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Matrix Operators on Orlicz Sequence Spaces

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Abstract. In this paper, we investigate matrix transformations acting between Orlicz sequence spaces. Using the structural properties of Orlicz functions satisfying the Δ_2-condition, we characterize the classes of infinite matrices mapping ℓ_ϕ into the Orlicz-type sequence spaces $\ell_{\phi,\infty}$ and $c(\phi)$. In particular, we establish necessary and sufficient conditions for a matrix operator $A = (a_{nk})$ to belong to $(\ell_\phi : \ell_{\phi,\infty})$ and $(\ell_\phi : c(\phi))$, expressed in terms of uniform boundedness of $\phi(a_{nk})$. Furthermore, we establish density properties of finitely supported sequences in the corresponding Orlicz sequence spaces. These results contribute to the understanding of operator theory on generalized sequence spaces, extending classical results for ℓ_p-spaces to the more flexible Orlicz setting.

1. Introduction

Matrix transformations between sequence spaces constitute an important area of functional analysis and operator theory, particularly in the study of boundedness, compactness, and structural properties of linear operators [18, 32]. Classical results concerning matrix mappings between ℓ_p-spaces provide a fundamental framework for the development of operator theory on sequence spaces. Nevertheless, many problems arising in modern analysis require more flexible spaces that extend beyond the classical Lebesgue-type setting [17].

Research on sequence spaces has been extensively developed by many authors (see, for example, [11]–[10]). One important approach in the construction of new sequence spaces is through matrix transformations

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[23]–[7]. This direction is closely related to the development of summability theory [6]. Matrix operators on classical sequence spaces such as ℓ_p, c_0, c, and bv have been extensively investigated by several authors [1, 2, 5, 7, 19, 20, 24, 27, 29].

A further line of research was initiated by Nur Kusnussa'adah [11], who introduced the Orlicz sequence space

$$\ell_\phi = \left\{ (x_n) \in \omega : \sum_{n=1}^{\infty} \phi(|x_n|) < \infty \right\},$$

where ϕ is an Orlicz function, and studied several topological and algebraic properties of the space. The definition of ℓ_ϕ is motivated by the classical space ℓ_p $(1 \leq p < \infty)$, in which the function $|x|^p$ appears as a particular example of an Orlicz function. Consequently, Orlicz sequence spaces provide a more flexible framework that extends the classical ℓ_p-theory to a broader class of sequence spaces.

Orlicz sequence spaces allow more general growth conditions than classical power-type spaces [16, 25]. Under the Δ_2-condition, these spaces retain sufficient structure to support a robust operator theory while accommodating a wider class of nonlinear behaviors [21]. Although matrix transformations on classical sequence spaces have been extensively studied, corresponding characterizations on Orlicz-type sequence spaces remain relatively limited, particularly for the operator classes $(\ell_\phi : \ell_{\phi,\infty})$ and $(\ell_\phi : c(\phi))$.

Such considerations naturally lead to the investigation of matrix transformations acting between Orlicz-type sequence spaces and their associated variants such as $\ell_{\phi,\infty}$ and $c(\phi)$ [18, 31, 32]. In particular, determining conditions under which an infinite matrix induces a bounded linear transformation provides insight into the relationship between the algebraic structure of matrices and the modular structure of Orlicz spaces [21, 25].

Furthermore, characterizations in terms of uniform boundedness of $\phi(a_{nk})$ extend classical criteria from ℓ_p-theory [17, 32]. They also emphasize the role of modular growth conditions in determining the behavior of matrix operators [21, 25]. These considerations motivate a systematic investigation of matrix transformations on Orlicz sequence spaces with the aim of extending classical results and developing a unified framework for generalized sequence spaces.

The modular structure of Orlicz sequence spaces prevents a direct application of classical ℓ_p techniques, since boundedness criteria must be formulated through modular growth conditions rather than power-type norm estimates.

Motivated by the above developments, in this paper we study matrix operators on Orlicz sequence spaces. We characterize the conditions under which an infinite matrix $A = (a_{nk})$ belongs to the operator classes

$$(\ell_\phi : \ell_{\phi,\infty}) \quad \text{and} \quad (\ell_\phi : c(\phi)).$$

The boundedness condition involving $\phi(a_{nk})$ naturally arises from modular estimates associated with the Orlicz modular. In particular, we establish necessary and sufficient conditions in terms of the uniform boundedness of $\phi(a_{nk})$.

We also show that such matrix operators define bounded linear transformations and discuss the role of the collection of all finite sequences in extending matrix operator representations. To the best of our knowledge, a systematic characterization of matrix operators between these Orlicz-type sequence spaces has not yet been established in the existing literature. The obtained results extend several classical theorems from ℓ_p-theory and contribute to the development of operator theory on Orlicz sequence spaces.

2. Some Sequence Spaces Defined by an Orlicz Function

In this study, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively. The symbol ω represents the collection of all real sequences (x_n) with $x_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. For any $(x_n), (y_n) \in \omega$ and $\alpha \in \mathbb{R}$, we define

$$(x_n) + (y_n) = (x_n + y_n), \qquad \alpha(x_n) = (\alpha x_n).$$

A sequence $(x_n) \in \omega$ is said to be finite if there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$. The collection of all finite sequences will be denoted by Θ.

A function $\phi : \mathbb{R} \to \mathbb{R}$ is called an Orlicz function if it is even, continuous, and satisfies the following conditions:

(i) $\phi(x) = 0 \Leftrightarrow x = 0$.
(ii) ϕ is non decreasing on $\mathbb{R}^+$, i.e. for any $x, y \in \mathbb{R}$ with $0 \leq x < y$ we have $\phi(x) \leq \phi(y)$.
(iii) $\lim_{x\to 0} \phi(x) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$.

For any Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$, the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x) = \sup\{|x|u - \phi(u) : u \geq 0\},$$

is also an Orlicz function and satisfies the Young inequality

$$|xy| \leq \phi(x) + \psi(y), \tag{2.1}$$

for every $x, y \in \mathbb{R}$. Any pair of Orlicz functions (ϕ, ψ) are said to be complementary if they satisfy the inequality 2.1.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Δ_2-condition, if there exists an $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in \mathbb{R}$.

Let $\phi : \mathbb{R} \to [0, \infty)$ be an Orlicz function. We define

$$\ell_\phi = \left\{ (x_n) \in \omega : \sum_{n=1}^{\infty} \phi(x_n) < \infty \right\},$$

$$\ell_{\phi,\infty} = \left\{ (x_n) \in \omega : \sup_{n \geq 1} \phi(x_n) < \infty \right\},$$

$$c(\phi) = \left\{ (x_n) \in \omega : \exists\, x \in \mathbb{R} \text{ with } \lim_{n \to \infty} \phi(x_n - x) = 0 \right\}.$$

If the Orlicz function ϕ satisfies the Δ_2-condition, then ℓ_ϕ, $\ell_{\phi,\infty}$, and $c(\phi)$ are all linear spaces over $\mathbb{R}$.

The following remarks describe some basic properties of the spaces ℓ_ϕ, $\ell_{\phi,\infty}$, and $c(\phi)$, as well as the relationships between $c(\phi)$ and the classical space c, and between $\ell_{\phi,\infty}$ and ℓ_∞.

Remark 2.1. Since ϕ is continuous and satisfies $\phi(t) = 0$ if and only if $t = 0$, we have

$$\phi(x_n - x) \to 0 \quad \Leftrightarrow \quad x_n \to x.$$

Consequently, $c(\phi)$ coincides with the classical space c of convergent sequences.

Moreover, since $\phi(t) \to \infty$ as $t \to \infty$, it follows that $\ell_{\phi,\infty} = \ell_\infty$. Nevertheless, we retain the notations $c(\phi)$ and $\ell_{\phi,\infty}$ throughout the paper for consistency with the Orlicz-space framework. In particular, the notation $c(\phi)$ reflects the fact that the convergence condition is formulated in terms of the underlying Orlicz function ϕ.

Remark 2.2. Since $c(\phi) = c$, the inclusions $\ell_\phi \subsetneq \ell_{\phi,\infty}$ and $\ell_\phi \subsetneq c$ are strict.

Consider the constant sequence $e = (1, 1, 1, \ldots)$. Then

$$\sup_{n \geq 1} \phi(e_n) = \phi(1) < \infty.$$

Therefore, $e \in \ell_{\phi,\infty}$. Moreover, since $e_n = 1$ for all $n \in \mathbb{N}$, we have $e_n \to 1$. Hence, $e \in c$.

On the other hand, since $e_n = 1$ for all $n \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \phi(e_n) = \sum_{n=1}^{\infty} \phi(1).$$

Because $\phi(1) > 0$, the series on the right-hand side diverges. Consequently,

$$\sum_{n=1}^{\infty} \phi(e_n) = \infty,$$

which implies that $e \notin \ell_\phi$. Therefore, both inclusions are proper.

As is well known, for any $1 \leq p < \infty$,

$$\ell_p = \left\{ (x_n) \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

is a normed space with respect to the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It can be easily shown that for any $(x_n) \in \ell_p$,

$$\|(x_n)\|_p = \inf \left\{ \varepsilon > 0 : \sum_{n=1}^{\infty} \left| \frac{x_n}{\varepsilon} \right|^p \leq 1 \right\}.$$

Notice that the function $t \mapsto |t|^p$ is a special case of an Orlicz function. For any $(x_n) \in \ell_p$ $(1 \leq p < \infty)$, the existence of $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty} \left| \frac{x_n}{\varepsilon} \right|^p \leq 1$$

is a key point in defining the so-called Luxemburg norm, and this existence is guaranteed by the convexity of $|t|^p$. By preserving the convexity property for a general Orlicz function ϕ, one can prove that the set

$$\left\{ \varepsilon > 0 : \sum_{n=1}^{\infty} \phi\left(\tfrac{x_n}{\varepsilon}\right) \leq 1 \right\}$$

is nonempty.

The previous discussion shows that the convexity of an Orlicz function plays a central role in the definition of Orlicz spaces, as it guarantees the well-definedness of the Luxemburg norm and ensures the subadditivity property required for a normed space. On the other hand, the Δ_2-condition provides essential growth control for the Orlicz function. Therefore, convexity and the Δ_2-condition are key elements in establishing the analytic and geometric foundations of Orlicz spaces.

Throughout this paper, unless otherwise stated, we assume that the Orlicz function ϕ is convex and satisfies the Δ_2-condition.

Under the assumptions mentioned above, it was shown in [11] that ℓ_ϕ is a Banach space with respect to the Luxemburg norm

$$\|x\|_{\ell_\phi} = \inf \left\{ \varepsilon > 0 : \sum_{n=1}^{\infty} \phi\left(\frac{x_n}{\varepsilon}\right) \leq 1 \right\},$$

and $c(\phi)$ is a Banach space with respect to the supremum norm

$$\|x\|_{c(\phi)} = \sup\{|x_k| : k \in \mathbb{N}\}.$$

One of the fundamental properties that describes the relationship between the spaces ℓ_ϕ and ℓ_ψ, for any pair of complementary Orlicz functions ϕ and ψ, is given in the following theorem (see [11] for the proof).

Theorem 2.3 (Orlicz–Hölder inequality)**.** *Let ϕ and ψ be a pair of complementary Orlicz functions. If $(x_n) \in \ell_\phi$ and $(y_n) \in \ell_\psi$, then*

$$\sum_{n=1}^{\infty} |x_n y_n| \leq 2\, \|(x_n)\|_{\ell_\phi} \|(y_n)\|_{\ell_\psi}.$$

The following theorem is a standard consequence of the duality theory of Orlicz sequence spaces (see, for example, [25]).

Theorem 2.4. *Let ϕ and ψ be complementary Orlicz functions satisfying the Δ_2-condition. Then the mapping $T : \ell_\psi \to (\ell_\phi)^*$ defined by*

$$(Ty)(x) = \sum_{k=1}^{\infty} y_k x_k,$$

is an isomorphism of Banach spaces.

3. Matrix Operators and Matrix Domains

Study of sequence spaces is closely related to the theory of infinite matrices [8]. Let $A = (a_{nk})$ and $B = (b_{nk})$ be two infinite matrices. We define the product AB as the matrix (c_{nk}), where

$$c_{nk} = \sum_{j=1}^{\infty} a_{nj} b_{jk}, \tag{3.1}$$

whenever the series in the right side of (3.1) exists for every $n, k \in \mathbb{N}$. An infinite matrix $B = (b_{nk})$ is called a right inverse (resp. left inverse) of an infinite matrix $A = (a_{nk})$ if $AB = I$ (resp. $BA = I$), where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

The matrix I is called the identity matrix. An infinite matrix $A = (a_{nk})$ is called a triangle matrix if $a_{nk} = 0$ whenever $k > n$ [32]. Every triangle matrix has a right inverse, and its inverse is also a triangle matrix. Moreover, if A triangle and B is a right inverse of A, then B is unique and $BA = AB = I$ (See [8]).

Let $A = (a_{nk})$ be any infinite matrix and $x \in \omega$. We define $A(x) = (A_n(x))$, where $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ for every $n \geq 1$, provided $\sum_{k=1}^{\infty} a_{nk}x_k$ exists for all $n \geq 1$. In fact, for any triangle matrix A and any $x \in \omega$, $A(x)$ exists [8].

Let X and Y be sequence spaces contained in ω. An infinite matrix A defines a matrix transformation from X to Y, denoted by $A : X \to Y$, if for every $x \in X$, $A(x)$ exists and $A(x) \in Y$ [4]. Every matrix transformation is a linear operator [12]. The collection of all bounded matrix transformations from X to Y is denoted by $(X : Y)$.

For later use, we recall the following lemma and theorem, which can be found in [12].

Lemma 3.1. *Let $X \subset \omega$ be a sequence space and A an infinite matrix. Then the set $X_A = \{x \in \omega : A(x) \in X\}$ is a normed space with respect to $\|x\|_{X_A} = \|A(x)\|_X$.*

Theorem 3.2. *Let $X, Y \subset \omega$ be sequence spaces. If A is an infinite matrix and U is a triangle matrix such that UA exists, then*

$$A \in (X : Y_U) \quad \Leftrightarrow \quad UA \in (X : Y).$$

In the study of matrix transformations between sequence spaces, it is often important to characterize when an infinite matrix $A = (a_{nk})$ maps one sequence space into another. Such characterizations are usually expressed in terms of conditions on the entries of A, involving boundedness, summability, or convergence properties.

For matrix transformations involving Orlicz sequence spaces, these conditions can be formulated naturally in terms of the generating Orlicz functions and their complementary functions. The next several results provide characterizations of matrix classes associated with the spaces ℓ_ϕ, $\ell_{\phi,\infty}$, $c(\phi)$, and ℓ_1.

We begin with a characterization of the matrix class $(\ell_\phi : \ell_{\phi,\infty})$.

Theorem 3.3. *Let ϕ and ψ be a pair of complementary Orlicz functions, and let $A = (a_{nk})$ be an infinite matrix. Then A defines a bounded linear operator from ℓ_ϕ into $\ell_{\phi,\infty}$ if and only if*

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad \text{for every } n \in \mathbb{N},$$

and $\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty$.

Proof. ($\Rightarrow$) Assume that $A : \ell_\phi \to \ell_{\phi,\infty}$ is a bounded linear operator. Since $\ell_{\phi,\infty} = \ell_\infty$, we may regard $A : \ell_\phi \to \ell_\infty$. For each fixed $n \in \mathbb{N}$, let

$$\pi_n : \ell_\infty \to \mathbb{R}, \qquad \pi_n(y) = y_n,$$

be the nth coordinate projection, and define $f_n = \pi_n \circ A$. Then $f_n \in (\ell_\phi)^*$ and

$$\|f_n\| \leq \|\pi_n\| \, \|A\| = \|A\|.$$

By Theorem 2.4, for each $n \in \mathbb{N}$ there exists a unique sequence

$$b^{(n)} = \left(b_k^{(n)}\right) \in \ell_\psi$$

such that

$$f_n(x) = \sum_{k=1}^{\infty} b_k^{(n)} x_k, \quad x \in \ell_\phi.$$

Since $e^{(k)} \in \ell_\phi$ for every $k \in \mathbb{N}$, we have

$$f_n\left(e^{(k)}\right) = \left(Ae^{(k)}\right)_n = a_{nk}.$$

Hence $b_k^{(n)} = a_{nk}$ for every $k \in \mathbb{N}$. Therefore,

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}).$$

By Theorem 2.4, the mapping $T : \ell_\psi \to (\ell_\phi)^*$ defined by

$$(Ty)(x) = \sum_{k=1}^{\infty} y_k x_k,$$

is an isomorphism. Hence T^{-1} is bounded. Therefore, there exists a constant $C > 0$ such that

$$\|y\|_{\ell_\psi} \leq C \, \|T(y)\|, \quad (y \in \ell_\psi).$$

Applying this to $y = (a_{nk})_{k\geq 1}$, we obtain

$$\|(a_{nk})_{k\geq 1}\|_{\ell_\psi} \leq C \, \|f_n\|.$$

for every $n \in \mathbb{N}$. Consequently,

$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} \leq C \sup_{n\geq 1} \|f_n\| \leq C\|A\|.$$

($\Leftarrow$) Assume that

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$

and

$$M := \sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty.$$

Let $x = (x_k) \in \ell_\phi$. By the Orlicz–Hölder inequality,

$$\sum_{k=1}^{\infty} |a_{nk} x_k| \leq 2 \, \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} \, \|x\|_{\ell_\phi}$$

for every $n \in \mathbb{N}$. Hence the series

$$\sum_{k=1}^{\infty} a_{nk} x_k$$

converges absolutely for every $n \in \mathbb{N}$. Therefore we may define

$$(Ax)_n := \sum_{k=1}^{\infty} a_{nk} x_k, \quad n \in \mathbb{N}.$$

Again by Orlicz–Hölder,

$$|(Ax)_n| \leq 2 \, \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} \, \|x\|_{\ell_\phi} \leq 2M \|x\|_{\ell_\phi}.$$

Taking the supremum over n yields

$$\|Ax\|_\infty \leq 2M \|x\|_{\ell_\phi}.$$

Since each coordinate of Ax is defined by a linear combination of the coordinates of x, the mapping A is linear. Hence $A : \ell_\phi \to \ell_\infty$ is a bounded linear operator. Since $\ell_{\phi,\infty} = \ell_\infty$, we conclude that $A : \ell_\phi \to \ell_{\phi,\infty}$ is a bounded linear operator. This completes the proof. □

Building upon this, we now turn to the characterization of matrix operators mapping ℓ_ϕ into the space $c(\phi)$. The following theorem provides the necessary and sufficient conditions for such mappings.

Theorem 3.4. *Let ϕ and ψ be a pair of complementary Orlicz functions. An infinite matrix $A = (a_{nk})$ belongs to $(\ell_\phi : c(\phi))$ if and only if*

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$
$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty,$$

and for each $k \in \mathbb{N}$ there exists $\alpha_k \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_{nk} = \alpha_k.$$

Proof. ($\Rightarrow$): Assume that $A \in (\ell_\phi : c(\phi))$. Since $c(\phi) \subset \ell_{\phi,\infty}$, it follows that $A \in (\ell_\phi : \ell_{\phi,\infty})$. Therefore, by Theorem 3.3,

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$

and

$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty.$$

Fix $k \in \mathbb{N}$. Since $e^{(k)} \in \ell_\phi$, we have

$$Ae^{(k)} = (a_{nk})_{n\geq 1} \in c(\phi).$$

Therefore the sequence $(a_{nk})_{n\geq 1}$ is convergent. Denote

$$\alpha_k = \lim_{n\to\infty} a_{nk}.$$

Thus the necessity is established.
($\Leftarrow$) Assume that

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$
$$M := \sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty,$$

and that

$$a_{nk} \to \alpha_k, \quad (n \to \infty)$$

for every $k \in \mathbb{N}$. Since

$$\|(a_{nk})_{k\geq 1}\|_{\ell_\psi} \leq M,$$

the definition of the Luxemburg norm yields

$$\sum_{k=1}^{\infty} \psi\left(\frac{|a_{nk}|}{M}\right) \leq 1, \quad (n \in \mathbb{N}).$$

By Fatou's lemma,

$$\begin{aligned}\sum_{k=1}^{\infty} \psi\left(\frac{|\alpha_k|}{M}\right) &= \sum_{k=1}^{\infty} \liminf_{n\to\infty} \psi\left(\frac{|a_{nk}|}{M}\right) \\ &\leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} \psi\left(\frac{|a_{nk}|}{M}\right) \\ &\leq 1.\end{aligned}$$

Hence $(\alpha_k)_{k\geq 1} \in \ell_\psi$ and $\|(\alpha_k)\|_{\ell_\psi} \leq M$. Define

$$L(x) = \sum_{k=1}^{\infty} \alpha_k x_k, \quad x \in \ell_\phi.$$

Since $(\alpha_k) \in \ell_\psi$, Theorem 2.4 implies that L is a bounded linear functional on ℓ_ϕ. Now define

$$B = (b_{nk}), \qquad b_{nk} = a_{nk} - \alpha_k.$$

Then

$$b_{nk} \to 0, \quad (n \to \infty)$$

for every $k \in \mathbb{N}$. Moreover,

$$\|(b_{nk})_{k\geq 1}\|_{\ell_\psi} \leq \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} + \|(\alpha_k)\|_{\ell_\psi},$$

and therefore

$$\sup_{n\geq 1} \|(b_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty.$$

Hence, by Theorem 3.3, $B \in (\ell_\phi : \ell_{\phi,\infty})$. If $x \in \Theta$, then

$$(Bx)_n = \sum_{k=1}^{m} b_{nk} x_k$$

for some $m \in \mathbb{N}$. Since $b_{nk} \to 0$ for every k,

$$(Bx)_n \to 0, \quad (n \to \infty).$$

Let $x \in \ell_\phi$ be arbitrary. Choose a sequence $\left(x^{(m)}\right) \subset \Theta$ such that

$$x^{(m)} \to x \quad \text{in } \ell_\phi.$$

Set

$$K = \sup_{n \geq 1} \left\| (b_{nk})_{k \geq 1} \right\|_{\ell_\psi}.$$

By Theorem 2.4,

$$\begin{aligned} |(Bz)_n| &= \left| \sum_{k=1}^{\infty} b_{nk} z_k \right| \\ &\leq 2 \left\| (b_{nk})_{k \geq 1} \right\|_{\ell_\psi} \|z\|_{\ell_\phi} \\ &\leq 2K \|z\|_{\ell_\phi} \end{aligned}$$

for every $z \in \ell_\phi$ and every $n \in \mathbb{N}$. Fix $\varepsilon > 0$. Choose m sufficiently large so that

$$K \left\| x - x^{(m)} \right\|_{\ell_\phi} < \frac{\varepsilon}{6}.$$

Since $x^{(m)} \in \Theta$, there exists $N \in \mathbb{N}$ such that

$$\left| \left(Bx^{(m)} \right)_n \right| < \frac{\varepsilon}{3}, \quad (n \geq N).$$

Hence, for $n \geq N$,

$$\begin{aligned} |(Bx)_n| &\leq \left| \left(B\left(x - x^{(m)} \right) \right)_n \right| + \left| \left(Bx^{(m)} \right)_n \right| \\ &\leq 2K \left\| x - x^{(m)} \right\|_{\ell_\phi} + \left| \left(Bx^{(m)} \right)_n \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Therefore,

$$(Bx)_n \to 0, \quad (n \to \infty).$$

Finally,

$$\begin{aligned} (Ax)_n &= \sum_{k=1}^{\infty} a_{nk} x_k \\ &= \sum_{k=1}^{\infty} (a_{nk} - \alpha_k) x_k + \sum_{k=1}^{\infty} \alpha_k x_k \\ &= (Bx)_n + L(x). \end{aligned}$$

Since $(Bx)_n \to 0$, we obtain $(Ax)_n \to L(x)$.

Thus Ax is convergent. Since ϕ is continuous,

$$\phi((Ax)_n) \to \phi(L(x)).$$

Hence $Ax \in c(\phi)$. Therefore $A \in (\ell_\phi : c(\phi))$. This proves the sufficiency. □

As a consequence of Theorem 3.4, we obtain a characterization of the class $(\ell_\phi : c_0(\phi))$.

Theorem 3.5. *Let ϕ and ψ be a pair of complementary Orlicz functions. An infinite matrix $A = (a_{nk})$ belongs to $(\ell_\phi : c_0(\phi))$ if and only if*

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$

$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty,$$

and

$$\lim_{n\to\infty} a_{nk} = 0, \quad (k \in \mathbb{N}).$$

Proof. ($\Rightarrow$): Assume that $A \in (\ell_\phi : c_0(\phi))$. Since $c_0(\phi) \subset c(\phi)$, it follows that $A \in (\ell_\phi : c(\phi))$. Hence, by Theorem 3.4,

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$

and

$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty.$$

Fix $k \in \mathbb{N}$. Since $e^{(k)} \in \ell_\phi$, we have

$$Ae^{(k)} = (a_{nk})_{n\geq 1} \in c_0(\phi).$$

Therefore,

$$\phi(a_{nk}) \to 0, \quad (n \to \infty).$$

Since every Orlicz function is continuous and $\phi(t) = 0$ if and only if $t = 0$, it follows that

$$a_{nk} \to 0, \quad (n \to \infty).$$

Thus the necessity is established.

($\Leftarrow$): Assume that

$$(a_{nk})_{k\geq 1} \in \ell_\psi, \quad (n \in \mathbb{N}),$$

$$\sup_{n\geq 1} \|(a_{nk})_{k\geq 1}\|_{\ell_\psi} < \infty,$$

and

$$a_{nk} \to 0, \quad (n \to \infty)$$

for every $k \in \mathbb{N}$.

Applying Theorem 3.4 with $\alpha_k = 0$ for all $k \in \mathbb{N}$, it follows that

$$(Ax)_n \to 0, \quad (n \to \infty)$$

for every $x \in \ell_\phi$. Hence

$$\phi((Ax)_n) \to \phi(0) = 0,$$

so that $Ax \in c_0(\phi)$. Therefore $A \in (\ell_\phi : c_0(\phi))$. This completes the proof. □

Concluding Remarks

We have obtained characterizations of the matrix classes

$$(\ell_\phi : \ell_{\phi,\infty}), \qquad (\ell_\phi : c(\phi)), \qquad (\ell_\phi : c_0(\phi)).$$

The results show that the behavior of matrix transformations from ℓ_ϕ is completely determined by the uniform ℓ_ψ-boundedness of the matrix rows together with suitable convergence conditions on the matrix entries.

More precisely, Theorem 3.3 characterizes the class $(\ell_\phi : \ell_{\phi,\infty})$ through the uniform boundedness of the row norms in ℓ_ψ. Theorem 3.4 shows that the additional existence of the limits

$$\lim_{n\to\infty} a_{nk}$$

for every fixed k characterizes the class $(\ell_\phi : c(\phi))$, while Theorem 3.5 identifies $(\ell_\phi : c_0(\phi))$ by requiring these limits to be zero.

Thus, the transition from $\ell_{\phi,\infty}$ to $c(\phi)$ and further to $c_0(\phi)$ is governed solely by the asymptotic behavior of the matrix columns, whereas the uniform ℓ_ψ-boundedness of the rows remains the fundamental boundedness condition throughout.

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