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On Stability, Data Dependence and a Generalized Functional Equation Using HR^*-Iteration

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Abstract. In this article, we enquire about two characteristic and one applicative aspect of recently proposed HR^*-iteration. For a first in the literature, different types of stability are unified and further, we establish the $\mathcal{H}$-stability of the iteration followed by a supporting non-trivial numerical example. Additionally, we derive a data dependence result for the HR^*-iteration. Finally, we assert the existence of a solution of a certain type of functional equation, which is a generalized form of a predator-prey model, by using the aforementioned iteration scheme.

1. Introduction

The existence and uniqueness of solutions of various real-world problems can be determined by different methods [16–18]. Fixed point theory functions as one such tool, which has several applicative prospects in different areas of mathematics, economics, social sciences, etc. Such applications have transitioned the field from pure mathematical to applicative in recent years, in the sense of approximating solutions to many real-world problems that can be posited as fixed point problems [1, 5, 10, 22, 24, 31, 32]. The approximation techniques rely on tools such as contractive mappings, non-expansive mappings and their generalizations, some of which are in [8, 24, 27], along with fixed point iterations, of which we have included a few in [1, 10, 13, 15, 19], see also [2–4, 23, 33] and references therein. Now, a natural question that arises

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is why this transition has demanded the study of aspects, like stability, data dependence, error estimation and other related notions. The answer to this is that these aspects or characteristics help to determine the impact of changes in the initial parameters on the final result, which involves approximating to a fixed point and determine the precision of the approximation techniques used as well. Therefore, these studies help in gauging the efficiency of the iterations used, to potentially enhance it for better approximations.

To address this need for studying these characteristics of an iteration in the context of a mapping, we have chosen HR^*-iteration [10], which is a recently proposed four-step iteration, along with weak contractions. We have chosen this iteration mainly for its recency, four steps for potential complications, relevance, as is evident from a steady shift to four-step iterations from three-step iterations by researchers [10, 13], rapid convergence than many notable iterations [10] and lack of literature on these aspects concerning this iteration. The characteristics of HR^*-iteration to be studied in this paper, are stability and data dependence and the application involves using HR^*-iteration to approximate solution to a generalized functional equation.

In this paper, a thorough review is provided in the first section for all the concepts related to stability scattered in the literature and we unify the relationships between each one of those. Furthermore, $\mathcal{H}$-stability for HR^*-iteration is proven, which according to the relationship determined implies all other types of stability. Also, a non-trivial example is provided to validate our finding. A result on data dependence is proven afterwards and exemplified in a non-trivial and constructive manner. Finally, in the context of fixed point theory, a detailed introduction on functional equations is provided in the last section before using HR^*-iteration to approximate a solution to a functional equation, which is a generalized form of a predator-prey model.

2. Preliminaries

Throughout this article, the set of natural numbers along with the set of real numbers are denoted by the standard notations. Also, the fixed point set of a certain operator, say $\mathcal{H}$, is denoted by $F(\mathcal{H})$. In this section are mentioned some preliminary definitions and results that will be used in this article. We begin with the definition of weak contractions.

Definition 2.1 ([6])**.** On a Banach space $\mathcal{W}$, assume $\mathcal{H}$ to be a self-map. If there is a $\sigma \in (0,1)$ and some constant $K \geq 0$, such that for all $u, v \in \mathcal{W}$,

$$\|\mathcal{H}u - \mathcal{H}v\| \leq \sigma \|u - v\| + K \|v - \mathcal{H}u\|, \tag{2.1}$$

then $\mathcal{H}$ is a weak contraction.

For weak contractions, the following theorem has been proved by Berinde in [6] for the existence and uniqueness of a fixed point.

Theorem 2.2 ([6]). *On a Banach space $\mathcal{W}$, assume $\mathcal{H}$ to be a self-map which satisfies (2.1). Then, for all $u, v \in \mathcal{W}$, if*

$$\|\mathcal{H}u - \mathcal{H}v\| \leq \sigma \|u - v\| + K \|u - \mathcal{H}u\|, \tag{2.2}$$

then $\mathcal{H}$ possesses a unique fixed point in $\mathcal{W}$.

The subsequent definition is about an approximate operator corresponding to a self-mapping.

Definition 2.3. For two self-mappings $\mathcal{H}$ and $\mathcal{R}$, defined on a non-empty subset $\mathcal{B}$ of a Banach space $\mathcal{W}$, $\mathcal{R}$ is referred to be an approximate operator of $\mathcal{H}$, if there exists a fixed $\epsilon > 0$ such that for all $u \in \mathcal{B}$, $\|\mathcal{R}u - \mathcal{H}u\| \leq \epsilon$.

The following lemmas play significant part in this article.

Lemma 2.4 ([8]). *If for a sequence of positive numbers (α_n),*

$$\alpha_{n+1} \leq \sigma\alpha_n + \epsilon_n, \quad n = 0, 1, 2, \ldots$$

where $\sigma \in [0, 1)$ and (ϵ_n) is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 ([26]). *If for a sequence of positive real numbers (α_n), the following inequality is satisfied*

$$\alpha_{n+1} \leq (1 - \sigma_n)\alpha_n,$$

where $\sigma_n \in (0, 1)$ and $\sum_{n=0}^{\infty} \sigma_n = \infty$, then $\lim_{n\to\infty} \alpha_n = 0$.

Now we recall the definition of HR^*-iteration [10]. For a non-empty subset $\mathcal{B}$ of a Banach space $\mathcal{W}$, let $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is a self-mapping. Then the iteration is defined as

$$\begin{cases} \alpha_0 \in \mathcal{B} \\ \alpha_{n+1} = (1 - a_n)\beta_n + a_n\mathcal{H}\beta_n \\ \beta_n = \mathcal{H}(\mathcal{H}\gamma_n) \\ \gamma_n = \mathcal{H}((1 - b_n)\theta_n + b_n\mathcal{H}\theta_n) \\ \theta_n = (1 - c_n)\alpha_n + c_n\mathcal{H}\alpha_n, \end{cases} \tag{2.3}$$

where (a_n), (b_n) and (c_n) are control sequences in $(0, 1)$.

3. Stability Unification: Relationships, $\mathcal{H}$-stability and Exemplification

After the pioneering notion of stability of a fixed point iteration (FP-iteration) in the Ph.D. thesis of A.M. Harder [11], several other types of stabilities have been proposed and studied by mathematicians [9, 20, 28–30]. Harder and Hicks [12] coined the following notion.

Definition 3.1 ([12])**.** Let $\mathcal{H} : \mathcal{W} \to \mathcal{W}$ is a mapping and $\alpha_{n+1} = f(\mathcal{H}, \alpha_n)$ an FP-iteration, such that $\alpha_n \to \alpha \in F(\mathcal{H})$, and assume (p_n) to be an arbitrary sequence in $\mathcal{W}$ and define

$$\epsilon_n = \|p_{n+1} - f(\mathcal{H}, p_n)\|.$$

Then FP-iteration is $\mathcal{H}$-stable iff

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} p_n = \alpha.$$

Timiş and Berinde [30] introduced a weaker form of stability which they referred to as weak $\mathcal{H}$-stability or weak stability with respect to $\mathcal{H}$ and defined it as follows.

Definition 3.2 ([30])**.** For a mapping $\mathcal{H} : \mathcal{W} \to \mathcal{W}$, assume (α_n) to be the sequence generated by an iteration procedure defined as $\alpha_0 \in \mathcal{W}$ and $\alpha_{n+1} = f(\mathcal{H}, \alpha_n)$. Let $\alpha_n \to \alpha \in F(\mathcal{H})$ and $(p_n) \subseteq \mathcal{W}$ be an approximate sequence of (α_n) such that

$$\epsilon_n = \|p_{n+1} - f(\mathcal{H}, p_n)\|.$$

Then the iteration is weakly $\mathcal{H}$-stable iff

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} p_n = \alpha.$$

Remark 3.3. It can be easily observed from the above definitions that if an iteration is $\mathcal{H}$-stable, then it is weakly $\mathcal{H}$-stable. But the converse does not generally hold, as exemplified by Timiş and Berinde in [30, Example 3.3].

Timiş [29] further weakened the notion of weak $\mathcal{H}$-stability to introduce weak ω^2-stability with respect to $\mathcal{H}$ by replacing the approximate sequences with the more precise equivalent sequences and defined it as follows.

Definition 3.4 ([29])**.** For a mapping $\mathcal{H} : \mathcal{W} \to \mathcal{W}$, assume (α_n) to be the sequence generated by an iteration procedure defined as $\alpha_0 \in \mathcal{W}$ and $\alpha_{n+1} = f(\mathcal{H}, \alpha_n)$. Let $\alpha_n \to \alpha \in F(\mathcal{H})$ and $(p_n) \subseteq \mathcal{W}$ be an equivalent sequence of (α_n) such that

$$\epsilon_n = \|p_{n+1} - f(\mathcal{H}, p_n)\|.$$

Then the iteration is weak ω^2-stable with respect to $\mathcal{H}$ or also referred to as ω^2-stable iff

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} p_n = \alpha.$$

Remark 3.5. Since every equivalent sequence is an approximate sequence, weak $\mathcal{H}$-stability implies weak ω^2-stability with respect to $\mathcal{H}$. But the converse is not true in general as demonstrated by Timiş in [29, Example 3.1].

The notions of almost $\mathcal{H}$-stability and summable almost $\mathcal{H}$-stability have been proposed by Osilike[20] and Berinde[9], respectively. Now we define these definitions.

Definition 3.6 ([20])**.** If $\mathcal{H} : \mathcal{W} \to \mathcal{W}$ is a mapping and $\alpha_{n+1} = f(\mathcal{H}, \alpha_n)$ an iteration, such that $\alpha_n \to \alpha \in F(\mathcal{H})$, assume (p_n) to be an arbitrary sequence in $\mathcal{W}$ to define

$$\epsilon_n = \|p_{n+1} - f(\mathcal{H}, p_n)\|.$$

Then the iteration is almost $\mathcal{H}$-stable iff

$$\sum_{n=0}^{\infty} \epsilon_n < \infty \quad \Rightarrow \quad \lim_{n\to\infty} p_n = \alpha.$$

Definition 3.7 ([9])**.** If $\mathcal{H} : \mathcal{W} \to \mathcal{W}$ is a mapping and $\alpha_{n+1} = f(\mathcal{H}, \alpha_n)$ an iteration, such that $\alpha_n \to \alpha \in F(\mathcal{H})$, assume (p_n) to be an arbitrary sequence in $\mathcal{W}$ to define

$$\epsilon_n = \|p_{n+1} - f(\mathcal{H}, p_n)\|.$$

Then the iteration is summable almost $\mathcal{H}$-stable iff

$$\sum_{n=0}^{\infty} \epsilon_n < \infty \quad \Rightarrow \quad \sum_{n=0}^{\infty} \|p_n - \alpha\| < \infty.$$

Using the following example, we infer that a summable almost $\mathcal{H}$-stable iterative sequence may not be almost $\mathcal{H}$-stable in general.

Example 3.8. Let $\mathcal{B} = [0, 1]$ and $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ be a mapping defined as $\mathcal{H}(u) = \frac{u+1}{8}$, $u \in \mathcal{B}$, which is clearly a contraction and consequently a weak contraction with $\sigma = \frac{1}{8}$ and $K \geq 0$ and possesses a fixed point, say $p = \frac{1}{7}$. Now, let $\alpha_{n+1} = \mathcal{H}\alpha_n$ be the Picard iteration. By using MATLAB R2021a, we observe that $\alpha_n \to \frac{1}{7}$ as $n \to \infty$. We now assume an arbitrary sequence $p_n = \frac{1}{n^2}$ such that

$$\begin{aligned} \epsilon_n &= \|p_{n+1} - f(\mathcal{H}, p_n)\| \\ &= \left\| \frac{1}{(n+1)^2} - \mathcal{H}(p_n) \right\| \end{aligned}$$

$$= \left\| \frac{1}{(n+1)^2} - \frac{p_n + 1}{8} \right\|$$
$$= \left\| \frac{1}{(n+1)^2} - \frac{\frac{1}{n^2} + 1}{8} \right\|$$
$$= \left\| \frac{1}{(n+1)^2} - \frac{1}{8n^2} - \frac{1}{8} \right\|.$$

This implies

$$\begin{aligned} \sum_{n=1}^{\infty} \epsilon_n &\leq \left| \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{8n^2} - \sum_{n=1}^{\infty} \frac{1}{8} \right| \\ &< \left| \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \right| \\ &< \infty, \end{aligned}$$

but

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \neq \frac{1}{7} = p.$$

Therefore, Picard iteration is not almost stable with respect to $\mathcal{H}$. On the other hand, using [9, Theorem 1], we can conclude that the Picard iteration is summable almost $\mathcal{H}$-stable.

Remark 3.9.

(i) Since $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $\lim_{n\to\infty} \epsilon_n = 0$, then every $\mathcal{H}$-stable iteration is almost $\mathcal{H}$-stable, but the converse may not hold [4, Remark, Page 161] as shown by Osilike [21, Example 1].

(ii) Since $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $\lim_{n\to\infty} \epsilon_n = 0$ and the set of arbitrary sequences contains the set of approximate sequences, then every weakly $\mathcal{H}$-stable iteration is almost $\mathcal{H}$-stable [4].

(iii) Since $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $\lim_{n\to\infty} \epsilon_n = 0$ and the set of arbitrary sequences contains the set of equivalent sequences, then every ω^2-stable iteration with respect to $\mathcal{H}$ is almost $\mathcal{H}$-stable.

(iv) Since $\sum_{n=0}^{\infty} \|p_n - \alpha\| < \infty$ implies $\lim_{n\to\infty} p_n = \alpha$, then every almost $\mathcal{H}$-stable iteration is summable almost $\mathcal{H}$-stable. But the converse need not be true as shown in Example 3.8.

From the above Remarks 3.3, 3.5 and 3.9, we can sum up the relationships between the different types of stabilities for a particular iteration as follows:

$\mathcal{H}$-stable $\Rightarrow$ weakly $\mathcal{H}$-stable $\Rightarrow$ ω^2-stable $\Rightarrow$ almost $\mathcal{H}$-stable $\Rightarrow$ summable almost $\mathcal{H}$-stable.

Although ω^2-stability of HR^*-iteration with respect to any weak contraction $\mathcal{H}$ has been established in [10], according to Timiş [29], it does not necessarily imply its $\mathcal{H}$-stability. Therefore, in the following, we establish the $\mathcal{H}$-stability of HR^*-iteration, where $\mathcal{H}$ is a weak contraction satisfying (2.2). However, before that, we establish a preliminary lemma as follows.

Lemma 3.10. *Assume $\mathcal{B}$ to be a non-empty, convex and closed subset of an arbitrary Banach space $\mathcal{W}$. Further assume $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ to be a weak contraction which satisfies* (2.2)*. Then the sequence (α_n) generated by HR^*-iteration scheme* (2.3) *converges strongly to a unique fixed point of $\mathcal{H}$, say α.*

Proof. We have, using Theorem 2.2, for all $n \in \mathbb{N}$, triangle inequality and (2.3),

$$\begin{aligned} \|\mathcal{H}\alpha_n - \alpha\| &= \|\mathcal{H}\alpha_n - \mathcal{H}\alpha\| \\ &= \|\mathcal{H}\alpha - \mathcal{H}\alpha_n\| \\ &\leq \sigma \|\alpha_n - \alpha\| + K \|\mathcal{H}\alpha - \alpha\| \\ &= \sigma \|\alpha_n - \alpha\| . \end{aligned} \tag{3.1}$$

We now have, from iterative scheme (2.3), triangle inequality and (3.1)

$$\begin{aligned} \|\theta_n - \alpha\| &= \|(1 - c_n)\alpha_n + c_n\mathcal{H}\alpha_n - \alpha\| \\ &\leq (1 - c_n) \|\alpha_n - \alpha\| + c_n \|\mathcal{H}\alpha_n - \alpha\| \\ &\leq (1 - c_n) \|\alpha_n - \alpha\| + \sigma c_n \|\alpha_n - \alpha\| \\ &= (1 - (1 - \sigma)c_n) \|\alpha_n - \alpha\| . \end{aligned} \tag{3.2}$$

By employing (3.1), triangle inequality and finally (3.2), we have

$$\begin{aligned} \|\gamma_n - \alpha\| &= \|\mathcal{H}((1 - b_n)\theta_n + b_n\mathcal{H}\theta_n) - \alpha\| \\ &\leq \sigma \|(1 - b_n)\theta_n + b_n\mathcal{H}\theta_n - \alpha\| \\ &\leq \sigma(1 - b_n) \|\theta_n - \alpha\| + \sigma^2 b_n \|\theta_n - \alpha\| \\ &= \sigma(1 - (1 - \sigma)b_n) \|\theta_n - \alpha\| \\ &\leq \sigma(1 - (1 - \sigma)b_n)(1 - (1 - \sigma)c_n) \|\alpha_n - \alpha\| . \end{aligned} \tag{3.3}$$

Using (3.3), we use the same approach to derive

$$\begin{aligned} \|\beta_n - \alpha\| &= \|\mathcal{H}(\mathcal{H}\gamma_n) - \alpha\| \\ &\leq \sigma \|\mathcal{H}\gamma_n - \alpha\| \\ &\leq \sigma^2 \|\gamma_n - \alpha\| \\ &\leq \sigma^3(1 - (1 - \sigma)b_n)(1 - (1 - \sigma)c_n) \|\alpha_n - \alpha\| . \end{aligned} \tag{3.4}$$

We finally have, using the same approach and the previous inequalities

(3.5)
$$\|\alpha_{n+1}-\alpha\| \leq \sigma^3(1-(1-\sigma)a_n)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\|\alpha_n-\alpha\|.$$

Since $\sigma, a_n, b_n, c_n \in (0,1)$, we have for all control sequences $v_n \in \{a_n, b_n, c_n\}$, $(1-(1-\sigma)v_n) \leq 1$. Therefore (3.5) can be written as

$$\|\alpha_{n+1}-\alpha\| \leq \sigma^3\|\alpha_n-\alpha\|.$$

Inductively, we have

$$\|\alpha_{n+1}-\alpha\| \leq \sigma^{3(n+1)}\|\alpha_0-\alpha\|.$$

Since $\sigma \in (0,1)$, (α_n) converges strongly to α, which completes the proof. □

The following is our main result on $\mathcal{H}$-stability of the iteration process (2.3).

Theorem 3.11. *Assume $\mathcal{B}$ to be a non-empty, convex and closed subset of an arbitrary Banach space $\mathcal{W}$. Further assume $\mathcal{H}: \mathcal{B} \to \mathcal{B}$ to be a weak contraction which satisfies* (2.2)*. Then, HR^*-iteration scheme* (2.3) *is $\mathcal{H}$-stable.*

Proof. Let (s_n) be an arbitrary sequence in $\mathcal{B}$ and (α_n) be the sequence generated by (2.3). Then by Lemma 3.10, (α_n) strongly converges to a unique fixed point, say α, of $\mathcal{H}$. Assume further that $\epsilon_n = \|s_{n+1}-f(\mathcal{H},s_n)\|$, for all $n \in \mathbb{N}$. Let $\lim_{n\to\infty}\epsilon_n = 0$. Then by iteration process (2.3) and (3.1), we have

$$\begin{aligned}
&\|s_{n+1}-\alpha\| \\
&\quad\leq \|s_{n+1}-f(\mathcal{H},s_n)\| + \|f(\mathcal{H},s_n)-\alpha\| \\
&\quad= \epsilon_n + \|f(\mathcal{H},s_n)-\alpha\| \\
&\quad= \epsilon_n + \|(1-a_n)t_n + a_n\mathcal{H}t_n - \alpha\| \\
&\quad\leq \epsilon_n + (1-a_n)\|t_n-\alpha\| + a_n\|\mathcal{H}t_n-\alpha\| \\
&\quad= \epsilon_n + (1-(1-\sigma)a_n)\|t_n-\alpha\| \\
&\quad\leq \epsilon_n + (1-(1-\sigma)a_n)\|\mathcal{H}(\mathcal{H}r_n)-\alpha\| \\
&\quad\leq \epsilon_n + \sigma^2(1-(1-\sigma)a_n)\|r_n-\alpha\| \\
&\quad= \epsilon_n + \sigma^2(1-(1-\sigma)a_n)\|\mathcal{H}((1-b_n)l_n + b_n\mathcal{H}l_n)-\alpha\| \\
&\quad\leq \epsilon_n + \sigma^3(1-(1-\sigma)a_n)(1-(1-\sigma)b_n)\|l_n-\alpha\| \\
&\quad\leq \epsilon_n + \sigma^3(1-(1-\sigma)a_n)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\cdot\|s_n-\alpha\|.
\end{aligned}$$

Since $\sigma \in (0,1)$ and $(a_n),(b_n),(c_n) \in (0,1)$, then using Lemma 2.4, we have

$$\lim_{n\to\infty} \|s_n - \alpha\| = 0 \quad \text{or} \quad \lim_{n\to\infty} s_n = \alpha.$$

Conversely, by assuming $\lim_{n\to\infty} s_n = \alpha$ and using the calculations done above, we have

$$\begin{aligned}
\epsilon_n &= \|s_{n+1} - f(\mathcal{H}, s_n)\| \\
&\leq \|s_{n+1} - \alpha\| + \|f(\mathcal{H}, s_n) - \alpha\| \\
&\leq \|s_{n+1} - \alpha\| \\
&\quad + \sigma^3(1-(1-\sigma)a_n)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n) \cdot \|s_n - \alpha\|.
\end{aligned}$$

Taking limits as $n \to \infty$ both side, we have $\lim_{n\to\infty} \epsilon_n = 0$, thus completing the proof. □

The following example validates Theorem 3.11.

Example 3.12. Let $\mathcal{B} = [0,1]$ and $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ be a mapping defined as $\mathcal{H}(u) = \sinh(\frac{u}{7})$, $u \in \mathcal{B}$. Since $\sinh(u) \leq u$ for all $u \in [0,1]$, it can be easily shown that $\mathcal{H}$ is a weak contraction that satisfies (2.2) with $\sigma = \frac{1}{7}$ and any $K \geq 0$. Also, zero is the unique fixed point of mapping $\mathcal{H}$. Now, we first exemplify the convergence of the sequence (α_n) generated by iteration (2.3) in context of the above defined mapping $\mathcal{H}$, to lay the framework for validating Theorem 3.11 and incidentally validate Lemma 3.10. By taking $(a_n) = (b_n) = (c_n) = \left(\frac{1}{n+2}\right)$, we have

$$\theta_n = \left(1 - \frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)$$

and

(3.6)

$$\begin{aligned}
\gamma_n &= \mathcal{H}\left(\left(1-\frac{1}{n+2}\right)\left(\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)\right)\right. \\
&\quad \left. + \frac{1}{n+2}\sinh\left(\frac{\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)}{7}\right)\right) \\
&= \sinh\left(\left(\frac{1}{7}\right)\left(\left(1-\frac{1}{n+2}\right)\left(\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)\right)\right.\right. \\
&\quad \left.\left. + \frac{1}{n+2}\sinh\left(\frac{\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)}{7}\right)\right)\right).
\end{aligned}$$

For convenience, we set

(3.7)

$$C = \left(\frac{1}{7}\right)\left(\left(1-\frac{1}{n+2}\right)\left(\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)\right)\right.$$
$$\left. + \frac{1}{n+2}\sinh\left(\frac{\left(1-\frac{1}{n+2}\right)\alpha_n + \frac{1}{n+2}\sinh\left(\frac{\alpha_n}{7}\right)}{7}\right)\right).$$

Using (3.7) and the fact that $\sinh(u) \leq u$, where u is positive, (3.6) can be rewritten as follows

$$\begin{aligned}\gamma_n &= \sinh(C),\\ \beta_n &= \mathcal{H}(\mathcal{H}\gamma_n)\\ &= \sinh\left(\frac{\sinh\left(\frac{\sinh(C)}{7}\right)}{7}\right),\end{aligned}$$

and

$$\begin{aligned}\alpha_{n+1} &= \left(1-\frac{1}{n+2}\right)\sinh\left(\frac{\sinh\left(\frac{\sinh(C)}{7}\right)}{7}\right)\\ &\quad + \frac{1}{n+2}\sinh\left(\left(\frac{1}{7}\right)\sinh\left(\frac{\sinh\left(\frac{\sinh(C)}{7}\right)}{7}\right)\right)\\ &\leq \left(1-\frac{1}{n+2}\right)\frac{C}{7^2} + \left(\frac{1}{n+2}\right)\frac{C}{7^3}\\ &= \frac{1}{7^2}\left(1-\frac{6}{7(n+2)}\right)C.\end{aligned} \tag{3.8}$$

Therefore, we have from (3.7) that

$$\begin{aligned}C &\leq \frac{1}{7}\left(1-\frac{1}{n+2}\right)^2\alpha_n + \frac{1}{7^2}\left(\frac{1}{n+2}\right)\left(1-\frac{1}{n+2}\right)\alpha_n\\ &\quad + \frac{1}{7^2}\left(1-\frac{1}{n+2}\right)\left(\frac{1}{n+2}\right)\alpha_n + \frac{1}{7^3}\frac{1}{(n+2)^2}\alpha_n\\ &= \frac{1}{7}\left(1+\frac{1}{(n+2)^2}-\frac{2}{n+2}+\frac{2}{7(n+2)}-\frac{2}{7(n+2)^2}+\frac{1}{7^2(n+2)^2}\right)\alpha_n\\ &= \left(\frac{1}{7}-\frac{12}{7^2(n+2)}+\frac{36}{7^3(n+2)^2}\right)\alpha_n.\end{aligned}$$

Therefore, from (3.8), we obtain

$$\begin{aligned}
\alpha_{n+1} &\leq \frac{1}{7^2}\left(1-\frac{6}{7(n+2)}\right)\left(\frac{1}{7}-\frac{12}{7^2(n+2)}+\frac{36}{7^3(n+2)^2}\right)\alpha_n \\
&= \frac{1}{7^2}\left(\frac{1}{7}-\frac{12}{7^2(n+2)}+\frac{36}{7^3(n+2)^2}-\frac{6}{7^2(n+2)}+\frac{72}{7^3(n+2)^2}\right. \\
&\quad \left.-\frac{216}{7^4(n+2)^3}\right)\alpha_n \\
&= \left(\frac{1}{7^3}-\frac{18}{7^4(n+2)}+\frac{108}{7^5(n+2)^2}-\frac{216}{7^6(n+2)^3}\right)\alpha_n \\
&= \left(1-\left(\frac{342}{343}+\frac{18}{7^4(n+2)}-\frac{108}{7^5(n+2)^2}+\frac{216}{7^6(n+2)^3}\right)\right)\alpha_n.
\end{aligned}$$

By taking $\mu_n = \left(\frac{342}{343}+\frac{18}{7^4(n+2)}-\frac{108}{7^5(n+2)^2}+\frac{216}{7^6(n+2)^3}\right)$, we observe that $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty}\mu_n = \infty$. Therefore the prerequisites of Lemma 2.5 are fulfilled and we have $\lim_{n\to\infty}\alpha_{n+1} = 0$.

We now proceed to showcase $\mathcal{H}$-stability of HR^*-iteration by a non-trivial example. We prove the converse first, by letting $s_n = \frac{1}{n+4} \to 0$ as $n \to \infty$. By assumption, we have

$$\epsilon_n = \|s_{n+1} - f(\mathcal{H}, s_n)\|. \tag{3.9}$$

Since the generated sequence is (s_n), therefore, we write the intermediate sequences involved in (2.3) as (t_n), (r_n) and (l_n) to acquire an iteration analogous to (2.3). Then, the substitution of terms involved in the acquired analogous iteration leads to

(3.10)

$$\begin{aligned}
&f(\mathcal{H}, s_n) \\
&= \left(1-\frac{1}{n+2}\right)t_n + \left(\frac{1}{n+2}\right)\mathcal{H}t_n \\
&= \left(1-\frac{1}{n+2}\right)\mathcal{H}(\mathcal{H}r_n) + \left(\frac{1}{n+2}\right)\mathcal{H}(\mathcal{H}(\mathcal{H}r_n)) \\
&= \left(1-\frac{1}{n+2}\right)\mathcal{H}\left(\mathcal{H}\left(\mathcal{H}\left(\left(1-\frac{1}{n+2}\right)l_n + \left(\frac{1}{n+2}\right)\mathcal{H}l_n\right)\right)\right) \\
&\quad + \left(\frac{1}{n+2}\right)\mathcal{H}\left(\mathcal{H}\left(\mathcal{H}\left(\mathcal{H}\left(\left(1-\frac{1}{n+2}\right)l_n + \left(\frac{1}{n+2}\right)\mathcal{H}l_n\right)\right)\right)\right) \\
&= \left(1-\frac{1}{n+2}\right)\mathcal{H}\left(\mathcal{H}\left(\mathcal{H}\left(\left(1-\frac{1}{n+2}\right)\left(\left(1-\frac{1}{n+2}\right)s_n\right.\right.\right.\right.
\end{aligned}$$

$$+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\Big)+\left(\frac{1}{n+2}\right)\mathcal{H}\Big(\Big(1-\frac{1}{n+2}\Big)s_n$$
$$+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\Big)\Big)\Big)\Big)$$
$$+\left(\frac{1}{n+2}\right)\mathcal{H}\Big(\mathcal{H}\Big(\mathcal{H}\Big(\mathcal{H}\Big(\Big(1-\frac{1}{n+2}\Big)\Big(\Big(1-\frac{1}{n+2}\Big)s_n$$
$$+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\Big)+\left(\frac{1}{n+2}\right)\mathcal{H}\Big(\Big(1-\frac{1}{n+2}\Big)s_n$$
$$+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\Big)\Big)\Big)\Big)\Big).$$

For convenience, we define

$$D=\left(1-\frac{1}{n+2}\right)\left(\left(1-\frac{1}{n+2}\right)s_n+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\right)+\left(\frac{1}{n+2}\right)\mathcal{H}\left(\left(1-\frac{1}{n+2}\right)s_n+\left(\frac{1}{n+2}\right)\mathcal{H}s_n\right)$$
$$=\left(\frac{n+1}{n+2}\right)^2 s_n+\left(\frac{n+1}{(n+2)^2}\right)\sinh\left(\frac{s_n}{7}\right)+\left(\frac{1}{n+2}\right)\sinh\left(\frac{\left(\frac{n+1}{n+2}\right)s_n+\left(\frac{1}{n+2}\right)\sinh\left(\frac{s_n}{7}\right)}{7}\right). \tag{3.11}$$

Therefore using (3.11) and (3.10), we rewrite (3.9) as

$$\epsilon_n=\left\|\frac{1}{n+5}-\left(1-\frac{1}{n+2}\right)(\mathcal{H}(\mathcal{H}(\mathcal{H}(D))))+\left(\frac{1}{n+2}\right)(\mathcal{H}(\mathcal{H}(\mathcal{H}(\mathcal{H}(D)))))\right\|. \tag{3.12}$$

Now, since $s_n=\frac{1}{n+4}$, we substitute it in (3.11) and by taking limits as $n\to\infty$, we have from (3.12) that $\lim\limits_{n\to\infty}\epsilon_n=0$. To prove the forward implication, we assume $\lim\limits_{n\to\infty}\epsilon_n=0$ for an arbitrary sequence (s_n), to obtain

$$\begin{aligned}\lim_{n\to\infty}\epsilon_n&=\lim_{n\to\infty}\|s_{n+1}-f(\mathcal{H},s_n)\|\\ \Rightarrow\quad 0&=\lim_{n\to\infty}\|s_{n+1}-f(\mathcal{H},s_n)\|\\ \Rightarrow\quad \lim_{n\to\infty}s_{n+1}&=\lim_{n\to\infty}f(\mathcal{H},s_n).\end{aligned} \tag{3.13}$$

Now, from (3.11), we have

$$\begin{aligned} D &= \left(\frac{n+1}{n+2}\right)^2 s_n + \left(\frac{n+1}{(n+2)^2}\right)\sinh\left(\frac{s_n}{7}\right) \\ &\quad + \left(\frac{1}{n+2}\right)\sinh\left(\frac{\left(\frac{n+1}{n+2}\right)s_n + \left(\frac{1}{n+2}\right)\sinh\left(\frac{s_n}{7}\right)}{7}\right) \\ &\leq \left(\frac{n+1}{n+2}\right)^2 s_n + \left(\frac{n+1}{(n+2)^2}\right)\left(\frac{s_n}{7}\right) + \left(\frac{1}{n+2}\right)\frac{1}{7}\left(\left(\frac{n+1}{n+2}\right)s_n \right. \\ &\quad \left. + \left(\frac{1}{n+2}\right)\left(\frac{s_n}{7}\right)\right). \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} D \leq \lim_{n\to\infty} s_n.$$

By utilizing it, we have, from (3.10)

$$\begin{aligned} \lim_{n\to\infty} f(\mathcal{H}, s_n) &\leq \lim_{n\to\infty}\left(\frac{1}{7^3}s_n + \frac{1}{7^4}s_n\right) \\ &= \lim_{n\to\infty}\frac{8}{7^4}s_n. \end{aligned}$$

Therefore, from (3.13), we have

$$\lim_{n\to\infty} s_{n+1} \leq \lim_{n\to\infty}\left(\frac{8}{7^4}\right)s_n.$$

This implies that $\lim_{n\to\infty} s_n = 0$.

4. Data Dependence

In this section, we procure a data dependence result of HR^*-iteration process.

Theorem 4.1. *Assume $\mathcal{H}$ to be a weak contraction satisfying* (2.2) *and $\mathcal{R}$ to be an approximate operator of $\mathcal{H}$. Further, consider (α_n) be a sequence defined by algorithm* (2.3) *for $\mathcal{H}$. Now, define an analogously generated sequence (ω_n) for $\mathcal{R}$ as follows*

$$\begin{aligned} \omega_{n+1} &= (1-a_n)\nu_n + a_n\mathcal{R}\nu_n \\ \nu_n &= \mathcal{R}(\mathcal{R}\kappa_n) \\ \kappa_n &= \mathcal{R}((1-b_n)\zeta_n + b_n\mathcal{R}\zeta_n) \\ \zeta_n &= (1-c_n)\omega_n + c_n\mathcal{R}\omega_n, \end{aligned} \tag{4.1}$$

where (a_n), (b_n) and (c_n) are sequences in $(0,1)$, for all $n \in \mathbb{N}$. If $\mathcal{H}\alpha = \alpha$ and $\mathcal{R}\omega = \omega$ such that $\omega_n \to \omega$ as $n \to \infty$, then we have

$$\|\alpha - \omega\| \leq \left(\frac{2+4\sigma}{1-\sigma}\right)\epsilon.$$

Proof. By using (2.3), (2.2), (4.1) and Definition 2.3, we have

(4.2)
$$\begin{aligned}
\|\theta_n - \zeta_n\| &= \|(1-c_n)\alpha_n + c_n\mathcal{H}\alpha_n - (1-c_n)\omega_n - c_n\mathcal{R}\omega_n\| \\
&\leq (1-c_n)\|\alpha_n - \omega_n\| + c_n\|\mathcal{H}\alpha_n - \mathcal{R}\omega_n\| \\
&= (1-c_n)\|\alpha_n - \omega_n\| + c_n\|\mathcal{H}\alpha_n - \mathcal{H}\omega_n + \mathcal{H}\omega_n - \mathcal{R}\omega_n\| \\
&\leq (1-c_n)\|\alpha_n - \omega_n\| + c_n\|\mathcal{H}\alpha_n - \mathcal{H}\omega_n\| + c_n\epsilon \\
&\leq (1-c_n)\|\alpha_n - \omega_n\| + \sigma c_n\|\alpha_n - \omega_n\| + c_nK\|\alpha_n - \mathcal{H}\alpha_n\| + c_n\epsilon \\
&= (1-(1-\sigma)c_n)\|\alpha_n - \omega_n\| + c_nK\|\alpha_n - \mathcal{H}\alpha_n\| + c_n\epsilon.
\end{aligned}$$

Using the same set of conditions along with (3.2), we have

(4.3)
$$\begin{aligned}
\|\gamma_n - \kappa_n\| &= \|\mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n) - \mathcal{R}((1-b_n)\zeta_n + b_n\mathcal{R}\zeta_n)\| \\
&\leq \|\mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n) - \mathcal{H}((1-b_n)\zeta_n + b_n\mathcal{R}\zeta_n)\| \\
&\quad + \|\mathcal{H}((1-b_n)\zeta_n + b_n\mathcal{R}\zeta_n) - \mathcal{R}((1-b_n)\zeta_n + b_n\mathcal{R}\zeta_n)\| \\
&\leq \sigma((1-b_n)\|\theta_n - \zeta_n\| + b_n\|\mathcal{H}\theta_n - \mathcal{R}\zeta_n\|) \\
&\quad + K\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n)\| + \epsilon.
\end{aligned}$$

By evaluating the two terms, respectively, in (4.3) by using (2.2), (3.1) and (3.2), we obtain

$$\begin{aligned}
&\sigma((1-b_n)\|\theta_n - \zeta_n\| + b_n\|\mathcal{H}\theta_n - \mathcal{R}\zeta_n\|) \\
&\quad \leq \sigma((1-b_n)\|\theta_n - \zeta_n\| + b_n\|\mathcal{H}\theta_n - \mathcal{H}\zeta_n\| + b_n\|\mathcal{H}\zeta_n - \mathcal{R}\zeta_n\|) \\
&\quad \leq \sigma((1-b_n)\|\theta_n - \zeta_n\| + b_n\sigma\|\theta_n - \zeta_n\| + b_nK\|\theta_n - \mathcal{H}\theta_n\| + b_n\epsilon) \\
&\quad = \sigma((1-(1-\sigma)b_n)\|\theta_n - \zeta_n\| + b_nK\|\theta_n - \mathcal{H}\theta_n\| + b_n\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
&K\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n)\| \\
&\quad = K\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \alpha - \mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n) + \alpha\| \\
&\quad \leq K\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \alpha\| + K\|\mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n) - \alpha\| \\
&\quad \leq K\|(1-b_n)\theta_n + b_n\alpha - b_n\alpha + b_n\mathcal{H}\theta_n - \alpha\| \\
&\qquad + \sigma K\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \alpha\| \\
&\quad \leq K((1-b_n)\|\theta_n - \alpha\| + b_n\|\mathcal{H}\theta_n - \alpha\|) \\
&\qquad + \sigma K((1-b_n)\|\theta_n - \alpha\| + b_n\|\mathcal{H}\theta_n - \alpha\|) \\
&\quad \leq K((1-b_n)\|\theta_n - \alpha\| + \sigma b_n\|\theta_n - \alpha\|)
\end{aligned}$$

$$
\begin{aligned}
&+ \sigma K((1-b_n)\left\|\theta_n-\alpha\right\| + \sigma b_n\left\|\theta_n-\alpha\right\|)\\
&= K(1-(1-\sigma)b_n)(1+\sigma)\left\|\theta_n-\alpha\right\|\\
&\leq K(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)(1+\sigma)\left\|\alpha_n-\alpha\right\|.
\end{aligned}
$$

Therefore by substituting (4.2), (4.3) can be written as

(4.4)

$$
\begin{aligned}
\left\|\gamma_n-\kappa_n\right\| &\leq \sigma((1-(1-\sigma)b_n)\left\|\theta_n-\zeta_n\right\| + b_nK\left\|\theta_n-\mathcal{H}\theta_n\right\| + b_n\epsilon)\\
&\quad + K(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\left\|\alpha_n-\alpha\right\|(1+\sigma)+\epsilon\\
&\leq \sigma((1-(1-\sigma)b_n)\left((1-(1-\sigma)c_n)\right\|\alpha_n-\omega_n\right\|\\
&\quad + c_nK\left\|\alpha_n-\mathcal{H}\alpha_n\right\| + c_n\epsilon) + b_nK\left\|\theta_n-\mathcal{H}\theta_n\right\| + b_n\epsilon)\\
&\quad + K(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\left\|\alpha_n-\alpha\right\|(1+\sigma)+\epsilon.
\end{aligned}
$$

We also have, using the priorly used set of conditions together with (3.3) and substituting (4.4)

(4.5)

$$
\begin{aligned}
\left\|\beta_n-\nu_n\right\| &= \left\|\mathcal{H}(\mathcal{H})\gamma_n-\mathcal{R}(\mathcal{R}\kappa_n)\right\|\\
&\leq \left\|\mathcal{H}(\mathcal{H}\gamma_n)-\mathcal{H}(\mathcal{R}\kappa_n)\right\| + \left\|\mathcal{H}(\mathcal{R}\kappa_n)-\mathcal{R}(\mathcal{R}\kappa_n)\right\|\\
&\leq \sigma\left\|\mathcal{H}\gamma_n-\mathcal{R}\kappa_n\right\| + K\left\|\mathcal{H}\gamma_n-\mathcal{H}(\mathcal{H}\gamma_n)\right\| + \epsilon\\
&\leq \sigma\left\|\mathcal{H}\gamma_n-\mathcal{H}\kappa_n\right\| + \sigma\left\|\mathcal{H}\kappa_n-\mathcal{R}\kappa_n\right\| + K\left\|\mathcal{H}\gamma_n-\alpha\right\|\\
&\quad + K\left\|\mathcal{H}(\mathcal{H}\gamma_n)-\alpha\right\| + \epsilon\\
&\leq \sigma^2\left\|\gamma_n-\kappa_n\right\| + \sigma K\left\|\gamma_n-\mathcal{H}\gamma_n\right\| + \sigma\epsilon + K\sigma\left\|\gamma_n-\alpha\right\|\\
&\quad + K\sigma^2\left\|\gamma_n-\alpha\right\| + \epsilon\\
&\leq \sigma^2\left\|\gamma_n-\kappa_n\right\| + \sigma K\left\|\gamma_n-\mathcal{H}\gamma_n\right\| + \sigma\epsilon\\
&\quad + K\sigma^2(1+\sigma)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\left\|\alpha_n-\alpha\right\| + \epsilon\\
&\leq \sigma^2[\sigma((1-(1-\sigma)b_n)\left((1-(1-\sigma)c_n)\right\|\alpha_n-\omega_n\right\|\\
&\quad + c_nK\left\|\alpha_n-\mathcal{H}\alpha_n\right\| + c_n\epsilon) + b_nK\left\|\theta_n-\mathcal{H}\theta_n\right\| + b_n\epsilon)\\
&\quad + K(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\left\|\alpha_n-\alpha\right\|(1+\sigma)+\epsilon]\\
&\quad + \sigma K\left\|\gamma_n-\mathcal{H}\gamma_n\right\| + \sigma\epsilon\\
&\quad + K\sigma^2(1+\sigma)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n)\left\|\alpha_n-\alpha\right\| + \epsilon.
\end{aligned}
$$

Due to structural similarity with (4.2), we can write directly

(4.6)

$$
\left\|\alpha_{n+1}-\omega_{n+1}\right\| \leq (1-(1-\sigma)a_n)\left\|\beta_n-\nu_n\right\| + a_nK\left\|\beta_n-\mathcal{H}\beta_n\right\| + a_n\epsilon.
$$

By substitution of (4.5), we rewrite (4.6) as

$$
\begin{aligned}
&\left\|\alpha_{n+1}-\omega_{n+1}\right\|\\
&\quad \leq (1-(1-\sigma)a_n)\left[\sigma^2[\sigma((1-(1-\sigma)b_n)\left((1-(1-\sigma)c_n\right)\right.\right.
\end{aligned}
$$

$$\cdot \|\alpha_n - \omega_n\| + c_n K \|\alpha_n - \mathcal{H}\alpha_n\| + c_n\epsilon) + b_n K \|\theta_n - \mathcal{H}\theta_n\|$$
$$+b_n\epsilon) + K(1-(1-\sigma)b_n)(1-(1-\sigma)c_n) \|\alpha_n - \alpha\| (1+\sigma) + \epsilon]$$
$$+\sigma K \|\gamma_n - \mathcal{H}\gamma_n\| + \sigma\epsilon + K\sigma^2(1+\sigma)(1-(1-\sigma)b_n)$$
$$\cdot(1-(1-\sigma)c_n) \|\alpha_n - \alpha\| + \epsilon] + a_n K \|\beta_n - \mathcal{H}\beta_n\| + a_n\epsilon.$$

Since $\sigma, a_n, b_n, c_n \in (0,1)$, by substituting the inequalities $\sigma^2 < \sigma$, $\sigma^3 < \sigma$, $(1-\sigma)v_n \leq v_n$ and $(1-(1-\sigma)v_n) \leq 1$, we have

(4.7)
$$\begin{aligned}\|\alpha_{n+1} - \omega_{n+1}\| \leq{}& \sigma \|\alpha_n - \omega_n\| + \sigma K \|\alpha_n - \mathcal{H}\alpha_n\| + \sigma K \|\theta_n - \mathcal{H}\theta_n\| \\ &+ \sigma K(1+\sigma) \|\alpha_n - \alpha\| + \sigma K \|\gamma_n - \mathcal{H}\gamma_n\| \\ &+ \sigma K(1+\sigma) \|\alpha_n - \alpha\| + K \|\beta_n - \mathcal{H}\beta_n\| + (2+4\sigma)\epsilon.\end{aligned}$$

From triangle inequality and (3.1), we have

$$\begin{aligned}\|\alpha_n - \mathcal{H}\alpha_n\| &\leq \|\alpha_n - \alpha\| + \|\alpha - \mathcal{H}\alpha_n\| \\ &\leq (1+\sigma) \|\alpha_n - \alpha\|.\end{aligned}$$

Similarly, we have, by using triangle inequality along with (3.2), (3.3) and (3.4) that

$$\|\theta_n - \mathcal{H}\theta_n\| \leq (1+\sigma)(1-(1-\sigma)c_n) \|\alpha_n - \alpha\|,$$
$$\|\gamma_n - \mathcal{H}\gamma_n\| \leq \sigma(1+\sigma)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n) \|\alpha_n - \alpha\|,$$
$$\|\beta_n - \mathcal{H}\beta_n\| \leq \sigma^3(1+\sigma)(1-(1-\sigma)b_n)(1-(1-\sigma)c_n) \|\alpha_n - \alpha\|.$$

As Lemma 3.10 establishes the strong convergence of the sequence (α_n), constructed by (2.3), we have $\lim_{n\to\infty} \alpha_{n+1} = \lim_{n\to\infty} \alpha_n = \alpha$. Also from the assumption, we have $\lim_{n\to\infty} \omega_{n+1} = \lim_{n\to\infty} \omega_n = \omega$. Finally, the substitution of values following (4.7) in (4.7) and applying limits to the expression ultimately yields

$$\begin{aligned}\lim_{n\to\infty} \|\alpha_{n+1} - \omega_{n+1}\| &\leq (2+4\sigma)\epsilon + \sigma \lim_{n\to\infty} \|\alpha_n - \omega_n\| \\ &= (2+4\sigma)\epsilon + \sigma \|\alpha - \omega\|\end{aligned}$$

then

$$\|\alpha - \omega\| \leq \left(\frac{2+4\sigma}{1-\sigma}\right)\epsilon. \qquad \square$$

Remark 4.2.
(i) For the proof of Theorem 4.1, the condition $\alpha_n, \beta_n, \gamma_n, \zeta_n \geq \frac{1}{2}$ is dropped to avoid redundance and secure a more sharper and refined estimate.
(ii) For an additional imposition $\lim_{n\to\infty} a_n = 0$ on (a_n) in Theorem 4.1, we attain a finer estimate as follows

$$\|\alpha - \omega\| \leq \frac{(1+4\delta)\epsilon}{1-\delta}.$$

The following is an example to showcase the validity of Theorem 4.1.

Example 4.3. Consider the interval $\mathcal{B} = [-1, 1]$, which is a subset of the Banach space $\mathcal{W} = \mathbb{R}$. Further, we define an operator $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{H}(a) = \begin{cases} \frac{1}{5}\tanh(\frac{a}{5}), & -1 \leq a < 0, \\ -\frac{1}{5}\tanh(\frac{a}{5}), & 0 \leq a \leq 1. \end{cases}$$

To establish that $\mathcal{H}$ is a weak contraction satisfying (2.2), it is sufficient to prove it to be a contraction, which can be easily checked to be so with $\sigma \in [\frac{1}{25}, 1)$ and a unique fixed point $\alpha = 0$. For the sake of completeness, we provide the proof as follows:
Case I: *If both a and b are in $[-1, 0)$.* Since $\tanh(a) \geq a$, then $\tanh(a) \leq -a$, where a is negative. Therefore, we have

$$\begin{aligned} \|\mathcal{H}(a) - \mathcal{H}(b)\| &= \frac{1}{5}\left\|\tanh\left(\frac{a}{5}\right) - \tanh\left(\frac{b}{5}\right)\right\| \\ &\leq \frac{1}{5}\left\|-\frac{a}{5} - \left(-\frac{b}{5}\right)\right\| \\ &= \frac{1}{25}\|a - b\|. \end{aligned}$$

Case II: *If both a and b are in $[0, 1]$.* Since $\tanh(a) \leq a$, where a is positive, then we have

$$\begin{aligned} \|\mathcal{H}(a) - \mathcal{H}(b)\| &= \frac{1}{5}\left\|-\tanh\left(\frac{a}{5}\right) - \left(-\tanh\left(\frac{b}{5}\right)\right)\right\| \\ &\leq \frac{1}{5}\left\|-\frac{a}{5} + \frac{b}{5}\right\| \\ &= \frac{1}{25}\|a - b\|. \end{aligned}$$

Case III: *If $a \in [-1, 0)$ and $b \in [0, 1]$.* Using properties of tanh mentioned in the above two cases, we have

$$\begin{aligned} \|\mathcal{H}(a) - \mathcal{H}(b)\| &= \frac{1}{5}\left\|\tanh\left(\frac{a}{5}\right) - \left(-\tanh\left(\frac{b}{5}\right)\right)\right\| \\ &\leq \frac{1}{5}\left\|-\frac{a}{5} + \frac{b}{5}\right\| \\ &= \frac{1}{25}\|a - b\|. \end{aligned}$$

TABLE 1. Approximated fixed point of $\mathcal{R}$ by using the iterative algorithm (4.8)

No. of iterations	Iterative algorithm (4.8)
1	-0.00201
2	-0.00200
3	-0.00200

We now define an operator $\mathcal{R} : \mathcal{B} \to \mathcal{B}$ as

$$\mathcal{R}(a) = \begin{cases} 0.04(a-0.09) + 0.19(a+0.2)^3 - 0.019(a-0.4)^5 \\ -0.0001(a+0.3)^7, & \text{if } -1 \leq a < 0, \\ -0.04a - 0.001(a-0.3)^3 - 0.0001(a+0.2)^5 \\ +0.01(a-0.6)^7, & \text{if } 0 \leq a \leq 1. \end{cases}$$

By employing MATLAB 2021a software, we get

$$\max_{a \in \mathcal{B}} \|\mathcal{H}(a) - \mathcal{R}(a)\| = 0.00319,$$

whence we have that, for all $a \in \mathcal{B}$ and a fixed $\epsilon = 0.00319 > 0$

$$\|\mathcal{H}(a) - \mathcal{R}(a)\| \leq 0.00319 = \epsilon.$$

Therefore, according to Definition 2.3, $\mathcal{R}$ is an approximate operator of $\mathcal{H}$. Additionally, $\omega = -0.002$ is the unique fixed point of $\mathcal{R}$ in $\mathcal{B}$ implying that $\|\alpha - \omega\| = 0.002$. If

$$\mathcal{R}(a) = 0.04(a-0.09) + 0.19(a+0.2)^3 - 0.019(a-0.4)^5 - 0.0001(a+0.3)^7,$$

and by choosing the control sequences (a_n), (b_n) and (c_n) respectively as 0.15, 0.45 and 0.93, then from (4.1), we have

(4.8)

$$\begin{cases} \omega_0 \in \mathcal{B} \\ \zeta_n = 0.07\omega_n + 0.93\left(0.04(a-0.09) + 0.19(a+0.2)^3 - 0.019(a-0.4)^5\right. \\ \left. -0.0001(a+0.3)^7\right) \\ \kappa_n = \mathcal{R}\left((1-0.45)\zeta_n + 0.45\mathcal{R}(\zeta_n)\right) \\ \nu_n = \mathcal{R}(\mathcal{R}(\kappa_n)) \\ \omega_{n+1} = (1-0.15)\nu_n + 0.15\mathcal{R}(\nu_n). \end{cases}$$

Using MATLAB 2021a, we observe that the sequence (ω_n) defined by (4.8) is converging to the fixed point -0.00200, as shown in Table 1. Now, from Theorem 4.1, we calculate the following estimate

$$\|\alpha - \omega\| \leq 0.00319\left(\frac{2 + 4 \times 0.04}{1 - 0.04}\right) = 0.00717.$$

5. Application to a Generalized Functional Equation

Functional equations arise in mathematical biological models to study the behaviour of a predator which feeds on two types of preys [5, 14, 25] as well as in mathematical psychology pertaining to the theory of learning, see [31, 32] and references therein. The functional equation which is used as a predator-prey model, with the predator feeding on categories of prey that amount to two, is given as

$$\xi(u) = u(\xi(1-\delta)u + \delta) + (1-u)\xi((1-\mu)u), \quad u \in [0,1], \tag{5.1}$$

where $0 < \delta \leq \mu < 1$. In (5.1), the solution is the ultimate probability ξ, which represents the chances of the event occurring when the predator targets a specific prey category. Notably, the initial probability associated with selecting this particular category is equal to u.

Over the years, for establishing the solution to (5.1) or similar functional equations using a fixed point approach, numerous attempts have been made, some of which are in [5, 7, 14, 25, 31, 32]. For the existence of the solution of (5.1) along with conditions $\xi(0) = 0$ and $\xi(1) = 1$, Istrăţescu, in [14], established a result using Banach contraction principle. It is important to note that Istrăţescu [14] allowed the construction of solution to (5.1) iteratively, which is neither in closed form nor for all values of δ. [14] further exposes the functions $p, q : [0,1] \to [0,1]$ to be $p(u) = (1-\delta)u + \delta$ and $q(u) = (1-\mu)u$, respectively, where $u \in [0,1]$ and are linear (contractions) that satisfy $p(1) = 1$ and $q(0) = 0$. Withstanding this, Berinde[5] considered the following functional equation, which generalizes (5.1) and is given as

$$\xi(u) = u\xi(p(u)) + (1-u)\xi(q(u)), \tag{5.2}$$

where $p, q : [0,1] \to [0,1]$ are contractions so that $p(1) = 1$ and $q(0) = 0$. Consider a set $\mathcal{W} = \{\xi \in C[0,1] : \xi(0) = 0, \xi(1) = 1\}$. Then the mapping

$$\|\xi\| = \sup_{h \neq k} \frac{|\xi(h) - \xi(k)|}{|h-k|}, \quad \xi \in \mathcal{W}, \tag{5.3}$$

is a norm on $\mathcal{W}$ and $(\mathcal{W}, \|.\|)$ is a Banach space. The following result has been established by Berinde and Khan [5] for the previously mentioned functional equation.

Theorem 5.1 ([5]). *Let p and q be contractions on $[0,1]$ (endowed with usual norm), such that the respective contraction coefficients, μ and δ, satisfy $\mu, \delta \in (0,1)$, $\mu \leq \delta$ and $2\mu < 1$. Then the functional equation (5.2) admits a unique solution $\bar{\xi}$ in $\mathcal{W}$ and the successively approximating sequence (ξ_n), defined by*

$$\xi_{n+1}(u) = u\xi_n(p(u)) + (1-u)\xi_n(q(u)), \quad u \in [0,1], n \geq 0$$

strongly converges to $\bar{\xi}$*, as* $n \to \infty$*, for any* $\xi_0 \in \mathcal{W}$*.*

Şahin [25] utilized this result to establish a similar result for the two step Picard-Krasnoselskii hybrid iteration process. We now establish the validity of this result for HR^*-iteration as follows.

Theorem 5.2. *Considering the same set of assumptions as in Theorem 5.1, the functional equation* (5.2) *possesses a unique solution* $\bar{\xi}$ *in* $\mathcal{W}$ *and the iterative sequence* (α_n) *generated by* (2.3) *strongly converges to* $\bar{\xi}$*.*

Proof. Assume (α_n) to be a sequence defined by (2.3) for the linear operator $\mathcal{H} : \mathcal{W} \to \mathcal{W}$ defined by

$$(\mathcal{H}\xi)(u) = u\xi(p(u)) + (1-u)\xi(q(u)), \quad u \in [0,1].$$

We claim that $\alpha_n \to \bar{\xi}$ as $n \to \infty$. Using the linearity of $\mathcal{H}$, (2.3) and (5.3), we have

$$\begin{aligned} \left\|\alpha_{n+1} - \bar{\xi}\right\| &= \left\|(1-a_n)\beta_n + a_n\mathcal{H}\beta_n - \bar{\xi}\right\| \\ &\leq (1-a_n)\left\|\beta_n - \bar{\xi}\right\| + a_n\left\|\mathcal{H}(\beta_n - \bar{\xi})\right\|. \end{aligned}$$

Similarly

(5.4)

$$\begin{aligned} \left\|\theta_n - \bar{\xi}\right\| &= \left\|(1-c_n)\alpha_n + c_n\mathcal{H}\alpha_n - \bar{\xi}\right\| \\ &\leq (1-c_n)\left\|\alpha_n - \bar{\xi}\right\| + c_n\left\|\mathcal{H}\alpha_n - \bar{\xi}\right\| \\ &= (1-c_n)\left\|\alpha_n - \bar{\xi}\right\| + c_n\left\|\mathcal{H}\left(\alpha_n - \bar{\xi}\right)\right\| \\ &= (1-c_n)\left\|\alpha_n - \bar{\xi}\right\| + c_n \sup_{h\neq k} \frac{\left|\mathcal{H}\left(\alpha_n - \bar{\xi}\right)(h) - \mathcal{H}\left(\alpha_n - \bar{\xi}\right)(k)\right|}{|h-k|}. \end{aligned}$$

Consider the quantity

$$|\Delta_{h,k}| = \frac{\left|\mathcal{H}\left(\alpha_n - \bar{\xi}\right)(h) - \mathcal{H}\left(\alpha_n - \bar{\xi}\right)(k)\right|}{|h-k|}, \quad h,k \in [0,1], h \neq k,$$

to be separately evaluated as

(5.5)

$$\begin{aligned} |\Delta_{h,k}| &= \frac{1}{|h-k|}\Big|h\left(\alpha_n - \bar{\xi}\right)(p(h)) + (1-h)\left(\alpha_n - \bar{\xi}\right)(q(h)) \\ &\quad -k\left(\alpha_n - \bar{\xi}\right)(p(k)) - (1-k)\left(\alpha_n - \bar{\xi}\right)(q(k))\Big| \\ &= \frac{1}{|h-k|}\Big|h\left(\alpha_n - \bar{\xi}\right)(p(h)) - h\left(\alpha_n - \bar{\xi}\right)(p(k)) \\ &\quad +(1-h)\left(\alpha_n - \bar{\xi}\right)(q(h)) - (1-h)\left(\alpha_n - \bar{\xi}\right)(q(k)) \\ &\quad + h\left(\alpha_n - \bar{\xi}\right)(p(k)) - k\left(\alpha_n - \bar{\xi}\right)(p(k)) \end{aligned}$$

$$
\begin{aligned}
&+(1-h)\left(\alpha_n-\bar{\xi}\right)(q(k))-(1-k)\left(\alpha_n-\bar{\xi}\right)(q(k))\big| \\
&\leq h.\frac{\left|\left(\alpha_n-\bar{\xi}\right)(p(h))-\left(\alpha_n-\bar{\xi}\right)(p(k))\right|}{|p(h)-p(k)|}.\frac{|p(h)-p(k)|}{|h-k|} \\
&+(1-h).\frac{\left|\left(\alpha_n-\bar{\xi}\right)(q(h))-\left(\alpha_n-\bar{\xi}\right)(q(k))\right|}{|q(h)-q(k)|}.\frac{|q(h)-q(k)|}{|h-k|} \\
&+\left|\frac{h-k}{h-k}\right|.\frac{\left|\left(\alpha_n-\bar{\xi}\right)(p(k))-\left(\alpha_n-\bar{\xi}\right)(p(1))\right|}{|p(k)-p(1)|}.\frac{|p(k)-p(1)|}{|k-1|}.|k-1| \\
&+\left|\frac{h-k}{h-k}\right|.\frac{\left|\left(\alpha_n-\bar{\xi}\right)(q(k))-\left(\alpha_n-\bar{\xi}\right)(q(0))\right|}{|q(k)-q(0)|}.\frac{|q(k)-q(0)|}{|k-0|}.|k-0|.
\end{aligned}
$$

As p and q are contractions with contraction coefficients μ and δ respectively, then we have

$$|p(h)-p(k)|\leq\mu|h-k|,\qquad |q(h)-q(k)|\leq\delta|h-k|,$$

and

$$|p(k)-p(1)|\leq\mu|k-1|,\qquad |q(k)-q(0)|\leq\delta|k-0|=\delta k.$$

Then, from (5.5) and (5.4), respectively, we have

$$
\begin{aligned}
|\Delta_{h,k}|\leq{}&\mu h\left\|\alpha_n-\bar{\xi}\right\|+\delta(1-h)\left\|\alpha_n-\bar{\xi}\right\|+\mu(1-k)\left\|\alpha_n-\bar{\xi}\right\| \\
&+\delta k\left\|\alpha_n-\bar{\xi}\right\|,\quad h\neq k,
\end{aligned}
$$

and

(5.6)

$$
\begin{aligned}
\left\|\theta_n-\bar{\xi}\right\|\leq{}&(1-c_n)\left\|\alpha_n-\bar{\xi}\right\|+\sup_{h\neq k}c_n[\mu h+\delta(1-h)+\mu(1-k)+\delta k] \\
&\cdot\left\|\alpha_n-\bar{\xi}\right\| \\
\leq{}&\sup_{h\neq k}\left((1-c_n)+c_n[\mu h+\delta(1-h)+\mu(1-k)+\delta k]\right)\left\|\alpha_n-\bar{\xi}\right\|.
\end{aligned}
$$

As $\mu,\delta\in(0,1)$, $\mu\leq\delta$ and $2\mu<1$, we have

$$
\begin{aligned}
\mu h+\delta(1-h)+\mu(1-k)+\delta k&=\mu+\delta+(\mu-\delta)h-(\mu-\delta)k \\
&\leq\mu+\delta+\mu-\delta=2\mu.
\end{aligned}
$$

Then, it follows from (5.6) that

$$\left\|\theta_n-\bar{\xi}\right\|\leq(1-(1-2\mu)c_n)\left\|\alpha_n-\bar{\xi}\right\|. \tag{5.7}$$

From (2.3) and (5.3), we also have

(5.8)

$$
\begin{aligned}
\left\|\gamma_n-\bar{\xi}\right\|&=\left\|\mathcal{H}((1-b_n)\theta_n+b_n\mathcal{H}\theta_n)-\bar{\xi}\right\| \\
&=\left\|\mathcal{H}((1-b_n)\theta_n+b_n\mathcal{H}\theta_n-\bar{\xi})\right\|
\end{aligned}
$$

$$= \sup_{h \neq k} \frac{|\mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \bar{\xi})(h) - \mathcal{H}((1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \bar{\xi})(k)|}{|h-k|}.$$

By evaluating it similarly as (5.5), (5.8) can be rewritten as

$$\begin{aligned} \left\|\gamma_n - \bar{\xi}\right\| &\leq 2\mu \left\|(1-b_n)\theta_n + b_n\mathcal{H}\theta_n - \bar{\xi}\right\| \\ &\leq 2\mu \left((1-b_n)\left\|\theta_n - \bar{\xi}\right\| + b_n \left\|\mathcal{H}(\theta_n - \bar{\xi})\right\|\right). \end{aligned}$$

By following the same process as in (5.5) and using (5.6), we have

$$\begin{aligned} \left\|\gamma_n - \bar{\xi}\right\| &\leq 2\mu \left((1-b_n)\left\|\theta_n - \bar{\xi}\right\| + 2\mu b_n \left\|\theta_n - \bar{\xi}\right\|\right) \\ &= 2\mu(1-(1-2\mu)b_n)\left\|\theta_n - \bar{\xi}\right\| \\ &\leq 2\mu(1-(1-2\mu)b_n)(1-(1-2\mu)c_n)\left\|\alpha_n - \bar{\xi}\right\|. \end{aligned} \tag{5.9}$$

Now, by again utilizing (2.3) and (5.3), we obtain

$$\begin{aligned} \left\|\beta_n - \bar{\xi}\right\| &= \left\|\mathcal{H}(\mathcal{H}\gamma_n) - \bar{\xi}\right\| \\ &= \left\|\mathcal{H}((\mathcal{H}\gamma_n) - \bar{\xi})\right\| \\ &= \sup_{h \neq k} \frac{|\mathcal{H}((\mathcal{H}\gamma_n) - \bar{\xi})(h) - \mathcal{H}((\mathcal{H}\gamma_n) - \bar{\xi})(k)|}{|h-k|}. \end{aligned}$$

With reference to (5.5), we utilize (5.9) to rewrite it as

$$\begin{aligned} \left\|\beta_n - \bar{\xi}\right\| &\leq 2\mu \left\|\mathcal{H}(\gamma_n - \bar{\xi})\right\| \\ &\leq 4\mu^2 \left\|\gamma_n - \bar{\xi}\right\| \\ &\leq 8\mu^3(1-(1-2\mu)b_n)(1-(1-2\mu)c_n)\left\|\alpha_n - \bar{\xi}\right\|. \end{aligned} \tag{5.10}$$

Again, from (2.3) and (5.3), we have

$$\begin{aligned} \left\|\alpha_{n+1} - \bar{\xi}\right\| &= \left\|(1-a_n)\beta_n + a_n\mathcal{H}\beta_n - \bar{\xi}\right\| \\ &\leq (1-a_n)\left\|\beta_n - \bar{\xi}\right\| + a_n\left\|\mathcal{H}(\beta_n - \bar{\xi})\right\| \\ &= (1-a_n)\left\|\beta_n - \bar{\xi}\right\| + a_n \sup_{h \neq k} \frac{|\mathcal{H}(\beta_n - \bar{\xi})(h) - \mathcal{H}(\beta_n - \bar{\xi})(k)|}{|h-k|}. \end{aligned}$$

With evaluation analogous to (5.5), we employ (5.10) to have

(5.11)

$$\begin{aligned} &\left\|\alpha_{n+1} - \bar{\xi}\right\| \\ &\quad\leq (1-(1-2\mu)a_n)\left\|\beta_n - \bar{\xi}\right\| \\ &\quad\leq (2\mu)^3(1-(1-2\mu)a_n)(1-(1-2\mu)b_n)(1-(1-2\mu)c_n)\cdot\left\|\alpha_n - \bar{\xi}\right\| \\ &\quad\vdots \\ &\quad\leq \left((2\mu)^3(1-(1-2\mu)a_n)(1-(1-2\mu)b_n)(1-(1-2\mu)c_n)\right)^n \cdot \left\|\alpha_1 - \bar{\xi}\right\|. \end{aligned}$$

Since $2\mu < 1$ and $(v_n) = \{a_n, b_n, c_n\} \in (0,1)$, $(2\mu)^3 < 1$ and $(1 - (1-2\mu)v_n) < 1$, (α_n) converges strongly to $\bar{\xi}$. Thus completing the proof. □

The following corollary can be stated if $p(u) = (1-\delta)u + \delta$ and $q(u) = (1-\mu)u$, for $u \in [0,1]$.

Corollary 5.3. *For p and q to be $p(u) = (1-\delta)u+\delta$ and $q(u) = (1-\mu)u$, $u \in [0,1]$, the functional equation* (5.1) *possesses a unique solution $\bar{\xi} \in \mathcal{W}$ and the sequence generated by iteration* (2.3) *strongly converges to $\bar{\xi}$.*

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