a soft continuous function, because it is a soft composition of two soft continuous functions. We have to prove that $((\pi_\rho)_\beta \circ \zeta_\eta)|_{(X,E)} = \gamma_\delta$. If (x_{e_i}, E) is a soft point in (X, E), then

$$(\pi_\rho)_\beta o\zeta_\eta\left(x_{e_i}, E\right)\tilde{=}(\pi_\rho)_\beta o\phi_\psi\left(x_{e_i}, E\right).$$

Let $\phi_\psi\left(x_{e_i}, E\right)\tilde{=}\left(F, E\right)$, where

$$F\left(\prod e_j\right) = \begin{cases} \phi((x_{e_i}(e_k))), & \text{if } e_k \in \psi^{-1}\left(\prod e_j\right), \\ \emptyset, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \phi(x), & \text{if } e_i \in \psi^{-1}\left(\prod e_j\right), \\ \emptyset, & \text{otherwise.} \end{cases}$$

If we take $(\pi_\rho)_\beta \circ \zeta_\eta (x_{e_i}, E) \tilde{=} (\pi_\rho)_\beta (F, E) \tilde{=} (G, E)$, then

$$
\begin{aligned}
G(e_{\beta_m}) &= \begin{cases} \pi_\beta (F(\prod e_j)), & \text{if } \prod e_j \in \rho_\beta^{-1}(e_{\beta_m}), \\ \emptyset, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \pi_\beta(\phi(x)), & \text{if } e_i \in \psi^{-1}(\prod e_j) \text{ and } \prod e_j \in \rho_\beta^{-1}(e_{\beta_m}), \\ \emptyset, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \pi_\beta(\phi(x)), & \text{if } e_i \in \psi^{-1}\left(\rho_\beta^{-1}(e_{\beta_m})\right), \\ \emptyset, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \phi_\beta(x), & \text{if } e_i \in \psi_\beta^{-1}(e_{\beta_m}), \\ \emptyset, & \text{otherwise,} \end{cases} \\
&= \phi_{\beta_{\psi_\beta}}(e_{\beta_m}).
\end{aligned}
$$

That is, $(\pi_\rho)_\beta \circ \zeta_\eta (x_{e_i}, E) \tilde{=} \phi_{\beta_{\psi_\beta}} (x_{e_i}, E)$. Thus

$$((\pi_\rho)_\beta o \zeta_\eta)|_{(X,E)} = (\phi_\psi)_\beta = \gamma_\delta. \qquad \square$$

If we take (X, τ, E) is soft completely regular space, then by Urysohn lemma for soft topological space, $\left\{(\phi_\psi)_\alpha : SS(X)_E \to SS([0,1])_E : \alpha \in I\right\}$ is the family of all soft continuous functions which seperates soft points and soft closed sets and the soft diagonal mapping $\Delta = \Delta_{\alpha \in I} (\phi_\psi)_\alpha$ is a soft embedding.

Theorem 4.22. *Let (Y, Σ, E_1) be a soft compact extension of (X, τ, E_1) and (C, σ, E_2) be a soft compact soft Hausdorff space. If $\gamma_\delta : SS(X)_{E_1} \to SS(C)_{E_2}$ is a soft continuous function, then there exists a soft continuous function $\zeta_\eta : SS(Y)_{E_1} \to SS(C)_{E_2}$ such that $\zeta_\eta|_{(X,E_1)} = \phi_\psi$.*

Proof. (C, E_2) be a soft compact soft Hausdorff space, then (C, E_2) is a soft normal space so that it is a soft completely regular space. Let $\left\{(\phi_\psi)_\alpha : \alpha \in I\right\}$ be the collection of all soft bounded soft continuous functions from $SS(C)_{E_2}$ to $SS([0,1])_{E_2}$. If $\phi_\psi : SS(C)_{E_2} \to SS(\prod [0,1])_{\prod E_2}$ is a soft function induced by $\phi(x) = (\phi_\alpha(x))_{\alpha \in I}$ and $\psi(x) = (\psi_\alpha(x))_{\alpha \in I}$, then by the above remark, $\phi_\psi : SS(C)_{E_2} \to SS(\prod [0,1])_{\prod E_2}$ is a soft embedding. If $\gamma_\delta : SS(X)_{E_1} \to SS(C)_{E_2}$ is a soft continuous function, then $(\phi_\psi)_\alpha \circ \gamma_\delta$ is a soft continuous function from $SS(X)_{E_1}$ to $SS([0,1])_{E_2}$. Also, $\phi_\psi \circ \gamma_\delta$ is a soft continuous function from $SS(X)_{E_1}$ to $SS(\prod [0,1])_{\prod E_2}$. Let $(\gamma_\delta)_\alpha = (\phi_\psi)_\alpha \circ \gamma_\delta$.

Since (X, E_1) is soft completely regular and (Y, E_1) is a soft compactification of (X, E_1), then by theorem 4.21, there is a soft continuous function $(\zeta_\eta)_\alpha$ from $SS(Y)_{E_1}$ to $SS([0,1])_{E_2}$ such that $(\zeta_\eta)_\alpha|_{(X,E_1)} = (\gamma_\delta)_\alpha$. Define $\zeta_\eta : SS(Y)_{E_1} \to SS(\prod [0,1])_{\prod E_2}$ induced by $\zeta : Y \to [0,1]$, $\zeta(x) = (\zeta_\alpha(x))_{\alpha\in I}$ and $\eta : E_1 \to E_2$, $\eta(x) = (\eta_\alpha(x))_{\alpha\in I}$. Then ζ_η is soft continuous. Now

$$\begin{aligned}\zeta_\eta(Y, E_1)\,\phi_{\beta_{\psi_\beta}} &= \zeta_\eta(\overline{(X, E_1)})\tilde{\subseteq}\overline{\zeta_\eta((X, E_1))}\\ &\tilde{=}\overline{\gamma_\delta((X, E_1))}\tilde{\subseteq}(C, E_2).\end{aligned}$$ □

Theorem 4.23. *Let (X, τ, E) be a soft completely regular space. If (Y_1, Σ_1, E) and (Y_2, Σ_2, E) are two soft compactifications of (X, τ, E) having the property " every soft bounded soft continuous function $\phi_\psi : SS(X)_E \to SS(R)_E$ can be uniquely extended as soft bounded soft continuous function $\zeta_\eta : SS(Y)_E \to SS(R)_E$", then (Y_1, E) is soft homeomorphic to (Y_2, E).*

Proof. Consider a soft function $(\phi_\psi)_1 : SS(X)_E \to SS(Y_1)_E$ induced by the functions $\phi_1 : X \to Y_1$, $\phi_1(x) = x$ and $\psi_1 : E \to E$, $\phi_1(e) = e$. Clearly $(\phi_\psi)_1$ is a soft continuous function from a soft completely regular space (X, E) to a soft compact soft Hausdorff space (Y_1, E). Since (Y_2, E) is a soft compactifications of (X, E), by previous theorem 4.22, there exists a soft continuous function $(\zeta_\eta)_1 : SS(Y_2)_E \to SS(Y_1)_E$ such that $(\zeta_\eta)_1|_{(X,E)} = (\phi_\psi)_1$. Similarly, if $(\phi_\psi)_2 : SS(X)_E \to SS(Y_2)_E$ is a soft function induced $\phi_2 : X \to Y_2$, $\phi_2(x) = x$ and $\psi_2 : E \to E$, $\phi_2(e) = e$. $(\phi_\psi)_2$ is a soft continuous function from (X, E) to a soft compact soft Hausdorff space (Y_2, E). Since (Y_1, E) is a soft compactifications of (X, E) and (X, E) is a soft completely regular space, by previous theorem 4.22, there exists a soft continuous function $(\zeta_\eta)_2 : SS(Y_1)_E \to SS(Y_2)_E$ such that $(\zeta_\eta)_2|_{(X,E)} = (\phi_\psi)_2$. Now $(\zeta_\eta)_1 \circ (\zeta_\eta)_2 : SS(Y_1)_E \to SS(Y_1)_E$ is a soft continuous function. If $(x_{e_i}, E)\,\tilde{\in}\,(X, E)$, $(\zeta_\eta)_2(x_{e_i}, E)\,\tilde{=}\,\zeta_{2_{\eta_2}}(x_{e_i}, E)\,\tilde{=}\,(F, E)$, where

$$\begin{aligned}F(e_j) &= \begin{cases}\zeta_2(x_{e_i}(e_k)), & \text{if } e_k \in \eta_2^{-1}(e_j),\\ \emptyset, & \text{otherwise,}\end{cases}\\ &= \begin{cases}\zeta_2(x), & \text{if } e_i \in \eta_2^{-1}(e_j),\\ \emptyset, & \text{otherwise,}\end{cases}\\ &= \begin{cases}x, & \text{if } e_i \in \eta_2^{-1}(e_j),\\ \emptyset, & \text{otherwise,}\end{cases}\end{aligned}$$

$$= x_{e_i}(e_j),$$

because $\eta_2(e) = e$.

This means

$$\begin{aligned}(\zeta_\eta)_2(x_{e_i}, E) &\tilde{=} (x_{e_i}, E).(\zeta_\eta)_1 o(\zeta_\eta)_2(x_{e_i}, E)\\ &= (\zeta_\eta)_1(F, E)\\ &= \zeta_{1_{\eta_1}}(x_{e_i}, E)\\ &= (G, E),\end{aligned}$$

where

$$\begin{aligned}G(e_l) &= \begin{cases}\zeta_1(x_{e_i}(e_k), & \text{if } e_k \in \eta_1^{-1}(e_l),\\ \emptyset, & \text{otherwise,}\end{cases}\\ &= \begin{cases}\zeta_1(x), & \text{if } e_i \in \eta_1^{-1}(e_l),\\ \emptyset, & \text{otherwise,}\end{cases}\\ &= \begin{cases}x, & \text{if } e_i \in \eta_1^{-1}(e_l),\\ \emptyset, & \text{otherwise,}\end{cases}\\ &= x_{e_i}(e_l),\end{aligned}$$

because $\eta_1(e) = e$.

Thus $(\zeta_\eta)_1 \circ (\zeta_\eta)_2(x_{e_i}, E) = (x_{e_i}, E)$. Since soft identity function $(I_i)_1 : SS(Y_1)_E \to SS(Y_1)_E$ induced by the identity functions $I_1 : Y_1 \to Y_1$ and $i_1 : E \to E$ is also a soft continuous extension function of $(\phi_\psi)_1$. By the uniqueness fo theorem 4.20, $(\zeta_\eta)_1 \circ (\zeta_\eta)_2 = (I_i)_1$. Similarly, we can prove $(\zeta_\eta)_2 \circ (\zeta_\eta)_1 = (I_i)_2$, where $(I_i)_2 : SS(Y_2)_E \to SS(Y_2)_E$ induced by the identity functions $I_2 : Y_2 \to Y_2$ and $i_2 : E \to E$. Thus $(\zeta_\eta)_1$ and $(\zeta_\eta)_2$ are soft homeomorphisms. □

Definition 4.24. Let (X, τ, E) be a soft completely regular space. There exists a soft compactification (Y, Σ, E) of (X, τ, E) having the property "If $\phi_\psi : SS(X)_E \to SS(\mathbb{R})_E$ is soft bounded soft continuous function, there exists a soft bounded soft continuous function $\zeta_\eta : SS(Y)_E \to SS(\mathbb{R})_E$ such that $\zeta_\eta|_{(X,E)} = \phi_\psi$". The soft compactification (Y, Σ, E) is the Stone-Cech compactification of (X, τ, E).

Conclusions

We have derived the tube lemma and Tychonoff theorem for soft topological spaces. we proved the soft product of soft competely regular space is soft completely regular by using the soft continuity of soft arithmetic

functions. Also, We derived a soft compactification for soft completely regular soft topological spaces.

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[1] Department of Mathematics, Alagappa University, Karaikudi-630 003, India.
Email address: kavyauma85@gmail.com

[2] Department of Mathematics, Alagappa University, Karaikudi-630 003, India.
Email address: jeyabalanr@alagappauniversity.ac.in

[3] Department of Basic Engineering, Alagappa Government Polytechnic College, Karaikudi-630 003, India.
Email address: ramkumarsolai@gmail.com