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Classification of Six-Dimensional Real Nilpotent Lie Bialgebras of Symplectic Type and Their Poisson-Lie Groups

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Abstract. In this paper, we classify all six-dimensional real nilpotent Lie bialgebras of symplectic type. The Poisson structures on all of the related six-dimensional Poisson-Lie groups are obtained. Some new integrable Hamiltonian systems for which the Poisson-Lie group plays the role of a phase space and its dual Lie group plays the role of a symmetry group of the system are obtained.

1. Introduction

The subject of classical integrable systems is closely connected to the geometry and to the representation theory of Poisson–Lie groups, together with their associated Lie bialgebras [11, 12]. Classical r-matrix and Yang–Baxter approaches provide another foundational route to Lie-bialgebra structures [7, 27]. In recent years, non-semisimple Lie algebras have proved to be significant in various physical applications. Earlier low-dimensional studies relevant to Lie-bialgebra and Poisson–Lie structures include classifications of triangular and three-dimensional Lie bialgebras, classifications of six-dimensional Manin triples and related Poisson–Lie constructions [15, 17, 19, 21, 26, 28], as well as classifications of Lie superbialgebras [13]. In this paper, we classify six-dimensional real nilpotent Lie bialgebras of symplectic type for which both the Lie algebra $\mathbf{g}$ and its dual $\tilde{\mathbf{g}}$ admit symplectic structures, up to the equivalence specified in Section 3. The classification is carried out by applying the established procedure previously used in [13]; no new classification method is introduced here. The principal contribution of the present work lies

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instead in the resulting classification in the six-dimensional real nilpotent symplectic setting. This should be distinguished from our earlier work [3]. In [3], compatible pairs of Poisson tensors were constructed directly on selected symplectic Lie groups and were used, through the Magri–Morosi framework, to obtain bi-Hamiltonian systems. In the six-dimensional nilpotent part of that work, explicit compatible Poisson pairs were displayed for $A_{6,1}$, $A_{6,7}$, $A_{6,9}$, $A_{6,24}$, $A_{6,25}$, $A_{6,26}$, $A_{6,27}$ and $A_{6,32}$. The present paper addresses a different classification problem: it first classifies Lie-bialgebra structures $(\mathbf{g}, \tilde{\mathbf{g}})$ for which both members are of symplectic type and then derives the multiplicative Poisson structures associated with those Lie-bialgebra data. Thus, the novelty claimed here is not the first consideration of six-dimensional nilpotent symplectic Lie groups, nor a new general method for constructing Poisson tensors, but the Lie-bialgebra classification and the Poisson–Lie structures attached to the resulting bialgebra representatives. The motivation for this classification is to construct Hamiltonian models in which the Lie group G integrating $\mathbf{g}$ serves as the phase space and the dual Lie group $\tilde{G}$ integrating $\tilde{\mathbf{g}}$ supplies the symmetry data (or vice versa).

The direct predecessor of the present classification is our earlier four-dimensional classification of real Lie bialgebras of symplectic type [2]. The six-dimensional calculation deliberately retains the same classification principle: one solves the Jacobi and mixed Jacobi identities, identifies the resulting dual Lie algebra and then reduces the solutions by automorphisms of the original algebra. The change from dimension four to dimension six is therefore a change in the scale and branching of the classification problem, not a new classification methodology. In dimension six the primal algebra ranges over the nilpotent symplectic types listed in Table 1. Moreover, before the Jacobi and mixed-Jacobi constraints are imposed, a skew bracket on a six-dimensional vector space has up to $\binom{6}{2}6 = 90$ structure-constant entries, compared with $\binom{4}{2}4 = 24$ in dimension four. This count is only an indication of the size of the polynomial systems, since the identities subsequently impose many relations. More importantly, the six-dimensional reduction contains generic and singular parameter branches that must be treated separately; the exceptional branches for $\mathbf{A}_{6,8}$ discussed in Section 3 provide an explicit example. Finally, the same underlying six-dimensional Lie algebra may occur with several inequivalent dual structures, so non-redundancy has to be checked at the Lie-bialgebra level rather than inferred from the isomorphism type of either member alone. Thus the additional six-dimensional work lies in organizing a substantially larger polynomial

case analysis, treating exceptional parameter loci and establishing completeness and non-equivalence of the retained representatives. We accordingly frame the result as a higher-dimensional classification obtained by an established method.

To place this problem in the current literature, we also note several developments after the classical references on which the basic definitions rely. Fischer's revised treatment of six-dimensional nilpotent symplectic Lie algebras [16] gives a modern classification framework for the symplectic side of the problem, while Ait Aissa and Mansouri [6] study additional symplectic/Novikov structures and include a classification of nilpotent six-dimensional Lie algebras admitting such structures. On the Lie-bialgebra side, recent classification work has continued in other settings: Abedin and Burban [4] study Lie bialgebras arising from solutions of the classical Yang–Baxter equation by algebro-geometric methods and Abedin, Maximov, Stolin and Zelmanov [5] classify topological Lie-bialgebra structures over formal current algebras. These works concern different classes of Lie algebras and do not provide a classification of finite-dimensional six-dimensional real nilpotent Lie bialgebras subject to the requirement that both $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are of symplectic type. Recent work also continues to exploit Poisson–Lie geometry in integrable systems and mathematical physics: see, for example, the review [24], the Poisson–Lie interpretation of pentagram-type integrable maps in [20] and Poisson reductions of integrable systems on doubles of compact Lie groups in [14]. In the low-dimensional Lie-group setting closest to our applications, compatible Poisson structures and bi-Hamiltonian systems were further studied in [18]. These results clarify the scope of the present contribution: our novelty claim is restricted to the finite-dimensional real nilpotent classification problem stated above and to the multiplicative Poisson–Lie tensors attached to its Lie-bialgebra representatives; we do not claim a new general theory of Lie bialgebras, Poisson–Lie groups, symplectic Lie algebras, or integrable systems. The outline of the paper is as follows. In Section 2, we recall the necessary definitions and notation. In Section 3, after presenting the relevant six-dimensional real nilpotent Lie algebras of symplectic type, we classify the corresponding Lie bialgebras by applying the established classification procedure described in [13]. In Section 4, we compute the multiplicative Poisson tensors on the corresponding Poisson–Lie groups using the map $\pi(g)$ between the Lie subalgebras $\mathbf{g}$ and $\tilde{\mathbf{g}}$. Finally, in Section 5, we apply the established Hamiltonian-realization procedure of [1] to two Lie-bialgebra examples obtained from the present classification. For one of these examples, the Lie group integrating $\mathbf{A}_{6,27}$ serves as the phase space, while

the dual Lie algebra $\mathbf{A}_{6,27.xiv}$ provides the symmetry algebra. Concluding remarks are given in Section 6.

2. Definitions and Notation

In this section, we recall standard definitions and propositions of Lie bialgebras [11, 12] (see for a review [23]). Let $\mathbf{g}$ be a finite-dimensional Lie algebra and $\mathbf{g}^*$ be its dual space with respect to a non-degenerate canonical pairing $(\cdot,\cdot)$.

Definition 2.1. A *Lie bialgebra* is a Lie algebra $\mathbf{g}$ with a skew-symmetric linear map $\delta : \mathbf{g} \to \mathbf{g} \otimes \mathbf{g}$ such that:
a) δ is a one-cocycle, i.e.:

$$\delta([X,Y]) = [\delta(X), 1\otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)], \quad \forall X, Y \in \mathbf{g},$$

where 1 is the identity map on $\mathbf{g}$.
b) $\delta^t : \mathbf{g}^* \otimes \mathbf{g}^* \to \mathbf{g}^*$ is a Lie bracket on $\mathbf{g}^*$:

$$(\xi \otimes \eta, \delta(X)) = (\delta^t(\xi \otimes \eta), X) = ([\xi,\eta]_{\mathbf{g}^*}, X) \qquad \forall X \in \mathbf{g};\ \xi, \eta \in \mathbf{g}^*. \tag{2.1}$$

The Lie bialgebra defined in this way will be denoted by $(\mathbf{g}, \mathbf{g}^*)$ or $(\mathbf{g}, \delta)$.

Proposition 2.2 ([26])**.** *If there exists an automorphism A of $\mathbf{g}$ such that*

$$\delta' = (A \otimes A) \circ \delta \circ A^{-1}, \tag{2.2}$$

then the one-cocycles δ and δ' of the Lie algebra $\mathbf{g}$ are equivalent. In this case, the two Lie bialgebras $(\mathbf{g}, \delta)$ and $(\mathbf{g}, \delta')$ are equivalent.

Definition 2.3. A Lie bialgebra $(\mathbf{g}, \mathbf{g}^*)$ is called a *coboundary* Lie bialgebra if there exists an element $r \in \mathbf{g} \otimes \mathbf{g}$ such that:

$$\delta(X) = [1 \otimes X + X \otimes 1, r], \quad \forall X \in \mathbf{g}. \tag{2.3}$$

Definition 2.4. A *Manin triple* is a triple of Lie algebras $(\mathbb{C}D, \mathbf{g}, \tilde{\mathbf{g}})$ with a non-degenerate ad-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}D$, so that

1. $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are Lie subalgebras of $\mathbb{C}D$,
2. $\mathbb{C}D = \mathbf{g} \oplus \tilde{\mathbf{g}}$ as a vector space,
3. $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isotropic with respect to $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle X_i, X_j\rangle = \left\langle \tilde{X}^i, \tilde{X}^j \right\rangle = 0, \qquad \left\langle X_i, \tilde{X}^j \right\rangle = \delta^j{}_i. \tag{2.4}$$

There is a one-to-one correspondence between Manin triples $(\mathbb{C}D, \mathbf{g}, \tilde{\mathbf{g}})$ with $\tilde{\mathbf{g}} = \mathbf{g}^*$ and Lie bialgebras $(\mathbf{g}, \mathbf{g}^*)$ [8]. If we choose the structure constants of Lie algebras $\mathbf{g}$ and $\tilde{\mathbf{g}}$ as follows:

$$[X_i, X_j] = f_{ij}{}^k X_k, \qquad \left[\tilde{X}^i, \tilde{X}^j\right] = \tilde{f}^{ij}{}_k \tilde{X}^k, \tag{2.5}$$

with them satisfying the following Jacobi identities:

$$f_{ij}{}^{k} f_{km}{}^{n} + f_{ik}{}^{n} f_{mj}{}^{k} + f_{jk}{}^{n} f_{im}{}^{k} = 0,$$

and

$$\tilde{f}^{ij}{}_{k}\ \tilde{f}^{km}{}_{n} + \tilde{f}^{im}{}_{k}\ \tilde{f}^{jk}{}_{n} + \tilde{f}^{jm}{}_{k}\ \tilde{f}^{ki}{}_{n} = 0, \tag{2.6}$$

then, ad-invariance of the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}D = \mathbf{g} \oplus \tilde{\mathbf{g}}$ implies that [8]-[23]

$$\left[X_i, \tilde{X}^j\right] = \tilde{f}^{jk}{}_{i} X_k + f_{ki}{}^{j} \tilde{X}^k,$$

where, using (2.4), (2.5) and (2.1) we obtain

$$\delta(X_i) = \tilde{f}^{jk}{}_{i} X_j \otimes X_k.$$

By applying the above relation in (1), one can obtain the following mixed Jacobi relation:

$$f_{kl}{}^{m} \tilde{f}^{ij}{}_{m} = f_{mk}{}^{i} \tilde{f}^{jm}{}_{l} - f_{ml}{}^{i} \tilde{f}^{jm}{}_{k} - f_{mk}{}^{j} \tilde{f}^{im}{}_{l} + f_{ml}{}^{j} \tilde{f}^{im}{}_{k}. \tag{2.7}$$

This relation can also be obtained from the Jacobi identity of $\mathbb{C}D$.

3. Classification of Six-Dimensional Real Nilpotent Lie Bialgebras of Symplectic Type

In this section, we classify six-dimensional real nilpotent Lie bialgebras of symplectic type for which both the Lie algebra $\mathbf{g}$ and its dual $\tilde{\mathbf{g}}$ admit symplectic structures. We emphasize that the classification procedure itself is not new: we apply the established method previously used in [13] for the classification of low-dimensional real Lie superbialgebras. The contribution of this section is the application of this procedure to the six-dimensional real nilpotent symplectic setting and the resulting classification of the corresponding Lie-bialgebra structures. We first recall the relevant six-dimensional real nilpotent Lie algebras admitting symplectic structures and then review the classification procedure used in the subsequent analysis.

3.1. Six-Dimensional Real Nilpotent Lie Algebras of Symplectic Type.

A symplectic structure on a $2n$-dimensional Lie algebra $\mathbf{g}$ is a two-form $\omega \in \Lambda^2 \mathbf{g}^*$ that is closed and nondegenerate. Here d denotes the Chevalley–Eilenberg (exterior) differential of the Lie algebra (see, for example, [10]). Equivalently,

$$d\omega = 0, \qquad \omega^n \neq 0 \in \Lambda^{2n} \mathbf{g}^*.$$

The latter condition is equivalent to maximal rank and, on the integrating Lie group, the corresponding left-invariant top-degree form is a volume form. We distinguish here between the classification of the

underlying six-dimensional real nilpotent Lie algebras and the determination of which of these Lie algebras admit symplectic structures. The notation for the underlying Lie algebras used in Table 1 follows the classification convention adopted from [25]; for a field-independent modern treatment of the classification of six-dimensional nilpotent Lie algebras, see also [9]. The symplectic admissibility of the Lie-algebra types retained in Table 1 is based on the classification of nilpotent symplectic Lie algebras of dimension at most six in [22], together with the later revised treatment of the six-dimensional case in [16]. More recent work such as [6] studies additional geometric algebra structures on nilpotent six-dimensional symplectic Lie algebras, but addresses a different classification problem. Thus, the citation to [25] concerns the underlying Lie-algebra classification and notation, not the existence of symplectic structures.

For explicit symplectic structures, we refer in particular to Theorem 5 of [22], where the six-dimensional nilpotent symplectic Lie algebras are presented together with closed nondegenerate two-forms and to Section 3 of [16], which gives a revised classification of the six-dimensional nilpotent symplectic case. These references therefore provide explicit representatives of the symplectic forms and the corresponding nondegeneracy conditions, rather than merely a list of Lie brackets.

As a direct verification in the notation of Table 1, consider for example $A_{6,15}$. If $\{e^1, \ldots, e^6\}$ denotes the basis dual to $\{X_1, \ldots, X_6\}$, its commutation relations in Table 1 give

$$de^1 = de^2 = 0, \qquad de^3 = -e^{12}, \qquad de^4 = -e^{13},$$

$$de^5 = -e^{12}, \qquad de^6 = -e^{14} - e^{25},$$

where $e^{ij} = e^i \wedge e^j$. The two-form

$$\omega = e^{16} + e^{24} + e^{35}$$

is closed, since $d\omega = 0$ and it is nondegenerate because

$$\omega^3 = \pm 6 e^{123456} \neq 0.$$

Thus, the symplectic property can also be checked directly from the structure equations in the notation used here.

Table 1: Six-dimensional real nilpotent Lie algebras of symplectic type.

g	Non-zero commutation relations	**g**	Non-zero commutation relations
$A_{6,1}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_5]=X_6$	$A_{6,2}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_5,\ [X_1,X_5]=X_6$
$A_{6,3}$	$[X_1,X_2]=X_6,\ [X_1,X_3]=X_4,\ [X_2,X_3]=X_5$	$A_{6,4}$	$[X_1,X_2]=X_5,\ [X_1,X_3]=X_6,\ [X_2,X_4]=X_6$
$A_{6,5}$	$[X_1,X_3]=X_5,\ [X_1,X_4]=X_6,\ [X_2,X_3]=kX_6$ $[X_2,X_4]=X_5,\ k=\pm1$	$A_{6,6}$	$[X_1,X_2]=X_6,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_5,\ [X_2,X_3]=X_5$
$A_{6,7}$	$[X_1,X_3]=X_4,\ [X_1,X_4]=X_5,\ [X_2,X_3]=X_6$	$A_{6,8}$	$[X_1,X_2]=X_3+X_5,\ [X_1,X_3]=X_4,\ [X_2,X_5]=X_6$
$A_{6,9}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_5]=X_6$ $[X_2,X_3]=X_6$	$A_{6,10}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_5,\ [X_1,X_4]=X_6$ $[X_2,X_3]=kX_6,\ [X_2,X_4]=X_5,\ k=\pm1$
$A_{6,11}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_5$ $[X_2,X_3]=X_6$	$A_{6,15}$	$[X_1,X_2]=X_3+X_5,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_6$ $[X_2,X_5]=X_6$
$A_{6,16}$	$[X_1,X_3]=X_4,\ [X_1,X_4]=X_5,\ [X_1,X_5]=X_6$ $[X_2,X_3]=X_5,\ [X_2,X_4]=X_6$	$A_{6,17}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_6$ $[X_2,X_5]=X_6$
$A_{6,18}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_6$ $[X_2,X_3]=X_5,\ [X_2,X_5]=kX_6,\ k\neq0$	$A_{6,19}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_5$ $[X_1,X_5]=X_6,\ [X_2,X_3]=X_6$
$A_{6,20}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4,\ [X_1,X_4]=X_5$ $[X_1,X_5]=X_6,\ [X_2,X_3]=X_5,\ [X_2,X_4]=X_6$	$A_{6,24}$	$[X_1,X_2]=X_3$
$A_{6,25}$	$[X_1,X_2]=X_3,\ [X_4,X_5]=X_6$	$A_{6,26}$	$[X_1,X_2]=X_3,\ [X_1,X_3]=X_4$
$A_{6,27}$	$[X_3,X_5]=X_1,\ [X_4,X_5]=X_2$	$A_{6,28}$	$[X_2,X_5]=X_1,\ [X_3,X_5]=X_2,\ [X_4,X_5]=X_3$
$A_{6,31}$	$[X_3,X_4]=X_1,\ [X_2,X_5]=X_1,\ [X_3,X_5]=X_2$	$A_{6,32}$	$[X_3,X_4]=X_1,\ [X_2,X_5]=X_1,\ [X_3,X_5]=X_2$ $[X_4,X_5]=X_3$

3.2. **Review of the Classification Method for Low-Dimensional Real Lie Bialgebras.** We briefly review the classification procedure employed in the present work. This procedure was previously used in [13] for the classification of low-dimensional real Lie superbialgebras and the same general strategy underlies our earlier four-dimensional symplectic Lie-bialgebra classification [2]; it is not a new methodological contribution of the present paper. Here, the established procedure is carried through the substantially larger six-dimensional nilpotent case in order to determine the admissible dual Lie-algebra structures and to classify the resulting Lie bialgebras of symplectic type up to equivalence, including the completeness, exceptional-branch and non-redundancy checks described below. For this purpose, we use the following adjoint representation:

$$(\mathbb{C}X_i)^{\ k}_{j}=-f_{ij}{}^{k},\qquad \left(\mathbb{C}Y^{k}\right)_{ij}=-f_{ij}{}^{k},$$
$$\left(\mathbb{C}\tilde{X}^{i}\right)^{\ j}_{k}=-\tilde{f}^{ij}_{\ \ k},\qquad \left(\mathbb{C}\tilde{Y}_{k}\right)^{ij}=-\tilde{f}^{ij}_{\ \ k},$$

for writing the matrix forms of equations (2.6) and (2.7) as follows, respectively,

(3.1)
$$\left(\mathbb{C}\tilde{X}^{i}\right)^{\ j}_{k}\mathbb{C}\tilde{X}^{k}+\mathbb{C}\tilde{X}^{i}\mathbb{C}\tilde{X}^{j}-\mathbb{C}\tilde{X}^{j}\mathbb{C}\tilde{X}^{i}=0,$$

(3.2)
$$\left(\mathbb{C}\tilde{X}^{i}\right)^{\ j}_{l}\mathbb{C}Y^{l}=-\left(\mathbb{C}\tilde{X}^{t}\right)^{j}\mathbb{C}Y^{i}+\mathbb{C}Y^{j}\mathbb{C}\tilde{X}^{i}-\mathbb{C}Y^{i}\mathbb{C}\tilde{X}^{j}+\left(\mathbb{C}\tilde{X}^{t}\right)^{i}\mathbb{C}Y^{j}.$$

Given the structure constants $\left(f_{ij}{}^{k}\right)$ of $\mathbf{g}$, we solve the matrix equations (3.1) and (3.2) for the dual structure constants $\tilde{f}^{ij}{}_{k}$, subject to the requirement that $(\mathbf{g}, \tilde{\mathbf{g}})$ is a Lie bialgebra. We carry out the classification in the following three steps, following the procedure of [13].

Step 1: Solving equations (3.1) and (3.2) and determining the Lie algebras $\mathbf{g}'$ which are isomorphic to the dual solutions $\tilde{\mathbf{g}}$.

Solving (3.1) and (3.2) determines the matrices $\mathbb{C}\tilde{X}^{i}$ and hence the dual structure constants, possibly with residual parameters and algebraic relations among them. To identify the isomorphism type of a resulting dual algebra $\tilde{\mathbf{g}}$, we compare it with a candidate algebra $\mathbf{g}'$ from Table 1. Under the change of basis

$$\tilde{X}'^{\,i} = C^{i}{}_{j}\tilde{X}^{j}, \qquad \left[\tilde{X}'^{\,i}, \tilde{X}'^{\,j}\right] = \tilde{f}'^{ij}{}_{k}\tilde{X}'^{\,k},$$

we have the following matrix equations for isomorphism:

$$C\left(C^{i}{}_{j}\tilde{X}^{j}_{(\tilde{\mathbf{g}})}\right) = \tilde{X}^{i}_{(\mathbf{g}')}C. \tag{3.3}$$

Solving (3.3) subject to $\det C \neq 0$ yields the additional parameter conditions under which the dual solution is isomorphic to $\mathbf{g}'$.

Step 2: Calculate the general form of the transformation matrices $B : \mathbf{g}' \to \mathbf{g}'.i$ such that $(\mathbf{g}, \mathbf{g}'.i)$ are Lie bialgebras.

In the second step, we transform the Lie bialgebra $(\mathbf{g}, \tilde{\mathbf{g}})$ to a Lie bialgebra $(\mathbf{g}, \mathbf{g}'.i)$ (where $\mathbf{g}'.i$ is isomorphic as a Lie algebra to $\mathbf{g}'$) with an automorphism of the Lie algebra $\mathbf{g}$. As the inner product (2.4) is invariant, we have $A^{-t} : \tilde{\mathbf{g}} \to \mathbf{g}'.i$

$$X'_{i} = A_{i}{}^{k}X_{k}, \qquad \tilde{X}'^{j} = \left(A^{-t}\right)^{j}{}_{l}\tilde{X}^{l}, \qquad \left\langle X'_{i}, \tilde{X}'^{j}\right\rangle = \delta^{j}{}_{i},$$

where A^{-t} is the inverse transpose of the same matrix $A \in \mathrm{Aut}(\mathbf{g})$. Thus, we have the following relation

$$\left(A^{-t}\right)^{i}{}_{k}\tilde{f}^{kl}_{(\tilde{\mathbf{g}})\,m}\left(A^{-t}\right)^{j}{}_{l} = f^{ij}_{(\mathbf{g}'.i)\,n}\left(A^{-t}\right)^{n}{}_{m}. \tag{3.4}$$

To determine the Lie bialgebras $(\mathbf{g}, \mathbf{g}'.i)$, we seek transformations $B : \mathbf{g}' \to \mathbf{g}'.i$ satisfying

$$B^{i}{}_{k}f^{kl}_{(\mathbf{g}')\,m}B^{j}{}_{l} = f^{ij}_{(\mathbf{g}'.i)\,n}B^{n}{}_{m}. \tag{3.5}$$

Eliminating $f^{ij}_{(\mathbf{g}'.i)\,n}$ between (3.4) and (3.5) gives the following matrix equation for B:

$$\left(A^{-t}\right)^{i}{}_{m}\mathbb{C}\tilde{X}^{t\,m}_{(\tilde{\mathbf{g}})}A^{-1} = (B^{t}A)^{-1}\left(B^{i}{}_{k}\mathbb{C}X^{t\,k}_{(\mathbf{g}')}\right)B^{t}. \tag{3.6}$$

Solving (3.6) subject to $\det B \neq 0$ gives the general admissible form of B.

Step 3: Calculate and classify the non-equivalent Lie bialgebras.

Having solved (3.6), we obtain the general form of the matrix B so that its elements are written in terms of the elements of matrices A, C and structure constants $\tilde{f}^{ij}_{(\tilde{\mathbf{g}})\ k}$. Now after substituting B in (3.5), we obtain the structure constants $f^{ij}_{(\mathbf{g}'.i)\ n}$ of the Lie algebra $\mathbf{g}'.i$ in terms of elements of matrices A and C and structure constants $\tilde{f}^{ij}_{(\mathbf{g})\ k}$. Then, it is checked whether it is possible to equalize the structure constants $f^{ij}_{(\mathbf{g}'.i)\ n}$ with each other and with ± 1 or not, such that $\det B \neq 0$, $\det A \neq 0$ and $\det C \neq 0$. In this way, we obtain matrices B_1, B_2, etc. Note that in obtaining B_i's we impose the condition $BB_i^{-1} \in \mathrm{Aut}^t(\mathbf{g})$. If this condition is not satisfied, then we cannot impose it on the structure constants, because B and B_i are not equivalent (see below).

Now, using isomorphism of matrices B_1, B_2, etc., we obtain Lie bialgebras $(\mathbf{g}, \mathbf{g}'.i)$, $(\mathbf{g}, \mathbf{g}'.ii)$, etc. On the other hand, there remains a question of whether these Lie bialgebras are equivalent. In order to answer this question, we use the matrix form of relation (2.2). Consider the two Lie bialgebras $(\mathbf{g}, \mathbf{g}'.i)$, $(\mathbf{g}, \mathbf{g}'.ii)$; then using

$$A(X_i) = A_i^{\ j} X_j,$$

the relation (2.2) will have the following matrix form:

$$A^t \left((A^t)^i_{\ k} \mathbb{C}X^{\ k}_{(\mathbf{g}'.i)} \right) = \mathbb{C}X^{\ i}_{(\mathbf{g}'.ii)} A^t. \tag{3.7}$$

On the other hand, the transformation matrix between $\mathbf{g}'.i$ and $\mathbf{g}'.ii$ is $B_2 B_1^{-1}$ such that $B_1 : \mathbf{g}' \to \mathbf{g}'.i$ and $B_2 : \mathbf{g}' \to \mathbf{g}'.ii$; then we have

$$\left(B_2 B_1^{-1}\right) \left(\left(B_2 B_1^{-1}\right)^i_{\ k} \mathbb{C}X^{\ k}_{(\mathbf{g}'.i)} \right) = \mathbb{C}X^{\ i}_{(\mathbf{g}'.ii)} \left(B_2 B_1^{-1}\right). \tag{3.8}$$

A comparison of (3.8) with (3.7) reveals that if $B_2 B_1^{-1} \in \mathrm{Aut}^t(\mathbf{g})$, then the Lie bialgebras $(\mathbf{g}, \mathbf{g}'.i)$ and $(\mathbf{g}, \mathbf{g}'.ii)$ are equivalent. In this way, we obtain non-equivalent classes of B_i's and we consider only one element of each class. We now apply this established procedure to the classification of six-dimensional real nilpotent Lie bialgebras of symplectic type.

3.3. **Classification of Six-Dimensional Real Nilpotent Lie Bialgebras of Symplectic Type.** (From Eqs. (3.1)–(3.2) to the representatives). To make the passage from the defining polynomial equations to Table 2 explicit, we organize every fixed-$\mathbf{g}$ computation into the following branch data:

solution branch of (3.1),(3.2)	dual type	isomorphism conditions
$\mathbb{C}S_r$	$\mathbf{g}'_r$	$\det C_r \neq 0$

$\cdots$	Aut(**g**)-reduction	retained representative
	$\mathbb{C}S_r/\operatorname{Aut}(\mathbf{g})$	$(\mathbf{g}, \mathbf{g}'_r.i)$

Here $\mathbb{C}S_r$ denotes a branch of the polynomial solution set, including its non-vanishing assumptions. The matrix equation (3.3) determines the possible abstract dual type $\mathbf{g}'_r$ and simultaneously records the conditions under which the isomorphism matrix C_r is nonsingular. Equations (3.4)–(3.6) then describe the induced automorphism action on the branch parameters. Parameters lying in the same Aut(**g**)-orbit are normalized or removed; the test (3.8) decides whether two normalized candidates are equivalent. Only the resulting orbit representatives are entered in Table 2. Thus the table is the final column of this branch-by-branch reduction, not the direct output of Eqs. (3.1)–(3.2).

The complete polynomial solution sets are too large to print for every **g**. Accordingly, the supplementary Mathematica/Maple archive accompanying the revised submission contains the case-by-case solution systems and transformation data. In the manuscript we give the full reduction pattern for $A_{6,8}$, because it displays both generic and singular branches and therefore illustrates the points at which information would be lost by considering only a generic solution.

Theorem 3.1. *Let* **g** *be one of the six-dimensional real nilpotent Lie algebras considered in Table 1 and let* $(\mathbf{g}, \tilde{\mathbf{g}})$ *be a Lie bialgebra for which both* **g** *and its dual Lie algebra* $\tilde{\mathbf{g}}$ *admit symplectic structures. Up to Lie-bialgebra equivalence induced by automorphisms of* **g**, *the resulting Lie-bialgebra structures are represented by the entries listed in Table 2. Conversely, each representative retained in Table 2 satisfies the Jacobi and mixed Jacobi identities defining a Lie-bialgebra structure.*

Completeness and Equivalence Verification. The classification summarized in Table 2 is obtained case by case by applying the three-step procedure described in Section 3.2 to each Lie algebra **g** under consideration. For a fixed **g**, the unknown structure constants $\tilde{f}^{ij}{}_{k}$ of the dual bracket are substituted into the Jacobi equations (3.1) and the mixed Jacobi equations (3.2). The resulting algebraic system is solved symbolically and the admissible solutions are subsequently compared with the candidate dual Lie algebras by solving the isomorphism equations (3.3) subject to the required nonsingularity condition $\det C \neq 0$.

For each admissible dual type, the automorphism group Aut(**g**) is then used to reduce the remaining structure constants to representatives of equivalence classes. In particular, two representatives obtained from transformation matrices B_1 and B_2 are identified only when the equivalence condition described in Section 3.2 is satisfied, namely when $B_2 B_1^{-1} \in \operatorname{Aut}^t(\mathbf{g})$. Thus, only one representative of each equivalence class is retained in Table 2.

These calculations were carried out symbolically on a case-by-case basis using Wolfram Mathematica notebooks and Maple worksheets. For each fixed $\mathbf{g}$, the computational sequence is: (i) substitute the unknown dual structure constants into the dual Jacobi equations; (ii) impose the mixed Jacobi equations, which are the coordinate form of the Lie-bialgebra 1-cocycle compatibility condition; (iii) solve the resulting polynomial system, with parameter branches separated whenever a quantity used as a divisor may vanish; (iv) solve the dual-algebra isomorphism equations with the nonsingularity condition on the change-of-basis matrix; and (v) compare candidate representatives using the automorphism criterion $B_j B_i^{-1} \in \mathrm{Aut}^t(\mathbf{g})$. Thus "Lie-bialgebra compatibility" is not an additional unchecked condition beyond the mixed Jacobi equations: in the present structure-constant formulation it is exactly the condition encoded by Eq. (3.2), together with the Jacobi identities on the two brackets.

The detailed computation for $(A_{6,8}, A_{6,27.vi})$ is presented below as an illustration of the procedure, while the same sequence of algebraic, isomorphism and equivalence tests is used for the remaining entries of Table 2. The computational archive used for the classification contains the case-by-case Mathematica and Maple files supporting these steps. Because some older notebooks store solved systems and transformation matrices rather than a uniform executable verification script, we use the archive as supplementary case-by-case evidence and do not claim that every retained row is certified by one single automated routine.

Parameter Reduction and Non-Equivalence of Representatives. For each fixed Lie algebra $\mathbf{g}$, the parameters appearing in the solutions of the Jacobi and mixed Jacobi equations are reduced only by transformations induced by $\mathrm{Aut}(\mathbf{g})$. More precisely, after the dual Lie-algebra isomorphism step, let B_i and B_j be two candidate transformation matrices. They represent the same Lie-bialgebra equivalence class precisely when the transformation relating them satisfies the automorphism criterion of Section 3.2, namely $B_j B_i^{-1} \in \mathrm{Aut}^t(\mathbf{g})$. Parameters that can be changed, rescaled, or normalized while this condition is satisfied are therefore removable orbit parameters and are not retained as classification parameters. Distinct representatives are retained when the corresponding automorphism-equivalence conditions do not identify them under an admissible nonsingular transformation.

The suffixes $\mathrm{i}, \mathrm{ii}, \ldots$ used in Table 2 label the Lie-bialgebra representatives obtained after this orbit reduction. In particular, they distinguish equivalence classes of Lie-bialgebra structures for a fixed $\mathbf{g}$ and should not be interpreted merely as different names for abstractly isomorphic dual Lie algebras. In the symbolic calculations, the equivalence step is

implemented case by case by constructing the transformation between candidate representatives, comparing it with the general form of $\mathrm{Aut}^t(\mathbf{g})$ and solving the resulting algebraic conditions together with the required nonsingularity assumptions. The $A_{6,8}$ calculation below illustrates this reduction explicitly: the parameter β is removable by the automorphism freedom, the remaining nonzero coefficients in the generic branch can be normalized and each normalization is made only when the corresponding transformation satisfies the automorphism-equivalence criterion.

To illustrate the case-by-case verification underlying Theorem 3.1, we now present the details of the calculation for the Lie bialgebra $(A_{6,8}, A_{6,27.vi})$.

An Example. One of the solutions of the Jacobi and mixed Jacobi identities (3.1) and (3.2) for the Lie algebra $A_{6,8}$ has the following form:

$$\tilde{f}^{34}_{\ 1} = \alpha, \qquad \tilde{f}^{46}_{\ 1} = \beta, \qquad \tilde{f}^{46}_{\ 2} = \gamma.$$

where α, β, γ are arbitrary constants.

At this stage, the parameter branches must be distinguished before introducing an isomorphism matrix containing divisions by the structure constants. The calculation leading to $A_{6,27.vi}$ corresponds to the generic branch

$$\alpha \neq 0, \qquad \gamma \neq 0. \tag{3.9}$$

Under the action induced by $\mathrm{Aut}(A_{6,8})$, the relevant parameters transform as

$$\alpha' = \frac{\alpha}{a_{11}^2 a_{22}^2}, \qquad \gamma' = \frac{\gamma}{a_{11}^3 a_{22}^2}, \qquad \beta' = \frac{a_{36}\alpha}{a_{11}^3 a_{22}^4} + \frac{\beta}{a_{11}^2 a_{22}^3}. \tag{3.10}$$

Since $\alpha \neq 0$ in the generic branch, the free automorphism parameter a_{36} can be chosen so that $\beta' = 0$. Thus, the vanishing of the coefficient of $\tilde{X}^1$ in $[\tilde{X}^4, \tilde{X}^6]$ used below is a normalization within the corresponding automorphism orbit rather than an additional restriction on the original solution.

The singular branches $\alpha = 0$ and/or $\gamma = 0$ must be considered separately and are not obtained by substituting zero values into the matrices displayed below. Indeed, if $\alpha\gamma = 0$ but $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, the derived algebra of the dual bracket generated by

$$\left[\tilde{X}^3, \tilde{X}^4\right] = \alpha\tilde{X}^1, \qquad \left[\tilde{X}^4, \tilde{X}^6\right] = \beta\tilde{X}^1 + \gamma\tilde{X}^2$$

is one-dimensional after a suitable change of basis and the resulting dual Lie algebra is of the one-bracket nilpotent isomorphism type $A_{6,24}$. If $\alpha = \beta = \gamma = 0$, the dual algebra is Abelian, namely $6A_1$. These dual types are treated separately in the case-by-case classification and occur among the $A_{6,8}$ entries of Table 2. Consequently, the divisions by α and

γ below are made only after the generic non-vanishing branch (3.9) has been specified.

For clarity, the branch structure of this solution family before the detailed generic reduction is

parameter locus	$\dim[\tilde{\mathbf{g}},\tilde{\mathbf{g}}]$	dual Lie-algebra type
$\alpha \neq 0,\ \gamma \neq 0$	2	$A_{6,27}$-type
$\alpha \neq 0,\ \gamma = 0$	1	$A_{6,24}$-type
$\alpha = 0,\ \gamma \neq 0$	1	$A_{6,24}$-type
$\alpha = 0,\ \gamma = 0,\ \beta \neq 0$	1	$A_{6,24}$-type
$\alpha = \beta = \gamma = 0$	0	$6A_1$

In the generic branch, Eq. (3.10) shows that β is removable because $\alpha \neq 0$; after this removal the two nonzero coefficients α' and γ' can be normalized within the admissible automorphism orbit, producing the $A_{6,27.vi}$ representative displayed below. On each singular non-Abelian branch the derived algebra is one-dimensional, so the dual is instead of $A_{6,24}$-type; the zero branch gives $6A_1$. This table separates the *solution branches* from the subsequent *orbit representatives*: a branch condition determines the possible dual isomorphism type, while the automorphism reduction determines which parameter values represent distinct Lie bialgebras.

For the generic branch, using (3.3) and the above normalization, the corresponding Lie algebra $\tilde{\mathbf{g}}$ is isomorphic to the Lie algebra $A_{6,27.vi}$, with the following isomorphism matrix

$$C = \begin{pmatrix} c_{11} & 0 & 0 & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 & 0 \\ c_{31} & c_{32} & -\dfrac{c_{11}c_{64}\alpha}{c_{22}\gamma} & 0 & 0 & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & -\dfrac{c_{22}\gamma}{c_{24}} & c_{64} \\ c_{51} & c_{52} & 0 & 0 & 0 & c_{56} \\ c_{61} & c_{62} & \dfrac{c_{64}(c_{11}\beta + c_{21}\gamma)}{c_{22}\gamma} & c_{64} & 0 & c_{66} \end{pmatrix},$$

where $c_{ij} \in \mathbb{R}$ are arbitrary elements of the isomorphism matrix, subject to its nonsingularity conditions. Here the representative has already been normalized so that the transformed coefficient β' vanishes, as explained above. Now, by substituting the above results and using the

following automorphism group element of the Lie algebra $A_{6,8}$, we have

$$A=\begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{11}a_{22} & a_{34} & 0 & a_{36} \\ 0 & 0 & 0 & a_{11}^2a_{22} & 0 & 0 \\ 0 & 0 & 0 & a_{11}a_{23}-a_{34} & a_{11}a_{12} & -a_{15}a_{22}-a_{36} \\ 0 & 0 & 0 & 0 & 0 & a_{11}a_{22}^2 \end{pmatrix},$$

where $a_{11}a_{22}\neq 0$ and a_{ij} are arbitrary elements of the A matrices. From (3.6), one can obtain the following form for the matrix B

$$B=\begin{pmatrix} \frac{a_{11}^2a_{22}^2b_{33}b_{45}}{\alpha} & \frac{a_{11}^2a_{22}^2b_{34}b_{45}}{\alpha} & 0 & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & 0 & 0 & 0 & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} \end{pmatrix},$$

where b_{ij} are arbitrary elements of the B matrices,

$$x=-\frac{a_{11}^2a_{36}b_{33}b_{45}}{\gamma}-\frac{a_{11}^3a_{22}^2b_{45}b_{63}}{\gamma}-\frac{a_{11}^3a_{22}b_{33}b_{45}\beta}{\alpha\gamma}$$

and

$$y=-\frac{a_{11}^2a_{36}b_{33}b_{45}}{\gamma}-\frac{a_{11}^3a_{22}^2b_{45}b_{63}}{\gamma}-\frac{a_{11}^3a_{22}b_{33}b_{45}\beta}{\alpha\gamma}.$$

Now, using (3.5) we have found the following commutation relations for the algebra $\mathbf{g}'.i$

$$\begin{aligned} \left[\tilde{X}^3,\tilde{X}^4\right] &= \frac{\alpha}{a_{11}^2a_{22}^2}\tilde{X}^1, \\ \left[\tilde{X}^4,\tilde{X}^6\right] &= \left(\frac{a_{36}\alpha}{a_{11}^3a_{22}^4}+\frac{\beta}{a_{11}^2a_{22}^3}\right)\tilde{X}^1+\frac{\gamma}{a_{11}^3a_{22}^2}\tilde{X}^2. \end{aligned}$$

On the other hand, choosing a_{36} so that the transformed coefficient β' in (3.10) vanishes, we obtain an equivalent representative. In the normalized notation used below this coefficient is set equal to zero and we have

$$B_1 = \begin{pmatrix} \frac{a_{11}^2 a_{22}^2 b_{33} b_{45}}{\alpha} & \frac{a_{11}^2 a_{22}^2 b_{34} b_{45}}{\alpha} & 0 & 0 & 0 & 0 \\ -\frac{a_{11}^3 a_{22}^2 b_{45} b_{63}}{\gamma} & -\frac{a_{11}^3 a_{22}^2 b_{45} b_{63}}{\gamma} & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & 0 & 0 & 0 & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} \end{pmatrix},$$

where, since $B_1 B^{-1} \in \mathrm{Aut}^t(\mathbf{g})$, B_1 is equivalent to B. Furthermore, by choosing $\frac{\alpha}{a_{11}^2 a_{22}^2} = \frac{\gamma}{a_{11}^3 a_{22}^2} = q \neq 0$, we have

$$B_2 = \begin{pmatrix} \frac{b_{33} b_{45}}{q} & \frac{b_{34} b_{45}}{q} & 0 & 0 & 0 & 0 \\ -\frac{b_{45} b_{63}}{q} & -\frac{b_{45} b_{63}}{q} & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & 0 & 0 & 0 & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} \end{pmatrix},$$

where, since $B_2 B_1^{-1} \in \mathrm{Aut}^t(\mathbf{g})$, B_2 is equivalent to B_1.

Now, by choosing $q = 1$, we have

$$B_3 = \begin{pmatrix} b_{33} b_{45} & b_{34} b_{45} & 0 & 0 & 0 & 0 \\ -b_{45} b_{63} & -b_{45} b_{63} & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & 0 & 0 & 0 & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} \end{pmatrix},$$

where, since $B_3 B_2^{-1} \in \mathrm{Aut}^t(\mathbf{g})$, B_3 is equivalent to B_2. So, we obtained the Lie bialgebra $(A_{6,8}, A_{6,27.vi})$ such that the commutation relations for $A_{6,27.vi}$ are as follows:

$$\left[\tilde{X}^3, \tilde{X}^4\right] = \tilde{X}^1, \qquad \left[\tilde{X}^4, \tilde{X}^6\right] = \tilde{X}^2.$$

Applying the verification procedure described above to the remaining admissible cases yields the representatives collected in Table 2. For each fixed $\mathbf{g}$, the table retains representatives after the corresponding Jacobi and mixed Jacobi equations, dual-algebra isomorphism conditions and automorphism-equivalence reductions have been imposed.[1]

[1]We consider only Lie bialgebras for which both the Lie algebra $\mathbf{g}$ and its dual $\tilde{\mathbf{g}}$ admit symplectic structures, according to Table 1.

Table 2: Six-dimensional real nilpotent Lie bialgebras of symplectic type. The suffixes i, ii, ... distinguish Lie-bialgebra representatives after reduction under Aut($\mathbf{g}$); they are not labels of abstract dual Lie-algebra isomorphism types alone.

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero commutation relations of $\tilde{\mathbf{g}}$	Comments
$A_{6,1}$	$6A_1$		
	$A_{6,1.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1+\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,4.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,6.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2+\tilde{X}^5, [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,7.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1+\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,8.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1+\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2+\tilde{X}^6,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,11.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6, [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,15.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2+\tilde{X}^6,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1$	
	$A_{6,19.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,24.i}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1$	
	$A_{6,26.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^3]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^1+\tilde{X}^2+\tilde{X}^5$	
	$A_{6,27.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^1, [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^5$	
	$A_{6,28.i}$	$[\tilde{X}^2,\tilde{X}^4]=\tilde{X}^5,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2+\tilde{X}^6,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1$	
$A_{6,2}$	$6A_1$		
	$A_{6,3.i}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,11.ii}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,24.ii}$	$[\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^1-\tilde{X}^2$	
	$A_{6,27.ii}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2+\tilde{X}^3$	
$A_{6,4}$	$6A_1$		
	$A_{6,2.i}$	$[\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2$	
	$A_{6,15.ii}$	$[\tilde{X}^1,\tilde{X}^5]=\tilde{X}^3+\tilde{X}^4, [\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,17.i}$	$[\tilde{X}^1,\tilde{X}^5]=\tilde{X}^3+\tilde{X}^4, [\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2-\tilde{X}^3+\tilde{X}^4$	
	$A_{6,19.ii}$	$[\tilde{X}^1,\tilde{X}^5]=\tilde{X}^3+\tilde{X}^4, [\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2-\tilde{X}^3+\tilde{X}^4, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,28.ii}$	$[\tilde{X}^1,\tilde{X}^5]=\tilde{X}^3+\tilde{X}^4, [\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2+\tilde{X}^4$	
$A_{6,5}$	$6A_1$		
	$A_{6,1.ii}$	$[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,2.ii}$	$[\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^2$	
	$A_{6,24.iii}$	$[\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^4$	
	$A_{6,25.i}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^4, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,26.ii}$	$[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,27.iii}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,28.iii}$	$[\tilde{X}^2,\tilde{X}^6]=-\tilde{X}^4$	
$A_{6,6}$	$6A_1$		
	$A_{6,3.ii}$	$[\tilde{X}^2,\tilde{X}^5]=2\tilde{X}^3,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^4$	
	$A_{6,6.ii}$	$[\tilde{X}^2,\tilde{X}^5]=-2\tilde{X}^3,\quad [\tilde{X}^2,\tilde{X}^6]=-\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,8.ii}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^3,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,15.iii}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^3, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,17.ii}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^5]=-\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,26.iii}$	$[\tilde{X}^2,\tilde{X}^5]=-\tilde{X}^1,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=2\tilde{X}^2+\tilde{X}^4$	
	$A_{6,27.iv}$	$[\tilde{X}^2,\tilde{X}^5]=-\tilde{X}^1+\tilde{X}^3,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1-\tilde{X}^3, [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^1$	
	$A_{6,32.i}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^3,\quad [\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
$A_{6,7}$	$6A_1$		
	$A_{6,1.iii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^6,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,8.iii}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=2\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,15.iv}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=2\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,24.iv}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2+\tilde{X}^3$	
	$A_{6,27.v}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^3,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2+\tilde{X}^3$	
	$A_{6,28.iv}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^3,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1+\tilde{X}^6$	
	$A_{6,32.ii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	

Table 2: (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero commutation relations of $\tilde{\mathbf{g}}$	Comments
$A_{6,8}$	$6A_1$		
	$A_{6,8.iv}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,15.v}$	$[\tilde{X}^1,\tilde{X}^4]=\tilde{X}^2,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^3+\tilde{X}^5$	
	$A_{6,24.v}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2$	
	$A_{6,26.iv}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^3+\tilde{X}^5$	
	$A_{6,27.vi}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2$	
$A_{6,9}$	$6A_1$		
	$A_{6,3.iii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,6.iii}$	$[\tilde{X}^3,\tilde{X}^6]=-\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,8.v}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1-\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^5,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2$	
	$A_{6,9.i}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,24.vi}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2$	
	$A_{6,27.vii}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1$	
$A_{6,10}$	$6A_1$		
	$A_{6,3.iv}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^6]=-2\tilde{X}^1,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,4.ii}$	$[\tilde{X}^3,\tilde{X}^5]=-\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^6]=-2\tilde{X}^1$	
	$A_{6,24.vii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1$	
	$A_{6,31.i}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,32.iii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=-2\tilde{X}^1,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3-\tilde{X}^4$	
$A_{6,11}$	$6A_1$		
	$A_{6,24.viii}$	$[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,26.v}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^6]=-\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^1+\tilde{X}^2$	
	$A_{6,27.viii}$	$[\tilde{X}^3,\tilde{X}^4]=-\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2$	
$A_{6,15}$	$6A_1$		
	$A_{6,3.v}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,27.ix}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6$	
$A_{6,16}$	$6A_1$		
	$A_{6,3.vi}$	$[\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2,\quad [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,4.iii}$	$[\tilde{X}^3,\tilde{X}^6]=-\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1$	
	$A_{6,6.iv}$	$[\tilde{X}^3,\tilde{X}^6]=2\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,8.vi}$	$[\tilde{X}^3,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1-\tilde{X}^2,\quad [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1+\tilde{X}^2, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,24.ix}$	$[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,27.x}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^3$	
$A_{6,17}$	$6A_1$		
	$A_{6,3.vii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,\quad [\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^3, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,8.vii}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^6]=2\tilde{X}^2, [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,15.vi}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^1,\quad [\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1, [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,17.iii}$	$[\tilde{X}^3,\tilde{X}^4]=-\tilde{X}^1$	
	$A_{6,24.x}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,26.vi}$	$[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^5, [\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2$	
	$A_{6,27.xi}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2+\tilde{X}^5$	
	$A_{6,32.iv}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^1, [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2, [\tilde{X}^4,\tilde{X}^6]=\tilde{X}^3+\tilde{X}^5$	
$A_{6,18}$	$6A_1$		
	$A_{6,18.i}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^3$	

Table 2: (Continued.)

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Non-zero commutation relations of $\tilde{\mathfrak{g}}$	Comments
$A_{6,19}$	$6A_1$		
	$A_{6,24.xi}$	$[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2$	
	$A_{6,26.vii}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^1,[\tilde{X}^3,\tilde{X}^5]=-\tilde{X}^1,[\tilde{X}^2,\tilde{X}^6]=-\tilde{X}^1,[\tilde{X}^3,\tilde{X}^6]=-\tilde{X}^1,[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,[\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^1,$ $[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2-\tilde{X}^3-\tilde{X}^4$	
	$A_{6,27.xii}$	$[\tilde{X}^2,\tilde{X}^5]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^6]=-\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^5]=-\tilde{X}^1,[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1,[\tilde{X}^5,\tilde{X}^6]=-\tilde{X}^2-\tilde{X}^3-\tilde{X}^4$	
$A_{6,20}$	$6A_1$		
	$A_{6,17.iv}$	$[\tilde{X}^3,\tilde{X}^5]=\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1,[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2,[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,24.xi}$	$[\tilde{X}^3,\tilde{X}^6]=\tilde{X}^1+\tilde{X}^2$	
	$A_{6,32.v}$	$[\tilde{X}^3,\tilde{X}^6]=2\tilde{X}^1,\ [\tilde{X}^4,\tilde{X}^5]=\tilde{X}^1,[\tilde{X}^4,\tilde{X}^6]=\tilde{X}^2,[\tilde{X}^5,\tilde{X}^6]=\tilde{X}^4$	
$A_{6,26}$	$6A_1$		
	$A_{6,24.xii}$	$[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2,[\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^2$	
	$A_{6,26.viii}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^5+\tilde{X}^6,[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^2$	
	$A_{6,27.xiii}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^2,[\tilde{X}^4,\tilde{X}^5]=\tilde{X}^5+\tilde{X}^6,[\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^5-\tilde{X}^6$	
	$A_{6,28.v}$	$[\tilde{X}^3,\tilde{X}^4]=\tilde{X}^6,[\tilde{X}^4,\tilde{X}^5]=-\tilde{X}^2+\tilde{X}^5+\tilde{X}^6,[\tilde{X}^4,\tilde{X}^6]=-\tilde{X}^5-\tilde{X}^6$	
$A_{6,27}$	$6A_1$		
	$A_{6,3.viii}$	$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^6$	
	$A_{6,6.v}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,7.ii}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,8.viii}$	$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^5$	
	$A_{6,9.ii}$	$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^3+\tilde{X}^4+\tilde{X}^5,[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,25.ii}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,27.xiv}$	$[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^3+\tilde{X}^4$	
$A_{6,31}$	$6A_1$		
	$A_{6,3.ix}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,4.iv}$	$[\tilde{X}^1,\tilde{X}^4]=\tilde{X}^5,[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^3$	
	$A_{6,6.vi}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^3,[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,8.ix}$	$[\tilde{X}^1,\tilde{X}^4]=\tilde{X}^3,[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^2+\tilde{X}^4$	
	$A_{6,11.iii}$	$[\tilde{X}^1,\tilde{X}^4]=\tilde{X}^3,[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^2$	
	$A_{6,17.v}$	$[\tilde{X}^2,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,24.xiii}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,27.xv}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^3$	
$A_{6,32}$	$6A_1$		
	$A_{6,15.vii}$	$[\tilde{X}^1,\tilde{X}^6]=-\tilde{X}^5$	
	$A_{6,26.ix}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^5$	
	$A_{6,27.xvi}$	$[\tilde{X}^1,\tilde{X}^6]=\tilde{X}^4$	
	$A_{6,32.vi}$	$[\tilde{X}^1,\tilde{X}^6]=-\tilde{X}^4$	

4. Calculation of Poisson Structures on Six-Dimensional Real Nilpotent Poisson–Lie Groups

We first distinguish the geometric structures used below. All Lie groups in this section are taken to be the connected, simply connected Lie groups integrating the nilpotent Lie algebras in Table 2. For a simply connected nilpotent Lie group, the exponential map $\exp:\mathfrak{g}\to G$ is a global diffeomorphism. Consequently, the exponential coordinates used to evaluate $a(g)$, $b(g)$, the invariant vector fields and the entries of Table 3 form a global coordinate system on G, rather than merely a chart near the identity. Thus the multiplicative Poisson bivector P_G determined by the Lie bialgebra is a globally defined smooth tensor on

G and the coordinate formulas in Table 3 are its expressions in these global coordinates.

Saying that a Lie algebra $\mathbf{g}$ is *of symplectic type* means that $\mathbf{g}$ admits a nondegenerate Chevalley–Eilenberg closed two-form; equivalently, the corresponding simply connected Lie group admits the associated left-invariant symplectic form. This symplectic form is not, in general, the inverse of the multiplicative Poisson tensor determined by a Lie-bialgebra structure $(\mathbf{g}, \tilde{\mathbf{g}})$. The latter is the *Poisson–Lie structure* P_G: it is multiplicative and its infinitesimal datum is the Lie cobracket (equivalently, the dual Lie algebra $\tilde{\mathbf{g}}$). A multiplicative Poisson tensor satisfies $P_G(e) = 0$ at the group identity and therefore, on a positive-dimensional Poisson–Lie group, cannot be a globally nondegenerate symplectic Poisson tensor. Only on an open subset on which P_G has full rank may its inverse be regarded as a local symplectic form. Throughout Table 3, the displayed brackets refer to the Poisson–Lie tensors associated with the Lie-bialgebra representatives; the phrase "of symplectic type" refers separately to the existence of symplectic forms on the underlying Lie algebras.

The Poisson structures calculated in this section should not be confused with the compatible Poisson pairs constructed in our previous work [3]. There, the six-dimensional calculation treated the eight underlying Lie algebras $A_{6,1}$, $A_{6,7}$, $A_{6,9}$, $A_{6,24}$, $A_{6,25}$, $A_{6,26}$, $A_{6,27}$ and $A_{6,32}$ and sought pairs (P, P') satisfying the compatibility equations used in the bi-Hamiltonian construction. In the present section, by contrast, the input is a specific Lie-bialgebra representative $(\mathbf{g}, \tilde{\mathbf{g}})$ from Table 2 and the Poisson tensor is the multiplicative Poisson–Lie tensor determined by that bialgebra through the Manin-triple data. Consequently, Table 3 is organized by pairs $(\mathbf{g}, \tilde{\mathbf{g}})$, including bialgebra representatives and underlying Lie algebras not treated in the six-dimensional compatible-Poisson table of [3]. For an underlying Lie group appearing in both papers, coincidence of a particular coordinate formula after specialization of parameters does not identify the two constructions: each entry in the present work is specified by its Lie-bialgebra/dual-algebra data. Some coordinate Poisson formulas in Table 3 may therefore coincide, after specialization of parameters, with tensors already occurring in [3]; the present contribution in such cases is their identification as multiplicative Poisson–Lie structures associated with the classified Lie-bialgebra representatives. We therefore do not claim that every coordinate Poisson formula in Table 3 is individually new as a Poisson tensor.

In order to compute the Poisson structures on six-dimensional real nilpotent Poisson–Lie groups, we employ the Manin triple construction together with the map $\pi(g)$ between the Lie subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. To

obtain the corresponding Poisson–Lie group, we use the following relations:

$$g^{-1}X_i g = a(g)_i^{\ j} X_j, \qquad g^{-1}\tilde{X}^i g = b(g)^{ij} X_j + d(g)^i_{\ j}\tilde{X}^j,$$

where $a(g)$ and $b(g)$ are the coefficient matrices arising from the adjoint action of g on the Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. The algebraic Poisson structure is then given by

$$\pi(g) = b(g)a^{-1}(g),$$

and the corresponding Poisson structure on the Lie group G can be expressed as

$$P_G(g) = \left(-b(g)a(g)^{-1}\right)^{ij} X_i^R \wedge X_j^R.$$

Equivalently, in terms of tensor components, we have

$$P^{kl} = \left(-b(g)a(g)^{-1}\right)^{ij} X_i^{R\,k} X_j^{R\,l}.$$

Accordingly, Table 3 should be read as the Poisson–Lie counterpart of the Lie-bialgebra classification in Table 2, rather than as a classification of arbitrary Poisson tensors on the underlying Lie groups. More precisely, its classification datum is the correspondence

$$(\mathbf{g}, \tilde{\mathbf{g}}) \longmapsto P_{(\mathbf{g},\tilde{\mathbf{g}})},$$

where the Poisson–Lie tensor is determined by the specified Lie-bialgebra structure. Thus, the same underlying Lie algebra $\mathbf{g}$ may occur in several rows with different dual algebras or different inequivalent Lie-bialgebra representatives, even when some resulting coordinate brackets have a form that also occurs in earlier compatible-Poisson calculations. This bialgebra-indexed organization and the resulting association between the representatives of Table 2 and their multiplicative Poisson–Lie structures, is the mathematical content added by Table 3 relative to the tables in [3]. We do not use Table 3 to claim that every displayed coordinate Poisson formula is, by itself, absent from the earlier literature.

Now, using the right-invariant vector fields as given in [2] and computing the adjoint matrices $a(g)$ and $b(g)$, we obtain the multiplicative Poisson tensors associated with the Lie-bialgebra representatives. The results are summarized in Table 3.

Global Versus Local Statements. The globality of P_G should be separated from later restrictions imposed for nondegeneracy or for particular realization functions. The entries of Table 3 themselves define the global multiplicative Poisson tensors on the simply connected nilpotent groups in the global coordinates just described. If a determinant or Pfaffian condition is imposed later, such as

$$x_3 x_4 (x_5 + x_6) \neq 0,$$

it specifies only the open full-rank locus of that already global Poisson tensor; on this locus P_G^{-1} is a symplectic form. It does not restrict the domain on which P_G exists. More generally, additional restrictions introduced later by particular Hamiltonian realizations or changes of variables-for example, conditions required by logarithms or denominators-apply only to the corresponding realization functions on their natural open domains and do not change the global domain of the underlying multiplicative Poisson bivector.

Verification of Table 3. Each entry of Table 3 is obtained from a Lie-bialgebra representative already retained in Table 2 by the Poisson–Lie construction used in [2]. At the structural level, the Jacobi identity for the resulting multiplicative Poisson bracket follows from the Lie-bialgebra compatibility conditions already imposed on the infinitesimal data. In the coordinate calculations, the displayed brackets were additionally checked by substituting their coordinate expressions into

$$J_{ijk}(x) = \sum_{\ell=1}^{6} \left(P^{i\ell}\partial_\ell P^{jk} + P^{j\ell}\partial_\ell P^{ki} + P^{k\ell}\partial_\ell P^{ij} \right), \quad 1 \le i < j < k \le 6,$$

and verifying $J_{ijk}(x) = 0$. This is the coordinate Poisson Jacobi identity, equivalently $[P, P] = 0$. The same symbolic-computation material used to generate the adjoint matrices and coordinate brackets records the case-by-case calculations underlying Table 3. The computational archive is provided as supplementary material with the revised submission so that the algebraic reductions and coordinate formulas can be inspected independently.

Table 3: Poisson brackets over the Poisson–Lie group G (using $\pi(g)$).

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{6,1}$	$A_{6,1.i}$	$\{x_2,x_4\}=x_1+x_5,\ \{x_3,x_4\}=\frac{3x_1^2}{2}+2x_1x_5+x_6,\ \{x_4,x_6\}=x_5$
	$A_{6,4.i}$	$\{x_2,x_4\}=qx_1,\ \ \{x_3,x_4\}=\frac{3qx_1^2}{2}+qx_5,\ \ \{x_3,x_6\}=qx_1,\ \ \{x_4,x_6\}=\frac{3qx_1^2}{2}$
	$A_{6,6.i}$	$\{x_2,x_4\}=x_1,\ \ \{x_3,x_4\}=\frac{3x_1^2}{2}+x_2+x_5,\ \ \{x_3,x_6\}=x_1,\ \ \{x_4,x_6\}=\frac{3x_1^2}{2}+x_5$
	$A_{6,7.i}$	$\{x_2,x_4\}=qx_1+qx_5,\ \{x_3,x_4\}=\frac{3qx_1^2}{2}+2qx_1x_5+qx_6,\ \{x_3,x_6\}=qx_1,\ \ \{x_4,x_6\}=\frac{3qx_1^2}{2}$
	$A_{6,8.i}$	$\{x_2,x_4\}=x_1+x_5,\ \{x_3,x_4\}=\frac{3x_1^2}{2}+x_2+2x_1x_5+x_6,\ \{x_3,x_6\}=x_1,\ \ \{x_4,x_6\}=\frac{3x_1^2}{2}$
	$A_{6,11.i}$	$\{x_2,x_4\}=x_5,\ \{x_3,x_4\}=2x_1x_5+x_6,\ \{x_3,x_6\}=x_1,\ \ \{x_4,x_6\}=\frac{3x_1^2}{2}+x_2$
	$A_{6,15.i}$	$\{x_2,x_4\}=x_5,\ \{x_3,x_4\}=x_2+2x_1x_5+x_6,\ \{x_3,x_6\}=x_1,\ \{x_4,x_5\}=x_1,\ \ \{x_4,x_6\}=3x_1^2$
	$A_{6,19.i}$	$\{x_2,x_4\}=x_5,\ \{x_3,x_4\}=2x_1x_5+x_6,\ \{x_3,x_6\}=x_1,\ \{x_4,x_5\}=x_1,\ \ \{x_4,x_6\}=3x_1^2+x_2$
	$A_{6,24.i}$	$\{x_3,x_4\}=x_1$
	$A_{6,26.i}$	$\{x_2,x_4\}=qx_1,\quad \{x_3,x_4\}=-qx_1+\frac{3qx_1^2}{2},\quad \{x_4,x_6\}=-qx_1+qx_2+qx_5$
	$A_{6,27.i}$	$\{x_2,x_4\}=x_1,\quad \{x_3,x_4\}=\frac{3x_1^2}{2}+x_5$
	$A_{6,28.i}$	$\{x_2,x_4\}=x_5,\quad \{x_3,x_4\}=x_2+2x_1x_5+x_6,\quad \{x_4,x_5\}=x_1,\quad \{x_4,x_6\}=\frac{3x_1^2}{2}$
$A_{6,2}$	$A_{6,3.i}$	$\{x_2,x_4\}=x_1+x_5,\ \{x_3,x_4\}=\frac{3x_1^2}{2}+2x_1x_5+x_6,\quad \{x_4,x_6\}=x_5$
	$A_{6,11.ii}$	$\{x_3,x_6\}=qx_2,\quad \{x_4,x_5\}=qx_1,\ \{x_4,x_6\}=\frac{3qx_1^2}{2}+2qx_1x_2+qx_3,\ \{x_5,x_6\}=\frac{7x_1^3}{6}+2qx_1^2x_2+2qx_1x_3+qx_4$
	$A_{6,24.ii}$	$\{x_5,x_6\}=-x_1-x_2$
	$A_{6,27.ii}$	$\{x_4,x_6\}=x_2,\quad \{x_5,x_6\}=x_2+2x_1x_2+x_3$
$A_{6,4}$	$A_{6,2.i}$	$\{x_4,x_5\}=-x_2,\ \ \{x_5,x_6\}=\frac{3x_1^2}{2}$
	$A_{6,15.ii}$	$\{x_1,x_5\}=x_3+x_4,\ \{x_4,x_5\}=-x_3,\ \{x_5,x_6\}=x_1+2x_2x_3+\frac{x_3^2}{2}+x_3x_4$
	$A_{6,17.i}$	$\{x_1,x_5\}=x_3+x_4,\ \{x_4,x_5\}=-x_2-x_3+x_4,\ \{x_5,x_6\}=\frac{3x_2^2}{2}+2x_2x_3+\frac{x_3^2}{2}-2x_2x_4+x_3x_4$
	$A_{6,19.ii}$	$\{x_1,x_5\}=x_3+x_4,\ \{x_4,x_5\}=-x_2-x_3+x_4,\ \{x_5,x_6\}=x_1+\frac{3x_2^2}{2}+2x_2x_3+\frac{x_3^2}{2}-2x_2x_4+x_3x_4$
	$A_{6,28.ii}$	$\{x_1,x_5\}=x_3+x_4,\ \{x_4,x_5\}=-x_2+x_4,\ \{x_5,x_6\}=\frac{3x_2^2}{2}+\frac{x_3^2}{2}-2x_2x_4+x_3x_4$
$A_{6,5}$	$A_{6,1.ii}$	$\{x_5,x_6\}=x_4$
	$A_{6,2.ii}$	$\{x_5,x_6\}=-x_2$
	$A_{6,24.iii}$	$\{x_5,x_6\}=-x_4$
	$A_{6,25.i}$	$\{x_2,x_6\}=x_4,\ \{x_5,x_6\}=x_2-\frac{x_4^2}{2}$
	$A_{6,26.ii}$	$\{x_5,x_6\}=x_2$
	$A_{6,27.iii}$	$\{x_2,x_6\}=x_4,\ \{x_5,x_6\}=-\frac{x_4^2}{2}$
	$A_{6,28.iii}$	$\{x_2,x_6\}=-x_4,\ \{x_5,x_6\}=\frac{x_4^2}{2}$
$A_{6,6}$	$A_{6,3.ii}$	$\{x_2,x_5\}=2x_3,\ \{x_2,x_6\}=x_1,\ \{x_5,x_6\}=-4x_1x_3-x_4$
	$A_{6,6.ii}$	$\{x_2,x_5\}=-2x_3,\ \{x_2,x_6\}=-x_1,\ \{x_3,x_5\}=x_1,\ \{x_4,x_5\}=\frac{3x_1^2}{2},\ \{x_5,x_6\}=4x_1x_3+x_4$
	$A_{6,8.ii}$	$\{x_2,x_5\}=x_3,\ \{x_2,x_6\}=x_1,\ \{x_4,x_6\}=x_1,\ \{x_5,x_6\}=\frac{3x_1^2}{2}+x_2-2x_1x_3$
	$A_{6,15.iii}$	$\{x_2,x_5\}=x_1,\ \{x_2,x_6\}=x_1,\ \{x_3,x_5\}=x_1,\ \{x_4,x_6\}=\frac{3x_1^2}{2}+x_3,\ \{x_5,x_6\}=x_4$
	$A_{6,17.ii}$	$\{x_2,x_6\}=x_1,\ \{x_3,x_5\}=-x_1,\ \{x_4,x_5\}=-\frac{3x_1^2}{2}+x_3,\ \{x_5,x_6\}=x_4$
	$A_{6,26.iii}$	$\{x_2,x_5\}=-x_1,\ \{x_2,x_6\}=x_1,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=-x_1,\ \{x_5,x_6\}=2x_2+x_4$
	$A_{6,27.iv}$	$\{x_2,x_5\}=-x_1+x_3,\ \{x_2,x_6\}=x_1,\ \{x_4,x_5\}=x_1-x_3,\ \{x_4,x_6\}=-x_1,\ \{x_5,x_6\}=-2x_1x_3$
	$A_{6,32.i}$	$\{x_2,x_5\}=x_3,\ \{x_2,x_6\}=x_1,\ \{x_3,x_5\}=x_1,\ \{x_4,x_5\}=\frac{3x_1^2}{2},\ \{x_5,x_6\}=x_2-2x_1x_3$
$A_{6,7}$	$A_{6,1.iii}$	$\{x_4,x_5\}=x_6,\ \{x_5,x_6\}=x_2$
	$A_{6,8.iii}$	$\{x_2,x_5\}=x_1,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=2x_3,\ \{x_5,x_6\}=4x_1x_3+x_4$
	$A_{6,15.iv}$	$\{x_2,x_5\}=x_1,\ \{x_4,x_6\}=2x_3,\ \{x_5,x_6\}=4x_1x_3+x_4$
	$A_{6,24.iv}$	$\{x_4,x_5\}=x_2+x_3$
	$A_{6,27.v}$	$\{x_4,x_5\}=x_3,\ \{x_5,x_6\}=x_2+x_3$
	$A_{6,28.iv}$	$\{x_2,x_5\}=x_3,\ \{x_4,x_5\}=x_1+x_6,\ \{x_5,x_6\}=\frac{x_3^2}{2}$
	$A_{6,32.ii}$	$\{x_4,x_5\}=x_1,\ \{x_4,x_6\}=x_3,\ \{x_5,x_6\}=2x_1x_3+x_4$
$A_{6,8}$	$A_{6,8.iv}$	$\{x_3,x_4\}=x_1,\ \{x_3,x_6\}=x_2,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=3x_1x_2+x_3$
	$A_{6,15.v}$	$\{x_1,x_4\}=x_2,\ \{x_3,x_4\}=-\frac{x_2^2}{2},\ \{x_3,x_6\}=x_2,\ \{x_4,x_5\}=x_1+\frac{x_2^2}{2},\ \{x_4,x_6\}=3x_1x_2+\frac{2x_2^3}{3}-x_3+x_5$
	$A_{6,24.v}$	$\{x_1,x_6\}=1,\ \{x_2,x_4\}=1,\ \{x_3,x_5\}=1,\ \{x_3,x_6\}=x_2,\ \{x_4,x_6\}=x_3-x_5,\ \{x_5,x_6\}=x_2$
	$A_{6,26.iv}$	$\{x_3,x_4\}=qx_2,\ \{x_4,x_6\}=-qx_3+qx_5$
	$A_{6,27.vi}$	$\{x_3,x_4\}=qx_1,\ \{x_4,x_6\}=qx_2$
$A_{6,9}$	$A_{6,3.iii}$	$\{x_4,x_5\}=x_2,\ \{x_4,x_6\}=2x_1x_2+x_3,\ \{x_5,x_6\}=x_1$
	$A_{6,6.iii}$	$\{x_3,x_6\}=-x_2,\ \{x_4,x_5\}=x_2,\ \{x_4,x_6\}=x_3,\ \{x_5,x_6\}=x_1$
	$A_{6,8.v}$	$\{x_3,x_6\}=x_1-x_2,\ \{x_4,x_5\}=x_2,\ \{x_4,x_6\}=\frac{3x_1^2}{2}+x_5,\ \{x_5,x_6\}=x_1+x_2$
	$A_{6,9.i}$	$\{x_3,x_6\}=x_2,\ \{x_4,x_5\}=-x_2,\ \{x_4,x_6\}=x_2,\ \{x_5,x_6\}=x_1$
	$A_{6,24.vi}$	$\{x_4,x_6\}=x_1+x_2$
	$A_{6,27.vii}$	$\{x_3,x_4\}=x_2,\ \{x_3,x_6\}=x_1,\ \{x_4,x_6\}=\frac{3x_1^2}{2}-\frac{3x_2^2}{2}$

Table 3: (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{6,10}$	$A_{6,3.iv}$	$\{x_4,x_5\}=x_2,\ \{x_4,x_6\}=-2x_1,\ \{x_5,x_6\}=-4x_1x_2+x_3$
	$A_{6,4.ii}$	$\{x_3,x_5\}=-x_1,\ \{x_4,x_5\}=x_2,\ \{x_4,x_6\}=-2x_1,\ \{x_5,x_6\}=-4x_1x_2+kx_1x_2$
	$A_{6,24.vii}$	$\{x_4,x_5\}=x_1,\ \{x_5,x_6\}=-\frac{3x_1^2}{2}$
	$A_{6,31.i}$	$\{x_3,x_6\}=x_1,\ \{x_5,x_6\}=\frac{3x_1^2}{2}$
	$A_{6,32.iii}$	$\{x_4,x_5\}=x_2,\ \{x_4,x_6\}=-2x_1,\ \{x_5,x_6\}=-4x_1x_2+x_3-x_4$
$A_{6,11}$	$A_{6,24.viii}$	$\{x_5,x_6\}=x_2$
	$A_{6,26.v}$	$\{x_2,x_6\}=x_1,\ \{x_3,x_6\}=-x_1+\frac{3x_1^2}{2},\ \{x_4,x_6\}=x_1-\frac{3x_1^2}{2}+\frac{7x_1^3}{6},$ $\{x_5,x_6\}=-x_1+\frac{3x_1^2}{2}-\frac{7x_1^3}{6}+\frac{5x_1^4}{8}+x_2$
	$A_{6,27.viii}$	$\{x_3,x_4\}=-x_1,\ \{x_3,x_5\}=-\frac{3x_1^2}{2},\ \{x_3,x_6\}=x_2,\ \{x_4,x_5\}=-\frac{7x_1^3}{6},\ \{x_4,x_6\}=3x_1x_2$ $\{x_5,x_6\}=\frac{7x_1^2x_2}{2}$
$A_{6,15}$	$A_{6,3.v}$	$\{x_4,x_6\}=x_5$
	$A_{6,27.ix}$	$\{x_3,x_4\}=x_6,\ \{x_3,x_6\}=2x_1x_6,\ \{x_4,x_6\}=2x_1^2x_6$
$A_{6,16}$	$A_{6,3.vi}$	$\{x_4,x_5\}=-x_1,\ \{x_4,x_6\}=-\frac{3x_1^2}{2}+x_2,\ \{x_5,x_6\}=-\frac{7x_1^3}{6}+3x_1x_2+x_3$
	$A_{6,4.iii}$	$\{x_3,x_6\}=-x_2,\ \{x_4,x_5\}=x_2,\ \{x_4,x_6\}=x_1,\ \{x_5,x_6\}=\frac{3x_1^2}{2}-3x_2^2$
	$A_{6,6.iv}$	$\{x_3,x_6\}=2x_1,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=\frac{9x_1^2}{2}+x_2,\ \{x_5,x_6\}=\frac{7x_1^3}{2}+3x_1x_2$
	$A_{6,8.vi}$	$\{x_3,x_5\}=x_1,\ \{x_3,x_6\}=x_1+\frac{3x_1^2}{2}-x_2,\ \{x_4,x_5\}=x_1+\frac{3x_1^2}{2}+x_2,\ \{x_4,x_6\}=\frac{7x_1^3}{3}+3x_1^2$ $\{x_5,x_6\}=\frac{7x_1^3}{3}+\frac{5x_1^4}{4}-3x_2^2+x_3$
	$A_{6,24.ix}$	$\{x_5,x_6\}=x_2$
	$A_{6,27.x}$	$\{x_4,x_6\}=x_1,\ \{x_5,x_6\}=\frac{3x_1^2}{2}+x_3$
$A_{6,17}$	$A_{6,3.vii}$	$\{x_4,x_5\}=x_1,\ \{x_4,x_6\}=x_1x_2-x_3,\ \{x_5,x_6\}=-\frac{3x_1^2}{2}+x_2$
	$A_{6,8.vii}$	$\{x_3,x_4\}=x_1,\ \{x_3,x_6\}=\frac{3x_1^2}{2}+2x_2,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=\frac{7x_1^3}{6}+5x_1x_2+x_3$ $\{x_5,x_6\}=-\frac{3x_1^2}{2}$
	$A_{6,15.vi}$	$\{x_2,x_6\}=x_1,\ \{x_3,x_4\}=x_1,\ \{x_3,x_6\}=3x_1^2+x_2,\ \{x_4,x_6\}=\frac{7x_1^3}{3}+2x_1x_2+x_3$
	$A_{6,17.iii}$	$\{x_3,x_4\}=-x_1,\ \{x_3,x_6\}=-\frac{3x_1^2}{2},\ \{x_4,x_6\}=-\frac{7x_1^3}{6}$
	$A_{6,24.x}$	$\{x_4,x_6\}=x_2$
	$A_{6,26.vi}$	$\{x_4,x_6\}=x_5,\ \{x_5,x_6\}=x_1+x_2$
	$A_{6,27.xi}$	$\{x_3,x_6\}=x_1,\ \{x_4,x_6\}=\frac{3x_1^2}{2}+x_2+x_5$
	$A_{6,32.iv}$	$\{x_3,x_4\}=x_1,\ \{x_3,x_6\}=x_1+\frac{3x_1^2}{2}+x_2,\ \{x_4,x_6\}=\frac{3x_1^2}{2}+\frac{7x_1^3}{6}+2x_1x_2+x_3+x_5$
$A_{6,18}$	$A_{6,18.i}$	$\{x_4,x_5\}=x_3,\ \{x_4,x_6\}=2kx_2x_3,\ \{x_5,x_6\}=-2x_1x_3$
$A_{6,19}$	$A_{6,24.xi}$	$\{x_5,x_6\}=x_1+x_2$
	$A_{6,26.vii}$	$\{x_2,x_5\}=x_1,\ \{x_2,x_6\}=-x_1+\frac{3x_1^2}{2},\ \{x_3,x_5\}=-x_1+\frac{3x_1^2}{2},\ \{x_3,x_6\}=-x_1-3x_1^2+\frac{7x_1^3}{3}$ $\{x_4,x_5\}=x_1-\frac{3x_1^2}{2}+\frac{7x_1^3}{6},\ \{x_4,x_6\}=-x_1-\frac{7x_1^3}{2}+\frac{15x_1^4}{8},$ $\{x_5,x_6\}=x_1-\frac{3x_1^2}{2}-\frac{15x_1^4}{8}+\frac{31x_1^5}{40}+x_2+x_1x_2-\frac{x_1^2x_2}{2}-x_3-x_4$
	$A_{6,27.xii}$	$\{x_2,x_5\}=x_1,\ \{x_2,x_6\}=-x_1+\frac{3x_1^2}{2},\ \{x_3,x_5\}=-x_1+\frac{3x_1^2}{2},\ \{x_3,x_6\}=x_1-3x_1^2+\frac{7x_1^3}{3}$ $\{x_4,x_5\}=-\frac{3x_1^2}{2}+\frac{7x_1^3}{6},\ \{x_4,x_6\}=\frac{3x_1^2}{2}-\frac{7x_1^3}{2}+\frac{15x_1^4}{8},$ $\{x_5,x_6\}=\frac{7x_1^3}{6}-\frac{15x_1^4}{8}+\frac{31x_1^5}{40}-x_2+x_1x_2-\frac{x_1^2x_2}{2}-x_3-x_4$
$A_{6,20}$	$A_{6,17.iv}$	$\{x_3,x_5\}=x_1,\ \{x_3,x_6\}=x_1+\frac{3x_1^2}{2},\ \{x_4,x_5\}=\frac{3x_1^2}{2},\ \{x_4,x_6\}=x_2+\frac{3x_1^2}{2}+\frac{7x_1^3}{3}$ $\{x_5,x_6\}=\frac{7x_1^3}{6}+\frac{5x_1^4}{4}+3x_1x_2+x_4$
	$A_{6,24.xi}$	$\{x_3,x_6\}=x_1+x_2,\ \{x_4,x_6\}=2x_1x_2,\ \{x_5,x_6\}=\frac{7x_1^3}{6}+x_1x_2+2x_1^2x_2+\frac{3x_2^2}{2}$
	$A_{6,32.v}$	$\{x_3,x_6\}=2x_1,\ \{x_4,x_5\}=x_1,\ \{x_4,x_6\}=\frac{9x_1^2}{2}+x_2,\ \{x_5,x_6\}=\frac{7x_1^3}{2}+3x_1x_2+x_4$
$A_{6,26}$	$A_{6,24.xii}$	$\{x_4,x_5\}=x_2,\ \{x_4,x_6\}=-x_2$
	$A_{6,26.viii}$	$\{x_3,x_4\}=qx_5+qx_6,\ \{x_4,x_5\}=qx_2$
	$A_{6,27.xiii}$	$\{x_3,x_4\}=qx_2,\ \{x_4,x_5\}=qx_5+qx_6,\ \{x_4,x_6\}=-qx_5-qx_6$
	$A_{6,28.v}$	$\{x_3,x_4\}=qx_6,\ \{x_4,x_5\}=-qx_2+qx_5+qx_6,\ \{x_4,x_6\}=-qx_5-qx_6$

Table 3: (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{6,27}$	$A_{6,3.viii}$	$\{x_1, x_2\} = x_6$
	$A_{6,6.v}$	$\{x_1, x_6\} = x_4$
	$A_{6,7.ii}$	$\{x_1, x_6\} = x_3$
	$A_{6,8.viii}$	$\{x_1, x_2\} = x_5$
	$A_{6,9.ii}$	$\{x_1, x_2\} = x_3 + x_4 + x_5,\ \{x_1, x_6\} = x_4$
	$A_{6,25.ii}$	$\{x_1, x_6\} = x_3$
	$A_{6,27.xiv}$	$\{x_1, x_3\} = x_3,\ \{x_2, x_4\} = x_4,\ \{x_5, x_6\} = x_5 + x_6$
$A_{6,31}$	$A_{6,3.ix}$	$\{x_1, x_6\} = x_4$
	$A_{6,4.iv}$	$\{x_1, x_4\} = x_5$
	$A_{6,6.vi}$	$\{x_1, x_6\} = x_3 - \frac{x_5^2}{2},\ \{x_2, x_6\} = x_5$
	$A_{6,8.ix}$	$\{x_1, x_4\} = x_3,\ \{x_1, x_6\} = x_2 + x_4$
	$A_{6,11.iii}$	$\{x_1, x_4\} = x_3,\ \{x_1, x_6\} = x_2$
	$A_{6,17.v}$	$\{x_1, x_6\} = -\frac{x_5^2}{2},\ \{x_1, x_6\} = x_5$
	$A_{6,24.xiii}$	$\{x_1, x_6\} = x_5$
	$A_{6,27.xv}$	$\{x_1, x_6\} = x_3$
$A_{6,32}$	$A_{6,15.vii}$	$\{x_1, x_6\} = -x_5$
	$A_{6,26.ix}$	$\{x_1, x_6\} = x_5$
	$A_{6,27.xvi}$	$\{x_1, x_6\} = x_4$
	$A_{6,32.vi}$	$\{x_1, x_6\} = -x_4$

5. Physical Application: An Example of Integrable Hamiltonian Systems via Symplectic Lie Bialgebras

In this section, following the formalism of [1], we examine two Hamiltonian realizations associated with symplectic Lie-bialgebra representatives. The connected, simply connected Lie group G integrating $\mathfrak{g}$ serves as the phase space, while $\tilde{\mathfrak{g}}$ provides the symmetry algebra.

We use the Hamiltonian terminology in the following precise sense. The functions Q_i produced by the realization procedure are *symmetry-realization functions*: their Poisson brackets reproduce the structure constants of the chosen symmetry Lie algebra. A *Hamiltonian* H is a separately specified function that generates the dynamics through $\dot{f} = \{f, H\}$. A function I is a *first integral* of that particular Hamiltonian system only when $\{I, H\} = 0$. A collection of functions is *in involution* when their mutual Poisson brackets vanish. A *Casimir* of a Poisson algebra is a function C satisfying $\{C, f\} = 0$ for every function f (equivalently, $P^\sharp dC = 0$); on a symplectic Poisson manifold such Casimirs are locally constant. These notions will not be used interchangeably below.

In particular, a symmetry-realization function Q_i is not called an invariant or a first integral merely because it belongs to the realization and no Q_i is called a Casimir unless the defining Casimir condition is verified. This construction is conceptually different from the bi-Hamiltonian systems of [3], which were obtained from compatible Poisson pairs and the Magri–Morosi procedure. In particular, the $A_{6,27}$ phase space itself already appeared in [3]; hence its reuse here is not claimed as new. The point of the present example is the Lie-bialgebra pairing with $A_{6,27.xiv}$ and the associated symmetry realization. Likewise, $A_{6,8}$ belongs to the underlying symplectic list used in [3], but it was not among the eight six-dimensional Lie algebras for which compatible Poisson pairs were displayed there. Claims concerning functional independence and Liouville integrability of the examples are addressed separately below. No new Hamiltonian-realization method is introduced in this section. The role of the examples is to apply the established construction to Lie-bialgebra representatives arising from the classification of Section 3; any claim of novelty therefore concerns the bialgebra/symmetry data to which the construction is applied, not the construction procedure itself.

Example 5.1 (Phase Space $\mathbf{A}_{6,27}$ with Symmetry Algebra $\mathbf{A}_{6,27.xiv}$)**.** For this example, the Poisson–Lie structure associated with the Lie bialgebra $(\mathbf{A}_{6,27}, \mathbf{A}_{6,27.xiv})$ (see Table 3) is given, in the coordinates used here, by the nonzero Poisson brackets

$$\{x_1, x_3\} = x_3, \qquad \{x_2, x_4\} = x_4, \qquad \{x_5, x_6\} = x_5 + x_6.$$

The corresponding Poisson matrix has determinant

$$\det P = \big[x_3 x_4 (x_5 + x_6)\big]^2.$$

The Poisson–Lie tensor itself is globally defined in the global nilpotent-group coordinates described in Section 4. Its rank is maximal only on the open subset

$$U = \{x \mid x_3 x_4 (x_5 + x_6) \neq 0\}.$$

On U, P^{-1} defines a symplectic two-form; outside U the Poisson tensor is degenerate. Thus, in the Hamiltonian discussion below, the relevant symplectic phase-space description is local (more precisely, it is restricted to an appropriate connected component of U), rather than a claim that the multiplicative Poisson–Lie tensor is globally symplectic on the whole group.

Applying the procedure of [1], the symmetry-realization functions Q_i take the form

$$\begin{aligned} Q_1 &= x_3 \ln(x_5 + x_6) - x_5, \\ Q_2 &= -x_4 \ln(x_5 + x_6) + x_5, \end{aligned}$$

$$Q_3 = x_3, \quad Q_4 = x_4, \quad Q_5 = x_5, \quad Q_6 = c,$$

where c is a constant.

These functions satisfy the Poisson brackets

$$\{Q_1, Q_2\} = Q_4 - Q_3, \qquad \{Q_1, Q_5\} = -Q_3, \qquad \{Q_2, Q_5\} = Q_4,$$

with all other brackets among these realization functions equal to zero. The logarithm appearing in Q_1 and Q_2 is the ordinary real logarithm. Accordingly, the Hamiltonian discussion in this example is restricted to

$$U_+ = \{x \mid x_3 \neq 0,\ x_4 \neq 0,\ x_5 + x_6 > 0\}.$$

This is the part of the full-rank Poisson domain $U = \{x \mid x_3x_4(x_5+x_6) \neq 0\}$ on which the logarithmic realization functions are real and smooth. We do not intend $\ln|x_5 + x_6|$ in the formulas above.

The commuting triples relevant to the integrability discussion are (Q_1, Q_3, Q_4) and (Q_3, Q_4, Q_5). Their functional independence can be verified explicitly on U_+. Put $s = x_5 + x_6$. Then

$$dQ_1 = \ln s\, dx_3 + \Big(\frac{x_3}{s} - 1\Big)\, dx_5 + \frac{x_3}{s}\, dx_6,$$

and therefore

$$dQ_1 \wedge dQ_3 \wedge dQ_4 = \Big(\frac{x_3}{s} - 1\Big)\, dx_5 \wedge dx_3 \wedge dx_4 + \frac{x_3}{s} dx_6 \wedge dx_3 \wedge dx_4.$$

The two coefficients cannot vanish simultaneously, since this would require both $x_3/s = 1$ and $x_3/s = 0$. Hence

$$dQ_1 \wedge dQ_3 \wedge dQ_4 \neq 0.$$

For the second triple,

$$dQ_3 \wedge dQ_4 \wedge dQ_5 = dx_3 \wedge dx_4 \wedge dx_5 \neq 0.$$

Thus both triples are functionally independent on U_+. Their mutual Poisson brackets vanish:

$$\{Q_1, Q_3\} = \{Q_1, Q_4\} = \{Q_3, Q_4\} = 0,$$
$$\{Q_3, Q_4\} = \{Q_3, Q_5\} = \{Q_4, Q_5\} = 0.$$

Consequently, the Hamiltonian choice must be made within one of these verified commuting families. More explicitly:

H	two independent commuting first integrals
Q_1	$Q_3,\ Q_4$
Q_3	$Q_1,\ Q_4$ (or $Q_4,\ Q_5$)
Q_4	$Q_1,\ Q_3$ (or $Q_3,\ Q_5$)
Q_5	$Q_3,\ Q_4$

Thus each of Q_1, Q_3, Q_4, Q_5 can be used as a Hamiltonian with the indicated first integrals on U_+. No corresponding claim is made for

Q_2: indeed, the displayed realization brackets give $\{Q_1, Q_2\} = Q_4 - Q_3$ and $\{Q_2, Q_5\} = Q_4$, so membership in the symmetry realization alone does not imply that a function can be chosen as a Hamiltonian with the same integrals. The constant $Q_6 = c$ is not counted as a nontrivial Hamiltonian or independent first integral. We do not refer to these functions as Casimirs of the symplectic phase-space Poisson algebra.

Example 5.2 (Phase Space $\mathbf{A}_{6,8}$ with Symmetry Algebra $\mathbf{A}_{6,24.v}$)**.** For this example, the Poisson structure used on $\mathbf{A}_{6,8}$ (see Table 3) is given by

$$\begin{aligned} &\{x_1, x_6\} = 1, \quad &&\{x_2, x_4\} = 1, \quad &&\{x_3, x_5\} = 1, \\ &\{x_3, x_6\} = x_2, \quad &&\{x_4, x_6\} = x_3 - x_5, \quad &&\{x_5, x_6\} = x_2. \end{aligned}$$

The determinant of the corresponding Poisson matrix is 1. Since the coordinates used here are global coordinates on the simply connected nilpotent group, this Poisson tensor is nondegenerate globally in this example.

Applying the established realization procedure of [1], the symmetry-realization functions are

$$\begin{aligned} Q_1 &= (x_3 - x_2x_1)^2, \\ Q_2 &= x_3 - x_2x_1, \\ Q_3 &= \exp(x_3 - x_2x_1), \\ Q_4 &= x_1, \\ Q_5 &= (x_3 - x_2x_1)x_6, \\ Q_6 &= \sin(x_3 - x_2x_1). \end{aligned}$$

They satisfy $\{Q_4, Q_5\} = Q_2$, with the remaining brackets required by the stated realization equal to zero. The realization functions obey the functional relations

$$Q_1 = Q_2^2, \qquad Q_3 = e^{Q_2}, \qquad Q_6 = \sin Q_2.$$

Thus, for the Hamiltonian analysis, functional independence cannot be inferred from the number of realization functions alone. Since the phase space is six-dimensional and symplectic, Liouville integrability requires three functionally independent functions in involution. We therefore choose the Hamiltonian and first integrals explicitly and verify both involutivity and functional independence.

A Liouville-integrable system is obtained by taking

$$H = Q_4 = x_1, \qquad I_1 = Q_2 = x_3 - x_2x_1, \qquad I_2 = x_2.$$

Direct calculation gives

$$\{Q_4, Q_2\} = \{Q_4, x_2\} = \{Q_2, x_2\} = 0,$$

so, for $H = Q_4$, both $I_1 = Q_2$ and $I_2 = x_2$ are first integrals and are mutually commuting. Moreover,

$$dH \wedge dI_1 \wedge dI_2 = dx_1 \wedge dx_3 \wedge dx_2 \neq 0,$$

so these three first integrals are functionally independent. Thus $H = Q_4$ defines a Liouville-integrable Hamiltonian system on this six-dimensional symplectic phase space.

There is also a second admissible choice using Q_5:

$$H = Q_5 = (x_3 - x_2x_1)x_6, \qquad I_1 = Q_2, \qquad I_2 = x_2.$$

For this choice, direct calculation gives

$$\{Q_5, Q_2\} = \{Q_5, x_2\} = \{Q_2, x_2\} = 0,$$

so Q_2 and x_2 are first integrals for $H = Q_5$ and commute with each other. Their independence together with H follows, for example, from

$$dQ_5 \wedge dQ_2 \wedge dx_2 = -Q_2\, dx_6 \wedge dx_3 \wedge dx_2 + \text{terms involving } dx_1,$$

so the wedge product is nonzero on the open subset $Q_2 \neq 0$. Here x_2 is an additional first integral of the Hamiltonian system and is not one of the six functions used to realize the symmetry algebra. Thus $H = Q_5$, together with $I_1 = Q_2$ and $I_2 = x_2$, provides a second Liouville-integrable system on the stated open subset.

6. Conclusion

The results of this work are most precisely stated as follows. Using the established classification procedure described in Section 3, we obtain the Lie-bialgebra representatives listed in Table 2 for the six-dimensional real nilpotent setting considered here, under the requirement that both $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are of symplectic type and modulo the Lie-bialgebra equivalence used in the classification. The completeness claim is supported by the branch-sensitive Jacobi and mixed-Jacobi analysis, the dual-isomorphism tests and the automorphism-based non-equivalence reductions described in Section 3 and in the accompanying computational material.

For each retained Lie-bialgebra representative, Table 3 records the associated multiplicative Poisson–Lie tensor on the connected, simply connected nilpotent group. These global Poisson tensors should be distinguished from the left-invariant symplectic forms associated with the symplectic-type Lie algebras and from the symplectic forms obtained by inverting a Poisson tensor only on its full-rank locus. We therefore do not claim that every corresponding Poisson–Lie tensor is globally symplectic.

The Hamiltonian section provides two worked applications of the classified structures rather than a new general construction method. In

Example 5.1, the admissible Hamiltonian choices are accompanied by explicitly verified independent commuting first integrals on the stated domain. In Example 5.2, the Liouville-integrable choices $H = Q_4$ and $H = Q_5$ are established using the commuting first integrals Q_2 and x_2; functional independence is verified explicitly, with the $H = Q_5$ system considered on the open subset $Q_2 \neq 0$. Thus the conclusions are restricted to the classification representatives, their associated Poisson–Lie tensors and the explicitly verified Hamiltonian systems established in the preceding sections.

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