Sahand Communications in Mathematical Analysis (SCMA) Vol. ··· No. ··· (20??), ···-···
http://scma.maragheh.ac.ir
DOI: ...

Order Isomorphisms Between Real Banach Algebras of Complex-Valued Continuous Functions with Involution

Davood Alimohammadi[1*] and Yasaman Adibdoost[2]

Abstract. Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. We first give some sufficient conditions that a composition operator from $C(X,\tau)$ to $C(Y,\eta)$ be an order real isomorphism. We next prove that if $T : C(X,\tau) \to C(Y,\eta)$ is an order real isomorphism with $|T(i\mathrm{Im}\, f)| \leq T(|\mathrm{Im}\, f|)$ for all $f \in C(X,\tau)$ and $\left|T^{-1}(i\mathrm{Im}\, h)\right| \leq T^{-1}(|\mathrm{Im}\, h|)$ for all $h \in C(Y,\eta)$, then T is a homeomorphism from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$, $T(1_X)$ and $T^{-1}(1_Y)$ are respectively nonvanishing positive functions on Y and X and there exists a bijective map $\Phi : Y_\eta \to X_\tau$ such that $T(f)(y) = T(1_X)(y)f(x)$ for all $y \in Y$, $x \in \Phi(\{y, \eta(y)\})$ and $f \in C(X,\tau)$ with $f(x) \in \mathbb{R}$, where $X_\tau = \{\{x, \tau(x)\} : x \in X\}$ and $Y_\eta = \{\{y, \eta(y)\} : y \in Y\}$. Finally, we prove that if $T : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ is an order real isomorphism then $T(1_X)$ and $T^{-1}(1_Y)$ are respectively nonvanishing positive functions on Y and X and there exists a homeomorphism $\varphi : Y \to X$ such that $T(f) = T(1_X) \cdot (f \circ \varphi)$ for all $f \in C_{\mathbb{R}}(X)$.

1. Introduction and Preliminaries

The symbol $\mathbb{K}$ denotes a scalar field which is $\mathbb{R}$ or $\mathbb{C}$. Let X be a nonempty set and let $\mathcal{F}_{\mathbb{K}}(X)$ denote the set of all $\mathbb{K}$-valued functions on X. It is clear that $\mathcal{F}_{\mathbb{K}}(X)$ is an algebra over $\mathbb{K}$ with the general addition, multiplication and scalar multiplication. Let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{F}_{\mathbb{C}}(X)$ over $\mathbb{K}$. The pointwise order $\leq$ in $\mathcal{A}(X)$ is defined by $f \leq g$ on X if and only if f and g are real-valued functions in $\mathcal{A}(X)$

2020 *Mathematics Subject Classification.* 46E05, 46J10, 47B65.
Key words and phrases. Order isomorphism, Order preserving map, Topological involution, Uniform norm.
Received: 2026-07-29, Accepted: 2026-10-03.
* Corresponding author.

and $f(x) \leq g(x)$ for all $x \in X$. A function $f \in F_{\mathbb{C}}(X)$ is called positive if f is a real-valued function on X and $0 \leq f(x)$ for all $x \in X$.

Let X and Y be nonempty sets and let $\mathcal{A}(X)$ and $\mathcal{B}(Y)$ respectively be linear subspaces of $\mathcal{F}_{\mathbb{C}}(X)$ and $\mathcal{F}_{\mathbb{C}}(Y)$ over $\mathbb{K}$ with the pointwise order $\leq$. A map $T : \mathcal{A}(X) \to \mathcal{B}(Y)$ is called an order preserving map from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ if $T(f) \leq T(g)$ on Y for all $f, g \in \mathcal{A}(X)$ with $f \leq g$ on X. An order isomorphism from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ is a bijective linear map T from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ over $\mathbb{K}$ such that T is an order preserving map from $\mathcal{A}(X)$ to $\mathcal{B}(Y)$ and T^{-1} is an order preserving map from $\mathcal{B}(Y)$ to $\mathcal{A}(X)$. Arden and De Cannière studied and characterized order isomorphisms between Fourier algebras in [3]. El Khatiri provided a complete description of order isomorphisms between the self-adjoint parts of C^*-algebras in [4]. In [10], Molnár proved that there is no bijective map between the set of all positive definite operators and the set of all self-adjoint operators on a Hilbert space with dimension greater than 1 which preserves the usual order in both directions.

Let X be a topological space. The set of all $f \in \mathcal{F}_{\mathbb{K}}(X)$ for which f is a continuous function on X, is denoted by $C_{\mathbb{K}}(X)$. Then $C_{\mathbb{K}}(X)$ is a subalgebra of $\mathcal{F}_{\mathbb{K}}(X)$ over $\mathbb{K}$. The set of all $f \in C_{\mathbb{K}}(X)$ for which f is a bounded function on X, is denoted by $C^b_{\mathbb{K}}(X)$. Notice that $\lambda_X \in C^b_{\mathbb{K}}(X)$, whenever $\lambda \in \mathbb{K}$ and λ_X is the constant function on X with value λ. It is known that $C^b_{\mathbb{K}}(X)$ is a subalgebra of $C_{\mathbb{K}}(X)$ and a commutative unital Banach algebra over $\mathbb{K}$ with unit 1_X and under the uniform norm $\|\cdot\|_X$ defined by

$$\|f\|_X = \sup\{\, |f(x)| : x \in X \,\}, \quad (f \in C^b_{\mathbb{K}}(X)).$$

The set of all $f \in C_{\mathbb{K}}(X)$ for which f vanishes at infinity, is denoted by $C_{0,\mathbb{K}}(X)$. It is known that $C_{0,\mathbb{K}}(X)$ is a uniformly closed subalgebra of $C^b_{\mathbb{K}}(X)$ and $1_X \notin C_{0,\mathbb{K}}(X)$ whenever X is not compact. Furthermore, $C_{0,\mathbb{K}}(X) = C^b_{\mathbb{K}}(X) = C_{\mathbb{K}}(X)$ if X is compact. We write $C(X)$, $C^b(X)$ and $C_0(X)$ instead of $C_{\mathbb{K}}(X)$, $C^b_{\mathbb{K}}(X)$ and $C_{0,\mathbb{K}}(X)$, respectively. In [5], Igarashi, Nakamura, Roy and Tanaka characterized order isomorphisms between $C_0(X)$ and $C_0(Y)$, where X and Y are locally compact Hausdorff spaces. In [9], Leung and Li characterized order isomorphisms between $C(X)$ and $C(Y)$, where X and Y are realcompact spaces (see [9, Theorem 4.1]). They also characterized order isomorphisms between certain sublattices of $C(X)$ and $C(Y)$, where X and Y are metric spaces (see [9, Theorem 4.5]).

Let X be a compact Hausdorff space. For any $f \in C_{\mathbb{K}}(X)$, the set of all $x \in X$ for which $f(x) \neq 0$ is denoted by $\operatorname{coz}(f)$. We denote by $\overline{\operatorname{coz}(f)}$ the closure of $\operatorname{coz}(f)$ in X. Let A be a subalgebra of $C_{\mathbb{K}}(X)$ over $\mathbb{K}$ which is a Banach algebra under a norm $\|\cdot\|$. Assume that A^*

is the dual space of $(A, \|\cdot\|)$ over $\mathbb{K}$. For any functional Λ in A^*, we define $\operatorname{supp}(\Lambda)$ to be the set of all points $x \in X$ such that for each open set U in X that contains x, there exists a function $f \in A$ with $\operatorname{coz}(f) \subseteq U$ such that $\Lambda(f) \neq 0$. In [6], Jiménez-Vargas and Villegas-Vallecillos studied order isomorphisms between little Lipschitz algebras over $\mathbb{K}$ and characterized their structures using the partition of the unity theorem in these algebras and the supports of certain functionals in the dual spaces of these algebras (see [6, Theorem 3.1]). Order isomorphisms between real little Lipschitz algebras with Lipschitz involution studied and characterized in [1]. Real little Lipschitz algebras with Lipschitz involution were first introduced in [2].

Let X be a nonempty set. A self-map τ of X is called an involution on X if $\tau(\tau(x)) = x$ for all $x \in X$. In this case, a subset E of X is called τ-invariant if $\tau(E) \subseteq E$ which implies that $\tau(E) = E$.

For a topological space X, a self-map τ of X is called a topological involution on X if τ is an involution on X and a continuous map from X to X. Clearly, in this case, τ is a homeomorphism from X onto X.

Let X be a compact Hausdorff space and let $\tau\colon X \longrightarrow X$ be a topological involution on X. The map $\tau^* : C(X) \longrightarrow C(X)$ defined by $\tau^*(f) = \overline{f} \circ \tau$, $f \in C(X)$, is an algebra involution on $C(X)$ which is called the algebra involution induced by τ on $C(X)$. Define

$$C(X, \tau) := \{f \in C(X) : \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a self-adjoint uniformly closed real subalgebra of $C(X)$ that contains 1_X and separates the points of X. In addition, we have $i1_X \notin C(X, \tau)$ and $C(X) = C(X, \tau) \oplus iC(X, \tau)$. It is clear that $\operatorname{Re}(f)$, $i\operatorname{Im}(f)$ and $|f|$ belong to $C(X, \tau)$ for all $f \in C(X, \tau)$. Furthermore, $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity map on X. The algebra $C(X, \tau)$ is called real Banach algebra of complex-valued continuous functions on X with involution. This algebra was first introduced by Kulkarni and Limaye in [7]. For further general facts about $C(X, \tau)$ and its certain subalgebras, we refer the reader to [8].

Let $(\mathcal{X}, \|\cdot\|)$ be a real linear normed space and let $\mathbb{C}$ be regarded as a real linear normed space equipped with the Euclidean norm $|\cdot|$. We denote by $\mathcal{X}'$ the set of all continuous real linear maps from $\mathcal{X}$ to $\mathbb{C}$ and call it the complex dual of $\mathcal{X}$. Every $\Lambda \in \mathcal{X}'$ is called a real linear complex-valued functional on $\mathcal{X}$.

Let X be a compact Hausdorff space, let τ be a topological involution on X and let A be a real subalgebra of $C(X, \tau)$ that contains 1_X and separates the points of X which is a Banach algebra under a norm $\|\cdot\|$ with $\|1_X\| = 1$. For $\Lambda \in A'$, we define $\operatorname{supp}(\Lambda)$ in (X, τ) to be the set of all points $x \in X$ such that for each τ-invariant open subset U of X that contains x, there exists a function $f \in A$ with $\operatorname{coz}(f) \subseteq U$ such that

$\Lambda(f) \neq 0$. For each $x \in X$, the map $\delta_x : A \longrightarrow \mathbb{C}$ defined by

$$\delta_x(f) = f(x), \quad (f \in A),$$

belongs to A' which is called the evaluation real linear complex-valued functional at x on A. In A', we can define the following order which we also denote by $\leq$: If $\Lambda \in A'$, let $0 \leq \Lambda$ if and only if $0 \leq \Lambda(f)$ for all $f \in A$ such that $0_X \leq f$ on X. Notice that, for each $x \in X$ we have $0 \leq \delta_x$.

In Section 2, we present and prove the partition of the unity theorem for $C(X,\tau)$-algebras. In Section 3, we first give some sufficient conditions that a weighted composition operator from $C(X,\tau)$ to $C(Y,\eta)$ be an order real isomorphism. We next obtain some properties of real linear order preserving maps between $C(X,\tau)$ and $C(Y,\eta)$. Using the partition of the unity theorem for $C(X,\tau)$-algebras and the sense of support for certain real linear complex-valued functionals in $C(X,\tau)'$, we determine the form of order linear isomorphisms $T\colon C(X,\tau) \longrightarrow C(Y,\eta)$, where $\big|T\big(i\operatorname{Im} f\big)\big| \leq T\big(|\operatorname{Im} f|\big)$ for all $f \in C(X,\tau)$ and $\big|T^{-1}\big(i\operatorname{Im} h\big)\big| \leq T^{-1}\big(|\operatorname{Im} h|\big)$ for all $h \in C(Y,\eta)$. Finally, we give a complete description of order real isomorphisms between $C_{\mathbb{R}}(X)$ and $C_{\mathbb{R}}(Y)$, where X and Y are compact Hausdorff spaces.

2. The Partition of the Unity Theorem in $C(X,\tau)$-Algebras

The main aim of this section is to present and prove the partition of unity theorem in $C(X,\tau)$-algebras. To reach this goal, we need the following lemma.

Lemma 2.1. *Let X be a compact Hausdorff space and let $\tau\colon X \longrightarrow X$ be a topological involution on X. Let K be a τ-invariant compact subset of X, let U be a τ-invariant open subset of X and let $K \subseteq U$. Then there exists a function $f \in C(X,\tau)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for all $x \in K$ and $\overline{\operatorname{coz}(f)} \subseteq U$.*

Proof. By Urysohn's lemma, there exists a function $g \in C(X)$ with $0 \leq g(x) \leq 1$ for all $x \in X$, $g(x) = 1$ for all $x \in K$ and $\overline{\operatorname{coz}(g)} \subseteq U$. Since $g \in C(X)$ and $0 \leq g(x) \leq 1$ for all $x \in X$, we deduce that $\tau^*(g) \in C(X)$, $0 \leq \tau^*(g)(x) \leq 1$ for all $x \in X$ and $\tau^*(g)(x) = 1$ for all $x \in K$. Take $f = g\,\tau^*(g)$. Then $f \in C(X)$, $\tau^*(f) = f$ and $\operatorname{coz}(f) \subseteq \operatorname{coz}(g)$. Therefore, $f \in C(X,\tau)$ and $\overline{\operatorname{coz}(f)} \subseteq \overline{\operatorname{coz}(g)} \subseteq U$. Furthermore, $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \in K$. Hence, the proof is complete. □

Theorem 2.2. *Let X be a compact Hausdorff space and let τ be a topological involution on X. Let the family $\{U_1, \ldots, U_n\}$ be an open covering of X with $\tau(U_j) = U_j$ for all $j \in \{1, \ldots, n\}$. Then there exist*

positive functions $f_1, \ldots, f_n$ *in* $C(X, \tau)$ *such that* $\sum_{j=1}^{n} f_j = 1_X$ *on* X *and* $\overline{\operatorname{coz}(f_j)} \subseteq U_j$ *for all* $j \in \{1, \ldots, n\}$.

Proof. Let $x \in X$ be given. Then there exists $l \in \{1, \ldots, n\}$ such that $x \in U_l$. Since X is a compact Hausdorff space, U_l is an open set in X and $x \in U_l$, there exists an open set V_x in X such that $x \in V_x$ and $\overline{V_x} \subseteq U_l$. Take $W_x = V_x \cup \tau(V_x)$. The W_x is a τ-invariant open subset of X that contains x and $\overline{W_x} \subseteq U_l$. It is clear that the collection $\{W_x : x \in X\}$ is an open covering of X. The compactness of X implies that there exists $x_1, \ldots, x_m \in X$ such that $X = \bigcup_{k=1}^{m} W_{x_k}$. Let $j \in \{1, \ldots, n\}$ be given and let H_j be the union of all $\overline{W_{x_k}}$ that $k \in \{1, \ldots, m\}$ and $\overline{W_{x_k}} \subseteq U_j$. Then H_j is a τ-invariant compact subset of X and $H_j \subseteq U_j$. By Lemma 2.1, there exists a function $h_j \in C(X, \tau)$ such that $0 \leq h_j(x) \leq 1$ for all $x \in X$, $h_j(x) = 1$ for all $x \in H_j$ and $\overline{\operatorname{coz}(h_j)} \subseteq U_j$. Let us define the following functions:

$$\begin{aligned} f_1 &= h_1, \\ f_2 &= (1_X - h_1)h_2, \\ &\vdots \\ f_n &= (1_X - h_1) \cdots (1_X - h_{n-1})h_n. \end{aligned}$$

Then for each $j \in \{1, \ldots, n\}$ we have $f_j \in C(X, \tau)$, $0 \leq f_j(x) \leq 1$ for all $x \in X$ and $\operatorname{coz}(f_j) \subseteq \operatorname{coz}(h_j)$ which implies that $\overline{\operatorname{coz}(f_j)} \subseteq U_j$. Let $x \in X$ be given. By induction, one can show that

$$\sum_{j=1}^{k} f_j(x) = 1 - \prod_{j=1}^{k} \big(1 - h_j(x)\big)$$

for all $k \in \{1, \ldots, n\}$. Therefore,

$$\sum_{j=1}^{n} f_j(x) = 1 - \prod_{j=1}^{n} \left(1 - h_j(x)\right). \tag{2.1}$$

Since $X = \bigcup_{k=1}^{m} W_{x_k}$ and $x \in X$, there exists $l \in \{1, \ldots, m\}$ such that $x \in W_{x_l}$. Let $j_0 \in \{1, \ldots, n\}$ satisfy $\overline{W_{x_l}} \subseteq H_{j_0}$. Then $x \in H_{j_0}$ and therefore $h_{j_0}(x) = 1$. According to (2.1), we obtain $\sum_{j=1}^{n} f_j(x) = 1$. Since this equality holds for every $x \in X$, we deduce that $\sum_{j=1}^{n} f_j = 1_X$ on X. Hence, the proof is complete. □

3. Order Isomorphisms Between $C(X,\tau)$-Algebras

We first give some sufficient conditions that a weighted composition operator between real Banach algebras of complex-valued continuous functions with involution on compact Hausdorff spaces be an order real isomorphism.

Theorem 3.1. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that μ is a nonvanishing positive function in $C(Y,\eta)$ and φ is a homeomorphism from Y to X with $\varphi\circ\eta=\tau\circ\varphi$ on Y. Let $T\colon C(X,\tau)\longrightarrow C(Y,\eta)$ be the map defined by*

$$T(f)=\mu\cdot(f\circ\varphi),\quad (f\in C(X,\tau)).$$

Then the following assertions hold.

(i) *T is well-defined; that is, $\mu\cdot(f\circ\varphi)\in C(Y,\eta)$ for all $f\in C(X,\tau)$.*
(ii) *T is an order real isomorphism from $C(X,\tau)$ onto $C(Y,\eta)$. Moreover, $T^{-1}(h)=\left(\frac{1}{\mu}\circ\varphi^{-1}\right)\cdot\left(h\circ\varphi^{-1}\right)$ for all $h\in C(Y,\eta)$.*
(iii) *$|T(f)|=T(|f|)$ for all $f\in C(X,\tau)$.*
(iv) *$|T(i\operatorname{Im} f)|=T(|\operatorname{Im} f|)$ for all $f\in C(X,\tau)$.*
(v) *$T(1_X)=\mu$ and so $T(1_X)$ is a nonvanishing positive function on Y.*
(vi) *$T^{-1}(1_Y)=\frac{1}{\mu}\circ\varphi^{-1}$ on X and so $T^{-1}(1_Y)$ is a nonvanishing positive function on X.*
(vii) *T is a homeomorphism from $(C(X,\tau),\|\cdot\|_X)$ to $(C(Y,\eta),\|\cdot\|_Y)$. Moreover, $\|T\|\leq\|T(1_X)\|_Y$ and $\left\|T^{-1}\right\|\leq\left\|T^{-1}(1_Y)\right\|_X$.*

Proof. To prove (i), let $f\in C(X,\tau)$ be given. Then $f\in C(X)$ and so $f\circ\varphi\in C(Y)$. According to $\tau^*(f)=f$ and $\varphi\circ\eta=\tau\circ\varphi$ on Y, we get

$$\begin{aligned}
\eta^*(f\circ\varphi)(y)&=(\overline{f\circ\varphi})\circ\eta)(y)\\
&=\overline{f\circ\varphi(\eta(y))}\\
&=\overline{f((\varphi\circ\eta)(y))}\\
&=\overline{f((\tau\circ\varphi)(y))}\\
&=\overline{f}((\tau\circ\varphi)(y))\\
&=\overline{f}(\tau(\varphi(y)))\\
&=(\overline{f}\circ\tau)(\varphi(y))\\
&=\tau^*(f)(\varphi(y))\\
&=f(\varphi(y))\\
&=(f\circ\varphi)(y)
\end{aligned}$$

for all $y \in Y$. This implies that $\eta^*(f \circ \varphi) = f \circ \varphi$. Therefore, we have $f \circ \varphi \in C(Y, \eta)$. Since μ and $f \circ \varphi$ are in $C(Y, \eta)$, we deduce that $\mu \cdot (f \circ \varphi) \in C(Y, \eta)$. Hence, (i) holds.

To prove (ii), we first show that T is a real linear map from $C(X, \tau)$ to $C(Y, \eta)$. The definition of T implies that $T(f+g) = T(f) + T(g)$ for all $f, g \in C(X, \tau)$ and that $T(rf) = rT(f)$ for all $r \in \mathbb{R}$ and $f \in C(X, \tau)$. Therefore, T is a real linear map.

To prove the injectivity of T, let $f \in C(X, \tau)$ with $T(f) = 0_Y$. Then $\mu \cdot (f \circ \varphi) = 0_Y$. This implies that $f \circ \varphi = 0_Y$ since μ is a nonvanishing function in $C(Y, \eta)$. Therefore, $f = 0_X$ since φ is a surjective map from Y to X. Hence, T is an injective map since T is a real linear map.

To prove the surjectivity of T, we first show that $k \circ \varphi^{-1} \in C(X, \tau)$ for all $k \in C(Y, \eta)$. Let $k \in C(Y, \eta)$ be given. Then $k \in C(Y)$ and so $k \circ \varphi^{-1} \in C(X)$ since φ^{-1} is a continuous map from X to Y. Since $\varphi \circ \eta = \tau \circ \varphi$ on Y and φ^{-1} is a bijective map from X to Y, we deduce that $\eta \circ \varphi^{-1} = \varphi^{-1} \circ \tau$ on X. Therefore, for each $x \in X$ we have

$$\begin{aligned}
\tau^* \left(k \circ \varphi^{-1}\right)(x) &= \left(\overline{k \circ \varphi^{-1}} \circ \tau\right)(x) \\
&= \overline{k \circ \varphi^{-1}}(\tau(x)) \\
&= \overline{\left(k \circ \varphi^{-1}\right)(\tau(x))} \\
&= \overline{k(\varphi^{-1}(\tau(x)))} \\
&= \overline{k((\varphi^{-1} \circ \tau)(x))} \\
&= \overline{k((\eta \circ \varphi^{-1})(x))} \\
&= \overline{(k \circ (\eta \circ \varphi^{-1}))(x)} \\
&= \overline{((k \circ \eta) \circ \varphi^{-1})(x)} \\
&= \overline{(k \circ \eta)\left(\varphi^{-1}(x)\right)} \\
&= \overline{k\left(\eta\left(\varphi^{-1}(x)\right)\right)} \\
&= \overline{k}\left(\eta\left(\varphi^{-1}(x)\right)\right) \\
&= \left(\overline{k} \circ \eta\right)\left(\varphi^{-1}(x)\right) \\
&= \eta^*(k)\left(\varphi^{-1}(x)\right) \\
&= k\left(\varphi^{-1}(x)\right) \\
&= \left(k \circ \varphi^{-1}\right)(x).
\end{aligned}$$

This implies that $\tau^*\left(k \circ \varphi^{-1}\right) = k \circ \varphi^{-1}$. Thus $k \circ \varphi^{-1} \in C(X, \tau)$. Since μ is a nonvanishing function in $C(Y, \eta)$, we deduce that $\frac{1}{\mu} \in C(Y, \eta)$. By the argument above, we have $\frac{1}{\mu} \circ \varphi^{-1} \in C(X, \tau)$.

We now prove that T is a surjection. Let $k \in C(Y,\eta)$ be given. Then $k \circ \varphi^{-1} \in C(X,\tau)$ by the argument above. Therefore, we have $\left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot \left(k \circ \varphi^{-1}\right) \in C(X,\tau)$ since $\left(\frac{1}{\mu} \circ \varphi^{-1}\right) \in C(X,\tau)$. Thus, by the definition of T we get

$$\begin{aligned} T\left(\left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot \left(k \circ \varphi^{-1}\right)\right) &= \mu \cdot \left(\left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot \left(k \circ \varphi^{-1}\right)\right) \circ \varphi \\ &= \mu \cdot \left(\left(\frac{1}{\mu} \circ \varphi^{-1}\right) \circ \varphi\right) \cdot \left(\left(k \circ \varphi^{-1}\right) \circ \varphi\right) \\ &= \mu \cdot \left(\frac{1}{\mu} \circ \left(\varphi^{-1} \circ \varphi\right)\right) \cdot \left(k \circ \left(\varphi^{-1} \circ \varphi\right)\right) \\ &= \mu \cdot \left(\frac{1}{\mu} \cdot k\right) \\ &= \left(\mu \cdot \frac{1}{\mu}\right) \cdot k \\ &= 1_Y \cdot k \\ &= k. \end{aligned}$$

Hence, T is a surjective map. Therefore, T is a bijective real linear map from $C(X,\tau)$ to $C(Y,\eta)$. This implies that T^{-1} is a real linear map from $C(Y,\eta)$ to $C(X,\tau)$.

We now show that T is an order preserving map from $C(X,\tau)$ to $C(Y,\eta)$. Let $f, g \in C(X,\tau)$ with $f \leq g$ on X. By the surjectivity of $\varphi\colon Y \longrightarrow X$, we deduce that $f \circ \varphi \leq g \circ \varphi$ on Y. This implies that $\mu \cdot (f \circ \varphi) \leq \mu \cdot (g \circ \varphi)$ on Y since μ is a nonvanishing positive function in $C(Y,\eta)$. Therefore, $T(f) \leq T(g)$ on Y. Hence, T is an order preserving map.

According to the definition and the bijectivity of map T from $C(X,\tau)$ to $C(Y,\eta)$ and the invertibility μ in $C(Y,\eta)$, we can easily show that

$$T^{-1}(h) = \left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot \left(h \circ \varphi^{-1}\right)$$

for all $h \in C(Y,\eta)$. We now prove that T^{-1} is an order preserving map from $C(Y,\eta)$ to $C(X,\tau)$. Let $h, k \in C(Y,\eta)$ with $h \leq k$ on Y. Then $T^{-1}(h), T^{-1}(k) \in C(X,\tau)$. In addition, $T\big(T^{-1}(h)\big) = h$ and $T\big(T^{-1}(k)\big) = k$. Therefore, $T\big(T^{-1}(h)\big) \leq T\big(T^{-1}(k)\big)$ on Y. According to the definition of T, we have

$$\mu \cdot \big(T^{-1}(h) \circ \varphi\big) \leq \mu \cdot \big(T^{-1}(k) \circ \varphi\big).$$

Therefore, $T^{-1}(h) \circ \varphi \leq T^{-1}(k) \circ \varphi$ on Y since μ is a nonvanishing positive function on Y. This implies that $T^{-1}(h) \leq T^{-1}(k)$ on X since $\varphi\colon Y \longrightarrow X$ is a surjective map from Y to X. Therefore, T^{-1} is an

order preserving map from $C(Y,\eta)$ to $C(X,\tau)$. Hence, T is an order isomorphism from $C(X,\tau)$ to $C(Y,\eta)$ and so (ii) holds.

To prove (iii), let $f \in C(X,\tau)$ be given. Then $|f| \in C(X,\tau)$. Since μ is a nonvanishing positive function on Y, according to the definition of T we get

$$|T(f)| = |\mu \cdot (f \circ \varphi)| = \mu \cdot |f \circ \varphi| = \mu \cdot (|f| \circ \varphi) = T(|f|).$$

Hence, (iii) holds.

To prove (iv), let $f \in C(X,\tau)$ be given. Clearly, $|i \operatorname{Im} f| = |\operatorname{Im} f|$ on X. Since $i \operatorname{Im} f \in C(X,\tau)$, by (iii) we have

$$|T(i \operatorname{Im} f)| = T(|i \operatorname{Im} f|) = T(|\operatorname{Im} f|).$$

Hence, (iv) holds.

Since $1_X \in C(X,\tau)$ and φ is a surjective map from Y to X, we have $1_X \circ \varphi = 1_Y$. Therefore,

$$T(1_X) = \mu \cdot (1_X \circ \varphi) = \mu \cdot 1_Y = \mu.$$

Thus, $T(1_X)(y) > 0$ for all $y \in Y$. Hence, (v) holds.

Since $1_Y \in C(Y,\eta)$ and $\varphi^{-1}\colon X \to Y$ is a surjective map, we have $1_Y \circ \varphi^{-1} = 1_X$. Therefore, by (ii), we have

$$T^{-1}(1_Y) = \left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot \left(1_Y \circ \varphi^{-1}\right) = \left(\frac{1}{\mu} \circ \varphi^{-1}\right) \cdot 1_X = \frac{1}{\mu} \circ \varphi^{-1}.$$

Thus, $T^{-1}(1_Y)(x) > 0$ for all $x \in X$. Hence, (vi) holds.

To prove (vii), let $f \in C(X,\tau)$ be given. By the definition of T and (v), we have $T(f) = T(1_X) \cdot (f \circ \varphi)$. Therefore, for each $y \in Y$ we get

$$|T(f)(y)| = |T(1_X)(y)| \cdot |f(\varphi(y))| \leq \|T(1_X)\|_Y \, \|f\|_X .$$

This implies that $\|T(f)\|_Y \leq \|T(1_X)\|_Y \, \|f\|_X$. Since this inequality holds for all $f \in C(X,\tau)$, we deduce that the real linear map T from $C(X,\tau)$ to $C(Y,\eta)$ is bounded and $\|T\| \leq \|T(1_X)\|_Y$. Therefore, T is a continuous map from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$.

Let $h \in C(Y,\eta)$ be given. By (ii), (v) and (iv), we get

$$T^{-1}(h) = \left(T^{-1}(1_Y)\right) \cdot \left(h \circ \varphi^{-1}\right).$$

Therefore, for each $x \in X$ we have

$$\begin{aligned} \left|T^{-1}(h)(x)\right| &= \left|T^{-1}(1_Y)(x) \cdot h\left(\varphi^{-1}(x)\right)\right| \\ &= \left|T^{-1}(1_Y)(x)\right| \left|h\left(\varphi^{-1}(x)\right)\right| \\ &\leq \left\|T^{-1}(1_Y)\right\|_X \|h\|_Y . \end{aligned}$$

This implies that $\left\|T^{-1}(h)\right\|_X \leq \left\|T^{-1}(1_Y)\right\|_X \|h\|_Y$. Since this inequality holds for all $h \in C(Y,\eta)$, we deduce that the real linear map T^{-1} from $C(Y,\eta)$ to $C(X,\tau)$ is bounded and $\left\|T^{-1}\right\| \leq \left\|T^{-1}(1_Y)\right\|_X$. Therefore,

T^{-1} is a continuous map from $(C(Y,\eta),\|\cdot\|_Y)$ to $(C(X,\tau),\|\cdot\|_X)$. Thus, T is a homeomorphism from $(C(X,\tau),\|\cdot\|_X)$ to $(C(Y,\eta),\|\cdot\|_Y)$. Hence, (vii) holds and the proof is complete. □

Corollary 3.2. *Let X and Y be compact Hausdorff spaces. Suppose that μ is a nonvanishing positive function in $C_{\mathbb{R}}(Y)$ and φ is a homeomorphism from Y to X. Let T be the map from $C_{\mathbb{R}}(X)$ to $C_{\mathbb{R}}(Y)$ defined by*

$$T(f)=\mu\cdot(f\circ\varphi),\quad (f\in C_{\mathbb{R}}(X)).$$

Then the following assertions hold.

(i) *T is well-defined, i.e., $\mu\cdot(f\circ\varphi)\in C_{\mathbb{R}}(Y)$ for all $f\in C_{\mathbb{R}}(X)$.*
(ii) *T is an order isomorphism from $C_{\mathbb{R}}(X)$ to $C_{\mathbb{R}}(Y)$. In addition, $T^{-1}(h)=\left(\frac{1}{\mu}\circ\varphi^{-1}\right)\cdot\left(h\circ\varphi^{-1}\right)$ for all $h\in C_{\mathbb{R}}(Y)$.*
(iii) *$|T(f)|=T(|f|)$ for all $f\in C_{\mathbb{R}}(X)$.*
(iv) *$T(1_X)=\mu$ and so $T(1_X)(y)>0$ for all $y\in Y$.*
(v) *$T^{-1}(1_Y)=\frac{1}{T(1_X)}\circ\varphi^{-1}$ and so $T^{-1}(1_Y)(x)>0$ for all $x\in X$.*
(vi) *T is a homeomorphism from $(C_{\mathbb{R}}(X),\|\cdot\|_X)$ to $(C_{\mathbb{R}}(Y),\|\cdot\|_Y)$.*

Proof. Let τ and η be the identity maps on X and Y, respectively. Then τ is a topological involution on X, η is a topological involution on Y, $C_{\mathbb{R}}(X)=C(X,\tau)$, $C_{\mathbb{R}}(Y)=C(Y,\eta)$, μ is a nonvanishing positive function in $C(Y,\eta)$ and φ is a homeomorphism from Y to X with $\varphi\circ\eta=\tau\circ\varphi$ on Y. Therefore, the assertions hold by Theorem 3.1. □

Here, we prove that if T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$ with $|T(i\operatorname{Im}f)|\le T(|\operatorname{Im}f|)$ for all $f\in C(X,\tau)$, then T is continuous. To this aim, we need the following lemma.

Lemma 3.3. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$. Then the following statements hold:*

(i) *$|T(u)|\le T(|u|)$ for all real-valued functions u in $C(X,\tau)$.*
(ii) *$|T(f)|\le 2T(|f|)$ for all $f\in C(X,\tau)$ with $|T(i\operatorname{Im}f)|\le T(|\operatorname{Im}f|)$.*
(iii) *$T(|f|)\le\|f\|_X T(1_X)$ for all $f\in C(X,\tau)$.*

Proof. To prove (i), let u be a real-valued function in $C(X,\tau)$. Then $|u|-u$ and $|u|+u$ are real-valued functions in $C(X,\tau)$, $0_X\le|u|-u$ on X and $0_X\le|u|+u$ on X. Since T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$, we deduce that

$$0_Y=T(0_X)\le T(|u|)-T(u)=T(|u|-u)$$

on Y which implies that

$$T(u)\le T(|u|)\quad\text{on } Y \tag{3.1}$$

and

$$0_Y = T(0_X) \leq T(|u|) + T(u) = T(|u| + u)$$

on Y which implies that

$$-T(u) \leq T(|u|) \quad \text{on } Y. \tag{3.2}$$

By (3.1) and (3.2), we get $|T(u)| \leq T(|u|)$ on Y. Hence, (i) holds.

To prove (ii), let $f \in C(X,\tau)$ with $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ on Y. According to $\operatorname{Re} f$ is a real-valued function in $C(X,\tau)$, by (i) we have

$$|T(\operatorname{Re} f)| \leq T(|\operatorname{Re} f|) \quad \text{on } Y. \tag{3.3}$$

Since $T : C(X,\tau) \to C(Y,\eta)$ is an order preserving real linear map, $\operatorname{Re} f$, $|\operatorname{Re} f|$, $|f|$, $i \operatorname{Im} f$ and $|i \operatorname{Im} f|$ are real-valued functions in $C(X,\tau)$, $|\operatorname{Re} f| \leq |f|$ on X, $|i \operatorname{Im} f| \leq |f|$ on X and $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ on Y, according to (3.3) we get

$$\begin{aligned} |T(f)| &= |T(\operatorname{Re} f + i \operatorname{Im} f)| \\ &= |T(\operatorname{Re} f) + T(i \operatorname{Im} f)| \\ &\leq |T(\operatorname{Re} f)| + |T(i \operatorname{Im} f)| \\ &\leq T(|\operatorname{Re} f|) + T(|\operatorname{Im} f|) \\ &\leq T(|f|) + T(|f|) \\ &= 2T(|f|). \end{aligned}$$

Hence, (ii) holds.

To prove (iii), let $f \in C(X,\tau)$. According to $|f|, \|f\|_X 1_X \in C(X,\tau)$, $|f| \leq \|f\|_X 1_X$ on X and $T\colon C(X,\tau) \longrightarrow C(Y,\eta)$ is an order preserving real linear map, we deduce that

$$T(|f|) \leq T(\|f\|_X 1_X) = \|f\|_X T(1_X).$$

Hence, (iii) holds and the proof is complete. □

Proposition 3.4. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$ with $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$. Then T is a continuous map from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$ and $\|T\| \leq 2\,\|T(1_X)\|_Y$.*

Proof. Let $f \in C(X,\tau)$ be given. Since $T\colon C(X,\tau) \longrightarrow C(Y,\eta)$ is an order preserving real linear map with $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$, by part (ii) of Lemma 3.3, we deduce that

$$|T(f)| \leq 2\,T(|f|) \quad \text{on } Y. \tag{3.4}$$

According to T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$, $f \in C(X,\tau)$ and $|f| \leq \|f\|_X 1_X$ on X, by part (iii) of Lemma

3.3, we get

$$T(|f|) \leq \|f\|_X \, T(1_X) \quad \text{on } Y. \tag{3.5}$$

By (3.4) and (3.5), we obtain $|T(f)| \leq 2\|f\|_X \, T(1_X)$ on Y. Therefore, for each $y \in Y$, we get

$$|T(f)(y)| \leq 2\|f\|_X \, |T(1_X)(y)| \leq 2\|T(1_X)\|_Y \, \|f\|_X .$$

This implies that

$$\|T(f)\|_Y \leq 2\|T(1_X)\|_Y \, \|f\|_X . \tag{3.6}$$

Since (3.6) holds for all $f \in C(X,\tau)$, we deduce that real linear map T from real linear normed space $(C(X,\tau), \|\cdot\|_X)$ to real linear normed space $(C(Y,\eta), \|\cdot\|_Y)$ is a bounded linear operator and $\|T\| \leq 2\|T(1_X)\|_Y$. Therefore, T is continuous from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$. Hence, the proof is complete. □

Using Theorem 2.2, Proposition 3.4 and the sense of support for certain real linear complex-valued functionals in $C(X,\tau)'$, we determine the form of order linear isomorphisms T from $C(X,\tau)$ to $C(Y,\eta)$. To reach this goal, we need the following lemmas.

Lemma 3.5. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$ with $1_Y \in T(C(X,\tau))$ and $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$. Then $T(1_X)$ is a nonvanishing positive function on Y.*

Proof. Since $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$, by part (ii) of Lemma 3.3 we have

$$|T(f)| \leq 2T(|f|) \tag{3.7}$$

for all $f \in C(X,\tau)$. According to $1_Y \in T(C(X,\tau))$, there exists a function $g \in C(X,\tau)$ such that

$$T(g) = 1_Y. \tag{3.8}$$

Since $g \in C(X,\tau)$ and (3.7) holds for all $f \in C(X,\tau)$, we conclude that

$$|T(g)| \leq 2T(|g|). \tag{3.9}$$

By part (iii) of Lemma 3.3, we have

$$T(|g|) \leq \|g\|_X \, T(1_X). \tag{3.10}$$

Since $1_X \in C(X,\tau)$, $0_X \leq 1_X$ on X and $T\colon C(X,\tau) \to C(Y,\eta)$ is an order preserving real linear map, we deduce that $0_Y = T(0_X) \leq T(1_X)$ on Y. We claim that $T(1_X)(y) > 0$ for all $y \in Y$. Otherwise, there

exists $y_0 \in Y$ such that $T(1_X)(y_0) = 0$. Therefore, by (3.8), (3.9) and (3.10) we get

$$1 = |1_Y(y_0)| = |T(g)(y_0)| \leq 2T(|g|)(y_0) \leq 2\,\|g\|_X\, T(1_X)(y_0) = 0,$$

which is impossible. Hence, our claim is justified and so the proof is complete. □

Lemma 3.6. *Let X be a compact Hausdorff space and let τ be a topological involution on X. Suppose that $\Lambda \in C(X,\tau)'$ with $0 \leq \Lambda$ and $|\Lambda(i \operatorname{Im} f)| \leq \Lambda(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$. Then $x \in \operatorname{supp}(\Lambda)$ if and only if for each τ-invariant open subset U of X that contains x, there exists a positive function $f \in C(X,\tau)$ with $\overline{\operatorname{coz}(f)} \subseteq U$ such that $\Lambda(f) > 0$.*

Proof. We first assume that $x \in \operatorname{supp}(\Lambda)$. Let U be a τ-invariant open subset of X that contains x. Then there exists a τ-invariant open subset W of X such that $x \in W$ and $\overline{W} \subseteq U$. Since $x \in \operatorname{supp}(\Lambda)$, there exists a function $g \in C(X,\tau)$ with $\overline{\operatorname{coz}(g)} \subseteq W$ such that $\Lambda(g) \neq 0$. According to $\Lambda \in C(X,\tau)'$, we deduce that $g \neq 0_X$. This implies that $\|g\|_X > 0$. From $\overline{\operatorname{coz}(g)} \subseteq W$ and $\overline{W} \subseteq U$, we get $\overline{\operatorname{coz}(g)} \subseteq U$. Since $\overline{\operatorname{coz}(g)}$ is a τ-invariant compact subset of X, U is a τ-invariant open subset of X and $\overline{\operatorname{coz}(g)} \subseteq U$, by Lemma 2.1 we deduce that there exists a function $f \in C(X,\tau)$ such that $0_X \leq f$ on X, $f(z) = 1$ for all $z \in \overline{\operatorname{coz}(g)}$ and $\overline{\operatorname{coz}(f)} \subseteq U$. If $z \in \overline{\operatorname{coz}(g)}$, then $f(z) = 1$ and so

$$|\operatorname{Re} g(z)| \leq |g(z)| \leq \|g\|_X = \|g\|_X\, f(z).$$

If $z \in X \setminus \overline{\operatorname{coz}(g)}$, then $g(z) = 0$ and so

$$|\operatorname{Re} g(z)| = 0 \leq \|g\|_X\, f(z).$$

Therefore, $|\operatorname{Re} g(z)| \leq \|g\|_X\, f(z)$ for all $z \in X$. Similarly, we can show that $|\operatorname{Im} g(z)| \leq \|g\|_X\, f(z)$ for all $z \in X$. Hence, for each $z \in X$ we have

$$|g(z)| \leq 2\,\|g\|_X\, f(z).$$

This implies that $|g| \leq 2\,\|g\|_X\, f$ on X. Therefore,

$$\Lambda(|g|) \leq 2\,\|g\|_X\, \Lambda(f) \tag{3.11}$$

since $|g|$, $2\,\|g\|_X\, f \in C(X,\tau)$ and $\Lambda \in C(X,\tau)'$ with $0 \leq \Lambda$. According to $-|g|$, $\operatorname{Re} g$, $|g| \in C(X,\tau)$, $-|g| \leq \operatorname{Re} g \leq |g|$ and $\Lambda \in C(X,\tau)'$ with $0 \leq \Lambda$, we deduce that

$$-\Lambda(|g|) = \Lambda(-|g|) \leq \Lambda(\operatorname{Re} g) \leq \Lambda(|g|).$$

It follows that

$$\Lambda(\operatorname{Re} g) \leq \Lambda(|g|). \tag{3.12}$$

Since $|\Lambda(i\operatorname{Im} h)| \leq \Lambda(|\operatorname{Im} h|)$ for all $h \in C(X,\tau)$ and $g \in C(X,\tau)$, we get

$$|\Lambda(i\operatorname{Im} g)| \leq \Lambda(|\operatorname{Im} g|). \tag{3.13}$$

According to $|\operatorname{Im} g|$, $|g| \in C(X,\tau)$, $|\operatorname{Im} g| \leq |g|$ on X and $\Lambda \in C(X,\tau)'$ with $0 \leq \Lambda$, we deduce that

$$\Lambda(|\operatorname{Im} g|) \leq \Lambda(|g|). \tag{3.14}$$

Since g, $\operatorname{Re} g$, $i\operatorname{Im} g \in C(X,\tau)$, $g = \operatorname{Re} g + i\operatorname{Im} g$ and $\Lambda \in C(X,\tau)'$, according to (3.12), (3.13) and (3.14) we get

$$\begin{aligned} |\Lambda(g)| &= |\Lambda(\operatorname{Re} g + i\operatorname{Im} g)| \\ &= |\Lambda(\operatorname{Re} g) + \Lambda(i\operatorname{Im} g)| \\ &\leq |\Lambda(\operatorname{Re} g)| + |\Lambda(i\operatorname{Im} g)| \\ &\leq \Lambda(|g|) + \Lambda(|g|). \end{aligned}$$

Therefore,

$$|\Lambda(g)| \leq 2\Lambda(|g|). \tag{3.15}$$

From (3.15) and (3.11), we get

$$|\Lambda(g)| \leq 2\,\|g\|_X\,\Lambda(f). \tag{3.16}$$

According to (3.16), $\Lambda(g) \neq 0$ and $2\,\|g\|_X > 0$, we deduce that $\Lambda(f) > 0$. Conversely, we now assume that for each τ-invariant open subset U of X that contains x, there exists a positive function $f \in C(X,\tau)$ with $\operatorname{coz}(f) \subseteq U$ such that $\Lambda(f) > 0$. The definition of $\operatorname{supp}(\Lambda)$ implies that $x \in \operatorname{supp}(\Lambda)$. Hence, the proof is complete. □

Lemma 3.7. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that T is a bounded real linear operator from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$ and $y \in Y$. Then $\delta_y \circ T \in C(X,\tau)'$ and $\|\delta_y \circ T\| \leq \|T\|$.*

Proof. Since δ_y is a real linear map from $C(Y,\eta)$ to $\mathbb{C}$ and T is a real linear map from $C(X,\tau)$ to $C(Y,\eta)$, we deduce that $\delta_y \circ T$ is a real linear map from $C(X,\tau)$ to $\mathbb{C}$. According to T is a bounded real linear map from $C(X,\tau)$ with the norm $\|\cdot\|_X$ to $C(Y,\eta)$ with the norm $\|\cdot\|_Y$, we have

$$|(\delta_y \circ T)(f)| = |\delta_y(T(f))| = |T(f)(y)| \leq \|T(f)\|_Y \leq \|T\|\ \|f\|_X$$

for all $f \in C(X,\tau)$. Therefore, $\delta_y \circ T \in C(X,\tau)'$ and $\|\delta_y \circ T\| \leq \|T\|$. Hence, the proof is complete. □

Theorem 3.8. *Let X and Y be compact Hausdorff spaces and let τ and η respectively be topological involutions on X and Y. Suppose that $X_\tau = \{\{x, \tau(x)\} : x \in X\}$ and $Y_\eta = \{\{y, \eta(y)\} : y \in Y\}$. Let T be an order real isomorphism from $C(X,\tau)$ to $C(Y,\eta)$ with $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$ and $\left|T^{-1}(i \operatorname{Im} h)\right| \leq T^{-1}(|\operatorname{Im} h|)$ for all $h \in C(Y,\eta)$. Then T is a homeomorphism from $(C(X,\tau), \|\cdot\|_X)$ to $(C(Y,\eta), \|\cdot\|_Y)$, $T(1_X)$ is a nonvanishing positive function on Y, $T^{-1}(1_Y)$ is a nonvanishing positive function on X and there exists a bijective map $\Phi\colon Y_\eta \longrightarrow X_\tau$ such that the following hold:*

(i) $T(f)(y) = T(1_X)(y)f(x)$ *for all* $y \in Y$, $x \in \Phi(\{y, \eta(y)\})$ *and* $f \in C(X,\tau)$ *with* $f(x) \in \mathbb{R}$.
(ii) $T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y)$ *for all* $x \in X$, $y \in \Phi^{-1}(\{x, \tau(x)\})$ *and* $h \in C(Y,\eta)$ *with* $h(y) \in \mathbb{R}$.
(iii) $\Phi(\{y, \eta(y)\})$ *is a singleton set if* $y \in Y$ *with* $\eta(y) = y$ *and* $\Phi^{-1}(\{x, \tau(x)\})$ *is a singleton set whenever* $x \in X$ *with* $\tau(x) = x$.

Proof. We divide the proof into ten following steps.

Step 1. *T is a homeomorphism from $C(X,\tau)$ with the $\|\cdot\|_X$-topology to $C(Y,\eta)$ with the $\|\cdot\|_Y$-topology.*

Since T is an order preserving linear map from $C(X,\tau)$ to $C(Y,\eta)$ with $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$ by Proposition 3.4 we deduce that T is a continuous map from $C(X,\tau)$ with the $\|\cdot\|_X$-topology to $C(Y,\eta)$ with the $\|\cdot\|_Y$-topology. According to T^{-1} is an order preserving real linear map from $C(Y,\eta)$ to $C(X,\tau)$, by Proposition 3.4 we conclude that T^{-1} is a continuous map from $C(Y,\eta)$ with the $\|\cdot\|_Y$-topology to $C(X,\tau)$ with the $\|\cdot\|_X$-topology. Therefore, T is a homeomorphism from $C(X,\tau)$ with the $\|\cdot\|_X$-topology to $C(Y,\eta)$ with the $\|\cdot\|_Y$-topology. Hence, Step 1 holds.

Step 2. *$T(1_X)$ and $T^{-1}(1_Y)$ are respectively nonvanishing positive functions on Y and on X.*

Since T is a bijective map from $C(X,\tau)$ to $C(Y,\eta)$, $1_Y \in C(Y,\eta)$ and $1_X \in C(X,\tau)$, we deduce that $1_Y \in T(C(X,\tau))$ and $1_X \in T^{-1}C(X,\tau)$. According to T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$, $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$ and $1_Y \in T(C(X,\tau))$, by Lemma 3.5 we deduce that $T(1_X)$ is a nonvanishing positive function on Y. According to T^{-1} is an order preserving real linear map from $C(Y,\eta)$ to $C(X,\tau)$, $\left|T^{-1}(i \operatorname{Im} h)\right| \leq T^{-1}(|\operatorname{Im} h|)$ for all $h \in C(Y,\eta)$ and $1_X \in T^{-1}(C(Y,\eta))$, by Lemma 3.5 we conclude that $T^{-1}(1_Y)$ is a nonvanishing positive function on X. Hence, Step 2 holds.

Step 3. *If $y \in Y$, then $\delta_y \circ T \in C(X,\tau)'$, $0 \leq \delta_y \circ T$, $|(\delta_y \circ T)(i \operatorname{Im} f)| \leq (\delta_y \circ T)(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$ and $\operatorname{supp}(\delta_y \circ T)$ is a τ-invariant nonempty subset of X.*

Let $y \in Y$ be given. By Step 1, T is a continuous map from $C(X,\tau)$ with the $\|\cdot\|_X$-topology to $C(Y,\eta)$ with the $\|\cdot\|_Y$-topology. Since T is a real linear map from $C(X,\tau)$ to $C(Y,\eta)$, by Lemma 3.7 we deduce that $\delta_y \circ T \in C(X,\tau)'$ and $\|\delta_y \circ T\| \leq \|T\|$. Let $f \in C(X,\tau)$ with $0_X \leq f$ on X. Then $0_Y \leq T(f)$ on Y. Since T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$. Therefore, $0 \leq T(f)(y)$ which implies that

$$0 \leq T(f)(y) = \delta_y(T(f)) = (\delta_y \circ T)(f).$$

Since the inequality above holds for all $f \in C(X,\tau)$, we deduce that $0 \leq \delta_y \circ T$. According to $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$ for all $f \in C(X,\tau)$, we deduce that

$$|(\delta_y \circ T)(i \operatorname{Im} f)| = |T(i \operatorname{Im} f)(y)| \leq T(|\operatorname{Im} f|)(y) = (\delta_y \circ T)(|\operatorname{Im} f|)$$

for all $f \in C(X,\tau)$. We now prove that $\operatorname{supp}(\delta_y \circ T)$ is nonempty. Let us suppose that $\operatorname{supp}(\delta_y \circ T)$ is empty. By the definition of $\operatorname{supp}(\delta_y \circ T)$, for each $x \in X$ there exists a τ-invariant open subset U_x of X that contains x such that $(\delta_y \circ T)(f) = 0$ for all $f \in C(X,\tau)$ satisfying that $\operatorname{coz}(f) \subseteq U_x$. It is clear that $\{U_x : x \in X\}$ is an open covering of X. The compactness of X implies that there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{j=1}^n U_{x_j}$. By Theorem 2.2, there exist functions $f_1, \ldots, f_n$ in $C(X,\tau)$ such that $\sum_{j=1}^n f_j = 1_X$ on X and $\operatorname{coz}(f_j) \subseteq U_{x_j}$ for all $j \in \{1, \ldots, n\}$. The surjectivity of $T : C(X,\tau) \longrightarrow C(Y,\eta)$ and $1_Y \in C(Y,\eta)$ imply that there exists a function $g_1 \in C(X,\tau)$ such that $1_Y = T(g_1)$. Let $j \in \{1, \ldots, n\}$ be given. Then $g_1 f_j \in C(X,\tau)$ and we have

$$\operatorname{coz}(g_1 f_j) \subseteq \operatorname{coz}(f_j) \subseteq U_{x_j}.$$

According to argument above, we conclude that

$$T(g_1 f_j)(y) = \delta_y(T(g_1 f_j)) = (\delta_y \circ T)(g_1 f_j) = 0. \tag{3.17}$$

Since $T : C(X,\tau) \longrightarrow C(Y,\eta)$ is a real linear mapping, $T(g_1) = 1_Y$ and (3.17) holds for all $j \in \{1, \ldots, n\}$, we deduce that

$$1 = T(g_1)(y) = T\Big(g_1 \sum_{j=1}^n f_j\Big)(y) = T\Big(\sum_{j=1}^n g_1 f_j\Big)(y) = \sum_{j=1}^n T(g_1 f_j)(y) = 0$$

which is impossible. Therefore, $\operatorname{supp}(\delta_y \circ T)$ is nonempty.

We now prove that $\operatorname{supp}(\delta_y \circ T)$ is τ-invariant. Let $x \in \operatorname{supp}(\delta_y \circ T)$ be given. Assume that U is a τ-invariant open subset of X that contains $\tau(x)$. Since $x \in \operatorname{supp}(\delta_y \circ T)$ and $x \in U$, by Lemma 3.6 there exists a positive function $f \in C(X,\tau)$ with $\operatorname{coz}(f) \subseteq U$ such that $(\delta_y \circ T)(f) > 0$. Therefore, by Lemma 3.6 we conclude that $\tau(x) \in \operatorname{supp}(\delta_y \circ T)$. Hence, $\operatorname{supp}(\delta_y \circ T)$ is a τ-invariant subset of X and so Step 3 holds.

Step 4. *For each $y \in Y$, there exists a unique element $\{x, \tau(x)\}$ of X_τ such that* $\operatorname{supp}(\delta_y \circ T) = \{x, \tau(x)\}$.

Let $y \in Y$ be given. By Step 3, $\delta_y \circ T \in C(X,\tau)'$, $0 \leq \delta_y \circ T$ and $\operatorname{supp}(\delta_y \circ T)$ is a τ-invariant nonempty subset of X. If $x \in \operatorname{supp}(\delta_y \circ T)$, then $\tau(x) \in \operatorname{supp}(\delta_y \circ T)$ and so $\{x, \tau(x)\} \subseteq \operatorname{supp}(\delta_y \circ T)$.

We now prove that if $x, z \in \operatorname{supp}(\delta_y \circ T)$ then $\{x, \tau(x)\} = \{z, \tau(z)\}$. Let us suppose that $x, z \in \operatorname{supp}(\delta_y \circ T)$ with $\{x, \tau(x)\} \neq \{z, \tau(z)\}$. Then $\{x, \tau(x)\}$ and $\{z, \tau(z)\}$ are disjoint closed sets in compact Hausdorff space X. Therefore, there exist disjoint open sets U and V in X such that $\{x, \tau(x)\} \subseteq U$ and $\{z, \tau(z)\} \subseteq V$. Clearly, $U \cap \tau(U)$ is a τ-invariant open subset of X, $x \in U \cap \tau(U)$, $V \cap \tau(V)$ is a τ-invariant open subset of X and $z \in V \cap \tau(V)$. Since $\delta_y \circ T \in C(X,\tau)'$ and $0 \leq \delta_y \circ T$, by Lemma 3.6 there exist two positive functions $f, g \in C(X,\tau)$ such that $\operatorname{coz}(f) \subseteq U \cap \tau(U)$, $(\delta_y \circ T)(f) > 0$, $\operatorname{coz}(g) \subseteq V \cap \tau(V)$ and $(\delta_y \circ T)(g) > 0$. Let $k = \min\{T(f), T(g)\}$. One can easily show that $k \in C(Y,\eta)$. In addition, it is clear that $0_Y \leq k \leq T(f)$ on Y, $0_Y \leq k \leq T(g)$ on Y and $k(y) > 0$. Take $h = T^{-1}(k)$. Then $h \in C(X,\tau) \setminus \{0_X\}$ since $k \neq 0_Y$ and T^{-1} is an injective real linear map from $C(Y,\eta)$ to $C(X,\tau)$. Therefore, there exists $x_0 \in X$ with $h(x_0) \neq 0$. On the other hand, $0_X \leq h \leq f$ and $0_X \leq h \leq g$ on X since T^{-1} is an order preserving real linear map from $C(Y,\eta)$ to $C(X,\tau)$, $k \in C(Y,\eta)$, $k = T(h)$, $0_Y \leq k \leq T(f)$ on Y and $0_Y \leq k \leq T(g)$ on Y. Therefore, $0 < h(x_0) \leq f(x_0)$ and $0 < h(x_0) \leq g(x_0)$ which imply that $x_0 \in \operatorname{coz}(f)$ and $x_0 \in \operatorname{coz}(g)$. Thus $x_0 \in U \cap V$. This contradicts to the disjointness of U and V. Therefore, we have proved that for each $y \in Y$ there exists a unique element $\{x, \tau(x)\}$ of X_τ such that $\operatorname{supp}(\delta_y \circ T) = \{x, \tau(x)\}$. Hence, Step 4 holds.

Step 4 permits us to define a mapping $\Phi : Y_\eta \longrightarrow X_\tau$ such that $\Phi(\{y, \eta(y)\}) = \operatorname{supp}(\delta_y \circ T)$ for any $y \in Y$.

Step 5. *Let $y \in Y$ and $f \in C(X,\tau)$ with $0_X \leq f$ on X. Then $T(f)(y) = 0$ if $\Phi(\{y, \eta(y)\}) \cap \overline{\operatorname{coz}(f)} = \emptyset$.*

Let $\Phi(\{y, \eta(y)\}) \cap \overline{\operatorname{coz}(f)} = \emptyset$. Then τ-invariant closed sets $\Phi(\{y, \eta(y)\})$ and $\overline{\operatorname{coz}(f)}$ in compact Hausdorff space X are disjoint. Therefore, there exists a τ-invariant open subset U of X with $\Phi(\{y, \eta(y)\}) \subseteq U$ such that $U \cap \overline{\operatorname{coz}(f)} = \emptyset$ which implies that $\operatorname{coz}(f) \subseteq X \setminus U$. By step 3, we have $\delta_y \circ T \in C(X,\tau)'$, $0 \leq \delta_y \circ T$ and $|(\delta_y \circ T)(i \operatorname{Im} f)| \leq (\delta_y \circ T)(|\operatorname{Im} f|)$. Assume that $\Phi(\{y, \eta(y)\}) = \{x, \tau(x)\}$. Then $x \in \operatorname{supp}(\delta_y \circ T)$ and U is a τ-invariant open subset of X that contains x. Therefore, by Lemma 3.6, there exists a function $g \in C(X,\tau)$ with $0_X \leq g$ and $\operatorname{coz}(g) \subseteq U$ with $T(g)(y) = (\delta_y \circ T)(g) > 0$. Clearly,

$$\operatorname{coz}(f) \cap \operatorname{coz}(g) = \emptyset. \tag{3.18}$$

To obtain a contradiction, we assume that $T(f)(y) \neq 0$. According to $f \in C(X,\tau)$, $0_X \leq f$ on X and T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$, we deduce that $0_Y = T(0_X) \leq T(f)$ on Y. Therefore $T(f)(y) > 0$. Define $k = \min\{T(f), T(g)\}$. Then $k \in C(Y,\eta)$, $0_Y \leq k \leq T(f)$, $0_Y \leq k \leq T(g)$ and $k(y) > 0$. Take $h = T^{-1}(k)$. Then $h \in C(X,\tau)$, $0_X \leq h \leq f$ and $0_X \leq h \leq g$ since T^{-1} is an order preserving real linear map from $C(Y,\eta)$ to $C(X,\tau)$. In addition, $h \neq 0_X$ since $T(h) = k \neq 0_Y = T(0_X)$ and T is an injection. Therefore, there exists a point $x_0 \in X$ such that $h(x_0) > 0$. This implies that $f(x_0) > 0$ and $g(x_0) > 0$. Therefore, $x_0 \in \operatorname{coz}(f) \cap \operatorname{coz}(g)$ which contradicts (3.18). Hence, $T(f)(y) = 0$ and so Step 5 holds.

Step 6. *If $y \in Y$ and $x \in \Phi(\{y, \eta(y)\})$, then $\ker(\delta_x)$ is a subset of $\ker(\delta_y \circ T)$.*

To obtain a contradiction, we assume that $y \in Y, x \in \Phi(\{y,\eta(y)\})$ and $\ker(\delta_x)$ is not a subset of $\ker(\delta_y \circ T)$. Then there exists a function $g \in C(X,\tau)$ such that $g(x) = 0$ and $T(g)(y) = (\delta_y \circ T)(g) \neq 0$. It is clear that $|g| \in C(X,\tau)$, $0_X \leq |g|$ and $|g|(x) = 0$. Since T is an order preserving real linear map from $C(X,\tau)$ to $C(Y,\eta)$, $g \in C(X,\tau)$ and $|T(i \operatorname{Im} f)| \leq T(|\operatorname{Im} f|)$, by part (ii) of Lemma 3.3 we deduce that

$$|T(g)(y)| \leq 2T(|g|)(y).$$

Therefore, $T(|g|)(y) > 0$. By Step 2, $T(1_X)$ is a nonvanishing positive function on Y. Thus $T(1_X)(y) > 0$. Take $\lambda = \frac{T(|g|)(y)}{2T(1_X)(y)}$. Then $\lambda \in \mathbb{R}$, $\lambda > 0$ and we have

$$T(\lambda 1_X)(y) = \lambda T(1_X)(y) = \frac{1}{2}T(|g|)(y) < T(|g|)(y). \tag{3.19}$$

Since $|g|$ is a real-valued continuous function at x and $\tau(x)$ and we have $|g|(x) = |g|(\tau(x)) = 0 < \lambda$, there exists a τ-invariant open subset U of X that contains x and $\tau(x)$ such that $|g|(z) < \lambda = (\lambda 1_X)(z)$ for all $z \in U$. Define $g_\lambda = \max\{|g|, \lambda 1_X\}$. Then $g_\lambda \in C(X,\tau)$, $g_\lambda(z) = \lambda$ for all $z \in U$ and $\lambda 1_X \leq g_\lambda$ on X. Define $f_\lambda = g_\lambda - \lambda 1_X$. Then $f_\lambda \in C(X,\tau)$ and $0_X \leq f_\lambda$ on X. Furthermore, $f_\lambda(z) = 0$ for all $z \in U$. This implies that $U \cap \operatorname{coz}(f_\lambda) = \emptyset$ and so $U \cap \overline{\operatorname{coz}(f_\lambda)} = \emptyset$ since U is an open set in X. Therefore, $\{x, \tau(x)\} \cap \overline{\operatorname{coz}(f_\lambda)} = \emptyset$ and so $\Phi(\{y,\eta(y)\}) \cap \overline{\operatorname{coz}(f_\lambda)} = \emptyset$ since $x \in \Phi(\{y,\eta(y)\})$ which implies that $\Phi(\{y,\eta(y)\}) = \{x, \tau(x)\}$. By Step 5, we deduce that $T(f_\lambda)(y) = 0$. Therefore,

$$T(g_\lambda)(y) - T(\lambda 1_X)(y) = T(g_\lambda - \lambda 1_X)(y) = T(f_\lambda)(y) = 0.$$

This implies that

$$T(g_\lambda)(y) = T(\lambda 1_X)(y). \tag{3.20}$$

By (3.19) and (3.20), we get

$$T(g_\lambda)(y) < T(|g|)(y). \tag{3.21}$$

Since $g_\lambda, |g| \in C(X,\tau)$, $0_X \leq |g| \leq g_\lambda$ on X and $T : C(X,\tau) \to C(Y,\eta)$ is an order preserving real linear map, we deduce that

$$0_Y = T(0_X) \leq T(|g|) \leq T(g_\lambda)$$

on Y and so $T(|g|)(y) \leq T(g_\lambda)(y)$ which contradicts (3.21). Hence, Step 6 holds.

Step 7. *$T(f)(y) = T(1_X)(y)f(x)$ for all $y \in Y$, $x \in \Phi(\{y, \eta(y)\})$ and $f \in C(X,\tau)$ with $f(x) \in \mathbb{R}$.*

Assume that $y \in Y$, $x \in \Phi(\{y,\eta(y)\})$ and $f \in C(X,\tau)$ with $f(x) \in \mathbb{R}$. Define $g = f - f(x)1_X$. Then $g \in C(X,\tau)$ and $\delta_x(g) = g(x) = 0$. Therefore, $g \in \ker(\delta_x)$. Since $y \in Y$ and $x \in \Phi(\{y,\eta(y)\})$, by Step 6 we have $\ker(\delta_x) \subseteq \ker(\delta_y \circ T)$. Therefore, $g \in \ker(\delta_y \circ T)$ and so we get

$$0 = (\delta_y \circ T)(g) = T(g)(y) = T(f - f(x)1_X)(y) = T(f)(y) - f(x)T(1_X)(y)$$

since $f(x) \in \mathbb{R}$ and $T : C(X,\tau) \to C(Y,\eta)$ is a real linear map. Hence, $T(f)(y) = T(1_X)(y)f(x)$ and so Step 7 holds.

Step 8. *$\Phi : Y_\eta \longrightarrow X_\tau$ is a bijection and*

$$T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y)$$

for all $x \in X$, $y \in \Phi^{-1}(\{x,\tau(x)\})$ and $h \in C(Y,\eta)$ with $h(y) \in \mathbb{R}$.

Since $T^{-1} : C(Y,\eta) \longrightarrow C(X,\tau)$ is an order preserving real linear map, from what has already been proved, we deduce that $T^{-1}(1_Y)$ is a nonvanishing positive function and there exists a map $\Psi : X_\tau \longrightarrow Y_\eta$ such that

$$T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y) \tag{3.22}$$

for all $x \in X$, $y \in \Psi(\{x,\tau(x)\})$ and $h \in C(Y,\eta)$ with $h(y) \in \mathbb{R}$. We now show that

$$\Phi \circ \Psi = I_{X_\tau}. \tag{3.23}$$

To obtain a contradiction, assume that there exists a point $u \in X$ such that $(\Phi \circ \Psi)(\{u,\tau(u)\}) \neq \{u,\tau(u)\}$. Take $\{v,\eta(v)\} = \Psi(\{u,\tau(u)\})$ and $\{z,\tau(z)\} = (\Phi \circ \Psi)(\{u,\tau(u)\})$, where $v \in Y$ and $z \in X$. Then we have $z \in \Phi(\{v,\eta(v)\})$. In addition, $\{u,\tau(u)\} \cap \{z,\tau(z)\} = \emptyset$. Since X is a compact Hausdorff space, $\{u,\tau(u)\}$ is a τ-invariant compact subset of X, $X \setminus \{z,\tau(z)\}$ is a τ-invariant open subset of X and $\{u,\tau(u)\}$ is a subset of $X \setminus \{z,\tau(z)\}$, by Lemma 2.1, there exists a function $g \in C(X,\tau)$ such that $0_X \leq g \leq 1_X$ on X, $g(u) = g(\tau(u)) = 1$ and $g(z) = 0$. Since $v \in Y$, $z \in \Phi(\{v,\eta(v)\})$ and $g \in C(X,\tau)$ with $g(z) \in \mathbb{R}$, by Step 7 we have

$$T(g)(v) = T(1_X)(v)g(z). \tag{3.24}$$

From (3.24) and $g(z) = 0$, we get $T(g)(v) = 0$. According to $u \in X$, $v \in \Psi(\{u, \tau(u)\})$, $T(g) \in C(Y, \eta)$ with $T(g)(v) \in \mathbb{R}$ and (3.22) holds for all $x \in X$, $y \in \Psi(\{x, \tau(x)\})$ and $h \in C(Y, \eta)$ with $h(y) \in \mathbb{R}$, we deduce that

$$T^{-1}(T(g))(u) = T^{-1}(1_Y)(u)T(g)(v). \tag{3.25}$$

From (3.25), $g(u) = 1$ and $T(g)(v) = 0$, we get $1 = 0$ which is impossible. Therefore, (3.23) holds.

We now prove that

$$\Psi \circ \Phi = I_{Y_\eta}. \tag{3.26}$$

To obtain a contradiction, assume that there exist $v \in Y$ such that $(\Psi \circ \Phi)(\{v, \eta(v)\}) \neq \{v, \eta(v)\}$. Take $\{u, \tau(u)\} = \Phi(\{v, \eta(v)\})$ and $\{w, \eta(w)\} = (\Psi \circ \Phi)(\{v, \eta(v)\})$, where $u \in X$ and $w \in Y$. Since Y is a compact Hausdorff space, $\{v, \eta(v)\}$ is an η-invariant compact subset of Y, $Y \setminus \{w, \eta(w)\}$ is an η-invariant open subset of Y and $\{v, \eta(v)\}$ is a subset of $Y \setminus \{w, \eta(w)\}$, by Lemma 2.1 there exists a function $k \in C(Y, \eta)$ such that $0_Y \leq k \leq 1_Y$, $k(v) = k(\eta(v)) = 1$, $k(w) = 0$. Since $u \in X$, $w \in \Psi(\{u, \tau(u)\})$, $k \in C(Y, \eta)$ with $k(w) \in \mathbb{R}$ and (3.22) holds for all $x \in X$, $y \in \Psi(\{x, \tau(x)\})$ and $h \in C(Y, \eta)$ with $h(y) \in \mathbb{R}$, we deduce that

$$T^{-1}(k)(u) = T^{-1}(1_Y)(u)k(w). \tag{3.27}$$

From (3.27) and $k(w) = 0$, we get $T^{-1}(k)(u) = 0$. According to $v \in Y$, $u \in \Phi(\{v, \eta(v)\})$ and $T^{-1}(k) \in C(X, \tau)$ with $T^{-1}(k)(u) \in \mathbb{R}$, by Step 7 we have

$$T(T^{-1}(k))(v) = T(1_X)(v)T^{-1}(k)(u). \tag{3.28}$$

From (3.28), $k(v) = 1$ and $T^{-1}(k)(u) = 0$, we get $1 = 0$ which is impossible. Therefore, (3.26) holds.

By (3.23) and (3.26), we deduce that $\Phi \colon Y_\eta \longrightarrow X_\tau$ is a bijective map and $\Psi = \Phi^{-1}$. Let $x \in X$, $y \in \Phi^{-1}(\{x, \tau(x)\})$ and $h \in C(Y, \eta)$ with $h(y) \in \mathbb{R}$. Then $y \in \Psi(\{x, \tau(x)\})$ since $\Psi = \Phi^{-1}$. Therefore, by (3.22) we have $T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y)$. Hence, Step 8 holds.

Step 9. *$\Phi(\{y, \eta(y)\})$ is a singleton whenever $y \in Y$ with $\eta(y) = y$.*

Let $y \in Y$ with $\eta(y) = y$ be given. Take $\{x, \tau(x)\} = \Phi(\{y, \eta(y)\})$, where $x \in X$. Let $f \in C(X, \tau)$ be arbitrary. Then $T(f) \in C(Y, \eta)$ with $T(f)(y) \in \mathbb{R}$ since $\eta(y) = y$. Thus $T(f)(y) = \overline{T(f)(y)}$. According to $x, \tau(x) \in X$, $\Phi^{-1}(\{x, \tau(x)\}) = \Phi^{-1}(\Phi(\{y, \eta(y)\}))$, $y \in \Phi^{-1}(\{x, \tau(x)\})$ and $T(f) \in C(Y, \eta)$ with $T(f)(y) \in \mathbb{R}$ and $T(f)(\eta(y)) \in \mathbb{R}$, by Step 8 we get

$$f(x) = T^{-1}(T(f))(x) = T^{-1}(1_Y)(x)T(f)(y) \tag{3.29}$$

and

$$f(\tau(x)) = T^{-1}(T(f))(\tau(x)) = T^{-1}(1_Y)(\tau(x))T(f)(y). \tag{3.30}$$

On the other hand, $T^{-1}(1_Y)$ is a real-valued function in $C(X,\tau)$. Hence,

$$T^{-1}(1_Y)(\tau(x)) = T^{-1}(1_Y)(x). \tag{3.31}$$

From (3.29), (3.30) and (3.31), we get $f(x) = f(\tau(x))$. According to this equality holds for all $f \in C(X,\tau)$ and $C(X,\tau)$ separates the points of X, we deduce that $x = \tau(x)$. Therefore, $\Phi(\{y, \eta(y)\}) = \{x\}$ and so $\Phi(\{y, \eta(y)\})$ is a singleton. Hence, Step 9 holds.

Step 10. *$\Phi^{-1}(\{x, \tau(x)\})$ is a singleton whenever $x \in X$ with $\tau(x) = x$.*

Step 10 can be proved similarly to Step 9.

In this way, the proof of Theorem 3.8 is complete. □

Applying Theorem 3.8, we give a complete description of order real isomorphisms between $C_{\mathbb{R}}(X)$-algebras. To reach this aim, we need the following lemma.

Lemma 3.9. *Let X be a compact Hausdorff space, let $\{x_\alpha\}_{\alpha\in J}$ be a net in X and let $x \in X$. If $\lim_\alpha f(x_\alpha) = f(x)$ in $\mathbb{C}$ for all $f \in C(X)$, then $\lim_\alpha x_\alpha = x$ in X.*

Proof. Let $\Delta(C(X))$ be the set of all nonzero multiplicative complex linear functionals on $C(X)$. We know that for each $z \in X$ the map $e_z : C(X) \longrightarrow \mathbb{C}$ defined by

$$e_z(f) = f(z), \quad (f \in C(X)),$$

belongs to $\Delta(C(X))$ and the map $E_X : X \longrightarrow \Delta(C(X))$ defined by

$$E_X(z) = e_z \quad (z \in X),$$

is a homeomorphism from X with the given topology to $\Delta(C(X))$ with the Gelfand topology. Suppose that $\lim_\alpha f(x_\alpha) = f(x)$ for all $f \in C(X)$. We prove that $\lim_\alpha x_\alpha = x$ in X. Let U be an open set in X and $x \in U$. Then $E_X(U)$ is an open set in $\Delta(C(X))$ with the Gelfand topology and $e_x \in E_X(U)$. This implies that there exists a positive real number ε and functions $f_1, \ldots, f_n \in C(X)$ such that

$$\left\{\varphi \in \Delta(C(X)) : \left|\widehat{f_j}(\varphi) - \widehat{f_j}(e_x)\right| < \varepsilon \ (j \in \{1, \ldots, n\})\right\} \subseteq E_X(U), \tag{3.32}$$

where $\widehat{f} : \Delta(C(X)) \to \mathbb{C}$ is the Gelfand transform of $f \in C(X)$ on $\Delta(C(X))$. Let $j \in \{1, \ldots, n\}$. Then $\lim_\alpha f_j(x_\alpha) = f_j(x)$ in $\mathbb{C}$. It follows that there exists $\beta_j \in J$ such that $|f_j(x_\alpha) - f_j(x)| < \varepsilon$ for all $\alpha \in J$

with $\alpha \geq \beta_j$. Assume that $\beta \in J$ with $\beta \geq \beta_j$ for all $j \in \{1, \ldots, n\}$. Let $\alpha \in J$ with $\alpha \geq \beta$. Then $\alpha \geq \beta_j$ for all $j \in \{1, \ldots, n\}$. Therefore,

$$|f_j(x_\alpha) - f_j(x)| < \varepsilon \tag{3.33}$$

for all $j \in \{1, \ldots, n\}$. On the other hand, for each $j \in \{1, \ldots, n\}$ we have

$$\left|\widehat{f_j}(e_{x_\alpha}) - \widehat{f_j}(e_x)\right| = |e_{x_\alpha}(f_j) - e_x(f_j)| = |f_j(x_\alpha) - f_j(x)|\,. \tag{3.34}$$

Since (3.34) and (3.33) hold for all $j \in \{1, \ldots, n\}$, we deduce that

$$\left|\widehat{f_j}(e_{x_\alpha}) - \widehat{f_j}(e_x)\right| < \varepsilon$$

for all $j \in \{1, \ldots, n\}$. This implies that

$$e_{x_\alpha} \in \left\{\varphi \in \Delta(C(X)) : \left|\widehat{f_j}(\varphi) - \widehat{f_j}(e_x)\right| < \varepsilon \;\; (j \in \{1, \ldots, n\})\right\}. \tag{3.35}$$

According to (3.35) and (3.32), we get $e_{x_\alpha} \in E_X(U)$. This implies that $x_\alpha = E_X^{-1}(e_{x_\alpha}) \in U$. Therefore, $\lim_\alpha x_\alpha = x$ in X. Hence, the proof is complete. □

Theorem 3.10. *Let X and Y be compact Hausdorff spaces and let T be an order isomorphism from $C_\mathbb{R}(X)$ to $C_\mathbb{R}(Y)$. Then $T(1_X)$ and $T^{-1}(1_Y)$ are respectively nonvanishing positive functions in $C_\mathbb{R}(Y)$ and $C_\mathbb{R}(X)$ and there exists a unique homeomorphism φ from Y to X such that $T(f) = T(1_X) \cdot (f \circ \varphi)$ for all $f \in C_\mathbb{R}(X)$ and $T^{-1}(h) = T^{-1}(1_Y) \cdot \left(h \circ \varphi^{-1}\right)$ for all $h \in C_\mathbb{R}(Y)$. In particular, $|T(f)| = T(|f|)$ for all $f \in C_\mathbb{R}(X)$, $T(1_X) = \frac{1}{T^{-1}(1_Y)} \circ \varphi$, $\left|T^{-1}(h)\right| = T^{-1}(|h|)$ for all $h \in C_\mathbb{R}(Y)$ and $T^{-1}(1_Y) = \frac{1}{T(1_X)} \circ \varphi^{-1}$.*

Proof. Let τ and η be the identity maps on X and Y, respectively. Then τ is a topological involution on X, $\{x, \tau(x)\} = \{x\}$ for all $x \in X$, $C_\mathbb{R}(X) = C(X, \tau)$, η is a topological involution on Y, $\{y, \eta(y)\} = \{y\}$ for all $y \in Y$, $C_\mathbb{R}(Y) = C(Y, \eta)$ and T is an order isomorphism from $C(X, \tau)$ to $C(Y, \eta)$. In addition, for each $f \in C(X, \tau)$ we have

$$\begin{aligned}
|T(i \operatorname{Im} f)| &= |T(i0_X)| \\
&= |T(0_X)| \\
&= |0_Y| \\
&= 0_Y \\
&= T(0_X) \\
&= T(|0_X|) \\
&= T(|\operatorname{Im} f|),
\end{aligned}$$

and for each $h \in C(Y,\eta)$ we have

$$\begin{aligned}\left|T^{-1}(i\operatorname{Im} h)\right| &= \left|T^{-1}(i0_Y)\right| \\ &= \left|T^{-1}(0_Y)\right| \\ &= |0_X| \\ &= 0_X \\ &= T^{-1}(0_Y) \\ &= T^{-1}(|0_Y|) \\ &= T^{-1}(|\operatorname{Im} h|).\end{aligned}$$

By Lemma 3.5, we deduce that $T(1_X)$ is a nonvanishing positive function in $C(Y,\eta) = C_{\mathbb{R}}(Y)$, $T^{-1}(1_Y)$ is a nonvanishing positive function in $C(X,\tau) = C_{\mathbb{R}}(X)$ and there exists a bijective map $\Phi : Y_\eta \longrightarrow X_\tau$ such that

$$T(f)(y) = T(1_X)(y)f(x)$$

for all $y \in Y$, $x \in \Phi(\{y,\eta(y)\}) = \Phi(\{y\})$ and $f \in C(X,\tau)$ with $f(x) \in \mathbb{R}$ and

$$T^{-1}(h)(x) = T^{-1}(1_Y)(x)h(y)$$

for all $x \in X$, $y \in \Phi^{-1}(\{x,\tau(x)\}) = \Phi^{-1}(\{x\})$ and $h \in C(Y,\eta)$ with $h(y) \in \mathbb{R}$. Furthermore, $\Phi(\{y\}) = \Phi(\{y,\eta(y)\})$ is a singleton for all $y \in Y$ and $\Phi^{-1}(\{x\}) = \Phi^{-1}(\{x,\tau(x)\})$ is a singleton for all $x \in X$. Define the map $\varphi : Y \longrightarrow X$ by

$$\{\varphi(y)\} = \Phi(\{y\}) = \Phi(\{y,\eta(y)\}), \quad (y \in Y).$$

The bijectivity of $\Phi : Y_\eta \longrightarrow X_\tau$ implies that φ is a bijective map. By the argument mentioned, we get $T(f)(y) = T(1_X)(y) \cdot f(\varphi(y))$ for all $f \in C_{\mathbb{R}}(X)$ and $y \in Y$ and $T^{-1}(h)(x) = T^{-1}(1_Y)(x)h\left(\varphi^{-1}(x)\right)$ for all $h \in C_{\mathbb{R}}(Y)$ and $x \in X$. Therefore,

$$T(f) = T(1_X) \cdot (f \circ \varphi) \tag{3.36}$$

for all $f \in C_{\mathbb{R}}(X)$ and

$$T^{-1}(h) = T^{-1}(1_Y) \cdot (h \circ \varphi^{-1}) \tag{3.37}$$

for all $h \in C_{\mathbb{R}}(Y)$. Since $T^{-1}(1_Y) \in C_{\mathbb{R}}(X)$ and (3.36) holds for all $f \in C_{\mathbb{R}}(X)$, we get $T(T^{-1}(1_Y)) = T(1_X) \cdot (T^{-1}(1_Y) \circ \varphi)$ and so $1_Y = T(1_X) \cdot (T^{-1}(1_Y) \circ \varphi)$, i.e., $T(1_X) = \frac{1}{T^{-1}(1_Y)\circ\varphi} = \frac{1}{T^{-1}(1_Y)} \circ \varphi$. Since $T(1_X) \in C_{\mathbb{R}}(Y)$ and (3.37) holds for all $h \in C_{\mathbb{R}}(Y)$, we get $T^{-1}(T(1_X)) = T^{-1}(1_Y)\cdot(T(1_X)\circ\varphi^{-1})$. This implies that $1_X = T^{-1}(1_Y)\cdot (T(1_X) \circ \varphi^{-1})$, i.e., $T^{-1}(1_Y) = \frac{1}{T(1_X)\circ\varphi^{-1}} = \frac{1}{T(1_X)} \circ \varphi^{-1}$.

We now show that φ is a continuous map from Y to X. Let $\{y_\alpha\}_{\alpha\in J}$ be a net in Y and let $y \in Y$ such that

$$\lim_\alpha y_\alpha = y \quad (\text{in } Y). \tag{3.38}$$

Let $f \in C(X)$ be given. Then $\operatorname{Re} f, \operatorname{Im} f \in C_{\mathbb{R}}(X)$. Therefore, we have $T(\operatorname{Re} f), T(\operatorname{Im} f) \in C_{\mathbb{R}}(Y)$ and

$$T(\operatorname{Re} f) = T(1_X)\cdot((\operatorname{Re} f)\circ\varphi), \qquad T(\operatorname{Im} f) = T(1_X)\cdot((\operatorname{Im} f)\circ\varphi).$$

This implies that $T(1_X)\cdot((\operatorname{Re} f)\circ\varphi),\ T(1_X)\cdot((\operatorname{Im} f)\circ\varphi) \in C_{\mathbb{R}}(Y)$. Hence,

$$T(1_X)\cdot((\operatorname{Re} f)\circ\varphi) + i\ T(1_X)\cdot((\operatorname{Im} f)\circ\varphi) \in C(Y). \tag{3.39}$$

On the other hand, we have

$$T(1_X)\cdot(f\circ\varphi) = T(1_X)\cdot((\operatorname{Re} f)\circ\varphi) + i\ T(1_X)\cdot((\operatorname{Im} f)\circ\varphi). \tag{3.40}$$

From (3.40) and (3.39), we get $T(1_X)\cdot(f\circ\varphi) \in C(Y)$. This implies that $f\circ\varphi \in C(Y)$ since $T(1_X)$ is a nonvanishing positive function in $C_{\mathbb{R}}(Y)$ and $C_{\mathbb{R}}(Y) \subseteq C(Y)$. Therefore, by (3.38) we have

$$\lim_\alpha (f\circ\varphi)(y_\alpha) = (f\circ\varphi)(y).$$

Hence,

$$\lim_\alpha f(\varphi(y_\alpha)) = f(\varphi(y)). \tag{3.41}$$

Since (3.41) holds for all $f \in C(X)$, by Lemma 3.9 we deduce that φ is a continuous map from Y to X.

According to Y is a compact space, X is a Hausdorff space and $\varphi : Y \longrightarrow X$ is a bijective continuous map, we conclude that φ is a homeomorphism from Y to X.

To prove the uniqueness of the map $\varphi\colon Y \longrightarrow X$, we assume that there exists a map $\psi\colon Y \longrightarrow X$ such that $T(f) = T(1_X)\cdot(f\circ\psi)$ for all $f \in C_{\mathbb{R}}(X)$. We show that $\psi = \varphi$ on Y. Let $y \in Y$ be given. Since $T(f) = T(1_X)\cdot(f\circ\varphi)$ for all $f \in C_{\mathbb{R}}(X)$, we deduce that

$$T(1_X)\cdot(f\circ\psi) = T(1_X)\cdot(f\circ\varphi)$$

for all $f \in C_{\mathbb{R}}(X)$. Therefore,

$$T(1_X)(y)f(\psi(y)) = T(1_X)(y)f(\varphi(y))$$

for all $f \in C_{\mathbb{R}}(X)$. This implies that

$$f(\psi(y)) = f(\varphi(y))$$

for all $f \in C_{\mathbb{R}}(X)$ since $T(1_X)(y) > 0$. Therefore,

$$\psi(y) = \varphi(y) \tag{3.42}$$

since $C_{\mathbb{R}}(X)$ separates the points of X. According to (3.42) holds for all $y \in Y$, we deduce that $\psi = \varphi$ on Y. Hence, the proof is complete. $\square$

As a direct consequence of Theorem 3.10 and Corollary 3.2, we deduce a Banach-Stone type result for $C_{\mathbb{R}}(X)$-algebras.

Corollary 3.11. *Let X and Y be compact Hausdorff spaces. The following are equivalent:*

(i) *$C_{\mathbb{R}}(X)$ is order real isomorphic to $C_{\mathbb{R}}(Y)$.*
(ii) *X is homeomorphic to Y.*

Proof. **(i) $\Rightarrow$ (ii):** Assume that (i) holds. Then there exists an order real isomorphism T from $C_{\mathbb{R}}(X)$ to $C_{\mathbb{R}}(Y)$. By Theorem 3.10, $T(1_X)$ is a nonvanishing positive function on Y and there exists a homeomorphism $\varphi\colon Y \longrightarrow X$ such that $T(f) = T(1_X)\cdot(f\circ\varphi)$ for all $f \in C_{\mathbb{R}}(X)$. Hence, X is homeomorphic to Y and so (ii) holds.

(ii) $\Rightarrow$ (i): Assume that X is homeomorphic to Y. Then there exists a homeomorphism φ from Y to X. Take $\mu = 1_Y$. Then $\mu \in C_{\mathbb{R}}(Y)$ and μ is a real-valued nonvanishing positive function on Y. By Corollary 3.2, the map $T\colon C_{\mathbb{R}}(X) \longrightarrow C_{\mathbb{R}}(Y)$ defined by

$$T(f) = \mu\cdot(f\circ\varphi)\,, \quad (f \in C_{\mathbb{R}}(X))$$

is an order real isomorphism from $C_{\mathbb{R}}(X)$ to $C_{\mathbb{R}}(Y)$. Therefore (i) holds. Hence, the proof is complete. □

Acknowledgment. The authors would like to thank the editor and the referee(s) for their valuable comments and suggestions.

References

1. E. Ahmadloo and D. Alimohammadi, *Order isomorphisms between real little Lipschitz algebras with Lipschitz involution*, The Journal of Analysis, 32 (6) (2024), pp. 3607-3629.
2. D. Alimohammadi and A. Ebadian, *Hedberg's theorem in real Lipschitz algebras*, Indian J. Pure Apll. Math., 32 (10) (2001), pp. 1475-1493.
3. W. Arendt and J. De Cannière, *Order isomorphisms of Fourier algebras*, J. Funct. Anal., 50 (1983), pp. 1-7.
4. Y. El Khatiri, *Order isomorphisms in C^*-algebras*, Linear Algebra and Its Applications, 743 (2026), pp. 181-210.
5. K. Igarashi, J. Nakamura, S. Roy and R. Tanaka, *On order isomorphisms of norm attainment sets of continuous functions*, Acta Math. Hungar., 175 (2) (2025), pp. 486-506.
6. A. Jiménez-Vargas and M. Villegas-Vallecillos, *Order isomorphisms of little Lipschitz algebras*, Houston J. Math., 43 (4) (2008), pp. 1185-1195.

7. S.H. Kulkarni and B.V. Limaye, *Gleason parts of real function algebras*, Canad. J. Math., 33 (1) (1981), pp. 181-200.
8. S.H. Kulkarni and B.V. Limaye, *Real Function Algebras*, Marcel Dekker, Inc., 1992.
9. D.H. Leung and L. Li, *Order isomorphisms on function spaces*, Studia Mathematica, 219 (2) (2013), pp. 123-138.
10. L. Molnár, *On the nonexistence of order isomorphisms between the sets of all self-adjoint and all positive definite operators*, Abstract and Applied Analysis, (2015), Article ID 434020, 6 pp.

---

[1] Department of Mathematics, Faculty of Science, Arak University, P.O.Box 3848177584, Arak, Iran.
Email address: d-alimohammadi@araku.ac.ir

[2] Department of Mathematics, Faculty of Science, Arak University, P.O.Box 3848177584, Arak, Iran.
Email address: yasmnadybdwst@gmail.com