Sahand Communications in Mathematical Analysis (SCMA) Vol. ··· No. ··· (20??), ···-···
http://scma.maragheh.ac.ir
DOI: ...

Inequalities for Convex and Related Functions and Further Results for Fractional Integrals

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Abstract. This paper establishes Hadamard-type inequalities for differentiable functions whose derivative modulus satisfies a $(g, h; \alpha - m)$-convexity condition. The results are obtained for extended generalized fractional integral operators whose kernels contain an extended Mittag-Leffler function. In contrast with earlier results that treat either the Mittag-Leffler fractional operator or a particular convexity class separately, the present approach combines the kernel parameters, the auxiliary weight g, the control function h, and the parameters α and m in one framework. We derive estimates based on a fractional integral identity, Jensen's inequality and Hölder's inequality, and we record explicit reductions to the $(g, h; m)$-convex case, the Riemann–Liouville fractional integral case, and the classical integral case. Illustrative examples and a notation table are included to clarify the hypotheses and to show how known inequalities are recovered from the main theorems.

1. Introduction

Fractional integrals are useful for generalizing concepts of ordinary integrals. Inequalities linked with usual integrals are studied very frequently for fractional and other kinds of integrals [7, 19]. Celebrated inequalities like Hölder inequality, Ostrowski inequality, Opial inequality, Grüss inequality, Hermite-Hadamard inequality, and many other such inequalities exist in literature for many kinds of fractional integrals. In the same context, convex functions are best tools to establish

2020 *Mathematics Subject Classification.* 26A51, 26A33, 33E12.
Key words and phrases. Convex function, Mittag–Leffler function, Integral operators, Fractional integral inequalities.
Received: 2025-12-26 Accepted: 2026-08-03.
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integral inequalities. A well known Jensen inequality is a nice presentation of convexity that reproduces several classical known inequalities by analyzing particular examples of convex functions, see [13, 16].

Many varieties of convexities can be found in literature which are used solely to extend already published results in the theory of inequalities. For instance h-convex [1, 14, 20], strongly convex, (α, m)-convex [12], harmonically convex, s-convex [15], and logarithmic convex functions were defined to generalize Jensen, and Hermite-Hadamard type inequalities, see [6, 10]. Also, all those convexities were utilized by different authors to establish fractional integral inequalities. For some recently published fractional integral inequalities utilizing several kinds of convex function, we refer the reader to [2, 3, 11] and references therein.

This paper investigates fractional integral inequalities by applying the extended Mittag-Leffler fractional operators are combined with the $(g, h; \alpha - m)$-convex structure. Compared with the Hermite–Hadamard inequalities for $(g, h; \alpha - m)$-convex functions in [10], the present results work with the extended fractional operators defined through the Mittag-Leffler kernel. Compared with the operator construction in [4] and the convex or m-convex inequalities in [9], the present estimates keep the more flexible $(g, h; \alpha - m)$-convex control and therefore contain several earlier inequalities as special cases. The main advantages of this formulation are: first, one can pass from the general Mittag-Leffler kernel to Riemann–Liouville fractional integrals by choosing the parameters appropriately; second, one can pass from $(g, h; \alpha - m)$-convexity to $(g, h; m)$-convexity and ordinary convexity by fixing α, m, g and h; third, the same identity gives both Jensen-type and Hölder-type estimates. These points clarify the mathematical contribution and the position of the results in relation to the available literature.

We start this article by listing some recent results that provide motivation to do this work. Focused inequalities are for fractional integrals which contain Mittag-Leffler functions in their kernels. The inequality (1.1) was proved for fractional integrals (1.8) & (1.9) containing the Mittag-Leffler function (1.7), by analyzing $(h, g; m)$-convex functions defined in [5]

$$f\left(\frac{a+b}{2}\right) C_{\underline{\omega},a^+}(b;p) \leq h\left(\frac{1}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\underline{\omega},a^+} fg\right)(b;p) + m^{\xi+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\underline{\omega},\frac{b}{m}-} fg\right)\left(\frac{a}{m};p\right)\right], \tag{1.1}$$

and

$$\underline{\omega} = \frac{\omega}{(b-a)^{\mu}}, \qquad \underline{\underline{\omega}} = \frac{m^{\mu}\omega}{(b-a)^{\mu}}.$$

The inequality (1.2) was proved in [9] by considering $|f'|$ convex function, where $k < \delta + \Re(\mu)$ and u is continuous function on $[a, b]$

(1.2)

$$\left| \frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{-\xi}} - \xi\int_a^b \frac{u(\zeta)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega\zeta^\mu;p)f(t)}{\left(\int_a^\zeta u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{1-\xi}}dt \right.$$
$$\left. -\xi\int_a^b \frac{u(\zeta)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega\zeta^\mu;p)f(t)}{\left(\int_\zeta^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{1-\xi}}d\zeta \right|$$
$$\leq \frac{(b-a)^{\xi+1}\|u\|_\infty^\xi S^\xi\left[|f'(a)|+|f'(b)|\right]}{\xi+1}.$$

The inequality (1.3) was proved in [10] by considering a class of so called $(g, h, \alpha - m)$-convex functions by using R-L fractional integrals

(1.3)

$$f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\tau+1)}{(mb-a)^\tau}\left[h\left(\frac{1}{2^\alpha}\right)I_{a^+}^\tau(fg)(mb) + m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)I_{b^-}^\tau(fg)\left(\frac{a}{m}\right)\right]$$
$$\leq M\left\{\left[h\left(\frac{1}{2^\alpha}\right)f(a)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right]\int_0^1 t^{\tau-1}h(t^\alpha)dt + m\left[h\left(\frac{1}{2^\alpha}\right)f(b)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right] \times\int_0^1 t^{\tau-1}h(1-t^\alpha)dt\right\},$$

where

$$M = \max\left(g(x)\right),$$
$$I_{a^+}^\tau f(x) = \frac{1}{\Gamma(\tau)}\int_a^x (x-t)^{\tau-1}f(\zeta)d\zeta,$$
$$I_{b^-}^\tau f(x) = \frac{1}{\Gamma(\tau)}\int_x^b (t-x)^{\tau-1}f(\zeta)d\zeta.$$

While a function f satisfying (1.4) is called $(g, h; \alpha - m)$-convex

(1.4)

$$f(\lambda a + m(1-\lambda)b) \leq h(\lambda^\alpha)g(a)f(a) + mh(1-\lambda^\alpha)g(b)f(b), \quad a, b \in I.$$

The function f is $(g, h; \alpha - m)$-concave if inequality (1.4) holds in reverse order. Inequalities (1.5) and (1.6) can be obtained from $(g, h; \alpha - m)$-convexity. These inequalities will be used to formulate the main results

of this paper.

$$f(\zeta) \leq h\left(\left(\frac{b-\zeta}{b-a}\right)^{\alpha}\right) fg(a) + mh\left(1-\left(\frac{b-\zeta}{b-a}\right)^{\alpha}\right) fg\left(\frac{b}{m}\right), \tag{1.5}$$

$$f(\zeta) \leq h\left(\left(\frac{\zeta-a}{b-a}\right)^{\alpha}\right) fg(b) + mh\left(1-\left(\frac{\zeta-a}{b-a}\right)^{\alpha}\right) fg\left(\frac{a}{m}\right). \tag{1.6}$$

From now on, we set $k_1(\zeta) = \left(\frac{\zeta-a}{b-a}\right)^{\alpha}$ and $k_2(\zeta) = \left(\frac{b-\zeta}{b-a}\right)^{\alpha}$.

Andrić et al. in [4] defined an extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(.;p)$ by the following equation:

$$E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\xi)} \frac{t^n}{(l)_{n\delta}}, \tag{1.7}$$

with conditions; $\mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Here β_p is the generalized beta function given by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt,$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol.

In [4] it can be found that $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)$ is absolutely convergent for $k < \delta + \Re(\mu)$, some more properties of this function were discussed in that reference. Let S denotes sum of series of absolute terms of the Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)$, then one has $\left|E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)\right| \leq S$, this inequality is also useful in establishing the results.

Integral operators in the following definition contain $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)$ in their kernels and were defined in [4].

Definition 1.1. Let $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the extended generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-} f$ are defined by:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x;p) = \int_a^x \frac{f(t) E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x-t)^{\mu};p)}{(x-t)^{1-\xi}} dt, \tag{1.8}$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-} f\right)(x;p) = \int_x^b \frac{f(t) E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(t-x)^{\mu};p)}{(t-x)^{1-\xi}} dt. \tag{1.9}$$

One can easily calculate that

(1.10)
$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+}1\right)(x;p) = (x-a)^{\xi}\, E^{\gamma,\delta,k,c}_{\mu,\xi+1,l}(w(x-a)^{\mu};p) := C_{\omega,a^+}(x;p),$$

(1.11)
$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-}1\right)(x;p) = (b-x)^{\xi}\, E^{\gamma,\delta,k,c}_{\mu,\xi+1,l}(w(b-x)^{\mu};p) := C_{\omega,b^-}(x;p).$$

The next section provides Hadamard-type inequalities for generalized convex functions.

TABLE 1. Main notation and standing assumptions used in the paper.

Symbol	Meaning and assumption
$[a,b]$	A compact interval with $a<b$.
f	A differentiable function on $[a,b]$; in the main theorems $\lvert f'\rvert$ or $\lvert f'\rvert^q$ satisfies the stated generalized convexity condition.
u	A continuous and nonnegative function on $[a,b]$, with $\|u\|_\infty = \sup_{\zeta\in[a,b]}\lvert u(\zeta)\rvert$.
g	A nonnegative auxiliary function appearing in the $(g,h;\alpha-m)$-convexity condition.
h	A nonnegative control function; in Theorem 2.4, h is additionally concave.
α, m	Parameters of the generalized convexity class, usually with $\alpha>0$ and $0<m\le 1$.
$k_1(\zeta), k_2(\zeta)$	$k_1(\zeta)=\left(\frac{\zeta-a}{b-a}\right)^{\alpha}$ and $k_2(\zeta)=\left(\frac{b-\zeta}{b-a}\right)^{\alpha}$.
$E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\cdot;p)$	The extended generalized Mittag-Leffler function defined in (1.7), taken as nonnegative for all admissible parameters.
S	The sum of absolute terms of the Mittag-Leffler series, so that $\lvert E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)\rvert\le S$.
$\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+}$, $\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-}$	Left- and right-sided extended generalized fractional integral operators defined in (1.8) and (1.9).

Example 1.2 (Classical convexity contained in the framework)**.** Let $g\equiv 1$, $h(t)=t$, $\alpha=1$ and $m=1$. Then (1.4) becomes

$$f(\lambda a+(1-\lambda)b)\le \lambda f(a)+(1-\lambda)f(b),$$

which is the usual convexity inequality. Hence the class considered in this paper contains the classical convex functions as a special case. For

instance, $f(x) = x^2$ on any interval $[a, b] \subset [0, \infty)$ satisfies the above condition.

Example 1.3 (Reduction to the Riemann–Liouville kernel)**.** If $p = 0$ and $\omega = 0$, then

$$E^{\gamma,\delta,k,c}_{\mu,\xi,l}(0; 0) = \frac{1}{\Gamma(\xi)}.$$

Consequently, (1.8) and (1.9) reduce to

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,0,a^+} f\right)(x; 0) = \frac{1}{\Gamma(\xi)} \int_a^x (x - t)^{\xi-1} f(t)dt,$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,0,b^-} f\right)(x; 0) = \frac{1}{\Gamma(\xi)} \int_x^b (t - x)^{\xi-1} f(t)dt.$$

Thus the extended operators used here contain the classical Riemann–Liouville fractional integrals. If, in addition, $\xi = 1$, the ordinary integral is recovered.

Remark 1.4 (Special cases)**.** The main estimates below may be read in several layers. Choosing $\alpha = 1$ gives the $(g, h; m)$-convex corollaries stated after each theorem. Choosing $g \equiv 1$, $h(t) = t$ and $m = 1$ gives the usual convex case. Choosing $p = 0$ and $\omega = 0$ reduces the Mittag-Leffler fractional operators to Riemann–Liouville fractional integrals. These reductions show explicitly how the present results extend, rather than merely restate, several known inequalities.

2. Hadamard Type Fractional Inequalities for Generalized Convexity

To establish the first result we need the following identity.

Lemma 2.1 ([9])**.** *Let $f \in C^1([a, b])$. Also, let $u : [a, b] \to \mathbb{R}$ be integrable, then we have*

(2.1)

$$\begin{aligned}
&\left(\int_a^b u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu; p)ds\right)^{\xi} [f(a) + f(b)] \\
&\quad - \xi \int_a^b \left(\int_a^\zeta u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu; p)ds\right)^{\xi-1} u(\zeta)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega\zeta^\mu; p)f(\zeta)d\zeta \\
&\quad - \xi \int_a^b \left(\int_\zeta^b u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu; p)ds\right)^{\xi-1} u(\zeta)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega\zeta^\mu; p)f(\zeta)d\zeta \\
&= \int_a^b \left(\int_a^\zeta u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu; p)ds\right)^{\xi} f'(\zeta)d\zeta
\end{aligned}$$

$$-\int_a^b \left(\int_\zeta^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^\xi f'(\zeta)d\zeta.$$

Applying identity (2.1), the following result can be proved.

Theorem 2.2. *Let $f \in C^1([a,b])$ with $a < b$, and assume that $|f'|$ belongs to the class of $(g,h;\alpha-m)$-convex functions. Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then for $k < \delta + \Re(\mu)$, we have*

(2.2)

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{-\xi}} - \xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \\
&\quad\le 2(b-a)^{\xi+1}(\|u\|_\infty S)^\xi \int_a^b \min\Bigg\{h(k_2(\zeta))\left(\left|f'\right| g\right)(a) \\
&\qquad + mh(1-k_2(\zeta))\left(\left|f'\right| g\right)\left(\frac{b}{m}\right), h(k_1(\zeta))\left(\left|f'\right| g\right)(b) \\
&\qquad + mh(1-k_1(\zeta))\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\Bigg\}d\zeta \\
&\quad\le \frac{2(b-a)^{\xi+1}}{(\|u\|_\infty S)^{-\xi}} \min\Bigg\{\left(\left|f'\right| g\right)(a)\int_a^b h(k_2(\zeta))d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)\int_a^b h(1-k_2(\zeta))d\zeta, \left(\left|f'\right| g\right)(b)\int_a^b h(k_1(\zeta))d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\int_a^b h(1-k_1(\zeta))d\zeta\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
S(\zeta) &= \frac{u(\zeta)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega\zeta^\mu;p)f(\zeta)}{\left(\int_a^\zeta u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{1-\xi}}, \\
\|u\|_\infty &= \sup_{\zeta\in[a,b]} |u(\zeta)|, \& \\
\mathbb{R}(\zeta) &= \frac{u(\zeta)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega\zeta^\mu;p)f(\zeta)}{\left(\int_\zeta^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{1-\xi}}.
\end{aligned}$$

Proof. The proof has three main steps. First, Lemma 2.1 converts the expression on the left into two weighted integrals involving f'. Second, the boundedness of u and the absolute convergence of the Mittag-Leffler kernel give a simple kernel estimate. Third, the $(g,h;\alpha-m)$-convexity of

$|f'|$ controls the remaining derivative integral by endpoint data. Lemma 2.1 implies the forthcoming inequality

$$\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}}-\xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \tag{2.3}$$
$$\leq \int_a^b \left|\int_a^{\zeta} u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right|^{\xi} \left|f'(\zeta)\right| d\zeta + \int_a^b \left|\int_{\zeta}^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right|^{\xi} \left|f'(\zeta)\right| d\zeta.$$

By using $\|u\|_{\infty}=\sup_{\zeta\in[a,b]}|u(\zeta)|$, and the absolute convergence of $E_{\mu,\xi,l}^{\gamma,\delta,k,c}(.;p)$, we have

$$\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}}-\xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \tag{2.4}$$
$$\leq \|u\|_{\infty}^{\xi} S^{\xi}\left[\int_a^b (\zeta-a)^{\xi}\left|f'(\zeta)\right| d\zeta+\int_a^b (b-\zeta)^{\xi}\left|f'(\zeta)\right| d\zeta\right]$$
$$= \|u\|_{\infty}^{\xi} S^{\xi}\int_a^b \left[(\zeta-a)^{\xi}+(b-\zeta)^{\xi}\right]\left|f'(\zeta)\right| d\zeta.$$

Above inequality further gives us

$$\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}}-\xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \tag{2.5}$$
$$\leq 2(b-a)\left[(b-a)\|u\|_{\infty} S\right]^{\xi}\int_a^b \left|f'(\zeta)\right| d\zeta.$$

Since $|f'|$ is $(g,h;\alpha-m)$-convex, we apply (1.5) and (1.6) to the derivative modulus. This gives the following two endpoint estimates:

$$\left|f'(\zeta)\right| \leq h(k_2(\zeta))\left(\left|f'\right| g\right)(a)+mh\left(1-k_2(\zeta)\right)\left(\left|f'\right| g\right)\left(\frac{b}{m}\right), \tag{2.6}$$

$$\left|f'(\zeta)\right| \leq h(k_1(\zeta))\left(\left|f'\right| g\right)(b)+mh(1-k_1(\zeta))\left(\left|f'\right| g\right)\left(\frac{a}{m}\right). \tag{2.7}$$

From inequalities (2.6) and (2.7), we can have

(2.8)

$$\int_a^b \left|f'(\zeta)\right| d\zeta \leq \int_a^b \min\left\{h(k_2(\zeta))\left(\left|f'\right| g\right)(a)+mh\left(1-k_2(\zeta)\right)\left|f'\left(\frac{b}{m}\right)\right|\right.$$

$$
\begin{aligned}
&\cdot g\left(\frac{b}{m}\right), h(k_1(\zeta))\left(\left|f'\right| g\right)(b) \\
&\quad + mh(1-k_1(\zeta))\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\Bigg\} d\zeta \\
&\leq \min\Bigg\{ \left(\left|f'\right| g\right)(a)\int_a^b h(k_2(\zeta))d\zeta \\
&\quad + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)\int_a^b h\left(1-k_2(\zeta)\right)d\zeta, \\
&\quad \left(\left|f'\right| g\right)(b)\int_a^b h(k_1(\zeta))d\zeta \\
&\quad + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\int_a^b h(1-k_1(\zeta))d\zeta\Bigg\}.
\end{aligned}
$$

By combining the above inequality with (2.5), one can get the inequality (2.2) as required. □

We give the special case for (g,h,m)-convex functions.

Corollary 2.3. *Fixing $\alpha = 1$, in Theorem 2.2 the following result holds:*

(2.9)

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}} - \xi\int_a^b \big(S(\zeta)+R(\zeta)\big)d\zeta\right| \\
&\quad \leq \frac{2(b-a)^{\xi+1}}{(\|u\|_\infty S)^{-\xi}}\int_a^b \min\Bigg\{ h\left(\frac{b-\zeta}{b-a}\right)\left(\left|f'\right| g\right)(a) \\
&\qquad + mh\left(\frac{\zeta-a}{b-a}\right)\left(\left|f'\right| g\right)\left(\frac{b}{m}\right), h\left(\frac{\zeta-a}{b-a}\right)\left(\left|f'\right| g\right)(b) \\
&\qquad + mh\left(\frac{b-\zeta}{b-a}\right)\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\Bigg\} d\zeta \\
&\quad \leq \frac{2(b-a)^{\xi+1}}{(\|u\|_\infty S)^{-\xi}}\min\Bigg\{ \left(\left|f'\right| g\right)(a)\int_a^b h\left(\frac{b-\zeta}{b-a}\right)d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)\int_a^b h\left(\frac{\zeta-a}{b-a}\right)d\zeta, \left(\left|f'\right| g\right)(b)\int_a^b h\left(\frac{\zeta-a}{b-a}\right)d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\int_a^b h\left(\frac{b-\zeta}{b-a}\right)d\zeta\Bigg\}.
\end{aligned}
$$

Next, we give the following result.

Theorem 2.4. *In addition to the suppositions of Theorem 2.2, further let h be concave, then we have*

(2.10)

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}} - \xi\int_a^b (S(\zeta)+R(\zeta))d\zeta \right| \\
&\quad \le \frac{2(b-a)^{\xi+1}}{(\|u\|_\infty S)^{-\xi}} \min\Bigg\{ \left(\left|f'\right| g\right)(a)\int_a^b h(k_2(\zeta))d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)\int_a^b h(1-k_2(\zeta))d\zeta, \left(\left|f'\right| g\right)(b)\int_a^b h(k_1(\zeta))d\zeta \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\int_a^b h(1-k_1(\zeta))d\zeta \Bigg\} \\
&\quad \le \frac{2(b-a)^{\xi+2}}{(\|u\|_\infty S)^{-\xi}} \min\Bigg\{ \left(\left|f'\right| g\right)(a)h\left(\frac{1}{\alpha+1}\right) \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)h\left(\frac{\alpha}{\alpha+1}\right), \left(\left|f'\right| g\right)(b)h\left(\frac{1}{\alpha+1}\right) \\
&\qquad + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)h\left(\frac{\alpha}{\alpha+1}\right) \Bigg\}.
\end{aligned}$$

Proof. The additional assumption that h is concave is used only at the last stage of the proof. It allows the integral averages of $h(k_1(\zeta))$ and $h(k_2(\zeta))$ to be estimated by Jensen's inequality. Therefore, from (2.8) we obtain

$$\begin{aligned}
&\int_a^b \left|f'(\zeta)\right| d\zeta \\
&\quad \le \min\Bigg\{ \left(\left|f'\right| g\right)(a)\int_a^b h(k_2(\zeta))d\zeta + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)\int_a^b h(1-k_2(\zeta))d\zeta, \\
&\qquad \left(\left|f'\right| g\right)(b)\int_a^b h(k_1(\zeta))d\zeta + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)\int_a^b h(1-k_1(\zeta))d\zeta \Bigg\} \\
&\quad \le (b-a)\min\Bigg\{ \left(\left|f'\right| g\right)(a)h\left(\frac{1}{\alpha+1}\right) + m\left(\left|f'\right| g\right)\left(\frac{b}{m}\right)h\left(\frac{\alpha}{\alpha+1}\right), \\
&\qquad \left(\left|f'\right| g\right)(b)h\left(\frac{1}{\alpha+1}\right) + m\left(\left|f'\right| g\right)\left(\frac{a}{m}\right)h\left(\frac{\alpha}{\alpha+1}\right) \Bigg\}.
\end{aligned}$$

Using the above inequality in (2.5), the required inequality can be obtained. □

We give the special case for (g,h,m)-convex functions.

Corollary 2.5. *Fixing $\alpha=1$, in Theorem 2.4 the following result holds:*

(2.11)

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}}-\xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \\
&\leq \frac{2(b-a)^{\xi+1}}{(\|u\|_\infty S)^{-\xi}}\min\left\{\left(\left|f'\right|g\right)(a)\int_a^b h\left(\frac{b-\zeta}{b-a}\right)d\zeta\right. \\
&\quad +m\left(\left|f'\right|g\right)\left(\frac{b}{m}\right)\int_a^b h\left(\frac{\zeta-a}{b-a}\right)d\zeta,\left(\left|f'\right|g\right)(b)\int_a^b h\left(\frac{\zeta-a}{b-a}\right)d\zeta \\
&\quad \left.+m\left(\left|f'\right|g\right)\left(\frac{a}{m}\right)\int_a^b h\left(\frac{b-\zeta}{b-a}\right)d\zeta\right\} \\
&\leq h\left(\frac{1}{2}\right)\frac{2(b-a)^{\xi+2}}{(\|u\|_\infty S)^{-\xi}} \\
&\quad \times\min\left\{\left|f'\right|g(a)+m\left|f'\right|g\left(\frac{b}{m}\right),\left|f'\right|g(b)+m\left|f'\right|g\left(\frac{a}{m}\right)\right\}.
\end{aligned}$$

Theorem 2.6. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be differentiable and assume that $|f'|^q$, $q>1$, belongs to the class of $(g,h;\alpha-m)$-convex functions, $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then for $k<\delta+\Re(\mu)$, we have*

(2.12)

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^{\mu};p)ds\right)^{-\xi}}-\xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \\
&\leq W\int_a^b\min\left\{\left[h(k_2(\zeta))\left(\left|f'\right|^q g\right)(a)+mh(1-k_2(\zeta))\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right)\right]^{1/q},\right. \\
&\quad \left.\left[h(k_1(\zeta))\left(\left|f'\right|^q g\right)(b)+mh(1-k_1(\zeta))\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)\right]^{1/q}\right\}d\zeta \\
&\leq W\min\left\{\int_a^b\left[\left(\left|f'\right|^q g\right)(a)h(k_2(\zeta))+m\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right)h(1-k_2(\zeta))\right]^{1/q}d\zeta,\right. \\
&\quad \left.\int_a^b\left[\left(\left|f'\right|^q g\right)(b)h(k_1(\zeta))+m\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)h(1-k_1(\zeta))\right]^{1/q}d\zeta\right\},
\end{aligned}$$

where

$$W = \frac{2(\|u\|_\infty S)^\xi}{(\xi p+1)^{1/p}}(b-a)^{(\xi p+1)/p},$$
$$\frac{1}{p}+\frac{1}{q}=1.$$

Proof. The proof follows the same structure as Theorem 2.2, but the last derivative integral is estimated in L^q. Applying Hölder's inequality on the right-hand side of (2.3), and using $\|u\|_\infty = \sup_{\zeta\in[a,b]} |u(\zeta)|$, and the absolute convergence of $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(.;p)$, one can get

(2.13)
$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu;p)ds\right)^{-\xi}} - \xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \\
&\leq \left\{\left(\int_a^b \left|\int_a^\zeta u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu;p)ds\right|^{\xi p} d\zeta\right)^{1/p}\right. \\
&\quad \left. + \left(\int_a^b \left|\int_\zeta^b u(s)E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega s^\mu;p)ds\right|^{\xi p} d\zeta\right)^{1/p}\right\} \left(\int_a^b \left|f'(\zeta)\right|^q d\zeta\right)^{1/q} \\
&\leq \|u\|^\xi_\infty S^\xi \left[\left(\int_a^b |t-a|^{\xi p}d\zeta\right)^{1/p} + \left(\int_a^b |b-t|^{\xi p}d\zeta\right)^{1/p}\right] \\
&\quad \cdot \left(\int_a^b \left|f'(\zeta)\right|^q d\zeta\right)^{1/q} \\
&\leq \frac{2(b-a)^{(\xi p+1)/p}\|u\|^\xi_\infty S^\xi}{(\xi p+1)^{1/p}} \left(\int_a^b \left|f'(\zeta)\right|^q d\zeta\right)^{1/q}.
\end{aligned}$$

Since $|f'(\zeta)|^q$ is $(g,h;\alpha-m)$-convex, one can have

(2.14)
$$\left|f'(\zeta)\right|^q \leq h(k_2(\zeta))\left(\left|f'\right|^q g\right)(a) + mh\left(1-k_2(\zeta)\right)\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right),$$

(2.15)
$$\left|f'(\zeta)\right|^q \leq h(k_1(\zeta))\left(\left|f'\right|^q g\right)(b) + mh(1-k_1(\zeta))\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right).$$

From inequalities (2.14) and (2.15), we have

(2.16)

$$\begin{aligned}
&\left(\int_a^b \left|f'(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}} \\
&\leq \int_a^b \min\left\{\left[h(k_2(\zeta))\left(\left|f'\right|^q g\right)(a) + mh\left(1-k_2(\zeta)\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right.\right. \\
&\quad \left.\left.\cdot g\left(\frac{b}{m}\right)\right]^{\frac{1}{q}}, \left[h(k_1(\zeta))\left(\left|f'\right|^q g\right)(b) + mh(1-k_1(\zeta))\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)\right]^{\frac{1}{q}}\right\} d\zeta \\
&\leq \min\left\{\int_a^b \left[\left(\left|f'\right|^q g\right)(a)h(k_2(\zeta)) + m\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right)h\left(1-k_2(\zeta)\right)\right]^{\frac{1}{q}} d\zeta,\right. \\
&\quad \left.\int_a^b \left[\left(\left|f'\right|^q g\right)(b)h(k_1(\zeta)) + m\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)h(1-k_1(\zeta))\right]^{\frac{1}{q}} d\zeta\right\}.
\end{aligned}$$

By combining the above inequality with (2.13), one can get the inequality (2.12) as required. □

We give the special case for (g,h,m)-convex functions.

Corollary 2.7. *Fixing $\alpha = 1$, in Theorem 2.6 the following result holds:*

(2.17)

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{\left(\int_a^b u(s)E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)ds\right)^{-\xi}} - \xi\int_a^b (S(\zeta)+R(\zeta))d\zeta\right| \\
&\leq \frac{2(b-a)^{(\xi p+1)/p}\left\|u\right\|_\infty^\xi S^\xi}{(\xi p+1)^{1/p}}\int_a^b \min\left\{\left[h\left(\frac{b-\zeta}{b-a}\right)\left(\left|f'\right|^q g\right)(a)\right.\right. \\
&\quad \left.+mh\left(\frac{\zeta-a}{b-a}\right)\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right)\right]^{1/q}, \\
&\quad \left.\left[h\left(\frac{\zeta-a}{b-a}\right)\left(\left|f'\right|^q g\right)(b) + mh\left(\frac{b-\zeta}{b-a}\right)\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)\right]^{1/q}\right\} d\zeta \\
&\leq \frac{2(b-a)^{(\xi p+1)/p}\left\|u\right\|_\infty^\xi S^\xi}{(\xi p+1)^{1/p}}\min\left\{\int_a^b \left[\left(\left|f'\right|^q g\right)(a)h\left(\frac{b-\zeta}{b-a}\right)\right.\right. \\
&\quad \left.+m\left(\left|f'\right|^q g\right)\left(\frac{b}{m}\right)h\left(\frac{\zeta-a}{b-a}\right)\right]^{1/q} d\zeta, \\
&\quad \left.\int_a^b \left[\left(\left|f'\right|^q g\right)(b)h\left(\frac{\zeta-a}{b-a}\right) + m\left(\left|f'\right|^q g\right)\left(\frac{a}{m}\right)h\left(\frac{b-\zeta}{b-a}\right)\right]^{1/q} d\zeta\right\}.
\end{aligned}$$

3. Conclusion

This paper developed Hadamard-type fractional integral inequalities for functions connected with the $(g, h; \alpha - m)$-convexity structure. The principal estimates were derived for extended generalized fractional integral operators involving Mittag-Leffler kernels. The obtained results include Jensen-type estimates, concavity-based refinements, and Hölder-type inequalities for the case in which $|f'|^q$ satisfies the generalized convexity condition. A notation table and illustrative examples were added to make the assumptions and parameter choices more transparent.

The results are useful because they provide a unified way to handle several fractional integral inequalities under one operator and one generalized convexity framework. By suitable choices of p, ω, ξ, α, m, g and h, the main results reduce to inequalities for Riemann–Liouville fractional integrals, ordinary integrals, $(g, h; m)$-convex functions and classical convex functions. Therefore, the paper gives both new estimates for the extended Mittag-Leffler setting and a clear path for recovering known inequalities.

The present work is theoretical in nature. Its limitations come from the number of parameters in the kernel and from the need to verify the generalized convexity assumptions in concrete problems. Future research may consider sharper constants, equality cases, multidimensional versions, quantum and conformable fractional operators, and applications to fractional differential equations, optimization problems and mathematical models in which convexity-type estimates are required.

Conflict of Interest: The authors declare that they have no competing or financial conflicts of interest.

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