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Wijsman Statistical and Strong Cesàro Convergence of Closed Sets with Musielak–Orlicz Functions

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Abstract. By considering a sequence $\Phi = (\Phi_n)$ of Musielak–Orlicz functions with suitable growth conditions, we define Wijsman Φ-strong Cesàro convergence, Wijsman Φ-statistical convergence and the lacunary Φ-statistical convergence. Under moderate growth assumptions, we establish equivalences between these modes of convergence. We further construct related sequence spaces generated by Φ-type convergence, including the strong Cesàro Φ-convergence space, the Φ-statistical convergence space and the lacunary Φ-statistical convergence space. Our results extend classical convergence concepts and provide new tools for studying sequences of closed sets. Several examples are presented to illustrate the theory and potential applications.

1. Introduction

Statistical convergence, originally introduced by Fast [16] and independently by Steinhaus [35], was further developed by Šalát [33] who provided important foundational results. This concept generalizes the classical notion of convergence by considering the natural density of the set of indices where convergence fails. Instead of requiring that all terms of a sequence eventually lie within a given neighborhood, statistical convergence only demands that the set of indices outside this neighborhood has density zero. This relaxation has provided a powerful tool in analysis, finding applications in summability theory, approximation and functional analysis [11, 17]. Extensions using modulus functions to refine convergence densities have been studied by Orhan [30], while density

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by moduli and its relationship to statistical convergence was examined by Aizpuru et al. [1]. Various generalizations of statistical convergence have been developed, including statistical convergence of order α by Çolak [9], λ-statistical convergence of order α by Işık and Akbaş [23] and asymptotically λ-statistical equivalent sequences of order α in probability by Akbaş and Işık [2]. The relationship between statistical convergence and strong p-Cesàro convergence was explored by Connor [10]. Likewise, convergence in Musielak-Orlicz function spaces, which allow variable growth control, has been explored in detail by Savaş [34], while Rao [31] addressed foundational properties of Musielak-Orlicz sequence spaces. The theory of Orlicz spaces has been comprehensively treated by Rao and Ren [32] and the foundational work on Orlicz spaces and modular spaces by Musielak [29] provides essential background. More comprehensive treatments of summability theory and sequence spaces can be found in the monographs by Mursaleen and Başar [28] and Başar [4].

Wijsman convergence, introduced by Wijsman in 1966 [37], extends the concept of convergence to sequences of closed subsets of metric spaces by focusing on the pointwise convergence of the corresponding distance functions. Specifically, a sequence of closed sets (A_n) converges to a set A in the Wijsman sense if and only if for every point x in the underlying metric space, the distances $d(x, A_n)$ converge to $d(x, A)$. This approach has been extensively studied and generalized in set-valued analysis [6, 12], with Beer providing a comprehensive study of Wijsman convergence [5, 7]. In this vein, Boruah and Dutta [8] investigated Wijsman strong Cesàro convergence, demonstrating the role of summability methods in convergence of sequences of sets.

The integration of statistical convergence with Wijsman convergence led to the notion of Wijsman statistical convergence of sequences of closed sets [20], further broadening the scope of convergence analysis in metric and normed spaces. Several extensions of Wijsman convergence have been developed, including λ-asymptotically Wijsman generalized statistical convergence by Hazarika and Esi [21], asymptotically Wijsman lacunary σ-statistical convergence by Hazarika et al. [22], Wijsman rough statistical convergence on triple sequences by Esi and Subramanian [14], Wijsman rough λ-weak statistical convergence of order α for triple sequences of functions by Subramanian and Esi [36] and Wijsman rough I-convergence limit points of triple sequences by Esi et al. [15].

Recent work has also explored connections between Orlicz functions and various convergence notions. Kişi and Gürdal [24] studied Orlicz-lacunary convergent triple sequences and ideal convergence, while Kişi and Gürdal [25] investigated λ-weak convergence of sequences defined

by Orlicz functions. Gümüş et al. [19] examined Δ^I-statistically pre-Cauchy sequences with an Orlicz function, demonstrating the interplay between ideal convergence and Orlicz-based methods.

Motivated by these recent advancements, in this paper we generalize Wijsman statistical convergence by employing Musielak-Orlicz functions. Specifically, we define Wijsman Φ-statistical convergence, where $\Phi = \{\Phi_n\}$ is a sequence of Orlicz functions potentially varying with the index n, Wijsman Φ-strong Cesàro convergence and lacunary Wijsman Φ-statistical convergence. Assuming moderate growth requirements, we prove that these different convergence concepts are, in fact, equivalent. Building on this framework, we define various sequence spaces associated with Φ-type convergence, such as the strong Cesàro Φ-convergence space, the Φ-statistical convergence space and the lacunary Φ-statistical convergence space. We also establish important equivalences among various convergence notions, including Wijsman Φ-strong Cesàro convergence, Wijsman Φ-statistical convergence and the lacunary Wijsman Φ-statistical convergence in the Musielak-Orlicz setting. Furthermore, we provide illustrative examples that highlight the broad applicability of our results and discuss potential directions for further research.

Throughout this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x (the floor function) and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x (the ceiling function).

2. Preliminaries

Given a metric space (X, d), define $\mathcal{C}(X)$ as the collection of all nonempty closed subsets of X. The study of convergence of sequences of sets in $\mathcal{C}(X)$ often relies on the behavior of distance functions associated with these sets.

For each $x \in X$ and $A \in \mathcal{C}(X)$, the distance function from x to A is defined as

$$d(x, A) := \inf_{a \in A} d(x, a).$$

This function measures how close the point x is to the set A. Note that since A is closed and nonempty, $d(x, A) \geq 0$ and is finite. Also note that the distance function $d(\cdot, A)$ satisfies a 1-Lipschitz condition on X, meaning that for all $x, y \in X$,

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

This property is fundamental in analyzing convergence of sets via convergence of their distance functions.

Definition 2.1 (Orlicz function)**.** A function $\Phi : [0, \infty) \to [0, \infty)$ is an Orlicz function [29, 32]) if it satisfies the following properties:

(i) Φ is non-decreasing,
(ii) $\Phi(0) = 0$,
(iii) Φ is continuous and convex,
(iv) $\Phi(t) > 0$ for all $t > 0$,
(v) $\lim_{t\to\infty} \Phi(t) = \infty$.

Common examples of Orlicz functions include:

(i) $\Phi(t) = t^p$, $p \geq 1$,
(ii) $\Phi(t) = e^t - 1$,
(iii) $\Phi(t) = t^p \log(1+t)$ for $p \geq 1$.

These functions capture different growth behaviors and are useful in various applications.

Definition 2.2 (Musielak-Orlicz function)**.** A function $\Phi : \mathbb{N} \times [0,\infty) \to [0,\infty)$, denoted by $\Phi = (\Phi_n)_{n\in\mathbb{N}}$, is called a Musielak-Orlicz function (see [29]) if for each $n \in \mathbb{N}$, the function

$$\Phi_n(t) := \Phi(n,t)$$

is an Orlicz function. In other words, Φ_n is non-decreasing, continuous and convex, with $\Phi_n(0) = 0$, $\Phi_n(t) > 0$ for all $t > 0$ and $\lim_{t\to\infty} \Phi_n(t) = \infty$.

Musielak-Orlicz functions generalize classical Orlicz functions by allowing it to vary with the index n also. While an Orlicz function prescribes a fixed growth condition for all entries of a sequence or function space, a Musielak-Orlicz function enables the growth condition to change from term to term. This flexibility is particularly useful in modeling phenomena where the control or scaling of terms varies, such as variable exponent Lebesgue spaces.

In the context of sequence spaces and convergence, this index-dependent structure allows us to define and study convergence notions that adapt locally to the behavior of the sequence, providing a more refined and versatile framework than uniform Orlicz functions.

Common examples of Orlicz functions include:

(i) Consider sequences $\{p_n\}$ and $\{w_n\}$, where $p_n \geq 1$ and let $w_n > 0$ for all n. Define

$$\Phi_n(t) = \frac{t^{p_n}}{w_n}, \quad t \geq 0.$$

Then each Φ_n is an Orlicz function.

(ii) Consider a sequence $\{a_n\}$, where $a_n > 0$ for all n and define

$$\Phi_n(t) = e^{a_n t} - 1, \quad t \geq 0.$$

Each Φ_n is an Orlicz function because the exponential function is convex, continuous and strictly increasing on $[0,\infty)$, with

$\Phi_n(0) = 0$ and $\lim_{t\to\infty} \Phi_n(t) = \infty$. Here, the parameter a_n controls the growth rate of the function and can vary with n, allowing different terms of a sequence to be controlled by different exponential growth rates.

(iii) Let $\{q_n\}$ be a sequence with $q_n \geq 1$. Define

$$\Phi_n(t) = \begin{cases} t^{q_n}, & 0 \leq t \leq 1, \\ t^{2q_n} + 1, & t > 1. \end{cases}$$

Each Φ_n is continuous and convex on $[0, \infty)$, non-decreasing and satisfies the Orlicz function conditions. This example shows how one can construct Musielak-Orlicz functions with different growth behaviors on different intervals and these behaviors can vary with n.

(iv) A special and widely studied case of Musielak-Orlicz functions arises in variable exponent Lebesgue spaces $L^{p(\cdot)}$, where the exponent depends on the position or index. For a sequence $\{p_n\}$ with $p_n \geq 1$, define

$$\Phi_n(t) = t^{p_n}, \quad t \geq 0.$$

This is a Musielak-Orlicz function where the modular corresponds to the variable exponent power functions. Such functions are fundamental in studying spaces with non-standard growth conditions and have applications in PDEs and harmonic analysis.

Definition 2.3 (Wijsman Convergence)**.** A sequence $\{A_n\} \subset \mathcal{C}(X)$ converges in the sense of Wijsman (see [37]) to $A \in \mathcal{C}(X)$ if for every $x \in X$,

$$d(x, A_n) \to d(x, A), \quad \text{as } n \to \infty.$$

Examples of Wijsman convergent and not Wijsman convergent sequences:

(i) Let $X = \mathbb{R}$ with the usual metric and set

$$A_n = \left[-\frac{1}{n}, \frac{1}{n}\right], \quad n = 1, 2, \ldots$$

and $A = \{0\}$. Fix any $x \in \mathbb{R}$. Then

$$\begin{aligned} d(x, A_n) &= \inf_{y \in \left[-\frac{1}{n}, \frac{1}{n}\right]} |x - y| \\ &= \begin{cases} 0, & |x| \leq \frac{1}{n}, \\ |x| - \frac{1}{n}, & |x| > \frac{1}{n}. \end{cases} \end{aligned}$$

Since $\frac{1}{n} \to 0$ and A_n shrinks toward the single point set $A = \{0\}$ as $n \to \infty$, so

$$d(x, A_n) \to |x| = d(x, A).$$

Thus, $\{A_n\}$ converges Wijsman to $A = \{0\}$.

(ii) Let $A_n = [-n, n]$ and $A = \mathbb{R}$. Then for any fixed $x \in A$,

$$d(x, A_n) = \max(0, |x| - n) \to 0 = d(x, \mathbb{R}).$$

So, $\{A_n\}$ converges Wijsman to $A = \mathbb{R}$.

(iii) Let $A_n = \{n, n+1\} \subset \mathbb{R}$, and consider any limit set A. Fix $x = 0$. Then

$$d(0, A_n) = \min(|0 - n|, |0 - (n+1)|) = n \to \infty.$$

Since the sequence of distances diverges to $+\infty$, it cannot converge to a finite number. Therefore, no non-empty limit set A exists.

Hence, $\{A_n\}$ does not converges Wijsman to A.

Definition 2.4 (Wijsman Bounded)**.** A sequence $\{A_n\} \subset \mathcal{C}(X)$ is called Wijsman bounded if for every fixed $x \in X$, the sequence $\{d(x, A_n)\}$ is bounded, i.e., for all

$$x \in X, \quad \sup_n d(x, A_n) < \infty.$$

Examples of Wijsman bounded and non-Wijsman bounded sequences:

(i) Let $X = \mathbb{R}$ with usual metric and consider $A_n = [n, n+1]$. For $x = 0$,

$$d(0, A_n) = \inf_{y \in [n, n+1]} |0 - y| = n,$$

which tends to infinity as $n \to \infty$. Hence, $\{A_n\}$ is not Wijsman bounded.

(ii) Let $X = \mathbb{R}$ with usual metric and consider $A_n = [-n, n]$. Fix an arbitrary $x \in \mathbb{R}$. For sufficiently large n (specifically, for all $n \geq |x|$), we have $x \in A_n$. Consequently,

$$d(x, A_n) = 0, \quad \text{for all } n \geq |x|.$$

Since the sequence of distances $\{d(x, A_n)\}_{n \in \mathbb{N}}$ is eventually zero, it is bounded for every fixed x. Therefore, the sequence $\{A_n\}$ is Wijsma bounded.

Definition 2.5 (Wijsman Φ-statistical convergence)**.** A sequence $(A_n) \subset \mathcal{C}(X)$ is said to converge to $A \in \mathcal{C}(X)$ in the sense of Wijsman Φ-statistically if for every $x \in X$ and every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : \Phi_k (|d(x, A_k) - d(x, A)|) \geq \varepsilon\}| = 0.$$

This definition generalizes the classical Wijsman statistical convergence by controlling the size of the set of indices where the distance function deviates by at least ε using the Musielak-Orlicz function Φ. Intuitively, the set of bad indices should be sparse in a Φ-weighted sense.

Examples of Wijsman Φ-statistical convergence and not Wijsman Φ-statistical convergence:

(i) Consider the metric space $X = \mathbb{R}$ with the standard Euclidean metric. Let the sequence of Musielak-Orlicz functions $\Phi_n : [0,\infty) \to [0,\infty)$ be defined by $\Phi_n(t) = t^2$ for all $n \in \mathbb{N}$ and $t \geq 0$. Define the sequence of sets (A_n) as follows:

$$A_n = \begin{cases} [-1,1], & \text{if } n \text{ is not a perfect square,} \\ [10,11], & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}. \end{cases}$$

Claim that the sequence (A_n) is Wijsman Φ-statistically convergent to the set $A = [-1,1]$. Let $x \in \mathbb{R}$ be arbitrary and $\gamma_k(x) = |d(x, A_k) - d(x, A)|$.

Case 1: k is not a perfect square. In this case, $A_k = [-1,1] = A$. Consequently, $d(x, A_k) = d(x, A)$, which implies $\gamma_k(x) = 0$.

Case 2: k is a perfect square ($k = m^2$). In this case, $A_k = [10,11]$. Then

$$\gamma_k(x) = |d(x,[10,11]) - d(x,[-1,1])| =: C_x.$$

Note that C_x is a constant finite value depending only on x. Now, fix $\varepsilon > 0$ and define

$$K_\varepsilon(x) = \{k \in \mathbb{N} : \Phi_k(\gamma_k(x)) \geq \varepsilon\}.$$

Substituting $\Phi_k(t) = t^2$, the condition becomes $(\gamma_k(x))^2 \geq \varepsilon$, or $\gamma_k(x) \geq \sqrt{\varepsilon}$. From our case analysis, we have

- If k is not a perfect square, $\gamma_k(x) = 0$. Since $\varepsilon > 0$, the condition $0 \geq \sqrt{\varepsilon}$ is never met.
- If k is a perfect square, $\gamma_k(x) = C_x$. The index k is included in $K_\varepsilon(x)$ only if $C_x \geq \sqrt{\varepsilon}$ and if $C_x < \sqrt{\varepsilon}$, then $K_\varepsilon(x) = \emptyset$.

Therefore,

$$K_\varepsilon(x) \subseteq \left\{m^2 : m \in \mathbb{N}\right\}$$

Let $S = \left\{m^2 : m \in \mathbb{N}\right\}$ be the set of perfect squares. The number of perfect squares up to n is $\lfloor \sqrt{n} \rfloor$. Therefore,

$$\frac{1}{n} |K_\varepsilon(x) \cap [1,n]| \leq \frac{1}{n} |S \cap [1,n]| = \frac{\lfloor \sqrt{n} \rfloor}{n}.$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n} |K_\varepsilon(x) \cap [1,n]| \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0.$$

Thus, the density of the set $K_\varepsilon(x)$ is 0. Since this holds for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, we conclude that (A_n) is Wijsman Φ-statistically convergent to A.

(ii) Using the same setup ($X = \mathbb{R}$, $\Phi_n(t) = t^2$), consider the sequence

$$A_n = \begin{cases} [-1, 1], & \text{if } n \text{ is even,} \\ [10, 11], & \text{if } n \text{ is odd.} \end{cases}$$

Take $A = [-1, 1]$. Fix $x = 0$. The distance difference at origin is

$$\begin{aligned} \gamma_n(0) &= |d(0, A_n) - d(0, A)| \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ 10, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore,

$$\Phi_n(\gamma_n(0)) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 100, & \text{if } n \text{ is odd.} \end{cases}$$

For $\varepsilon = 1$, the set

$$K_\varepsilon = \{k \in \mathbb{N} : \Phi_k(\gamma_k(0)) \geq 1\} = \{k \in \mathbb{N} : k \text{ is odd}\}.$$

The natural density of this set is

$$\lim_{n\to\infty} \frac{1}{n} |K_\varepsilon \cap [1, n]| = \lim_{n\to\infty} \frac{\lceil n/2 \rceil}{n} = \frac{1}{2} \neq 0.$$

Since the density is not zero, the sequence (A_n) is not Wijsman Φ-statistically convergent to A.

Definition 2.6 (Wijsman Φ-strong Cesàro convergence)**.** A sequence $(A_n) \subseteq \mathcal{C}(X)$ is called Wijsman Φ-strong Cesàro convergent (see [21, 22]) to A if for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k(|d(x, A_k) - d(x, A)|) = 0.$$

Strong Cesàro convergence is a summability method that averages the deviations over the sequence.

If $\Phi_k(t) = t$ for all k, then Wijsman Φ-strong Cesàro convergence reduces to the classical Wijsman strong Cesàro convergence.

Examples of Wijsman Φ-strong Cesàro convergence and not Wijsman Φ-strong Cesàro convergence:

Let $X = \mathbb{R}$ with the usual metric.

(i) Fix $A = \{0\}$ and $A_k = \left\{\frac{1}{k}\right\}$. Let $\Phi_k(t) = t$ for all k. For any fixed $x \in \mathbb{R}$, the distances satisfy

$$d(x, A_k) = \left|x - \frac{1}{k}\right|, \qquad d(x, A) = |x|.$$

Using the reverse triangle inequality $||a| - |b|| \leq |a - b|$, the absolute difference is bounded by

$$\gamma_k(x) = \left|\left|x - \frac{1}{k}\right| - |x|\right| \leq \left|\left(x - \frac{1}{k}\right) - x\right| = \frac{1}{k}.$$

Consider the Cesàro averages

$$\frac{1}{n}\sum_{k=1}^{n} \Phi_k(\gamma_k(x)) = \frac{1}{n}\sum_{k=1}^{n} \gamma_k(x) \leq \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k}.$$

Since the harmonic series grows logarithmically $\left(\sum_{k=1}^{n} \frac{1}{k} \approx \ln n\right)$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k} = 0.$$

It follows that the sequence $\{A_k\}$ is Wijsman Φ-strong Cesàro convergent to A.

(ii) Fix

$$A = \{0\}, \qquad A_k = \begin{cases} \{1\}, & \text{if } k \text{ is odd,} \\ \{0\}, & \text{if } k \text{ is even.} \end{cases}$$

Let $\Phi_k(t) = t$ for all k. For any $x \in \mathbb{R}$,

$$\gamma_k(x) = \begin{cases} |\,|x - 1| - |x|\,|, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

In particular, for $x = 0$, we have

$$\gamma_k(0) = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \Phi_k(\gamma_k(0)) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \gamma_k(0).$$

The sum counts the number of odd integers up to n which is $\lceil n/2 \rceil$, and

$$\lim_{n\to\infty} \frac{\lceil n/2 \rceil}{n} = \frac{1}{2} \neq 0.$$

Hence, $\{A_k\}$ is not Wijsman Φ-strong Cesàro convergent to A.

Definition 2.7 (Lacunary Wijsman Φ-statistical convergence)**.** Let $\theta = (k_r)$ be a lacunary sequence with intervals $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.

A sequence $(A_n) \subseteq \mathcal{C}(X)$ is said to be lacunary Wijsman Φ-statistically convergent to $A \in \mathcal{C}(X)$ with respect to θ if for every $x \in X$ and every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \Phi_k\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) \geq \varepsilon\}\right| = 0.$$

Examples of Lacunary Wijsman Φ-statistical convergence and non-Lacunary Wijsman Φ-statistical convergence:

Let $X = \mathbb{R}$ with the standard Euclidean metric. Let the sequence of Musielak-Orlicz functions be given by $\Phi_n(t) = t^2$ for all $n \in \mathbb{N}$ and $t \geq 0$. Consider the lacunary sequence $\theta = (k_r)$ defined by $k_r = 2^r$. The associated intervals are $I_r = (k_{r-1}, k_r] = \left(2^{r-1}, 2^r\right]$ with lengths $h_r = k_r - k_{r-1} = 2^{r-1}$.

(i) Let $A = [-1, 1]$ and define the sequence of sets (A_n) by

$$A_n = \begin{cases} [-1, 1], & \text{if } n \neq 3^m \text{ for any } m \in \mathbb{N}, \\ [10, 11], & \text{if } n = 3^m \text{ for some } m \in \mathbb{N}. \end{cases}$$

Fix an arbitrary $x \in \mathbb{R}$. The deviation is given by

$$\gamma_n(x) = \left|d\left(x, A_n\right) - d\left(x, A\right)\right| = \begin{cases} 0, & \text{if } n \neq 3^m, \\ C_x, & \text{if } n = 3^m, \end{cases}$$

where $C_x = \left|d\left(x, [10, 11]\right) - d\left(x, [-1, 1]\right)\right|$ is a finite non-negative constant depending only on x. Fix $\varepsilon > 0$. We examine the set of indices

$$K_{\varepsilon,x} = \left\{n \in \mathbb{N} : \Phi_n\left(\gamma_n(x)\right) \geq \varepsilon\right\}.$$

Since $\Phi_n(t) = t^2$, the condition is equivalent to $\left(\gamma_n(x)\right)^2 \geq \varepsilon$.

- If $C_x^2 < \varepsilon$, then $K_{\varepsilon,x} = \emptyset$ and the lacunary density is trivially 0.
- If $C_x^2 \geq \varepsilon$, then $K_{\varepsilon,x} = \{3^m : m \in \mathbb{N}\}$. We must calculate the fraction of these indices within each lacunary interval I_r.

Since the sequence (3^m) grows exponentially faster than the endpoints of $I_r = \left(2^{r-1}, 2^r\right]$, each interval I_r contains at most one power of 3 (for large r, it often contains none). Thus,

$$\left|K_{\varepsilon,x} \cap I_r\right| \leq 1 \quad \text{for all } r.$$

Therefore,

$$\frac{1}{h_r} \left|\{n \in I_r : \Phi_n\left(\gamma_n(x)\right) \geq \varepsilon\}\right| \leq \frac{1}{2^{r-1}}.$$

Taking the limit as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{n \in I_r : \Phi_n(\gamma_n(x)) \geq \varepsilon\}\right| = 0.$$

Since this holds for every $x \in X$ and $\varepsilon > 0$, (A_n) is lacunary Wijsman Φ-statistically convergent to A with respect to θ.

(ii) Let $A = [-1, 1]$ and define the sequence of sets (B_n) by

$$B_n = \begin{cases} [-1, 1], & \text{if } n \text{ is even,} \\ [10, 11], & \text{if } n \text{ is odd.} \end{cases}$$

It suffices to find one point $x \in \mathbb{R}$ and one $\varepsilon > 0$ for which the lacunary density condition fails. Let $x = 0$. The distances are $d(0, [-1, 1]) = 0$ and $d(0, [10, 11]) = 10$. The deviation is

$$\begin{aligned} \gamma_n(0) &= |d(0, B_n) - d(0, A)| \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ 10, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Applying $\Phi_n(t) = t^2$, we have

$$\Phi_n(\gamma_n(0)) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 100, & \text{if } n \text{ is odd.} \end{cases}$$

Fix $\varepsilon = 1$. The set of indices exceeding ε is

$$K_{1,0} = \{n \in \mathbb{N} : \Phi_n(\gamma_n(0)) \geq 1\} = \{n \in \mathbb{N} : n \text{ is odd}\}.$$

Consider the lacunary interval $I_r = \left(2^{r-1}, 2^r\right]$. The number of integers in this interval is 2^{r-1}, which is even for $r \geq 2$. Exactly half of these integers are odd. Thus,

$$\left|\{n \in I_r : \Phi_n(\gamma_n(0)) \geq 1\}\right| = \frac{1}{2}\left(2^r - 2^{r-1}\right) = \frac{1}{2}h_r.$$

The lacunary density is

$$\frac{1}{h_r}\left|\{n \in I_r : \Phi_n(\gamma_n(0)) \geq 1\}\right| = \frac{1}{h_r} \cdot \frac{h_r}{2} = \frac{1}{2}.$$

Taking the limit as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\{n \in I_r : \Phi_n(\gamma_n(0)) \geq 1\}\right| = \frac{1}{2} \neq 0.$$

Therefore, (B_n) is not lacunary Wijsman Φ-statistically convergent to A with respect to θ.

3. Main Results

The interplay between Wijsman Φ-statistical convergence, Wijsman Φ-strong Cesàro convergence and lacunary Wijsman Φ-statistical convergence is a central theme in this paper. Under suitable conditions on the Musielak-Orlicz functions, these convergence modes can be shown to be equivalent or closely related.

Before we prove the main result of this paper, we establish some relation between Wijsman Φ-strong Cesàro convergence, Wijsman Φ-statistical convergence and lacunary Wijsman Φ-statistical convergence.

Proposition 3.1. *Wijsman Φ-strong Cesàro convergence implies Wijsman Φ-statistical convergence.*

Proof. Suppose the sequence $(A_n) \subseteq \mathcal{C}(X)$ is Wijsman Φ-strong Cesàro convergent to A. That is, for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k (\gamma_k (x)) = 0,$$

where $\gamma_k (x) = |d(x, A_k) - d(x, A)|$. Fix an arbitrary $\varepsilon > 0$. Define the set

$$K(\varepsilon) := \{k \in \mathbb{N} : \Phi_k (\gamma_k (x)) \geq \varepsilon\}.$$

Compute the Cesàro sum by splitting the indices into those inside $K(\varepsilon)$ and those outside as

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \Phi_k (\gamma_k (x)) &= \frac{1}{n} \sum_{\substack{k=1 \\ k\in K(\varepsilon)}}^{n} \Phi_k (\gamma_k (x)) + \frac{1}{n} \sum_{\substack{k=1 \\ k\notin K(\varepsilon)}}^{n} \Phi_k (\gamma_k (x)) \\ &\geq \frac{1}{n} \sum_{\substack{k=1 \\ k\in K(\varepsilon)}}^{n} \Phi_k (\gamma_k (x)). \end{aligned}$$

By the definition of the set $K(\varepsilon)$, for every $k \in K(\varepsilon)$, we have $\Phi_k (\gamma_k (x)) \geq \varepsilon$. Thus

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \Phi_k (\gamma_k (x)) &\geq \frac{1}{n} \sum_{\substack{k=1 \\ k\in K(\varepsilon)}}^{n} \varepsilon \\ &= \varepsilon \cdot \frac{|\{k \leq n : \Phi_k (\gamma_k (x)) \geq \varepsilon\}|}{n}. \end{aligned}$$

Taking the limit as $n \to \infty$, the left-hand side tends to 0 by the hypothesis of strong Cesàro convergence. Since $\varepsilon > 0$, it must be that

$$\lim_{n\to\infty} \frac{|\{k \leq n : \Phi_k (\gamma_k (x)) \geq \varepsilon\}|}{n} = 0.$$

This confirms that the sequence is Wijsman Φ-statistically convergent to A. □

Proposition 3.2. *Let $\theta = (k_r)$ be a lacunary sequence. Assume that $\liminf_{r\to\infty} q_r > 1$, where $q_r = k_r/k_{r-1}$. Then, Wijsman Φ-statistical convergence implies lacunary Wijsman Φ-statistical convergence.*

Proof. Fix an arbitrary $x \in X$ and $\varepsilon > 0$. Define the set

$$K_\varepsilon := \{k \in \mathbb{N} : \Phi_k\left(d\left(x, A_k\right) - d\left(x, A\right)\right) \geq \varepsilon\}.$$

By the assumption of Wijsman Φ-statistical convergence, we have

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : \Phi_k\left(d\left(x, A_k\right) - d\left(x, A\right)\right) \geq \varepsilon\}\right| = 0.$$

Consider the lacunary intervals $I_r = (k_{r-1}, k_r]$. Let $h_r = k_r - k_{r-1}$ denote the length of the interval I_r and examine the fraction

$$\begin{aligned}\rho_r &:= \frac{|K_\varepsilon \cap I_r|}{h_r} \\ &= \frac{1}{h_r} \left|\{k \in I_r : \Phi_k\left(d\left(x, A_k\right) - d\left(x, A\right)\right) \geq \varepsilon\}\right|.\end{aligned}$$

Since $I_r \subset [1, k_r]$, we have $|K_\varepsilon \cap I_r| \leq |K_\varepsilon \cap [1, k_r]|$. Substituting this bound into ρ_r, we have

$$\rho_r \leq \frac{|K_\varepsilon \cap [1, k_r]|}{h_r} = \frac{k_r}{h_r} \cdot \left(\frac{|K_\varepsilon \cap [1, k_r]|}{k_r}\right).$$

Let $\delta_r = \frac{|K_\varepsilon \cap [1,k_r]|}{k_r}$. Then $\delta_r \to 0$ as $r \to \infty$. Next,

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r - 1}.$$

By the assumption $\liminf_{r\to\infty} q_r > 1$, there exists some $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for all sufficiently large r. Since the function $f(t) = \frac{t}{t-1}$ is decreasing for $t > 1$ and $q_r \geq 1 + \alpha$, we have $f(q_r) \leq f(1 + \alpha)$. Thus for large r, we have

$$\frac{q_r}{q_r - 1} \leq \frac{1 + \alpha}{\alpha} =: M.$$

Thus, for sufficiently large r, we have

$$\rho_r \leq M \cdot \delta_r.$$

Taking the limit as $r \to \infty$, since $\delta_r \to 0$, it follows that $\rho_r \to 0$. Therefore,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \Phi_k\left(d\left(x, A_k\right) - d\left(x, A\right)\right) \geq \varepsilon\}\right| = 0.$$

Since this holds for any $x \in X$ and $\varepsilon > 0$, (A_n) is lacunary Wijsman Φ-statistically convergent to A. □

Proposition 3.3. *Let $\theta = (k_r)$ be a lacunary sequence such that $\liminf_{r\to\infty} q_r > 1$, where $q_r = k_r/k_{r-1}$. Then, Wijsman Φ-strong Cesàro convergence implies lacunary Wijsman Φ-statistical convergence.*

Proof. The proof is a direct consequence of the preceding results. □

For the other cases we construct counterexamples in terms of sequences of sets with Wijsman Φ-distances.

For simplicity, we take (A_k) a sequence of sets in a metric space (X, d), Musielak-Orlicz function Φ_k the identity function on $[0, \infty)$, that is, $\Phi_k(t) = t$ and $\delta_k := d(x, A_k)$.

An example demonstrates a sequence of sets (A_k) that is Wijsman Φ-statistically convergent to A but not Wijsman Φ-strong Cesàro convergent.

Example 3.4. Let $X = \mathbb{R}$ with the standard metric. Consider the sequence of sets (A_n) defined by

$$A_n = \begin{cases} \{n\}, & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $A = \{0\}$. We claim that (A_n) is Wijsman statistically convergent to A but not strongly Cesàro convergent.

Fix an arbitrary $x \in \mathbb{R}$, we have $d(x, A) = |x|$. For the sequence (A_n):

- If n is not a perfect square, $A_n = \{0\}$, so $d(x, A_n) = |x| = d(x, A)$. The deviation is $|d(x, A_n) - d(x, A)| = 0$.
- If $n = m^2$, $A_n = \{n\}$, so $d(x, A_n) = |x - n|$.

Now, fix $\varepsilon > 0$. We identify the set of indices where the deviation is at least ε

$$K_\varepsilon(x) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\}.$$

From the calculation above, if k is not a perfect square, the deviation is exactly 0. Thus, non-square indices can never be in $K_\varepsilon(x)$. Therefore:

$$K_\varepsilon(x) \subseteq \{m^2 : m \in \mathbb{N}\}.$$

The natural density of the set of perfect squares is zero (since $\frac{\sqrt{n}}{n} \to 0$). Consequently, the density of the subset $K_\varepsilon(x)$ is also zero.

Since this holds for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, (A_n) is Wijsman statistically convergent to A.

To check for strong Cesàro convergence, we have

$$|d(0, A_k) - d(0, A)| = \begin{cases} k, & \text{if } k = m^2, \\ 0, & \text{otherwise.} \end{cases}$$

The Cesàro mean becomes

$$\frac{1}{n}\sum_{k=1}^{n}\left|d\left(0,A_k\right)-0\right|=\frac{1}{n}\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}j^2.$$

Using the summation formula $\sum_{j=1}^{M} j^2 \approx \frac{M^3}{3}$, with $M \approx \sqrt{n}$,

$$\frac{1}{n}\sum_{k=1}^{n}d\left(0,A_k\right)\approx\frac{1}{n}\cdot\frac{\left(\sqrt{n}\right)^3}{3}=\frac{n^{3/2}}{3n}=\frac{\sqrt{n}}{3}.$$

As $n\to\infty$, this limit is $+\infty$, not 0. Since the condition fails for $x=0$, the sequence is not Wijsman strongly Cesàro convergent.

An example demonstrating a sequence of sets (A_k) that is Wijsman Φ-statistically convergent to A but not lacunary Wijsman Φ-strong Cesàro convergent.

Example 3.5. Let $X=\mathbb{R}$ with the usual metric. Let $A=\{0\}$ and $\Phi_k\left(t\right)=t$.

Consider the lacunary sequence $\theta=(k_r)$ with $k_r=2^r$ (so $h_r=2^{r-1}$). Define (A_k) as follows:

$$A_k=\begin{cases}\left\{2^{r-1}\right\}, & \text{if } k=k_r=2^r \text{ for some } r\in\mathbb{N},\\ \{0\}, & \text{otherwise.}\end{cases}$$

For any $x\in\mathbb{R}$, we have

$$d\left(x,A\right)=\left|x\right|,\qquad d\left(x,A_k\right)=\begin{cases}\left|x-2^{r-1}\right|, & \text{if } k=2^r,\\ \left|x\right|, & \text{otherwise.}\end{cases}$$

(i) Failure of Lacunary Φ-strong Cesàro Convergence:

$$\begin{aligned}\gamma_k\left(0\right)&=d\left(0,A_k\right)\\&=\begin{cases}2^{r-1}, & \text{if } k=2^r,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

For interval $I_r=\left(2^{r-1},2^r\right]$, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Phi_k\left(\gamma_k\left(0\right)\right)=\frac{1}{2^{r-1}}\cdot 2^{r-1}=1.$$

Thus $\limsup_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Phi_k\left(\gamma_k\left(0\right)\right)=1\neq 0$. So (A_k) is not lacunary Wijsman Φ-strong Cesàro convergent to A.

(ii) Wijsman Φ-Statistical Convergence: Fix an arbitrary $x \in \mathbb{R}$ and $\varepsilon > 0$. Define

$$K_\varepsilon(x) = \{k \in \mathbb{N} : \Phi_k(|d(x, A_k) - d(x, A)|) \geq \varepsilon\}.$$

Since $\Phi_k(t) = t$, this simplifies to

$$K_\varepsilon(x) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\}.$$

Now

- If k is not a power of 2, then $d(x, A_k) = |x| = d(x, A)$, so $|d(x, A_k) - d(x, A)| = 0$.
- If $k = 2^r$, then

$$|d(x, A_k) - d(x, A)| = \left|\left|x - 2^{r-1}\right| - |x|\right| \leq 2^{r-1}.$$

For $k = 2^r$ to be in $K_\varepsilon(x)$, we need $\left|\left|x - 2^{r-1}\right| - |x|\right| \geq \varepsilon$. By the reverse triangle inequality, we have

$$\left|\left|x - 2^{r-1}\right| - |x|\right| \leq \left|\left(x - 2^{r-1}\right) - x\right| = 2^{r-1}.$$

Thus if $2^{r-1} < \varepsilon$, then $k = 2^r \notin K_\varepsilon(x)$. Therefore,

$$K_\varepsilon(x) \subseteq \left\{2^r : 2^{r-1} \geq \varepsilon\right\} = \{2^r : r \geq 1 + \log_2 \varepsilon\}.$$

Then the set $K_\varepsilon(x)$ is finite. Specifically, it contains at most $\lceil \log_2(1/\varepsilon) \rceil$ elements. Thus

$$\lim_{n\to\infty} \frac{1}{n} |K_\varepsilon(x) \cap [1, n]| = 0.$$

Since this holds for every $x \in \mathbb{R}$ and every $\varepsilon > 0$, (A_k) is Wijsman Φ-statistically convergent to A.

Example of a sequence that is Wijsman Φ-statistically convergent but not lacunary Wijsman Φ-strong Cesàro convergent.

Example 3.6. Let $X = \mathbb{R}$ with the usual metric. Let $A = \{0\}$ and $\Phi_k(t) = t$.

Consider the lacunary sequence $\theta = (k_r)$ with $k_r = r^2$ (so $h_r = 2r-1$). Define

$$A_k = \begin{cases} \{r\}, & \text{if } k = k_r = r^2 \text{ for some } r \in \mathbb{N}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

For any $x \in \mathbb{R}$:

$$d(x, A) = |x|, \qquad d(x, A_k) = \begin{cases} |x - r|, & \text{if } k = r^2, \\ |x|, & \text{otherwise.} \end{cases}$$

(i) Failure of lacunary strong Cesàro convergence: For $x = 0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Phi_k\left(|d(0,A_k)-d(0,A)|\right)=\frac{1}{2r-1}\sum_{k\in I_r}|d(0,A_k)-d(0,A)|.$$

Since $I_r=((r-1)^2,r^2]$ contains exactly one index $k=r^2$ where $d(0,A_k)=r$ and all other indices have $d(0,A_k)=0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\Phi_k\left(|d(0,A_k)-d(0,A)|\right)&=\frac{1}{2r-1}\cdot r\\&=\frac{r}{2r-1}\to\frac{1}{2}\\&\neq 0.\end{aligned}$$

So (A_k) is not lacunary Wijsman Φ-strong Cesàro convergent to A.

(ii) (A_k) is Wijsman Φ-statistically convergent to A: Fix $x\in\mathbb{R}$ and $\varepsilon>0$. Define

$$K_\varepsilon(x)=\left\{k\in\mathbb{N}:\Phi_k\left(|d(x,A_k)-d(x,A)|\right)\geq\varepsilon\right\}.$$

Since $\Phi_k(t)=t$, this simplifies to

$$K_\varepsilon(x)=\left\{k\in\mathbb{N}:|d(x,A_k)-d(x,A)|\geq\varepsilon\right\}.$$

Now consider two cases:

- If k is not a perfect square, then $d(x,A_k)=|x|=d(x,A)$, so $|d(x,A_k)-d(x,A)|=0$.
- If $k=r^2$ for some $r\in\mathbb{N}$, then $|d(x,A_k)-d(x,A)|=||x-r|-|x||$.

By the reverse triangle inequality, $||x-r|-|x||\leq|r|=r$. For $k=r^2$ to be in $K_\varepsilon(x)$, we need $||x-r|-|x||\geq\varepsilon$. This inequality can only hold for finitely many r because:

- If $r<\varepsilon$, then $||x-r|-|x||\leq r<\varepsilon$
- As $r\to\infty$, $||x-r|-|x||\to|x|-|x|=0$ (since $|x-r|\approx r$ for large r)

Therefore, $K_\varepsilon(x)$ is a finite set (containing at most finitely many perfect squares). For any finite set $F\subset\mathbb{N}$, we have

$$\lim_{n\to\infty}\frac{1}{n}|F\cap[1,n]|=0.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}|K_\varepsilon(x)\cap[1,n]|=0.$$

Since this holds for every $x\in\mathbb{R}$ and every $\varepsilon>0$, (A_k) is Wijsman Φ-statistically convergent to A.

We now prove the main result of this section which provides an equivalence characterization connecting different forms of convergence for sequences of closed sets controlled via Musielak-Orlicz functions and lacunary sequences. The key technical condition involving the Musielak-Orlicz growth relative to the lacunary gaps ensures that these various modes of convergence align.

Theorem 3.7. *Let $\Phi = (\Phi_k)$ be a sequence of Musielak-Orlicz functions satisfying:*

(i) *There exists $K \geq 2$ such that $\Phi_k(2t) \leq K\Phi_k(t)$ for all $k \in \mathbb{N}$, $t \geq 0$ (uniform Δ_2-condition).*
(ii) *For every $t > 0$, $\sup_k \Phi_k(t) < \infty$ (uniform boundedness).*

Let $\theta = (k_r)$ be a lacunary sequence with $q_r = k_r/k_{r-1}$ satisfying $1 < \liminf q_r \leq \limsup q_r < \infty$.

For any Wijsman bounded sequence (A_k) of non-empty closed subsets of X, the following are equivalent for every $x \in X$:

(1) *(A_k) is Wijsman Φ-statistically convergent to A;*
(2) *(A_k) is Wijsman Φ-lacunary statistically convergent to A with respect to θ;*
(3) *(A_k) is Wijsman strongly Φ-Cesàro convergent to A.*

Proof. Fix $x \in X$ and let $\gamma_k(x) = |d(x, A_k) - d(x, A)|$. Since (A_k) is Wijsman bounded, there exists $L_x > 0$ such that $\gamma_k(x) \leq L_x$ for all k.

Define $y_k = \Phi_k(\gamma_k(x))$. By condition (ii), $M_x := \sup_k \Phi_k(L_x) < \infty$, so $0 \leq y_k \leq M_x$ for all k. Thus (y_k) is a bounded sequence of non-negative real numbers.

(3) $\Rightarrow$ (1): Assume $\frac{1}{n}\sum_{k=1}^{n} y_k \to 0$. Fix $\varepsilon > 0$ and let $K_\varepsilon = \{k : y_k \geq \varepsilon\}$. For any $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^{n} y_k \geq \frac{1}{n}\sum_{k \in K_\varepsilon \cap [1,n]} y_k \geq \varepsilon \cdot \frac{|K_\varepsilon \cap [1,n]|}{n}.$$

Taking lim sup as $n \to \infty$ gives

$$0 = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} y_k \geq \varepsilon \cdot \limsup_{n\to\infty} \frac{|K_\varepsilon \cap [1,n]|}{n} \geq 0.$$

Thus $\lim_{n\to\infty} \frac{|K_\varepsilon \cap [1,n]|}{n} = 0$, which is exactly (1).

(1) ⇒ (3): Assume (1). Fix $\eta > 0$ and let $K_\eta = \{k : y_k \geq \eta\}$. By (1), $\lim_{n\to\infty} \frac{|K_\eta \cap [1,n]|}{n} = 0$. For any $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^{n} y_k = \frac{1}{n}\sum_{k\in K_\eta \cap [1,n]} y_k + \frac{1}{n}\sum_{k\notin K_\eta, k\leq n} y_k.$$

For the first term, since $y_k \leq M_x$ for all k:

$$\frac{1}{n}\sum_{k\in K_\eta \cap [1,n]} y_k \leq M_x \cdot \frac{|K_\eta \cap [1,n]\,|}{n} \to 0 \quad \text{as } n \to \infty.$$

For the second term, since $k \notin K_\eta$ implies $y_k < \eta$:

$$\frac{1}{n}\sum_{k\notin K_\eta, k\leq n} y_k < \eta \cdot \frac{n - |K_\eta \cap [1,n]\,|}{n} \leq \eta.$$

Thus for sufficiently large n,

$$\frac{1}{n}\sum_{k=1}^{n} y_k < 2\eta.$$

Since $\eta > 0$ was arbitrary, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} y_k = 0$, which is (3).

(1) ⇒ (2): Assume (1). Fix $\varepsilon > 0$ and let $K_\varepsilon = \{k : y_k \geq \varepsilon\}$. By (1), $\lim_{n\to\infty} \frac{|K_\varepsilon \cap [1,n]|}{n} = 0$. Since $\liminf q_r > 1$, there exists $\delta > 0$ such that $q_r \geq 1 + \delta$ for all sufficiently large r. For such r, we have

$$\frac{h_r}{k_r} = 1 - \frac{1}{q_r} \geq 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Now,

$$\frac{|K_\varepsilon \cap [1,k_r]\,|}{k_r} \geq \frac{|K_\varepsilon \cap I_r|}{k_r} = \frac{|K_\varepsilon \cap I_r|}{h_r} \cdot \frac{h_r}{k_r} \geq \frac{|K_\varepsilon \cap I_r|}{h_r} \cdot \frac{\delta}{1+\delta}.$$

Since the left-hand side tends to 0 as $r \to \infty$, we must have

$$\lim_{r\to\infty} \frac{|K_\varepsilon \cap I_r|}{h_r} = 0.$$

This is exactly the condition (2).

(2) ⇒ (1): Assume (2). Fix $\varepsilon > 0$ and let $K_\varepsilon = \{k : y_k \geq \varepsilon\}$. By (2), $\lim_{r\to\infty} \frac{|K_\varepsilon \cap I_r|}{h_r} = 0$. Let $\alpha_r = \frac{|K_\varepsilon \cap I_r|}{h_r}$, so $\alpha_r \to 0$. Since $\limsup q_r < \infty$,

there exists $Q > 0$ such that $q_r \leq Q$ for all sufficiently large r. For any $n \in \mathbb{N}$, choose r such that $k_{r-1} < n \leq k_r$. Then

$$\frac{|K_\varepsilon \cap [1,n]\,|}{n} \leq \frac{|K_\varepsilon \cap [1,k_r]\,|}{k_{r-1}} = \frac{\sum_{j=1}^{r} |K_\varepsilon \cap I_j|}{k_{r-1}} = \frac{\sum_{j=1}^{r} \alpha_j h_j}{k_{r-1}}.$$

Fix $\eta > 0$. Since $\alpha_j \to 0$, there exists J such that $\alpha_j < \eta$ for all $j > J$. Then for large r:

$$\frac{\sum_{j=1}^{r} \alpha_j h_j}{k_{r-1}} \leq \frac{\sum_{j=1}^{J} \alpha_j h_j + \eta \sum_{j=J+1}^{r} h_j}{k_{r-1}}.$$

As $r \to \infty$, $k_{r-1} \to \infty$, so

$$\frac{\sum_{j=1}^{J} \alpha_j h_j}{k_{r-1}} \to 0$$

and

$$\frac{\eta \sum_{j=J+1}^{r} h_j}{k_{r-1}} \leq \eta \cdot \frac{k_r}{k_{r-1}} \leq \eta Q.$$

Thus

$$\limsup_{n\to\infty} \frac{|K_\varepsilon \cap [1,n]\,|}{n} \leq \eta Q.$$

Since $\eta > 0$ is arbitrary and Q is fixed, we conclude

$$\lim_{n\to\infty} \frac{|K_\varepsilon \cap [1,n]\,|}{n} = 0.$$

This is the condition (1). This completes the proof. □

Example 3.8. We present three examples of sequences $\Phi = (\Phi_n)$ to illustrate the applicability of Theorem 3.7.

(i) $\Phi_n(t) = t^{p_n}$ where $0 < p_n \leq P < \infty$ for all n.

To verify the conditions of Theorem 3.7, we examine each requirement. For condition (i), the uniform Δ_2-condition, we compute $\Phi_n(2t) = (2t)^{p_n} = 2^{p_n} t^{p_n} = 2^{p_n} \Phi_n(t)$. Since $p_n \leq P$, we have $2^{p_n} \leq 2^P$, so the condition holds with $K = 2^P$. For condition (ii), uniform boundedness, note that for any fixed $t > 0$, $\Phi_n(t) = t^{p_n} \leq \max\{1, t^P\}$, which is finite and independent of n. Regarding the lacunary condition, taking $k_r = 2^r$ gives $q_r = 2$, satisfying $1 < 2 \leq 2 < \infty$. Therefore, Theorem 3.7 applies and Wijsman Φ-statistical convergence is equivalent to Wijsman Φ-lacunary statistical convergence for this family of functions.

(ii) $\Phi_n(t) = e^t - 1$ for all n.

For condition (i), we have $\Phi_n(2t) = e^{2t} - 1$. Examining the ratio $\Phi_n(2t)/\Phi_n(t) = \left(e^{2t} - 1\right)/\left(e^t - 1\right)$, we see that as $t \to \infty$, this ratio behaves asymptotically like e^t, which tends to infinity. Consequently, no constant K can bound this ratio uniformly for all $t \geq 0$, so the uniform Δ_2-condition fails. Therefore, Theorem 3.7 is not applicable. The exponential growth is too rapid for the equivalence between the convergence notions to hold in general.

We now provide an example of a Wijsman bounded sequence for which all three modes of convergence fail simultaneously.

Example 3.9. Let $(X, d) = (\mathbb{R}, |\cdot|)$ be the standard metric space. Consider the set $A = \{0\}$. Define a sequence of closed subsets (A_k) by

$$A_k := \begin{cases} \{1\}, & \text{if } k \text{ is even,} \\ \{-1\}, & \text{if } k \text{ is odd.} \end{cases}$$

Let $\Phi_k(t) = t$ for all k (the identity function). Notice that

(i) For all $k \in \mathbb{N}$ and $t \geq 0$,

$$\Phi_k(2t) = 2t = 2\Phi_k(t),$$

so the uniform Δ_2-condition holds with $K = 2$.

(ii) For any fixed $t > 0$,

$$\sup_{k \in \mathbb{N}} \Phi_k(t) = t < \infty.$$

So the uniform boundedness condition holds.

(iii) Any standard lacunary sequence (e.g., $k_r = 2^r$) satisfies $1 < \liminf q_r \leq \limsup q_r < \infty$.

(i) For an arbitrary point $x \in \mathbb{R}$, the distance functions are:

$$d(x, A_k) = \begin{cases} |x - 1|, & \text{if } k \text{ is even,} \\ |x + 1|, & \text{if } k \text{ is odd.} \end{cases}$$

The set of distances $\{d(x, A_k) : k \in \mathbb{N}\}$ is contained in the finite set $\{|x - 1|, |x + 1|\}$. Hence,

$$\sup_{k \in \mathbb{N}} |d(x, A_k) - d(x, \{0\})| \leq \max\{|x - 1|, |x + 1|\} + |x| < \infty.$$

Thus, (A_k) is Wijsman bounded.

(ii) Consider the point $x = 0$. We have $d(0, A_k) = 1$ for all k. The Cesàro means are

$$\frac{1}{n} \sum_{k=1}^{n} \Phi_k(d(0, A_k)) = \frac{1}{n} \sum_{k=1}^{n} 1 = 1.$$

Since the limit is 1, the sequence does not converge to $\{0\}$ in the Wijsman Φ-strong Cesàro sense.

(iii) For any $\varepsilon \in (0,1]$, the set of bad indices is

$$K(\varepsilon) := \{k \in \mathbb{N} : \Phi_k(d(0,A_k)) \geq \varepsilon\} = \{k \in \mathbb{N} : 1 \geq \varepsilon\} = \mathbb{N}.$$

The natural density of this set, denoted by $\delta(K(\varepsilon))$, is

$$\delta(K(\varepsilon)) = \lim_{n\to\infty} \frac{|K(\varepsilon) \cap [1,n]|}{n} = 1 \neq 0.$$

Thus, Wijsman Φ-statistical convergence to $\{0\}$ fails.

(iv) Let $\theta = (k_r)$ be any lacunary sequence with intervals $I_r = (k_{r-1}, k_r]$. Since $d(0,A_k) = 1$ for all k, every index in every interval is a "bad" index for $\varepsilon \in (0,1]$.

$$\frac{1}{h_r}|\{k \in I_r : \Phi_k(d(0,A_k)) \geq \varepsilon\}| = \frac{1}{h_r}|I_r| = 1.$$

Since the limit as $r \to \infty$ is 1 (not 0), lacunary Wijsman Φ-statistical convergence also fails.

4. Additional Results and Applications

The next theorem provides a necessary and sufficient condition for a sequence of closed sets in a metric space to be Wijsman Φ-statistically convergent.

Theorem 4.1. *Let (A_n) be a sequence in $\mathcal{C}(X)$ and $A \in \mathcal{C}(X)$. Suppose that (A_n) is Wijsman bounded. Assume that the Musielak-Orlicz functions satisfy the following two conditions:*

(i) *For every $t > 0$, $\sup_k \Phi_k(t) < \infty$.*
(ii) *$m := \inf_k \Phi_k(1) > 0$.*

Then (A_n) is Wijsman Φ-statistically convergent to A if and only if for every point $x \in X$ and any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right) = 0.$$

Proof. Fix an arbitrary $x \in X$. Since (A_n) is Wijsman bounded, there exists $L_x > 0$ such that $|\gamma_k(x)| \leq L_x$ for all k.

First assume that (A_n) is Wijsman Φ-statistically convergent to A. Fix $\varepsilon > 0$ and let $\eta > 0$ be arbitrary. Define:

$$K_1(\eta) := \left\{k \in \mathbb{N} : \Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right) \geq \eta\right\},$$

$$K_2(\eta) := \left\{k \in \mathbb{N} : \Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right) < \eta\right\}.$$

By Φ-statistical convergence,

$$\lim_{n\to\infty}\frac{|K_1(\eta)\cap[1,n]|}{n}=0$$

for each $\eta>0$. Split the Cesàro sum into two terms as

$$\frac{1}{n}\sum_{k=1}^{n}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)=S_1(n)+S_2(n),$$

where $S_1(n)=\frac{1}{n}\sum\limits_{k\in K_1(\eta)\cap[1,n]}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)$ and $S_2(n)$ is the complementary sum. For $k\in K_1(\eta)$, using monotonicity of Φ_k and $|\gamma_k(x)|\le L_x$:

$$\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)\le\Phi_k\left(\frac{L_x}{\varepsilon}\right)\le\sup_j\Phi_j\left(\frac{L_x}{\varepsilon}\right)=:M<\infty.$$

Thus $S_1(n)\le M\cdot\frac{|K_1(\eta)\cap[1,n]|}{n}\to 0$ as $n\to\infty$. For $k\in K_2(\eta)$, by definition $\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)<\eta$, so $S_2(n)<\eta$. Therefore,

$\limsup_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)\le\eta$. Since $\eta>0$ is arbitrary, the limit is 0.

Conversely, assume that for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)=0.$$

We need to show that for every $\delta>0$, the set

$$F_\delta:=\{k:\Phi_k(|\gamma_k(x)|)\ge\delta\}$$

has density zero. Fix $\delta>0$. Since Φ_k are continuous at 0 and $\Phi_k(0)=0$, there exists $\varepsilon>0$ such that for all k,

$$\Phi_k(t)<\delta\quad\text{whenever } 0\le t<\varepsilon.$$

This implies that

$$F_\delta\subseteq E_\varepsilon:=\{k:|\gamma_k(x)|\ge\varepsilon\}.$$

Now, for $k\in E_\varepsilon$, we have $\frac{|\gamma_k(x)|}{\varepsilon}\ge 1$. By monotonicity of Φ_k:

$$\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)\ge\Phi_k(1)\ge m:=\inf_j\Phi_j(1)>0.$$

Thus for any n, we have

$$\frac{1}{n}\sum_{k=1}^{n}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)\ge\frac{1}{n}\sum_{k\in E_\varepsilon\cap[1,n]}\Phi_k\left(\frac{|\gamma_k(x)|}{\varepsilon}\right)\ge m\cdot\frac{|E_\varepsilon\cap[1,n]|}{n}.$$

Taking the limit as $n \to \infty$, the left-hand side tends to 0 by assumption, we have

$$0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k \left(\frac{|\gamma_k(x)|}{\varepsilon} \right) \geq m \cdot \limsup_{n\to\infty} \frac{|E_\varepsilon \cap [1,n]|}{n} \geq 0.$$

Terefore,

$$\lim_{n\to\infty} \frac{|E_\varepsilon \cap [1,n]|}{n} = 0.$$

Since $F_\delta \subseteq E_\varepsilon$, we have

$$\frac{|F_\delta \cap [1,n]|}{n} \leq \frac{|E_\varepsilon \cap [1,n]|}{n} \to 0, \quad \text{as } n \to \infty.$$

Hence (A_n) is Wijsman Φ-statistically convergent to A.

□

Proposition 4.2. *Suppose $\Phi = (\Phi_n)$ and $\Psi = (\Psi_n)$ are Musielak-Orlicz functions such that*

$$\Phi_n(t) \leq C\Psi_n(t)$$

for some constant $C > 0$ and all $n \in \mathbb{N}, t \geq 0$. Then, Wijsman Ψ-statistical convergence implies Wijsman Φ-statistical convergence.

We now extends the classical Strong Law of Large Numbers (SLLN) to the context of random sets.

5. Strong Law of Large Numbers for Random Sets

We begin by recalling fundamental concepts from probability theory that will be essential for our analysis.

Definition 5.1 (Random Variable)**.** A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable function $X : \Omega \to \mathbb{R}$.

Definition 5.2 (Independence)**.** A collection $\{X_n\}_{n=1}^{\infty}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is independent if for any finite subset of indices $\{n_1, \ldots, n_k\}$ and any Borel sets $B_1, \ldots, B_k \subseteq \mathbb{R}$,

$$\mathbb{P}(X_{n_1} \in B_1, \ldots, X_{n_k} \in B_k) = \prod_{i=1}^{k} \mathbb{P}(X_{n_i} \in B_i).$$

Definition 5.3 (Identically Distributed)**.** Random variables $\{X_n\}$ are identically distributed if for every Borel set $B \subseteq \mathbb{R}$,

$$\mathbb{P}(X_n \in B) = \mathbb{P}(X_1 \in B), \quad \text{for all } n \in \mathbb{N}.$$

Definition 5.4 (Expectation and Integrability)**.** The expectation of a random variable X is defined by the Lebesgue integral

$$\mathbb{E}[X] := \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega).$$

If $\mathbb{E}[|X|] < \infty$, the random variable X is said to be integrable.

Theorem 5.5 (Kolmogorov's Strong Law of Large Numbers)**.** *Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) integrable random variables. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = \mathbb{E}[X_1] \quad \textit{almost surely.}$$

For a proof of Theorem 5.5, see [13].

5.1. **Random Closed Sets and Their Properties.** Let (X, d) be a complete separable metric space and let $\mathcal{F}(X)$ denote the family of non-empty closed subsets of X.

Definition 5.6 (Random Closed Set)**.** A map $A : \Omega \to \mathcal{F}(X)$ is called a random closed set if for every open subset $U \subseteq X$, the set

$$\{\omega \in \Omega : A(\omega) \cap U \neq \emptyset\}$$

is measurable (i.e., belongs to $\mathcal{F}$). Equivalently, A is a random closed set if and only if for every $x \in X$, the map $\omega \mapsto d(x, A(\omega))$ is a random variable (see [26]).

For comprehensive treatments of random sets, see [26, 27].

Definition 5.7 (Distance Function of a Random Set)**.** For a fixed $x \in X$ and a random closed set A, the distance function d is defined by

$$d(x, A(\omega)) := \inf_{y \in A(\omega)} d(x, y).$$

This function is measurable and hence a random variable (see [27]).

Definition 5.8 (Wijsman Φ-Strong Cesàro Almost Sure Convergence)**.** A sequence $\{A_n\}$ of random closed sets is said to Wijsman Φ-strong Cesàro converge almost surely to a random closed set A if, for almost every $\omega \in \Omega$ and all $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k\left(\left|d(x, A_k(\omega)) - d(x, A(\omega))\right|\right) = 0,$$

where $\Phi = (\Phi_k)$ is a sequence of Musielak-Orlicz functions.

Theorem 5.9 (Strong Law of Large Numbers for Distance Functions)**.** *Let (X, d) be a complete separable metric space and let $\{A_n\}$ be a sequence of independent and identically distributed random closed sets. Fix $x \in X$ and assume that the random variable $Y_1(\omega) := d(x, A_1(\omega))$ is integrable, i.e., $\mathbb{E}[|Y_1|] < \infty$. Then:*

(i) *For almost every $\omega \in \Omega$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k(\omega)) = \mathbb{E}[d(x, A_1)].$$

(ii) *Moreover, if Ψ is an Orlicz function such that $\mathbb{E}[\Psi(d(x, A_1))] < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Psi(d(x, A_k(\omega))) = \mathbb{E}[\Psi(d(x, A_1))] \quad \textit{almost surely.}$$

Proof. Define $Y_k(\omega) := d(x, A_k(\omega))$ for each $k \in \mathbb{N}$. Since the A_k are i.i.d. random closed sets, the sequence $\{Y_k\}$ consists of i.i.d. random variables.

Part (1): Direct application of Kolmogorov's Strong Law of Large Numbers (Theorem 5.5) to the i.i.d. sequence $\{Y_k\}$ yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k = \mathbb{E}[Y_1] \quad \text{almost surely.}$$

Part (2): Define $Z_k(\omega) := \Psi(Y_k(\omega))$. Then Z_k are i.i.d. Given $\mathbb{E}[|Z_1|] < \infty$, Theorem 5.5 implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Z_k = \mathbb{E}[Z_1] = \mathbb{E}[\Psi(Y_1)] \quad \text{almost surely.} \qquad \square$$

6. Spaces Defined by Wijsman Φ-type Convergences

In this section, we define sequence spaces characterized by Wijsman Φ-type convergences. While the motivation comes from Wijsman convergence of sets (where $f_n(x) = d(x, A_n)$ for sequences of closed sets (A_n)), we state the definitions for general sequences of functions $f_n : X \to \mathbb{R}$.

Definition 6.1. A sequence of functions (f_n) is said to belong to the class $\mathcal{C}_\Phi^W$ if for every $x \in X$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k(|f_k(x)|) = 0.$$

Definition 6.2. A sequence of functions (f_n) is said to belong to the class $\mathcal{S}_\Phi^W$ if for every $\varepsilon > 0$ and every $x \in X$:

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \Phi_k\left(|f_k\left(x\right)|\right) \ge \varepsilon\}\right| = 0.$$

Equivalently, the sequence $(\Phi_k\left(|f_k\left(x\right)|\right))$ converges to 0 in density.

Definition 6.3. Let $\theta = (k_r)$ be a lacunary sequence (i.e., an increasing integer sequence with $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$). Let $I_r = (k_{r-1}, k_r]$. A sequence of functions (f_n) is said to belong to the class $\mathcal{L}_\Phi^W\left(\theta\right)$ if for every $\varepsilon > 0$ and every $x \in X$:

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \Phi_k\left(|f_k\left(x\right)|\right) \ge \varepsilon\}\right| = 0.$$

To ensure that the sets $\mathcal{C}_\Phi^W$, $\mathcal{S}_\Phi^W$ and $\mathcal{L}_\Phi^W\left(\theta\right)$ form linear spaces and possess desirable topological properties, we assume the sequence $\Phi = (\Phi_n)$ satisfies the following standard conditions for Musielak-Orlicz functions:

(i) Global Uniform Δ_2-condition: There exists a constant $K > 0$ such that for all $n \in \mathbb{N}$ and all $t \ge 0$,

$$\Phi_n\left(2t\right) \le K\Phi_n\left(t\right).$$

(ii) Subadditivity (follows from convexity and Δ_2): If Φ_n satisfies the Δ_2-condition, there exists a constant $C \ge 1$ such that for all $u, v \ge 0$:

$$\Phi_n\left(u+v\right) \le C\left(\Phi_n\left(u\right) + \Phi_n\left(v\right)\right).$$

(iii) Homogeneity-like Bound: For any scalar $\alpha \in \mathbb{R}$, there exists a constant $M_\alpha > 0$ such that for all $t \ge 0$:

$$\Phi_n\left(|\alpha|t\right) \le M_\alpha\Phi_n\left(t\right).$$

These conditions guarantee that if $f, g \in \mathcal{X} = \mathcal{C}_\Phi^W$, $\mathcal{S}_\Phi^W$ or $\mathcal{L}_\Phi^W\left(\theta\right)$, then $f + g$ and αf are also in $\mathcal{X}$.

For a set $A \subseteq \mathbb{N}$, denote its natural density by $\delta\left(A\right) := \lim_{n\to\infty} \frac{1}{n}|A \cap \{1, \ldots, n\}|$ whenever the limit exists.

Theorem 6.4. *The space $\mathcal{S}_\Phi^W$ is a vector space over $\mathbb{R}$.*

Proof. Let (f_n) and (g_n) be two sequences in $\mathcal{S}_\Phi^W$ and let $\alpha \in \mathbb{R}$. We must show that $(f_n + g_n) \in \mathcal{S}_\Phi^W$ and $(\alpha f_n) \in \mathcal{S}_\Phi^W$.

1. Closure under Addition: Fix $x \in X$ and $\varepsilon > 0$. By the triangle inequality, $|\left(f_n + g_n\right)\left(x\right)| \le |f_n\left(x\right)| + |g_n\left(x\right)|$. Since Φ_n is non-decreasing and satisfies the subadditivity condition, there exists $C \ge 1$ such that

$$\Phi_n\left(|f_n\left(x\right) + g_n\left(x\right)|\right) \le \Phi_n\left(|f_n\left(x\right)| + |g_n\left(x\right)|\right)$$

$$\leq C\left(\Phi_n\left(|f_n\left(x\right)|\right)+\Phi_n\left(|g_n\left(x\right)|\right)\right).$$

Define $K_\varepsilon := \{k \in \mathbb{N} : \Phi_k\left(|f_k\left(x\right)+g_k\left(x\right)|\right) \geq \varepsilon\}$. If $k \in K_\varepsilon$, then

$$C\left(\Phi_k\left(|f_k\left(x\right)|\right)+\Phi_k\left(|g_k\left(x\right)|\right)\right) \geq \varepsilon$$

implies

$$\Phi_k\left(|f_k\left(x\right)|\right)+\Phi_k\left(|g_k\left(x\right)|\right) \geq \frac{\varepsilon}{C}.$$

If both $\Phi_k\left(|f_k\left(x\right)|\right) < \frac{\varepsilon}{2C}$ and $\Phi_k\left(|g_k\left(x\right)|\right) < \frac{\varepsilon}{2C}$, their sum would be $< \frac{\varepsilon}{C}$, a contradiction. Hence, at least one of the inequalities must hold:

$$\Phi_k\left(|f_k\left(x\right)|\right) \geq \frac{\varepsilon}{2C} \quad \text{or} \quad \Phi_k\left(|g_k\left(x\right)|\right) \geq \frac{\varepsilon}{2C}.$$

Therefore,

$$K_\varepsilon \subseteq \left\{k : \Phi_k\left(|f_k\left(x\right)|\right) \geq \frac{\varepsilon}{2C}\right\} \cup \left\{k : \Phi_k\left(|g_k\left(x\right)|\right) \geq \frac{\varepsilon}{2C}\right\}.$$

Since $\left(f_n\right), \left(g_n\right) \in \mathcal{S}_\Phi^W$, both sets on the right have density zero. Therefore, $\delta\left(K_\varepsilon\right) = 0$, proving $\left(f_n + g_n\right) \in \mathcal{S}_\Phi^W$.

2. Closure under Scalar Multiplication: If $\alpha = 0$, then $\alpha f_n = 0$ trivially belongs to $\mathcal{S}_\Phi^W$. Assume $\alpha \neq 0$. By the homogeneity condition, there exists a constant $K_\alpha > 0$ such that for all $n \in \mathbb{N}$ and $t \geq 0$, we have

$$\Phi_n\left(|\alpha| t\right) \leq K_\alpha \Phi_n\left(t\right).$$

Now consider the set $L_\varepsilon := \{k : \Phi_k\left(|\alpha f_k\left(x\right)|\right) \geq \varepsilon\}$. If $k \in L_\varepsilon$, applying the above inequality, we have $\varepsilon \leq \Phi_k\left(|\alpha f_k\left(x\right)|\right) \leq K_\alpha \Phi_k\left(|f_k\left(x\right)|\right)$. Therefore,

$$\Phi_k\left(|f_k\left(x\right)|\right) \geq \frac{\varepsilon}{K_\alpha}.$$

Thus, we have

$$L_\varepsilon \subseteq \left\{k : \Phi_k\left(|f_k\left(x\right)|\right) \geq \frac{\varepsilon}{K_\alpha}\right\}.$$

Since $\left(f_n\right) \in \mathcal{S}_\Phi^W$, the set on the right has density zero. Consequently, $\delta\left(L_\varepsilon\right) = 0$. Hence, $\left(\alpha f_n\right) \in \mathcal{S}_\Phi^W$. □

Remark 6.5. The same arguments, with natural density replaced by Cesàro averages (using linearity of limits) or lacunary densities, show that $\mathcal{C}_\Phi^W$ and $\mathcal{L}_\Phi^W(\theta)$ are also vector spaces under the same conditions on Φ.

We now demonstrate that the spaces $\mathcal{C}_\Phi^W$, $\mathcal{S}_\Phi^W$ and $\mathcal{L}_\Phi^W(\theta)$ can be endowed with paranormed structures. Since Wijsman convergence is defined pointwise relative to $x \in X$, we consider a family of paranorms $\{g_x\}_{x\in X}$, where each g_x acts on the sequence of distances $\xi = (d(x, A_k))_{k\in\mathbb{N}}$.

Recall that a *paranorm* $g : V \to [0, \infty)$ on a vector space V is a function satisfying:

(i) $g(\mathbf{0}) = 0$;
(ii) $g(-\mathbf{u}) = g(\mathbf{u})$ (symmetry);
(iii) $g(\mathbf{u} + \mathbf{v}) \leq g(\mathbf{u}) + g(\mathbf{v})$ (subadditivity);
(iv) Continuity of scalar multiplication: If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ and $g(\mathbf{u}_n - \mathbf{u}) \to 0$, then $g(\lambda_n \mathbf{u}_n - \lambda \mathbf{u}) \to 0$.

Paranorm for $\mathcal{C}_\Phi^W$: Define the functional:

$$\rho_{\mathcal{C}}(\xi) := \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi_k(|\xi_k|).$$

and $g_{\mathcal{C}}$ as the Luxemburg norm:

$$g_{\mathcal{C}}(\xi) := \inf \{\lambda > 0 : \rho_{\mathcal{C}}(\xi/\lambda) \leq 1\}.$$

Theorem 6.6. *The functional $g_{\mathcal{C}}$ is a paranorm on $\mathcal{C}_\Phi^W$ and convergence $g_{\mathcal{C}}(\xi) \to 0$ is equivalent to $\rho_{\mathcal{C}}(\xi) \to 0$, which corresponds to strong Wijsman Φ-Cesàro convergence.*

Proof. We break the proof into two main parts: first, verifying the paranorm axioms for $g_{\mathcal{C}}$ and second, proving the equivalence between these convergences.

Part 1: $g_{\mathcal{C}}$ is a paranorm. We verify the four axioms of a paranorm. Let ξ, η be sequences in $\mathcal{C}_\Phi^W$.

(i) If $\xi = \mathbf{0}$, then $\xi_k = 0$ for all k. Since $\Phi_k(0) = 0$, we have $\rho_{\mathcal{C}}(\mathbf{0}/\lambda) = 0 \leq 1$ for all $\lambda > 0$. Therefore, the infimum is 0, so $g_{\mathcal{C}}(\mathbf{0}) = 0$.
(ii) Since $\Phi_k(|-\xi_k|) = \Phi_k(|\xi_k|)$, it trivially follows that $\rho_{\mathcal{C}}(-\xi/\lambda) = \rho_{\mathcal{C}}(\xi/\lambda)$. Thus, $g_{\mathcal{C}}(-\xi) = g_{\mathcal{C}}(\xi)$.
(iii) Subadditivity. We must show $g_{\mathcal{C}}(\xi + \eta) \leq g_{\mathcal{C}}(\xi) + g_{\mathcal{C}}(\eta)$. Let $u, v > 0$ be such that $g_{\mathcal{C}}(\xi) < u$ and $g_{\mathcal{C}}(\eta) < v$. By definition, $\rho_{\mathcal{C}}(\xi/u) \leq 1$ and $\rho_{\mathcal{C}}(\eta/v) \leq 1$. By the convexity of Φ_k, we have

$$\Phi_k\left(\frac{|\xi_k + \eta_k|}{u+v}\right) \leq \Phi_k\left(\frac{u}{u+v}\frac{|\xi_k|}{u} + \frac{v}{u+v}\frac{|\eta_k|}{v}\right)$$
$$\leq \frac{u}{u+v}\Phi_k\left(\frac{|\xi_k|}{u}\right) + \frac{v}{u+v}\Phi_k\left(\frac{|\eta_k|}{v}\right).$$

Summing from $k = 1$ to n, dividing by n and taking the limit superior, we obtain

$$\rho_{\mathcal{C}}\left(\frac{\xi+\eta}{u+v}\right) \leq \frac{u}{u+v}\rho_{\mathcal{C}}\left(\frac{\xi}{u}\right) + \frac{v}{u+v}\rho_{\mathcal{C}}\left(\frac{\eta}{v}\right)$$

1.

By definition of the infimum, $g_{\mathcal{C}}(\xi+\eta) \leq u+v$. Taking the infimum over all such $u > g_{\mathcal{C}}(\xi)$ and $v > g_{\mathcal{C}}(\eta)$ yields $g_{\mathcal{C}}(\xi+\eta) \leq g_{\mathcal{C}}(\xi) + g_{\mathcal{C}}(\eta)$.

(iv) Continuity of scalar multiplication. We need to show: If $\lambda_n \to \lambda$ in $\mathbb{R}$ and $g_{\mathcal{C}}(\xi_n - \xi) \to 0$, then $g_{\mathcal{C}}(\lambda_n\xi_n - \lambda\xi) \to 0$. By the subadditivity, we have

$$g_{\mathcal{C}}(\lambda_n\xi_n - \lambda\xi) \leq g_{\mathcal{C}}(\lambda_n(\xi_n - \xi)) + g_{\mathcal{C}}((\lambda_n - \lambda)\xi).$$

Since $g_{\mathcal{C}}(\xi_n - \xi) \to 0$ and λ_n is bounded, by the properties of Φ, there exists $M > 0$ such that $g_{\mathcal{C}}(\lambda_n(\xi_n - \xi)) \leq M g_{\mathcal{C}}(\xi_n - \xi) \to 0$. Again, $\lambda_n - \lambda \to 0$ and $g_{\mathcal{C}}(\xi) < \infty$, we have $g_{\mathcal{C}}((\lambda_n - \lambda)\xi) = |\lambda_n - \lambda|\, g_{\mathcal{C}}(\xi) \to 0$. Therefore, $g_{\mathcal{C}}(\lambda_n\xi_n - \lambda\xi) \to 0$.

Part 2: Equivalence of convergence $g_{\mathcal{C}}(\xi) \to 0 \iff \rho_{\mathcal{C}}(\xi) \to 0$. First suppose that $g_{\mathcal{C}}(\xi) \to 0$. Let $0 < \varepsilon < 1$. For $g_{\mathcal{C}}(\xi)$ sufficiently small, we have $g_{\mathcal{C}}(\xi) < \varepsilon$, which implies $\rho_{\mathcal{C}}(\xi/\varepsilon) \leq 1$. By the convexity of Φ_k and the fact that $\Phi_k(0) = 0$, we have

$$\rho_{\mathcal{C}}(\xi) = \rho_{\mathcal{C}}\left(\varepsilon\frac{\xi}{\varepsilon}\right) \leq \varepsilon\rho_{\mathcal{C}}\left(\frac{\xi}{\varepsilon}\right) \leq = \varepsilon.$$

Since this holds for arbitrarily small ε, we conclude $\rho_{\mathcal{C}}(\xi) \to 0$.

Conversely, suppose that $\rho_{\mathcal{C}}(\xi) \to 0$. We want to show $g_{\mathcal{C}}(\xi) \to 0$. Let $\varepsilon > 0$. We need to show that eventually $g_{\mathcal{C}}(\xi) < \varepsilon$, which is equivalent to showing $\rho_{\mathcal{C}}(\xi/\varepsilon) \leq 1$. Since Φ satisfies the uniform Δ_2-condition ($\Phi_k(2t) \leq K\Phi_k(t)$), there exists a constant $M = M(\varepsilon) > 0$ such that for all k and t,

$$\Phi_k\left(\frac{t}{\varepsilon}\right) \leq M\Phi_k(t).$$

Therefore,

$$\rho_{\mathcal{C}}\left(\frac{\xi}{\varepsilon}\right) \leq M\rho_{\mathcal{C}}(\xi).$$

Since $\rho_{\mathcal{C}}(\xi) \to 0$, we can choose ξ sufficiently close to $\mathbf{0}$ such that $\rho_{\mathcal{C}}(\xi/\varepsilon) \leq 1$, which implies $g_{\mathcal{C}}(\xi) \leq \varepsilon$. Since ε was arbitrary, $g_{\mathcal{C}}(\xi) \to 0$.

Therefore, $g_{\mathcal{C}}(\xi) \to 0$ if and only if the distance sequence strongly Wijsman Φ-Cesàro converges to 0. □

Quasi-Paranorms for Statistical and Lacunary Convergence.

Theorem 6.7. *The functionals $g_{\mathcal{S}}$ and $g_{\mathcal{L}}$ defined by:*

$$g_{\mathcal{S}}(\xi) := \inf\left\{\varepsilon > 0 : \limsup_{n\to\infty} \frac{1}{n}\left|\{k \leq n : \Phi_k(|\xi_k|) \geq \varepsilon\}\right| \leq \varepsilon\right\},$$

$$g_{\mathcal{L}}(\xi) := \inf\left\{\varepsilon > 0 : \limsup_{r\to\infty} \frac{1}{h_r}\left|\{k \in I_r : \Phi_k(|\xi_k|) \geq \varepsilon\}\right| \leq \varepsilon\right\},$$

are quasi-paranorms on $\mathcal{S}_\Phi^W$ and $\mathcal{L}_\Phi^W(\theta)$, respectively.

Proof. We prove the result for $g_{\mathcal{S}}$; the proof for $g_{\mathcal{L}}$ is analogous with n replaced by r and averages over I_r.

1. Symmetry: Since $\Phi_k(|-\xi_k|) = \Phi_k(|\xi_k|)$, we have $g_{\mathcal{S}}(-\xi) = g_{\mathcal{S}}(\xi)$.
2. Positive property: Clearly $g_{\mathcal{S}}(\xi) \geq 0$ for all ξ. For the zero sequence $\mathbf{0}$, we have $\Phi_k(0) = 0$ for all k, so for any $\varepsilon > 0$, the set $\{k : 0 \geq \varepsilon\}$ is empty. Thus $g_{\mathcal{S}}(\mathbf{0}) = 0$.
3. Quasi-subadditivity: We show there exists $K \geq 1$ such that $g_{\mathcal{S}}(\xi + \eta) \leq K(g_{\mathcal{S}}(\xi) + g_{\mathcal{S}}(\eta))$. Let $\delta > 0$ be arbitrary. Choose:
$$u = g_{\mathcal{S}}(\xi) + \delta, \text{ and } v = g_{\mathcal{S}}(\eta) + \delta.$$
Since $u > g_{\mathcal{S}}(\xi)$ and $v > g_{\mathcal{S}}(\eta)$, by definition of $g_{\mathcal{S}}$ we have
$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : \Phi_k(|\xi_k|) \geq u\}| \leq u,$$
$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : \Phi_k(|\eta_k|) \geq v\}| \leq v.$$
By the subadditivity of Φ_k, there exists $C \geq 1$ such that $\Phi_k(s+t) \leq C(\Phi_k(s) + \Phi_k(t))$ for all $k, s, t \geq 0$. Consider $\varepsilon = 2C(u+v) = 2C(g_{\mathcal{S}}(\xi) + g_{\mathcal{S}}(\eta) + 2\delta)$. For any k with $\Phi_k(|\xi_k + \eta_k|) \geq \varepsilon$, we have
$$2C(u+v) \leq \Phi_k(|\xi_k + \eta_k|) \leq C(\Phi_k(|\xi_k|) + \Phi_k(|\eta_k|)),$$
which implies $2(u+v) \leq \Phi_k(|\xi_k|) + \Phi_k(|\eta_k|)$. Therefore, either $\Phi_k(|\xi_k|) \geq u+v$ or $\Phi_k(|\eta_k|) \geq u+v$ (or both). Thus
$$\{k : \Phi_k(|\xi_k + \eta_k|) \geq \varepsilon\} \subseteq \{k : \Phi_k(|\xi_k|) \geq u+v\} \cup \{k : \Phi_k(|\eta_k|) \geq u+v\}.$$
Since $u+v \geq u$ and $u+v \geq v$, we have:
$$\{k : \Phi_k(|\xi_k|) \geq u+v\} \subseteq \{k : \Phi_k(|\xi_k|) \geq u\},$$
$$\{k : \Phi_k(|\eta_k|) \geq u+v\} \subseteq \{k : \Phi_k(|\eta_k|) \geq v\}.$$
Therefore,
$$\{k : \Phi_k(|\xi_k + \eta_k|) \geq \varepsilon\} \subseteq \{k : \Phi_k(|\xi_k|) \geq u\} \cup \{k : \Phi_k(|\eta_k|) \geq v\}.$$
Thus we have
$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : \Phi_k(|\xi_k + \eta_k|) \geq \varepsilon\}| \leq \limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : \Phi_k(|\xi_k|) \geq u\}| + \limsup_{n\to\infty} \frac{1}{n} |\{k \leq n : \Phi_k(|\eta_k|) \geq v\}|$$

$$\begin{aligned}&\leq u + v\\ &= (g_{\mathcal{S}}(\xi) + \delta) + (g_{\mathcal{S}}(\eta) + \delta)\\ &= g_{\mathcal{S}}(\xi) + g_{\mathcal{S}}(\eta) + 2\delta.\end{aligned}$$

Therefore, $g_{\mathcal{S}}(\xi+\eta) \leq \max\{\varepsilon,\ g_{\mathcal{S}}(\xi) + g_{\mathcal{S}}(\eta) + 2\delta\}$. Since $\delta > 0$ was arbitrary, letting $\delta \to 0$, we have

$$g_{\mathcal{S}}(\xi+\eta) \leq 2C(g_{\mathcal{S}}(\xi) + g_{\mathcal{S}}(\eta)).$$

Thus, $g_{\mathcal{S}}$ satisfies quasi-subadditivity with constant $K = 2C$.

4. Continuity of scalar multiplication: Let $\lambda_n \to \lambda$ in $\mathbb{R}$ and suppose $g_{\mathcal{S}}(\xi_n - \xi) \to 0$. We need to show $g_{\mathcal{S}}(\lambda_n\xi_n - \lambda\xi) \to 0$. By the properties of Φ, for any $\alpha \in \mathbb{R}$, there exists $M_\alpha > 0$ such that $\Phi_k(|\alpha t|) \leq M_\alpha \Phi_k(t)$ for all $k, t \geq 0$. Write

$$\lambda_n\xi_n - \lambda\xi = \lambda_n(\xi_n - \xi) + (\lambda_n - \lambda)\xi.$$

By quasi-subadditivity, we have

$$g_{\mathcal{S}}(\lambda_n\xi_n - \lambda\xi) \leq K\left(g_{\mathcal{S}}(\lambda_n(\xi_n - \xi)) + g_{\mathcal{S}}((\lambda_n - \lambda)\xi)\right).$$

Using

$$\begin{aligned}g_{\mathcal{S}}(\lambda_n(\xi_n - \xi)) &\leq M_M \cdot g_{\mathcal{S}}(\xi_n - \xi)\\ &\to 0.\end{aligned}$$

and

$$\begin{aligned}g_{\mathcal{S}}((\lambda_n - \lambda)\xi) &\leq M_1 \cdot |\lambda_n - \lambda| \cdot g_{\mathcal{S}}(\xi)\\ &\to 0,\end{aligned}$$

we have $g_{\mathcal{S}}(\lambda_n\xi_n - \lambda\xi) \to 0$. □

Example that the positive definiteness property may fails.

Example 6.8 (Failure of Positive Definiteness)**.** Consider the sequence $\xi = (\xi_k)$ defined by

$$\xi_k = \begin{cases}1 & \text{if } k \text{ is a perfect square,}\\ 0 & \text{otherwise.}\end{cases}$$

Then $\xi \neq \mathbf{0}$, but $g_{\mathcal{S}}(\xi) = 0$ because the set of perfect squares has density zero and $g_{\mathcal{L}}(\xi) = 0$ for any lacunary sequence θ because the set of perfect squares intersects each lacunary interval in at most finitely many points. This demonstrates that $g_{\mathcal{S}}$ and $g_{\mathcal{L}}$ are not norms or proper paranorms, but quasi-paranorms.

Theorem 6.9 (Completeness of $\mathcal{S}_\Phi^W$)**.** *Assume the Musielak–Orlicz functions $\Phi_k : [0,\infty) \to [0,\infty)$ satisfy the Δ_2-condition uniformly in k and the following uniform growth conditions:*

(i) *There exists* $p > 0$ *and* $c > 0$ *such that* $\Phi_k(u) \geq c \cdot u^p$ *for all* $k \in \mathbb{N}$ *and all* $u \geq 0$.
(ii) *There exists* $u_0 > 0$ *and* $M > 0$ *such that* $\sup_{k\in\mathbb{N}} \Phi_k(u_0) \leq M$.

Then the quasi-paranormed space $\left(\mathcal{S}_\Phi^W, g_\mathcal{S}\right)$ *is complete.*

Proof. Let $(\xi^{(m)})_{m\in\mathbb{N}}$ be a Cauchy sequence in $\left(\mathcal{S}_\Phi^W, g_\mathcal{S}\right)$. That is, for every $\varepsilon > 0$, there exists N_ε such that for all $m, n \geq N_\varepsilon$, we have

$$g_\mathcal{S}\left(\xi^{(m)} - \xi^{(n)}\right) < \varepsilon.$$

Choose a strictly increasing sequence $(m_j)_{j\in\mathbb{N}}$ such that for all $j \geq 1$, we have

$$g_\mathcal{S}\left(\xi^{(m_{j+1})} - \xi^{(m_j)}\right) < 2^{-j}.$$

By definition of $g_\mathcal{S}$, for each j, there exists δ_j with $0 < \delta_j < 2^{-j}$ such that:

$$\limsup_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \Phi_k\left(\left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right|\right) \geq \delta_j\right\}\right| \leq \delta_j.$$

Define $A_j = \left\{k : \Phi_k\left(\left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right|\right) \geq \delta_j\right\}$. Then $\delta(A_j) \leq \delta_j < 2^{-j}$. Since $\sum_{j=1}^{\infty} \overline{\delta}(A_j) \leq \sum_{j=1}^{\infty} 2^{-j} < \infty$, the density Borel–Cantelli lemma gives

$$\delta\left(\limsup_{j\to\infty} A_j\right) = 0.$$

Define $Z = \limsup_{j\to\infty} A_j = \{k : k \in A_j \text{ for infinitely many } j\}$. Then $\overline{\delta}(Z) = 0$. For $k \notin Z$, there exists n_k such that $k \notin A_j$ for all $j \geq n_k$. Thus for $j \geq n_k$, we have

$$\Phi_k\left(\left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right|\right) < \delta_j < 2^{-j}.$$

By the uniform lower bound assumption in the hypothesis, we have $\Phi_k(u) \geq c \cdot u^p$, which implies

$$\left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right| \leq \left(\frac{\Phi_k\left(\left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right|\right)}{c}\right)^{1/p} < \left(\frac{2^{-j}}{c}\right)^{1/p}.$$

Therefore, for $k \notin Z$, we have

$$\sum_{j=n_k}^{\infty} \left|\xi_k^{(m_{j+1})} - \xi_k^{(m_j)}\right| \leq \sum_{j=n_k}^{\infty} \left(\frac{2^{-j}}{c}\right)^{1/p} < \infty.$$

This shows that $\left(\xi_k^{(m_j)}\right)_j$ is a Cauchy sequence in $\mathbb{R}$ for every $k \notin Z$. Define

$$\xi_k = \begin{cases} \lim\limits_{j\to\infty} \xi_k^{(m_j)} & \text{if } k \notin Z, \\ 0 & \text{if } k \in Z. \end{cases}$$

Claim that $\xi^{(m_j)} \to \xi$ in $g_{\mathcal{S}}$. Fix $\varepsilon > 0$. Choose J large enough such that

(i) $2^{-J} < \varepsilon/2$

(ii) $\sum\limits_{j=J}^{\infty} \left(\frac{2^{-j}}{c}\right)^{1/p} < \min\left(\varepsilon/2, u_0\right)$

For $j \geq J$ and $k \notin Z$, we have

$$\left|\xi_k^{(m_j)} - \xi_k\right| \leq \sum_{\ell=j}^{\infty} \left|\xi_k^{(m_{\ell+1})} - \xi_k^{(m_\ell)}\right| < \min\left(\varepsilon/2, u_0\right).$$

Now, by convexity of Φ_k and assumption in the hypothesis, for any $u \in [0, u_0]$, we have

$$\Phi_k(u) = \Phi_k\left(\frac{u}{u_0} \cdot u_0 + \left(1 - \frac{u}{u_0}\right) \cdot 0\right) \leq \frac{u}{u_0} \cdot \Phi_k(u_0) \leq \frac{u}{u_0} \cdot M.$$

Thus for $j \geq J$ and $k \notin Z$:

$$\Phi_k\left(\left|\xi_k^{(m_j)} - \xi_k\right|\right) \leq \frac{M}{u_0} \cdot \left|\xi_k^{(m_j)} - \xi_k\right| < \frac{M}{u_0} \cdot \frac{\varepsilon}{2}.$$

Choose ε small enough so that $\frac{M}{u_0} \cdot \frac{\varepsilon}{2} < \varepsilon$, i.e., $\varepsilon < \frac{2u_0}{M}$. For such ε, we have

$$\Phi_k\left(\left|\xi_k^{(m_j)} - \xi_k\right|\right) < \varepsilon \quad \text{for all } k \notin Z.$$

Therefore, for $j \geq J$, we have

$$\left\{k : \Phi_k\left(\left|\xi_k^{(m_j)} - \xi_k\right|\right) \geq \varepsilon\right\} \subseteq Z.$$

Since $\delta(Z) = 0$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \Phi_k\left(\left|\xi_k^{(m_j)} - \xi_k\right|\right) \geq \varepsilon\right\}\right| = 0.$$

This implies $g_{\mathcal{S}}\left(\xi^{(m_j)} - \xi\right) \to 0$ as $j \to \infty$. Since $(\xi^{(m)})$ is Cauchy and has a subsequence converging to ξ, the entire sequence converges to ξ. For any $\varepsilon > 0$, choose N such that for $m, n \geq N$, $g_{\mathcal{S}}\left(\xi^{(m)} - \xi^{(n)}\right) < \varepsilon/2$. Choose j such that $m_j \geq N$ and $g_{\mathcal{S}}\left(\xi^{(m_j)} - \xi\right) < \varepsilon/2$. Then for $m \geq N$, we have

$$\begin{aligned} g_{\mathcal{S}}\left(\xi^{(m)} - \xi\right) &\leq K\left(g_{\mathcal{S}}\left(\xi^{(m)} - \xi^{(m_j)}\right) + g_{\mathcal{S}}\left(\xi^{(m_j)} - \xi\right)\right) \\ &< K\left(\varepsilon/2 + \varepsilon/2\right) = K\varepsilon, \end{aligned}$$

where K is the quasi-subadditivity constant of $g_{\mathcal{S}}$. Since $\varepsilon > 0$ was arbitrary, $\xi^{(m)} \to \xi$ in $g_{\mathcal{S}}$. Claim that $\xi \in \mathcal{S}_\Phi^W$. Since $\xi^{(N)} \in \mathcal{S}_\Phi^W$ for some N, we have $g_{\mathcal{S}}\left(\xi^{(N)}\right) < \infty$. By quasi-subadditivity property, we have

$$g_{\mathcal{S}}(\xi) \leq K\left(g_{\mathcal{S}}\left(\xi^{(N)}\right) + g_{\mathcal{S}}\left(\xi - \xi^{(N)}\right)\right) < \infty.$$

Thus $\xi \in \mathcal{S}_\Phi^W$. Therefore, every Cauchy sequence in $\left(\mathcal{S}_\Phi^W, g_{\mathcal{S}}\right)$ converges to an element of $\mathcal{S}_\Phi^W$, proving completeness. Similarly, we can prove that $(\mathcal{L}_\Phi^W(\theta), g_{\mathcal{L}})$ and $(\mathcal{C}_\Phi^W, g_{\mathcal{C}})$ are also complete. The proofs follow on the same line by replacing asymptotic density with lacunary density and Cesàro means, respectively. □

Theorem 6.10 (Inclusion relations among $\mathcal{C}_\Phi^W$, $\mathcal{S}_\Phi^W$ and $\mathcal{L}_\Phi^W$)**.** *Let $\theta = (k_r)$ be a lacunary sequence. The following inclusion relations hold:*

(i) $\mathcal{C}_\Phi^W \subseteq \mathcal{S}_\Phi^W$.
(ii) *If* $\liminf_{r\to\infty} q_r > 1$, *then* $\mathcal{S}_\Phi^W \subseteq \mathcal{L}_\Phi^W$.

Consequently, under the condition $\liminf_{r\to\infty} q_r > 1$, *we have*

$$\mathcal{C}_\Phi^W \subseteq \mathcal{S}_\Phi^W \subseteq \mathcal{L}_\Phi^W.$$

Proof. The inclusions follow directly from the Proposition 3.1, Proposition 3.2 and Proposition 3.3. □

Theorem 6.11 (Equivalence under bounded lacunary ratios)**.** *Let $\Phi = (\Phi_k)$ be a sequence of Musielak-Orlicz functions satisfying:*

(i) *There exists $K \geq 2$ such that $\Phi_k(2t) \leq K\Phi_k(t)$ for all $k \in \mathbb{N}$, $t \geq 0$.*
(ii) *For every $t > 0$, $\sup_k \Phi_k(t) < \infty$.*

Let $\theta = (k_r)$ be a lacunary sequence with $q_r = k_r/k_{r-1}$ satisfying

$$1 < \liminf_{r\to\infty} q_r \leq \limsup_{r\to\infty} q_r < \infty.$$

Then

(i) *The Wijsman lacunary statistical summability space coincides with the Wijsman statistical summability space: $\mathcal{S}_\Phi^W = \mathcal{L}_\Phi^W$.*
(ii) *For any Wijsman bounded sequence (A_k) of non-empty closed subsets of X, all three spaces coincide: $\mathcal{C}_\Phi^W = \mathcal{S}_\Phi^W = \mathcal{L}_\Phi^W$.*

Proof. Both statements follow directly from Theorem 3.7. □

7. Conclusion and Future Work

7.1. **Conclusion.** In this paper, we systematically studied Wijsman Φ-strong Cesàro convergence ($\mathcal{C}_\Phi^W$), Wijsman Φ-statistical convergence

$(\mathcal{S}_\Phi^W)$ and Wijsman Φ-lacunary statistical convergence $(\mathcal{L}_\Phi^W)$. We established that all three spaces $(\mathcal{C}_\Phi^W, g_\mathcal{C})$, $\left(\mathcal{S}_\Phi^W, g_\mathcal{S}\right)$ and $(\mathcal{L}_\Phi^W, g_\mathcal{L})$ are quasi-paranormed spaces. Under appropriate conditions on the Musielak-Orlicz functions Φ_k (convexity, Δ_2-condition, uniform lower bound and uniform normalization), we proved that these spaces are complete, making them Fréchet spaces. We established the fundamental inclusion relations: $\mathcal{C}_\Phi^W \subseteq \mathcal{S}_\Phi^W \subseteq \mathcal{L}_\Phi^W$, provided that the lacunary sequence θ satisfies $\liminf_{r\to\infty} q_r > 1$. These inclusions are strict in general, demonstrating a hierarchy of convergence notions. Under the stronger conditions that Φ_k satisfies a uniform Δ_2-condition and uniform boundedness and when the lacunary sequence has bounded ratios ($1 < \liminf q_r \leq \limsup q_r < \infty$), we proved that:

- $\mathcal{S}_\Phi^W = \mathcal{L}_\Phi^W$
- For Wijsman bounded sequences, all three spaces coincide: $\mathcal{C}_\Phi^W = \mathcal{S}_\Phi^W = \mathcal{L}_\Phi^W$

This provides a complete characterization of when these different convergence modes are equivalent.

7.2. **Future Work.** Several interesting directions for future research emerge from this work:

(i) **Weaker Conditions on Φ_k:** Some of our results require several technical conditions on the Musielak-Orlicz functions. It would be valuable to investigate whether these conditions can be relaxed while still preserving these properties. In particular,
   - Can the uniform Δ_2-condition be weakened to a local or asymptotic version?
   - Is the uniform normalization condition necessary, or can it be replaced by a growth condition?
   - What minimal conditions guarantee that the spaces are linear or have other desirable algebraic properties?

(ii) **Other Summability Methods:** The results developed here can be extended to other summability methods:
   - λ-statistical convergence and λ-lacunary statistical convergence
   - Matrix summability methods (Cesàro matrices, Hausdorff matrices, etc.)
   - Ideal convergence and filter convergence with Musielak-Orlicz functions.
   - Double sequences and multi-dimensional extensions.

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