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Statistical Convergence and its Effects on Some Property of Analysis

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Abstract. In this paper, we investigate topological properties of the subspace $\mathcal{C}_{\text{st}}^0$, of the statistical convergence sequences of $\mathcal{C}_{\text{st}}$ which are statistically convergen to zero. We show that it is a closed hyperplane subspace of $\mathcal{C}_{\text{st}}$ and give some results on its structure. We give a sufficient condition for Riemann integrability in terms of statistical convergence of sample values. Also we show that uniform statistical convergence of Riemann sums over all tag choices fully implies classical Riemann integrability. Finally, we bridge statistical convergence and ordinary Cesàro mean convergence for bounded linear operator iterates, establishing that all statistical orbit limits are fixed points.

1. Introduction

The concept of statistical convergence of the real-valued sequences was independently introduced by H. Fast [7] and H. Steinhaus [17].

After that, the statistical convergence has been enterd to many branch of mathematics (see, e.g, [2–4, 12, 13, 15, 19]).

Let $A \subseteq \mathbb{N}$. The natural density of A, d(A), is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right|,$$

where $|.|$ denotes the cardinal of a finite set.

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to have a property P for almost all n (abbreviated as a.a.n.) if the set of all n for which x_n fails to have property P has natural density zero.

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Definition 1.1 ([8])**.** The real-valued sequence $\{x_n\}_n$ is statistically convergent to a real number l, if for every $\epsilon > 0$,

$$d\left(\{k \in \mathbb{N} : |x_k - l| \geq \epsilon\}\right) = 0.$$

In this case we write,

$$\operatorname*{st–lim}_{n\to\infty} x_n = l.$$

It is clear that every convergent sequence is statistically convergent; however, as the following example demonstrates, the converse is not true.

Example 1.2. Let $\alpha \neq \beta$ in $\mathbb{R}$ and $\{x_n\}_n$ be a sequence such that,

$$x_n = \begin{cases} \alpha + \dfrac{1}{n}, & n \text{ is not a perfect square,} \\ \beta, & n \text{ is a perfect square.} \end{cases}$$

Since the set of perfect squares $K = \left\{1^2, 2^2, 3^2, \ldots\right\}$ has natural density zero (i.e. $d(K) = \lim\limits_{n\to\infty} \frac{\lfloor\sqrt{n}\rfloor}{n} = 0$), then $\operatorname*{st–lim}\limits_{n\to\infty} x_n = \alpha$, while $\{x_n\}_n$ does not converge in the usual sense.

Example 1.2 shows that for every $\alpha \in \mathbb{R}$, there exists a sequence $\{x_n\}_n$ that statistically converges to α, but dose not convergent in the usual sense.

The following theorem states a characterization of statistical convergence based on the results in [8, 11, 12].

Theorem 1.3. *A sequence $x = \{x_n\}_{n=1}^{\infty}$ is statistically convergent to L if and only if there exists $H = \{n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N}$ with $d(H) = 1$ such that $\lim\limits_{k\to\infty} x_{n_k} = L$, i.e.*

$$\lim_{\substack{n\to\infty \\ n\in H}} x_n = L.$$

Lemma 1.4. *Let X and Y be Banach spaces and let $A : X \to Y$ be a continuous linear operator. If*

$$\operatorname*{st–lim}_{n\to\infty} x_n = x_0$$

in X, then

$$\operatorname*{st–lim}_{n\to\infty} Ax_n = Ax_0$$

in Y.

Proof. The same proof of in [16, Lemma 2.2] can extend to continuous operators between Banach spaces. □

A result similar to [4, Theorem 2.2] can be stated for normed space as the following lemma

Lemma 1.5. *Let $\{x_n\}_n$ be a bounded sequence in a normed space X. If*

$$\operatorname*{st–lim}_{n\to\infty} x_n = x_0,$$

then

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} x_k - x_0 \right\| = 0.$$

Proof. Let $\{x_n\}_n$ be a bounded sequence in X. Then there exists $M > 0$ such that $\|x_n - x_0\| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and define

$$K_\varepsilon = \{k \in \mathbb{N} : \|x_k - x_0\| \geq \varepsilon\}.$$

By statistical convergence, $d(K_\varepsilon) = 0$. For every $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \frac{1}{n}\sum_{k=0}^{n-1} x_k - x_0 \right\| &\leq \frac{1}{n}\sum_{k=0}^{n-1} \|x_k - x_0\| \\ &\leq \varepsilon + M\frac{|K_\varepsilon \cap \{0,\ldots,n-1\}|}{n}. \end{aligned}$$

Taking the upper limit as $n \to \infty$ gives

$$\limsup_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} x_k - x_0 \right\| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the desired conclusion follows. □

Remark 1.6. The sequence

$$\sigma_n := \frac{1}{n}\sum_{k=0}^{n-1} x_k$$

is called Cesàro mean sequence of $\{x_n\}_n$. Lemma 1.5 shows that whenever $\{x_n\}_n$ is bounded and statistically convergent, its Cesàro mean sequence ordinary converges in norm to the same limit.

Let $\mathcal{C}_{\text{st}}$ be the real vector space of statistically convergent sequences.

Definition 1.7 ([16])**.** For any $x, y \in \mathcal{C}_{\text{st}}$, define

$$\langle x, y\rangle = \operatorname*{st–lim}_{n\to\infty} x_n y_n.$$

$\mathcal{C}_{\text{st}}$ with together $\langle ., .\rangle$ is a semi-inner product, because it does not have the positive definiteness property. Indeed, for the non-zero sequence $x = \{1/n\} \in \mathcal{C}_{\text{st}}$,

$$\langle x, x\rangle = \operatorname*{st–lim}_{n\to\infty} \frac{1}{n^2} = 0.$$

They also define,

$$x \sim y \text{ if and only if } \operatorname*{st–lim}_{n\to\infty} x_n = \operatorname*{st–lim}_{n\to\infty} y_n, \quad \text{where } x, y \in \mathcal{C}_{\text{st}}.$$

Clearly the relation $\sim$ is an equivalence relation on $\mathcal{C}_{\text{st}}$. Therefore

$$\widetilde{\mathcal{C}}_{\text{st}} := \mathcal{C}_{\text{st}}/\sim,$$

is the quotient space of $\mathcal{C}_{\text{st}}$. For $x \in \mathcal{C}_{\text{st}}$, let $[x]$ be its equivalence class.

Their approach, which parallels the classical construction of L^p-spaces via quotienting by equivalence classes, provides a natural setting for studying topological and algebraic properties of statistically convergent sequences.

Put

$$\langle [x], [y] \rangle_{\widetilde{\mathcal{C}}_{\text{st}}} = \operatorname*{st–lim}_{n\to\infty} x_n y_n, \quad \text{where } x, y \in \mathcal{C}_{\text{st}}.$$

Then $\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{C}}_{\text{st}}}$ is a real inner product on $\widetilde{\mathcal{C}}_{\text{st}}$. Moreover,

$$\langle [x], [x] \rangle_{\widetilde{\mathcal{C}}_{\text{st}}} = 0,$$

implies that $\operatorname*{st–lim}_{n\to\infty} x_n = 0$, so $x \sim 0$ and $[x] = [0]$.

On the original sequence space $\mathcal{C}_{\text{st}}$, the expression

$$d_J(x, y) = \operatorname*{st–lim}_{n\to\infty} |x_n - y_n|,$$

is a pseudo-metric rather than a metric. It becomes a metric after passing to equivalence classes. The quotient space of $\mathcal{C}_{\text{st}}$ endowed with this inner product is a Hilbert space (see [16]).

Let $\mathcal{C}^0_{\text{st}}$ be the non-trivial subspace of $\mathcal{C}_{\text{st}}$ is defined by,

$$\mathcal{C}^0_{\text{st}} = \left\{ x = \{x_n\} \in \mathcal{C}_{\text{st}} : \operatorname*{st–lim}_{n\to\infty} x_n = 0 \right\}.$$

In this paper, we investigate the interplay between statistical convergence and three areas of analysis: the structure of statistically convergent sequence spaces, Riemann integrability and the ergodic behavior of linear operators. In the second section, by studying the subspace $\mathcal{C}^0_{\text{st}}$ of $\mathcal{C}_{\text{st}}$, we show that it is a closed hyperplane in the space of all statistically convergent sequences. Moreover, we prove that the quotient space $\mathcal{C}_{\text{st}}/\mathcal{C}^0_{\text{st}}$ is isometrically isomorphic to $\mathbb{R}$. In section three, we turn to Riemann integration and introduce a sufficient condition for Riemann integrability in terms of statistical convergence of Riemann sums over all possible tag choices; this demonstrates that a statistical convergence condition is strong enough to force classical integrability. Finally in section 4, we bridge the concept of statistical convergence with the theory of mean ergodic operators. This result establishes a direct connection between pointwise statistical convergence of iterates and the global ergodic behavior of the operator.

2. Structural Properties of $\mathcal{C}^0_{\mathrm{ST}}$

In view of the significance of the sequences in $\mathcal{C}^0_{\mathrm{st}}$ among all sequences in $\mathcal{C}_{\mathrm{st}}$, we now examine some properties of the subspace $\mathcal{C}^0_{\mathrm{st}}$.

Lemma 2.1. *Let d_J be the pseudo-metric on $\mathcal{C}_{st}$. Then,*

$$x \in \mathcal{C}^0_{st} \quad \text{if and only if} \quad d_J(x,0) = 0.$$

Proof. If $x = \{x_n\} \in \mathcal{C}^0_{\mathrm{st}}$, then

$$\operatorname*{st–lim}_{n\to\infty} x_n = 0.$$

By Lemma 1.4,

$$\operatorname*{st–lim}_{n\to\infty} |x_n| = 0,$$

and therefore,

$$d_J(x,0) = \operatorname*{st–lim}_{n\to\infty} |x_n| = 0.$$

Conversely, if $d_J(x,0) = 0$, then

$$\operatorname*{st–lim}_{n\to\infty} |x_n| = 0,$$

which implies

$$\operatorname*{st–lim}_{n\to\infty} x_n = 0.$$

Thus, $x \in \mathcal{C}^0_{st}$. This completes the proof. □

Proposition 2.2. *$\mathcal{C}^0_{st}$ is a closed linear subspace of $\mathcal{C}_{st}$ with respect to d_J.*

Proof. Let $\{x^{(m)}\}_m \subseteq \mathcal{C}^0_{st}$ be a sequence. Suppose $d_J(x^{(m)}, x) \to 0$ as $m \to \infty$. Since $d_J(x^{(m)}, 0) = 0$ for every m, by Lemma 2.1,

$$d_J(x,0) \le d_J\left(x, x^{(m)}\right) + d_J(x^{(m)}, 0) = d_J\left(x, x^{(m)}\right) \to 0,$$

as $m \to \infty$. Therefore $x \in \mathcal{C}^0_{st}$. □

Example 2.3. The function $f : \mathcal{C}_{\mathrm{st}} \to \mathbb{R}$, defined by $f(x) = \operatorname*{st–lim}_{n\to\infty} x_n$, is a non-zero linear functional and

$$\ker f = \mathcal{C}^0_{\mathrm{st}}.$$

Moreover, f is a surjective continuous function with respect to d_J.

Proof. Linearity of f follows from the linearity of statistical limit. Since $f(1,1,\ldots) = 1$, we have $f \neq 0$ and by definition of f, $\ker f = \mathcal{C}^0_{\mathrm{st}}$. Surjectivity of f follows from Example1.2. Also

$$|f(x) - f(y)| = |\text{st–lim}(x_n - y_n)| \le \text{st–lim}\,|x_n - y_n| = d_J(x,y).$$

Thus f is 1-Lipschitz and so is continuous. □

Corollary 2.4. *The space $\mathcal{C}^0_{st}$ is a hyperplane in the vector space $\mathcal{C}_{st}$.*

Proof. Since $f \neq 0$ and $\ker f = \mathcal{C}_{\mathrm{st}}^0$, the first isomorphism theorem gives,

$$\mathcal{C}_{\mathrm{st}}/\mathcal{C}_{\mathrm{st}}^0 \cong \mathbb{R}.$$

Hence

$$\dim\left(\mathcal{C}_{st}/\mathcal{C}_{st}^0\right) = 1.$$

Therefore $\mathcal{C}_{\mathrm{st}}^0$ is a hyperplane. □

Theorem 2.5 (A Direct-Sum Decomposition)**.** *There exists $y \in \mathcal{C}_{st} \setminus \mathcal{C}_{st}^0$ with $f(y) = 1$ such that*

$$\mathcal{C}_{st} = \mathcal{C}_{st}^0 \oplus \operatorname{span}\{y\}.$$

Proof. Take $y = (1, 1, \ldots)$. Then $f(y) = 1$. For $x \in \mathcal{C}_{st}$, put $t = f(x)$. Then

$$f(x - ty) = f(x) - tf(y) = 0,$$

and so $x - ty \in \mathcal{C}_{st}^0$. Hence

$$x = (x - ty) + ty.$$

If $z \in \mathcal{C}_{st}^0 \cap \operatorname{span}\{y\}$, then $z = cy$ and $0 = f(z) = c$, so $z = 0$. Thus the proof is complete. □

Note that the vector y in Theorem 2.5 is not unique. In fact any y satisfying $f(y) = 1$ gives the same direct-sum decomposition. For any $y' \neq 0$ with $f(y') = 1$, we have

$$f(y - y') = 0,$$

so $y \sim y'$. Thus the equivalence class $[y]$ is unique.

Theorem 2.6 (Isometric Identification with $\mathbb{R}$)**.** *The quotient Hilbert space*

$$\widetilde{\mathcal{C}_{st}} = \mathcal{C}_{st}/\sim$$

is isometrically isomorphic to $\mathbb{R}$.

Proof. Define $T : \widetilde{\mathcal{C}_{st}} \to \mathbb{R}$,

$$T([x]) = \operatorname*{st\text{–}lim}_{n\to\infty} x_n.$$

The map is well-defined by the definition of $\sim$. It is linear by linearity of statistical limits. If $T([x]) = T([y])$, then

$$\operatorname*{st\text{–}lim}_{n\to\infty} x_n = \operatorname*{st\text{–}lim}_{n\to\infty} y_n,$$

so $x \sim y$ and $[x] = [y]$. Hence T is injective.

For any $a \in \mathbb{R}$, the constant sequence $x_n = a$ satisfies $T([x]) = a$. Thus T is onto.

Finally,

$$\|[x]\|_{\widetilde{\mathcal{C}_{st}}} = \left|\operatorname*{st\text{–}lim}_{n\to\infty} x_n\right| = |T([x])| = \|T([x])\|_{\mathbb{R}}.$$

Therefore T is an isometric isomorphism. □

3. Statistical Convergence and Riemann Integrability

In this section, we demonstrate that when statistical convergence is applied to the sequence of Riemann sums uniformly across all possible tag choices, it becomes sufficiently robust to force classical Riemann integrability.

Definition 3.1 (Uniform Partition and Riemann Sum [1])**.** Let $f \in B[0,1]$ be a bounded real-valued function on $[0,1]$. For each $n \in \mathbb{N}$, let

$$P_n = \left\{ x_i = \frac{i}{n} : i = 0, 1, \ldots, n \right\}$$

denote the uniform partition of $[0,1]$. A choice of tags for P_n is a sequence $T^{(n)} = \left(t_1^{(n)}, \ldots, t_n^{(n)}\right)$ such that $t_i^{(n)} \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, n$. The Riemann sum associated with P_n and $T^{(n)}$ is given by

$$S_n\left(f, T^{(n)}\right) = \frac{1}{n} \sum_{i=1}^{n} f\left(t_i^{(n)}\right).$$

Lemma 3.2. *Let f be a bounded function on $[0,1]$ and let P_n be the uniform partition. For each $n \in \mathbb{N}$ and each $\varepsilon > 0$, there exists a tag sequence $T_U^{(n)}$ such that*

$$U_n(f) - \varepsilon < S_n\left(f, T_U^{(n)}\right) \leq U_n(f),$$

where $U_n(f)$ is the upper Darboux sum of f with respect to P_n.

Proof. For each $i \in \{1, \ldots, n\}$, by the definition of supremum, there exists $t_i^{(n)} \in [x_{i-1}, x_i]$ such that

$$M_i^{(n)} - \varepsilon < f\left(t_i^{(n)}\right) \leq M_i^{(n)},$$

where $M_i^{(n)} = \sup_{x \in [x_{i-1}, x_i]} f(x)$. Summing over i and dividing by n yields the desired inequality. □

Theorem 3.3 (Statistical Characterization of Riemann Integrability)**.** *Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Suppose there exists a constant $l \in \mathbb{R}$ such that for **every** sequence of tag choices $T = \{T^{(n)}\}_{n=1}^{\infty}$, the sequence of Riemann sums $\{S_n\left(f, T^{(n)}\right)\}_{n=1}^{\infty}$ is statistically convergent to l, i.e.,*

$$\operatorname*{st\text{–}lim}_{n \to \infty} S_n\left(f, T^{(n)}\right) = l.$$

Then f is Riemann integrable in the classical sense on $[0,1]$ and

$$\int_0^1 f(x)\,dx = l.$$

Proof. Since f is bounded on $[0,1]$, the upper and lower Darboux integrals $U(f)$ and $L(f)$ are finite real numbers. For each uniform partition P_n, let $U_n(f)$ and $L_n(f)$ be the upper and lower Darboux sums:

$$U_n(f) = \frac{1}{n}\sum_{i=1}^{n} M_i^{(n)}, \qquad M_i^{(n)} = \sup_{x\in[x_{i-1},x_i]} f(x),$$

$$L_n(f) = \frac{1}{n}\sum_{i=1}^{n} m_i^{(n)}, \qquad m_i^{(n)} = \inf_{x\in[x_{i-1},x_i]} f(x).$$

It is a standard result of classical analysis that,

$$\lim_{n\to\infty} U_n(f) = U(f), \qquad \lim_{n\to\infty} L_n(f) = L(f).$$

We first show that $U(f) = l$. Fix $\varepsilon > 0$. By Lemma 3.2, for each $n \in \mathbb{N}$, there exists a tag sequence $T_U^{(n)}$ such that

$$U_n(f) - \varepsilon < S_n\left(f, T_U^{(n)}\right) \le U_n(f).$$

By taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} U(f) - \varepsilon &\le \liminf_{n\to\infty} S_n\left(f, T_U^{(n)}\right) \\ &\le \limsup_{n\to\infty} S_n\left(f, T_U^{(n)}\right) \\ &\le U(f). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the sequence $\left\{S_n\left(f, T_U^{(n)}\right)\right\}_{n=1}^{\infty}$ converges ordinarily to $U(f)$,

$$\lim_{n\to\infty} S_n\left(f, T_U^{(n)}\right) = U(f).$$

Since ordinary convergence implies statistical convergence, we have

$$\operatorname{st–lim}_{n\to\infty} S_n\left(f, T_U^{(n)}\right) = U(f).$$

By the hypothesis, the statistical limit must be l for every tag sequence. Hence,

$$U(f) = l.$$

By similar argument using the infimum $m_i^{(n)}$, there exists a tag sequence $T_L^{(n)}$ such that,

$$\operatorname{st–lim}_{n\to\infty} S_n\left(f, T_L^{(n)}\right) = L(f).$$

Applying the hypothesis again yields

$$L(f) = l.$$

Therefore, $U(f) = L(f) = l$. By Darboux's integrability criterion, f is Riemann integrable on $[0,1]$ with $\int_0^1 f(x)\,dx = l$. □

Recall that the Cantor set, $\mathcal{C} \subseteq [0,1]$ is defined as:

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n,$$

where $C_0 = [0,1]$ and

$$C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n\right).$$

Proposition 3.4. *Let $\mathcal{C}$ be the Cantor set in $[0,1]$ and let $I_{n,k} = \left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $k = 1, \ldots, n$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \in \{1,\ldots,n\} : I_{n,k} \cap \mathcal{C} \neq \emptyset\}\right| = 0.$$

Proof. The Cantor set $\mathcal{C}$ has Lebesgue measure zero and is compact [5]. Hence, for every $\varepsilon > 0$, there exists a finite collection of open intervals $\{J_1, \ldots, J_N\}$ such that

$$\mathcal{C} \subseteq \bigcup_{j=1}^{N} J_j, \qquad \sum_{j=1}^{N} |J_j| < \varepsilon.$$

If $I_{n,k} \cap \mathcal{C} \neq \emptyset$, then there exists at least one j such that $I_{n,k} \cap J_j \neq \emptyset$. For each fixed j, the number of intervals $I_{n,k}$ intersecting J_j is at most $n|J_j| + 2$. Therefore,

$$\left|\{k : I_{n,k} \cap \mathcal{C} \neq \emptyset\}\right| \leq \sum_{j=1}^{N} (n|J_j| + 2) < n\varepsilon + 2N.$$

Dividing by n and taking lim sup gives

$$\limsup_{n\to\infty} \frac{1}{n} \left|\{k : I_{n,k} \cap \mathcal{C} \neq \emptyset\}\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the limit is zero. □

Here we are ready to give an example to guarantee Theorem 3.2.

Example 3.5. Let $\chi_{\mathcal{C}} : [0,1] \to \{0,1\}$ be the characteristic function of the Cantor set:

$$\chi_{\mathcal{C}}(x) = \begin{cases} 1, & x \in \mathcal{C}, \\ 0, & x \notin \mathcal{C}. \end{cases}$$

Then $\chi_{\mathcal{C}}$ satisfies the hypothesis of Theorem 3.3 with $l = 0$. Hence, $\chi_{\mathcal{C}}$ is Riemann integrable on $[0, 1]$ and

$$\int_0^1 \chi_{\mathcal{C}}(x)\, dx = 0.$$

In fact for any sequence of tag choices $\{T^{(n)}\}_{n=1}^{\infty}$ with

$$t_i^{(n)} \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

Riemann sum is

$$S_n\left(\chi_{\mathcal{C}}, T^{(n)}\right) = \frac{1}{n}\sum_{i=1}^{n} \chi_{\mathcal{C}}\left(t_i^{(n)}\right).$$

For each n, if $\chi_{\mathcal{C}}\left(t_i^{(n)}\right) = 1$, then $t_i^{(n)} \in \mathcal{C}$ and hence $I_{n,i} \cap \mathcal{C} \neq \emptyset$. Therefore,

$$\left\{i : \chi_{\mathcal{C}}\left(t_i^{(n)}\right) = 1\right\} \subseteq \left\{i : I_{n,i} \cap \mathcal{C} \neq \emptyset\right\}.$$

Consequently,

$$0 \leq S_n\left(\chi_{\mathcal{C}}, T^{(n)}\right) \leq \frac{1}{n}\left|\left\{i : I_{n,i} \cap \mathcal{C} \neq \emptyset\right\}\right|,$$

and the right-hand side tends to zero by Proposition 3.4. Thus

$$\lim_{n\to\infty} S_n\left(\chi_{\mathcal{C}}, T^{(n)}\right) = 0.$$

Since the tag sequence was arbitrary, the hypothesis of Theorem is satisfied with $l = 0$. Therefore, $\chi_{\mathcal{C}}$ is Riemann integrable with integral 0.

4. From Statistical Convergence of Iterates to Mean Ergodicity

Now we are going to present the fixed point property for a bounded linear operator under a statistical convergence condition on its iterates.

let X be a Banach space and $A \in B(X)$ be a bounded linear operator, with $A^0 := I$ and $A^n := A \circ A^{n-1}$ for $n \in \mathbb{N}$. The Cesàro mean of the iterates of A is defined by

$$M_n := \frac{1}{n}\sum_{k=0}^{n-1} A^k, \quad (n \in \mathbb{N}).$$

Recall that A is called mean ergodic if the sequence of Cesàro mean $\{M_n x\}_{n=1}^{\infty}$ converges in the norm topology of X for every $x \in X$ (see, e.g., [6, 9]).

Equivalently, there exists a bounded linear operator $P \in B(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} A^k x = Px, \quad (x \in X),$$

where P is a projection onto the fixed space $\ker(I - A)$. In the classical mean ergodic theorem, due to von Neumann[18] and later generalized by Riesz[14], Yosida[20] and Kakutani[10], it is known that if A is power bounded (i.e., $\sup_{n\in\mathbb{N}} \|A^n\| < \infty$) and X is reflexive, then A is mean ergodic. However, in our setting, we do not assume reflexivity; instead, we derive mean ergodicity from a statistical convergence assumption on the orbits.

Theorem 4.1. *Let X be a Banach space and $A \in B(X)$ be a power bounded operator, i.e.,*

$$\sup_{n\in\mathbb{N}} \|A^n\| < \infty.$$

Suppose there exists $x \in X$ such that the sequence $\{A^n x\}_{n=1}^{\infty}$ is statistically convergent to some $x_* \in X$. Then x_* is a fixed point of A, i.e.,

$$Ax_* = x_*.$$

Proof. For power bounded operator A, there exists $C > 0$ such that

$$\|A^n\| \leq C, \quad (n \geq 0).$$

So, for every $x \in X$,

$$\|A^n x\| \leq C\|x\|,$$

and then the orbit $\{A^n x\}$ is bounded.

By assumption,

$$\operatorname*{st–lim}_{n\to\infty} A^n x = x_*.$$

Applying Lemma 1.5 to the bounded sequence $A^n x$ yields

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} A^k x - x_* \right\| = 0.$$

Let $f \in X^*$ be arbitrary. By Lemma 1.4 applied to the continuous linear functional f,

$$\operatorname*{st–lim}_{n\to\infty} f\left(A^n x\right) = f(x_*).$$

Since A is power bounded, the sequence $\{f\left(A^n x\right)\}$ is a bounded sequence that statistically converges to $f(x_*)$. By Remark 1.6,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(A^k x\right) = f(x_*).$$

By the linearity of f we have,

$$\lim_{n\to\infty} f\left(\frac{1}{n}\sum_{k=0}^{n-1} A^k x\right) = f(x_*).$$

Now, using the continuity of A and the above limit gives,

$$f(Ax_*) = \lim_{n\to\infty} f\left(A(\frac{1}{n}\sum_{k=0}^{n-1} A^k x)\right) = \lim_{n\to\infty} f\left(\frac{1}{n}\sum_{k=0}^{n-1} A^{k+1} x\right).$$

Thus,

$$f(Ax_* - x_*) = \lim_{n\to\infty} f\left(\frac{1}{n}\sum_{k=0}^{n-1} \left(A^{k+1} - A^k\right) x\right).$$

But,

$$\frac{1}{n}\sum_{k=0}^{n-1} \left(A^{k+1} - A^k\right) x = \frac{1}{n}\left(A^n x - x\right).$$

By power boundedness of A, $\frac{1}{n}\left(A^n x - x\right) \to 0$ as $n \to \infty$. Hence,

$$f(Ax_* - x_*) = \lim_{n\to\infty} f\left(\frac{1}{n}\left(A^n x - x\right)\right) = 0.$$

Since $f \in X^*$ was arbitrary, the Hahn-Banach theorem implies,

$$Ax_* - x_* = 0.$$

Therefore, $Ax_* = x_*$. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, if the hypothesis holds for* ***every*** $x \in X$ *(with possibly different limits* $x_0(x) \in X$*), then* A *is mean ergodic. More precisely, for every* $x \in X$*,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} A^k x = x_0(x),$$

and $x_0(x) \in \mathrm{Fix}(A)$.

Proof. By Theorem 4.1, for each $x \in X$, the statistical limit $x_0(x)$ satisfies $Ax_0(x) = x_0(x)$. Applying Remark1.6 to the bounded sequence $\{A^n x\}$ gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} A^k x = x_0(x),$$

for any $x \in X$. Hence A is mean ergodic. □

Remark 4.3. Theorem 4.1 shows that statistical convergence of even a single orbit implies a fixed point for the operator. When this property holds uniformly for all vectors, the operator becomes mean ergodic. This demonstrates how a pointwise statistical condition can lead to a global ergodic property.

The Corollary 4.2 is related to the statistical ergodic results obtained by Oğuz [13]. The two approaches, however, start from different assumptions and pursue different directions.

Oğuz in[13], by considering statistical convergence of Cesàro means

$$M_n(A)x = \frac{1}{n}\sum_{k=0}^{n-1} A^k x,$$

establishes statistical mean and uniform ergodic results under operator-theoretic boundedness conditions.

Here with different approach on the statistical convergence of the orbit of the power bounded linear operator A, we obtain the ordinary norm convergence of the Cesàro means. Furthermore, we showed that the statistical limit is in fact a fixed point of the operator A.

Thus, the present result may be viewed as a complementary functional-analytic direction: statistical convergence of the iterates is used as the starting point, while ordinary mean ergodicity and the fixed-point property are obtained as consequences.

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