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Jacobson Radical and Representations in Certain Topological Algebras

Majid Sabet[1], Ali Morovatpoor[2*] and Ali Yousefi Pirbasti[3]

Abstract. Fundamental strongly sequential topological algebras (FSS-algebras) form a proper extension of locally bounded algebras. In this paper, we extend several classical results from Banach algebra theory to the setting of complete metrizable FSS-algebras. These include a generalization of the unitization technique, as well as extensions of results concerning the spectral radius and the radius of boundedness. We prove that the quotient algebra $A/\mathrm{J}(A)$ inherits the FSS structure from A. We also analyze the structure of irreducible modules associated with maximal ideals in a topological framework. Their connection with the corresponding representations is examined in detail. In addition, we establish supplementary results on epimorphisms and radicals. The results provide a coherent foundation for further research on representations and the structure of non-Banach topological algebras.

1. Introduction

Topological algebras play an essential role in functional analysis and operator theory, particularly in the study of spectra, radicals and structural properties [2, 3, 7, 8]. The notions of strongly sequential and fundamental topological algebras, originally motivated by Michael's problem on automatic continuity and Cohen's factorization theorem, respectively, have been introduced and studied in the literature [4, 12]. Their combination gives rise to the class of fundamental strongly sequential algebras (FSS-algebras), which properly extends the class of locally bounded algebras [5, Theorem 3.5 and Example 3.6]. This broader

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* Corresponding author.

framework provides a natural setting for investigating how far classical results from Banach algebra theory remain valid beyond the normed setting, particularly in view of recent developments on topological radicals and their interaction with spectra, modules and representations [1, 9].

The class of FSS-algebras is itself recent [5, Theorem 3.5 and Example 3.6]. Within this framework, we establish several new results, including the preservation of the FSS structure under quotient by the Jacobson radical (Proposition 3.14), the topological analysis of irreducible modules via $A - M$ (Proposition 3.19), the boundedness of intertwiners (Proposition 3.22) and the continuity of the induced representation with respect to the topology t_u (Theorem 3.25). The remaining results extend classical theorems from Banach algebra theory [8, Chapter 3] to complete metrizable FSS-algebras.

The aim of this paper is to extend several classical results on the Jacobson radical, the spectral radius, irreducible modules and their associated representations from Banach and locally bounded algebras to complete metrizable FSS-algebras. We prove that the quotient algebra $A/J(A)$ inherits the FSS structure, investigate the topological structure of irreducible modules arising from maximal ideals and their associated representations and establish additional results concerning epimorphisms and radicals (Propositions 3.15 and 3.27).

2. Preliminaries

For clarity, we first recall the basic definitions used throughout this paper.

A complex vector space A is called a topological vector space if it is equipped with a Hausdorff topology such that

(a) the addition map $(x, y) \mapsto x + y$ from $A \times A$ to A is continuous,
(b) the scalar multiplication map $(\alpha, x) \mapsto \alpha x$ from $\mathbb{C} \times A$ to A is continuous.

Such a topological vector space is called a topological algebra if the multiplication

$$A \times A \longrightarrow A, \quad (x, y) \longmapsto xy,$$

is continuous.

A topological vector space A is called metrizable if its topology is generated by a metric. Similarly, a topological algebra A is metrizable if its underlying vector space admits a compatible metric. In this paper, all vector spaces and algebras are assumed to be metrizable.

A topological algebra A is called strongly sequential if there exists a neighborhood U of zero such that, for every $x \in U$, the sequence $(x^n)_n$ converges to zero in A (see [12, p. 413, Definition (i)]).

Let A be a topological algebra. An element $x \in A$ is called bounded if there exists a nonzero complex number z such that

$$\lim_{n\to\infty} \left(z^{-1}x\right)^n = 0.$$

The radius of boundedness of x is defined by

$$\beta(x) := \inf\left\{r > 0 : \lim_{n\to\infty} \left(r^{-1}x\right)^n = 0\right\}.$$

This notion originates in the context of locally convex algebras, where it is proved to be equivalent to the definition based on bounded sets (see [2, Proposition 2.14 (iii)]). In the setting of metrizable topological algebras considered throughout this paper, the sequential definition is likewise equivalent to the bounded-set definition. This equivalence follows from the general result established in [15, Section 2, p. 281], where the quantities $\beta(x)$, $\beta'(x)$ and $\beta''(x)$ are introduced and shown to coincide. Consequently, the two characterizations of boundedness are used interchangeably throughout this paper, as in the subsequent works [3, 5].

Remark 2.1. The radius of boundedness β is positively homogeneous. That is, for any scalar $\alpha \in \mathbb{C}$ and any element $x \in A$,

$$\beta(\alpha x) = |\alpha|\,\beta(x).$$

This property follows directly from the definition of β (see [15, p. 281, Prop. (i)]).

Remark 2.2. Let A be a strongly sequential topological algebra and $a \in A$. There exists a neighborhood U_0 of zero such that, for every $x \in U_0$, the sequence $\left(x^n\right)_n$ converges to zero in A. Since $\left(\frac{1}{n}\right)_n \to 0$, the sequence $\left(\frac{1}{n}a\right)_n$ also converges to zero. Hence, there exists $N \in \mathbb{N}$ such that $\frac{1}{n}a \in U_0$ for all $n \geq N$. In particular, $\frac{1}{N}a \in U_0$, which implies

$$\left(\frac{1}{N}a\right)^n \to 0.$$

Therefore,

$$\beta\Big(\frac{1}{N}a\Big) \leq 1 \quad \text{and thus} \quad \beta(a) \leq N.$$

This shows that every element of a strongly sequential topological algebra is bounded.

A topological algebra A is called fundamental if there exists a constant $b > 1$ such that, for any sequence $(x_n)_n$ in A, the convergence

$$b^n(x_{n+1} - x_n) \to 0$$

implies that $(x_n)_n$ is a Cauchy sequence [4, Definition 2.1, Definition 2.3].

In this paper, we use the abbreviation FSS-algebra for a fundamental strongly sequential topological algebra.

3. Main Results

3.1. **Ideals and the Jacobson Radical.** In this section, we recall the basic definitions and results concerning ideals and the Jacobson radical. We also summarize relevant propositions on the spectrum and its properties and conclude with two new results.

A left ideal E of an algebra A is called modular if there exists an element $e \in A$ (a right modular unit for E) such that

$$a - ae \in E \quad \text{for all} \quad a \in A.$$

The intersection of all maximal modular right ideals coincides with the intersection of all maximal modular left ideals. This two-sided ideal is called the Jacobson radical of A and is denoted by $J(A)$. This definition agrees with the classical definition for rings; see, for example, [10, Chapter 1, Theorem 1.2.2], where equivalent characterizations in terms of maximal regular ideals are provided.

An algebra A is said to be semisimple if $\mathrm{J}(A) = \{0_A\}$[10, p. 16].

Throughout this paper, $\mathrm{Inv}(A)$ denotes the set of all invertible elements of a unital topological algebra A, while $\mathrm{Sing}(A)$ denotes the set of all non-invertible elements.

In a topological algebra A, the quasi-product of two elements x and y is defined as $x \circ y := x + y - xy$. An element $x \in A$ is called quasi-invertible if there exists $y \in A$ such that $x \circ y = y \circ x = 0_A$.

In this case, y is the quasi-inverse of x, denoted x^0. The set of all quasi-invertible elements of A is denoted by $\text{q-Inv}(A)$ and the set of non-quasi-invertible elements by $\text{q-Sing}(A)$.

A topological algebra A is called a Q-algebra if $\text{q-Inv}(A)$ is an open subset of A, a notion originally introduced by Kaplansky for topological rings [13, p. 810].

In this paper, we adopt the definition of quasi-inverses given in [8]. Some other references, such as [7], use a slightly different definition, which is nevertheless equivalent in the context of our results.

Lemma 3.1. *Let (X, d_X) and (Y, d_Y) be complete metrizable topological vector spaces. Then the product space $X \times Y$, equipped with the metric*

$$d((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2),$$

is a complete metrizable topological vector space.

Proof. Since d_X and d_Y are complete, the product metric d is also complete on $X \times Y$; see, for example, the proof of Theorem 2.15 in [16]. □

If A is a complex algebra (not necessarily with unit), its unitization is defined by $A_1 := A \oplus \mathbb{C}$, with operations for $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$

given by

$$\begin{aligned}(x,\alpha)+(y,\beta)&=(x+y,\alpha+\beta),\\ \beta(x,\alpha)&=(\beta x,\beta\alpha),\\ (x,\alpha)(y,\beta)&=(xy+\alpha y+\beta x,\alpha\beta).\end{aligned}$$

This construction yields an algebra with unit element $(0_A, 1)$.

If (A, d) is a metric algebra, defining

$$d_{A_1}((x,\alpha),(y,\beta)) := d(x,y)+|\alpha-\beta|$$

makes A_1 a metric algebra. Moreover, completeness and fundamentality of A are preserved in A_1.

Convergence in d_{A_1} is componentwise. That means $(x_n,\alpha_n)\to(x,\alpha)$ if and only if $x_n\to x$ in (A,d) and $\alpha_n\to\alpha$ in $\mathbb{C}$. Using this, the multiplication in A_1 is jointly continuous

$$\begin{aligned}(x_n,\alpha_n)(y_n,\beta_n)&=(x_ny_n+\alpha_ny_n+\beta_nx_n,\alpha_n\beta_n)\\ &\to(xy+\alpha y+\beta x,\alpha\beta)\\ &=(x,\alpha)(y,\beta).\end{aligned}$$

Hence, A_1 is a metrizable topological algebra.

Proposition 3.2. *Let (A,d) be a metrizable complex algebra and let $A_1 = A\oplus\mathbb{C}$ be its unitization equipped with the operations and the metric d_{A_1} defined above.*

(1) *If A is complete, then A_1 is complete.*
(2) *If A is fundamental, then A_1 is fundamental.*

Proof. Algebraically, A_1 is an algebra with unit $(0_A,1)$ and multiplication is jointly continuous with respect to d_{A_1}, so A_1 is a metrizable topological algebra.
(1) If A is complete, then d_{A_1} is the product metric of d on A and the absolute value on $\mathbb{C}$. By Lemma 3.1, the product of complete metrizable spaces is complete, so (A_1,d_{A_1}) is complete.
(2) Assume A is fundamental. Let $(x_n,\alpha_n)_n$ be a sequence in A_1 and $b>1$ such that

$$b^n\left((x_{n+1},\alpha_{n+1})-(x_n,\alpha_n)\right)\to(0_A,0)\text{ in }A_1.$$

By definition of d_{A_1}, this is equivalent to

$$b^n(x_{n+1}-x_n)\to 0_A\text{ in }A,\quad b^n(\alpha_{n+1}-\alpha_n)\to 0\text{ in }\mathbb{C}.$$

Since A is fundamental, $(x_n)_n$ is Cauchy in A. Also, $\mathbb{C}$ is a normed algebra and therefore fundamental [4, p. 54], so $(\alpha_n)_n$ is Cauchy in $\mathbb{C}$. Hence, $(x_n,\alpha_n)_n$ is Cauchy in A_1, proving that A_1 is fundamental. □

In this paper, we consider two notions of the spectra. For an algebra A, the classical spectrum $\sigma_A(a)$ is defined as follows. For a general algebra A (possibly without unit), the quasi-spectrum $\sigma'_A(a)$ is defined via its unitization, as given in the subsequent definition. Proposition 3.3 shows that, for algebras with unit, the two definitions coincide on all nonzero values.

Let A be an algebra with unit and $a \in A$. The spectrum of a is defined as [7, Definition 1.7.1]

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \in \mathrm{Sing}(A)\}.$$

For an arbitrary algebra A (with or without unit) and $a \in A$, the quasi-spectrum of a is defined by $\sigma'_A(a) = \sigma_{A_1}((a,0))$, where A_1 denotes the unitization of A. By this definition, $\sigma'_A(a)$ always contains 0 and is therefore non-empty (see [7, Definition 1.7.5 and lemma 1.7.6]).

Proposition 3.3 ([7, p. 58, Proposition 1.7.21])**.** *Let A be an arbitrary algebra and $a \in A$. Then the quasi-spectrum of a satisfies*

$$\sigma'_A(a) = \left\{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}a \in \text{q-Sing}(A)\right\} \cup \{0_A\}.$$

Let A be an arbitrary algebra (with or without unit). For $a \in A$, the spectral radius of a is defined by

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma'_A(a)\}.$$

Propositions 3.5 and 3.6 have been established for algebras with unit (see [3, Theorem 2.1 and Cororally 2.2]). Using the unitization method (see Proposition 3.2), we extend these results to arbitrary algebras, including those without a unit.

Proposition 3.4 ([8, Section 3, p. 16. Proposition 5])**.** *Let A be an algebra.*

(i) *An element x of A has the quasi-inverse y if and only if $(0_A, 1) - (x, 0)$ has the inverse $(0_A, 1) - (y, 0)$ in $A_1 = A \oplus \mathbb{C}$.*
(ii) *If A has a unit element, an element x of A has the quasi-inverse y if and only if $1 - x$ has the inverse $1 - y$.*

Proposition 3.5. *Let a be an element of a complete metrizable fundamental algebra A such that $\beta(a) < 1$. Then a is quasi-invertible and its quasi-inverse is given by*

$$a^0 = -\sum_{n=1}^{\infty} a^n.$$

Proof. By Proposition 3.2, the unitization A_1 is a complete metrizable fundamental algebra with unit. Since

$$\beta((a,0)) = \beta(a) < 1,$$

Proposition 2.1 of [3] implies that $(0_A, 1) - (a, 0)$ is invertible in A_1. Assume that the inverse is of the form (c, α), where $\alpha \in \mathbb{C}$ and $c \in A$. A direct computation then yields

$$\begin{aligned}((0_A, 1) - (a, 0))^{-1} &= (c, \alpha) \\ &= (0_A, 1) + \left(\sum_{n=1}^{\infty} (a, 0)^n\right) \\ &= (0_A, 1) + \left(\sum_{n=1}^{\infty} a^n, 0\right) \\ &= \left(\sum_{n=1}^{\infty} a^n, 1\right).\end{aligned}$$

Consequently, $\alpha = 1$ and

$$c = \sum_{n=1}^{\infty} a^n.$$

If we set $c = -b$, then

$$(c, \alpha) = (-b, 1) = (0_A, 1) - (b, 0).$$

Since the inverse of $(0_A, 1) - (a, 0)$ in A_1 is $(0_A, 1) - (b, 0)$, Part (i) of Proposition 3.4 implies that b is the quasi-inverse of a. Therefore,

$$a^{\circ} = b = -c = -\sum_{n=1}^{\infty} a^n.$$

This completes the proof. □

Proposition 3.6. *Let A be a complete metrizable fundamental algebra in which every element is bounded. Then, for each $a \in A$,*

$$\rho(a) \leq \beta(a).$$

In particular, the spectrum of every element of A is bounded.

Proof. Let $a \in A$. We show that $\rho(a) \leq \beta(a)$. Let $\lambda \in \sigma'_A(a)$ with $\lambda \neq 0$. We claim that $|\lambda| \leq \beta(a)$.

Assume, to the contrary, that $\beta(a) < |\lambda|$. By the positive homogeneity of β (Remark 2.1),

$$\beta\left(\lambda^{-1}a\right) = \left|\lambda^{-1}\right| \beta(a) = \frac{\beta(a)}{|\lambda|} < 1.$$

Hence, by Proposition 3.5, $\lambda^{-1}a \in \text{q-Inv}(A)$. By Proposition 3.3, this implies that $\lambda \notin \sigma'_A(a)$, a contradiction.

Therefore, $|\lambda| \leq \beta(a)$ for every nonzero $\lambda \in \sigma'_A(a)$ and consequently $\rho(a) \leq \beta(a)$. □

Combining Proposition 3.6 with Remark 2.2, we obtain the following consequence.

Corollary 3.7. *Let A be a complete metrizable FSS-algebra. Then, for every $a \in A$, $\rho(a) \leq \beta(a)$.*

This shows that, in complete metrizable FSS-algebras, the boundedness of the spectrum of each element follows directly from the boundedness of the elements.

Proposition 3.8 ([7, Corollary 7.1.9])**.** *In every Q-algebra, every maximal ideal is closed.*

Proposition 3.9 ([5, Proposition 3.1])**.** *Let A be a complete metrizable algebra in which every element is bounded. Then the spectral radius function β is continuous at 0_A if and only if A is strongly sequential.*

Proposition 3.10. *Let A be a complete metrizable FSS-algebra. Then every maximal ideal of A is closed.*

Proof. Since A is a complete metrizable FSS-algebra, Corollary 3.7 yields

$$\rho(a) \leq \beta(a), \quad a \in A.$$

Because A is strongly sequential, Proposition 3.9 implies that the spectral radius function β is continuous at 0_A. Hence, by the above inequality, ρ is also continuous at 0_A.

A classical result in the spectral theory states that ρ is continuous at 0_A if and only if A is a Q-algebra (see [7, Lemma 3.6.8]; essentially, because the set $\{x : \rho(x) \leq 1\}$ has nonempty interior if and only if ρ is continuous at zero). Therefore, A is a Q-algebra. By Proposition 3.8, every maximal ideal of a Q-algebra is closed. This completes the proof. □

Remark 3.11. (Quotient Spaces: Metric and Completeness). Let A be a metrizable topological algebra with a fixed invariant metric d compatible with its topology.

- **Quotient by a closed subspace (topological vector space).** Let I be a closed subspace of A (not necessarily an ideal). We denote the corresponding quotient vector space by $A - I$ (see [8, Chapter 1, Section 9, Definition 6]). It is endowed with the quotient metric

  $$\rho_I(a + I, b + I) = \inf\{d(a - b, x) : x \in I\}, \quad a, b \in A.$$

  This metric is invariant, compatible with the quotient topology and turns $A - I$ into a complete metrizable topological vector

space whenever A is complete and I is closed (see [16, Proposition 1.41]).
A sequence $(a_n + I)_n$ converges to $a + I$ in $A - I$ if and only if there exists a sequence $(x_n) \subset I$ such that

$$a_n - a - x_n \to 0 \quad \text{in } A.$$

- **Quotient by a closed two-sided ideal (topological algebra).**
  Let I be a closed two-sided ideal of A. The corresponding quotient algebra is denoted by A/I (see [8, Sec. 19, Def. 8]). As a vector space, A/I coincides with $A-I$ considered as a quotient space and it is equipped with the same quotient metric $\rho_{_I}$.
  The algebraic operations are inherited from A and defined by

$$\begin{aligned}(a + I) + (b + I) &= (a + b) + I,\\ \lambda(a + I) &= (\lambda a) + I,\\ (a + I)(b + I) &= ab + I,\end{aligned}$$

  for all $a, b \in A$ and $\lambda \in \mathbb{C}$.
  Since I is a two-sided ideal, these operations are well-defined. Their continuity follows from the continuity of the corresponding operations on A and the openness of the canonical projection

$$\pi : A \to A/I.$$

  Consequently, A/I is a complete metrizable topological algebra.

In the sequel, all topological properties of a quotient space such as convergence, neighborhoods and continuity of the module action are understood with respect to the quotient metric $\rho_{_I}$ and the topology it induces. The notation $A - I$ will be used when the quotient is considered merely as a vector space, whereas A/I will denote the quotient algebra.

Proposition 3.12. *Let A be a complete metrizable FSS-algebra and let M be a closed two-sided ideal of A. Then the quotient algebra A/M (equipped with the quotient metric $\rho_{_M}$ as in Remark 3.11) is a complete metrizable FSS-algebra.*

Proof. By Remark 3.11, the quotient A/M, equipped with the metric $\rho_{_M}$, is a complete metrizable topological algebra. Indeed, completeness follows from the fact that A is complete and M is closed, while the continuity of the algebraic operations on A/M follows from the openness of the canonical quotient map and the continuity of the corresponding operations on A. It remains to verify that quotient algebra A/M satisfies the FSS property. To this end, we first show that A/M is fundamental.

Assume that A is fundamental. Let $b > 1$ and let $(a_n + M)_n$ be a sequence in A/M such that

$$\rho_{_M}\left(b^n((a_{n+1} + M) - (a_n + M)), 0\right) \to 0.$$

Then, for any $\varepsilon > 0$, there exists N such that for every $n \geq N$, there exists $c_n \in M$ satisfying

$$d\left(b^n(a_{n+1} - a_n + c_n), 0\right) < \varepsilon.$$

We define a sequence $(m_n)_n$ in M recursively by

$$m_1 = 0, \quad m_{n+1} = c_1 + c_2 + \cdots + c_n, \quad (n \geq 1).$$

Hence, for every $n \geq N$ we obtain

$$\begin{aligned} &d\left(b^n((a_{n+1} + m_{n+1}) - (a_n + m_n)), 0\right) \\ &\quad = d\left(b^n(a_{n+1} - a_n + m_{n+1} - m_n), 0\right) \\ &\quad = d\left(b^n(a_{n+1} - a_n + c_n), 0\right) \\ &\quad < \varepsilon. \end{aligned}$$

By the assumption and the fact that A is fundamental, the sequence $(a_n + m_n)_n$ is Cauchy in A. Hence, for every $\varepsilon > 0$, there exists N such that for all $s > r \geq N$,

$$d(a_s + m_s, a_r + m_r) < \varepsilon.$$

Therefore,

$$\begin{aligned} \rho_{_M}(a_s + M, a_r + M) &= \inf\{d(a_s - a_r, m) : m \in M\} \\ &\leq d(a_s - a_r, m_r - m_s) \\ &= d(a_s + m_s, a_r + m_r) \\ &< \varepsilon. \end{aligned}$$

Finally, we show that A/M is strongly sequential. Since A is an FSS-algebra, there exists a neighbourhood U_0 of 0 in A such that $x^n \to 0$ for every $x \in U_0$.

Because the canonical projection π is open, $\pi(U_0)$ is a neighbourhood of 0 in A/M. Let $y \in \pi(U_0)$. Then there exists $x \in U_0$ such that $\pi(x) = y$. Since the multiplication on the quotient algebra is defined by $\pi(a)\pi(b) = \pi(ab)$, it follows by induction that

$$(\pi(x))^n = \pi(x^n), \quad n \in \mathbb{N}.$$

Since $x^n \to 0$ in A and π is continuous, it follows that

$$y^n = \pi(x^n) \to \pi(0) = 0 \quad \text{in } A/M.$$

Thus, every element of $\pi(U_0)$ converges to 0 under successive powers and hence A/M is strongly sequential.

Combining the above steps, we conclude that A/M is a complete metrizable FSS-algebra. □

In the next proposition, we recall a basic result concerning the Jacobson radical, which plays an important role in the study of quasi-invertible elements. This result appears in [8, Chapter 3, p. 125, Proposition 16]; here we state only its formulation.

Proposition 3.13 ([8, Chapter 3, p. 125, Proposition 16])**.** *Let A be an algebra. The following assertions hold:*

(1) $J(A) \subseteq \text{q-Inv}(A)$*;*
(2) $J(A) = \{\, q \in A : Aq \subseteq \text{q-Inv}(A) \,\}$.

In Banach algebras, the basic results concerning the Jacobson radical and the spectral radius are well established (see [8, Chapter 3, Section 25]). The following two propositions show that these classical results extend to the more general class of complete metrizable FSS-algebras.

Proposition 3.14. *Let A be a complete metrizable FSS-algebra. Then the quotient algebra $A/\mathrm{J}(A)$ is a complete metrizable FSS-algebra and is semisimple.*

Proof. By Proposition 3.10, every maximal ideal of A is closed. Hence the Jacobson radical $\mathrm{J}(A)$, being the intersection of all maximal ideals, is closed. It then follows from Proposition 3.12 and Remark 3.11 that the quotient algebra $A/\mathrm{J}(A)$ is a complete metrizable FSS-algebra. Moreover, it is well known that $A/\mathrm{J}(A)$ is semisimple (see [10, Theorem 1.2.4] or [8, Section 24, Proposition 21]). □

Proposition 3.15. *Let A be a complete metrizable FSS-algebra.*

(a) *If $q \in \mathrm{J}(A)$ and $a \in A$, then*

$$\rho(aq) = \rho(qa) = \rho(q) = 0.$$

(b) *If either $\rho(aq) = 0$ for every $a \in A$, or $\rho(qa) = 0$ for every $a \in A$, then $q \in \mathrm{J}(A)$.*

Proof. (a) Let $q \in \mathrm{J}(A)$. Since $\mathrm{J}(A)$ is a two-sided ideal of A, it suffices to show that $\rho(q) = 0$. By the definition of the spectral radius, it is enough to prove that $\sigma'_A(q)$ contains no nonzero elements (note that $0 \in \sigma'_A(q)$ always).

Suppose, for the sake of contradiction, that there exists $\lambda \in \sigma'_A(q) \setminus \{0\}$. Because $q \in \mathrm{J}(A)$ and $\mathrm{J}(A)$ is an ideal, we have $\lambda^{-1}q \in \mathrm{J}(A)$. By part (a) of Proposition 3.13, we know that $\mathrm{J}(A) \subseteq q\text{-Inv}(A)$ and consequently $\lambda^{-1}q \in q\text{-Inv}(A)$. This contradicts Proposition 3.3. Therefore, $\sigma'_A(q) = \{0\}$ and hence $\rho(q) = 0$, as claimed.

(b) Suppose that for every $a \in A$, $\rho(aq) = 0$. If $aq \in$ q-Sing(A), then by Proposition 3.3, $1 \in \sigma'_A(aq)$, which implies $\rho(aq) \geq 1$, a contradiction. Hence, $aq \in$ q-Inv(A). Therefore, $Aq \subseteq$ q-Inv(A) and by Proposition 3.13, it follows that $q \in \mathrm{J}(A)$.

Thus, the proof is complete. □

3.2. **Modules and Representations.** A classical approach to studying topological algebras is through their modules. Modules provide a framework to examine algebraic properties together with the underlying topology. This approach is closely linked to representation theory and enables the analysis of ideals and radicals from both algebraic and topological perspectives. In the following, we first review the basic definitions of modules and then discuss their connection to representations of topological algebras.

Let A be an algebra and let M be a vector space. The space M is called a left A-module if for every complex number β and for all elements $a, a_1, a_2 \in A$ and $m, m_1, m_2 \in M$, the following relations hold [7, p. 416]

$$(\beta a_1 + a_2)m = \beta(a_1 m) + a_2 m,$$
$$a(\beta m_1 + m_2) = \beta(a m_1) + a m_2,$$
$$(a_1 a_2)m = a_1(a_2 m).$$

Let A be a topological algebra and M a topological vector space. The space M is called a topological left A-module if

(a) M is a left A-module.
(b) The map $(a, m) \mapsto am$, for $a \in A$ and $m \in M$, is continuous .

This definition coincides with the standard notion of a topological module over a topological ring (see [6, Definition 1.1.24] or [7, Definition 9.3.1]). Clearly, every left ideal of A is a topological left A-module over A.
In the following, we recall the basic notion of a representation, which is central to the study of module structures.

Let X be a complex vector space. A homomorphism $\pi : A \longrightarrow L(X)$, from an algebra A into $L(X)$, the space of all linear operators on X, is called a representation of A on X.

If π is a representation, then X becomes a left A-module via the action

$$ax = \pi(a)x, \quad a \in A,\ x \in X.$$

Conversely, every left A-module X defines a representation of A on X by

$$\pi(a)x := ax, \quad a \in A,\ x \in X,$$

(see [8, Chapter 3, Section 24]). This map is a well-defined homomorphism and is called the representation induced by the module multiplication.

The kernel of the representation π is denoted by $\ker(\pi)$ and is given by

$$\begin{aligned}\ker(\pi) &= \{a \in A : \pi(a) = 0\} \\ &= \{a \in A : \pi(a)x = 0 \text{ for all } x \in X\} \\ &= \{a \in A : ax = 0 \text{ for all } x \in X\} \\ &= \{a \in A : aX = \{0\}\}.\end{aligned}$$

This kernel coincides with the annihilator of the module X, a basic concept in ring theory (see [11, p. 417]).

To further investigate the structure of modules in this topological framework, we introduce the notions of cyclic vectors and irreducible modules.

A left A-module X is called nontrivial if $AX \neq \{0\}$. A module X is irreducible (or simple) if it is nontrivial and its only left A-submodules are $\{0\}$ and X itself (see [11, p. 416, Definition 1.1]).

For $x_0 \in X$, the set $\ker(x_0) = \{a \in A : ax_0 = 0\}$ is a left ideal of A.

An element x_0 of a left A-module X is called a cyclic vector if $Ax_0 = X$ (see [8, Chapter 3, Section 24, Definition 3]).

We also define

$$\mathrm{id}(x_0) = \{e \in A : ex_0 = x_0\},$$

the set of all elements of A that act as an identity on x_0.

Both $\ker(x_0)$ and $\mathrm{id}(x_0)$ play a fundamental role in analyzing the structure of modules.

The next propositions summarize some of its key properties.

Proposition 3.16 ([8, Chapter 3, Section 24, Proposition 4])**.** *Let X be a left A-module and let $x_0 \in X \setminus \{0\}$.*

(a) *Every element of* $\mathrm{id}(x_0)$ *acts as a right modular unit for the left ideal* $\ker(x_0)$.
(b) *If x_0 is a cyclic vector, then* $\mathrm{id}(x_0)$ *is nonempty and* $\ker(x_0)$ *is a modular left ideal.*
(c) *If X is irreducible, then x_0 is a cyclic vector and* $\ker(x_0)$ *is a maximal modular left ideal.*

The above proposition illustrates the connection between cyclic vectors and the structure of their corresponding left ideals.

In what follows, we recall that the notation $A-I$ refers to the quotient vector space defined in Remark 3.11. That is, $A - I$ is a vector space, but not necessarily an algebra.

Proposition 3.17 ([8, Chapter 3, Section 24, Proposition 5])**.** *Suppose A is an algebra such that*

1. *$J \neq A$ is a modular left ideal of A and e is a right modular unit for J;*
2. *$x_0 := e + J$ and $X := A - J$ denotes the left A-module with the action*

$$a \cdot (b + J) = ab + J, \quad a, b \in A.$$

Then

(a) *x_0 is a cyclic vector and* $\ker(x_0) = J$;
(b) *if J is a maximal modular left ideal, then X is irreducible.*

The above result indicates that irreducible modules can be naturally derived from maximal modular left ideals.

A key approach to studying modules and representations of topological algebras involves analyzing quotient modules. In this context, left ideals play a central role, as forming quotients by them yields simpler yet topologically rich structures. The continuity of module multiplication is essential to ensure that the quotient retains a topological module structure.

The next lemma establishes this fundamental property, showing that the quotient of an algebra by a closed left ideal is naturally a topological left module. Subsequently, the following proposition uses this tool to analyze irreducible modules in detail, demonstrating that they can be equivalently reconstructed from quotient modules. These results together provide a clearer and more accessible framework for studying representations and module structures in complete metrizable FSS-algebras.

Lemma 3.18. *Let A be a metrizable algebra and let L be a closed left ideal of A. Consider the quotient vector space $Y := A - L$ equipped with the quotient metric $\rho_{_L}$ as defined in Remark 3.11 and let $\pi : A \to Y$ denote the canonical projection $\pi(a) = a + L$. Then Y becomes a topological left A-module with respect to the action*

$$a \cdot (b + L) := ab + L, \quad a, b \in A.$$

Proof. The fact that Y inherits the structure of a topological A-module from the quotient construction is standard [6, Proposition 1.5.32]. Here, we provide an explicit verification of continuity in our metrizable setting.

Clearly, Y is a left A-module. It remains to show that module multiplication $(a, b + L) \mapsto a(b + L)$ is continuous.

First, suppose that $a_n \to 0$ in A (with respect to the metric d) and $b_n + L \to L$ in Y (with respect to $\rho_{_L}$). By the characterization of convergence in quotient spaces (Remark 3.11), there exist $l_n \in L$ such that $d(b_n - l_n, 0_A) \to 0$.

Since the multiplication in A is continuous, we have $a_n(b_n - l_n) \to 0_A$ in A. Moreover, since L is a left ideal, $a_n l_n \in L$. Hence,

$$\begin{aligned} a_n(b_n + L) &= a_n b_n + L \\ &= a_n(b_n - l_n) + L \\ &\longrightarrow L \quad \text{in } Y, \end{aligned}$$

where the convergence follows from $a_n(b_n - l_n) \to 0_A$ and the continuity of the canonical projection.

Next, suppose that $a_n \to a$ in A and $b_n + L \to b + L$ in Y. Then

$$\begin{aligned} a_n(b_n + L) - a(b + L) &= (a_n b_n - ab) + L \\ &= \big[(a_n - a)(b_n - b) + (a_n - a)b + a(b_n - b)\big] + L \\ &= [(a_n - a)(b_n - b) + L] + [(a_n - a)b + L] \\ &\quad + [a(b_n - b) + L] \\ &\longrightarrow L \quad \text{in } Y. \end{aligned}$$

Therefore, module multiplication is continuous. This completes the proof. □

Proposition 3.19. *Let A be a complete metrizable algebra and let X be an irreducible topological left A-module. Fix $x_0 \in X \setminus \{0\}$ and define*

$$M := \ker(x_0) = \{a \in A : ax_0 = 0\}.$$

Consider the quotient space $A - M$ endowed with the quotient metric ρ_M as in Remark 3.11. Define the map $U : A - M \to X$ by

$$U(a + M) = ax_0, \quad a \in A.$$

The following assertions hold:

(a) *The quotient space $A-M$ is a complete metrizable left A-module.*
(b) *The mapping U is an A-module isomorphism from $A - M$ onto X.*
(c) *If, in addition, X is complete and metrizable as a topological vector space, then U is a homeomorphism.*

Proof. According to Proposition 3.16, $M = \ker(x_0)$ is a maximal modular left ideal. Since the module multiplication is continuous, the map

$$a \mapsto ax_0$$

is continuous. Hence $M = \ker(x_0)$ is closed.

Following the argument in the proof of Lemma 3.18, the quotient space $A - M$ (with the quotient metric ρ_M from Remark 3.11) is a topological left A-module with module multiplication defined by

$$a(b + M) = ab + M, \quad a, b \in A.$$

(a) Because A is a complete metrizable algebra and M is a closed left ideal, the standard result from functional analysis (see [16, Theorem 1.41]) ensures that the vector space $A - M$ is a complete metrizable left A-module (Here, completeness is understood with respect to $\rho_{_M}$).

(b) Let $a_1 + M$ and $a_2 + M$ be equal in $A - M$. Then $a_1 - a_2 \in M = \ker(x_0)$, so $(a_1 - a_2)x_0 = 0$. Hence, $a_1x_0 = a_2x_0$ and thus $U(a_1 + M) = U(a_2 + M)$. This shows that U is well-defined.

Linearity of U follows immediately: for $a_1, a_2 \in A$ and $\gamma \in \mathbb{C}$, we have

$$\begin{aligned} U((a_1 + M) + (a_2 + M)) &= (a_1 + a_2)x_0 \\ &= a_1x_0 + a_2x_0 \\ &= U(a_1 + M) + U(a_2 + M), \end{aligned}$$

and

$$\begin{aligned} U(\gamma(a + M)) &= U(\gamma a + M) \\ &= (\gamma a)x_0 \\ &= \gamma(ax_0) \\ &= \gamma\, U(a + M). \end{aligned}$$

Since X is irreducible, Proposition 3.16 shows that x_0 is cyclic. Hence U is surjective. If $U(a + M) = ax_0 = 0$, then $a \in \ker(x_0) = M$ and thus $a + M = 0_{A-M}$. Hence, U is injective and therefore bijective.

Finally, U preserves the module action: for all $a, b \in A$,

$$\begin{aligned} U(b(a + M)) &= U(ba + M) \\ &= (ba)x_0 \\ &= b(ax_0) \\ &= b\,(U(a + M)). \end{aligned}$$

Consequently, U is an A-module isomorphism.

(c) Suppose that X is a complete metrizable left A-module. Let $(a_n + M)_n$ be a sequence in $A - M$ converging to $a + M$ with respect to $\rho_{_M}$. We show that $(U(a_n + M))_n$ converges to $U(a + M)$ in X.

Since $a_n + M \to a + M$ in $A - M$ with respect to $\rho_{_M}$, by the characterization of convergence in quotient spaces (Remark 3.11), there exist a sequences $(m_n)_n \subseteq M$ such that $d(a_n - a, m_n) \to 0$. Define $b_n = a_n - a - m_n$. Then it follows that

$$a_nx_0 = (a_n - a)x_0 + ax_0$$

$$= (b_n + m_n)x_0 + ax_0$$
$$= b_n x_0 + ax_0.$$

Since $b_n \to 0$ in A with respect to d and the module multiplication is continuous, we have $b_n x_0 \to 0_X$ in X. Hence, $a_n x_0 \to ax_0$, showing that $U(a_n + M) \to U(a + M)$.

Therefore, U is a continuous linear bijection from the complete metrizable space $A - M$ onto the complete metrizable space X. Since both $A - M$ and X are complete metrizable topological vector spaces, the Open Mapping Theorem (see [16, Theorem 2.11]) implies that U is an open mapping. Hence, its inverse U^{-1} is continuous. This proves that U is a topological module isomorphism. □

In view of proposition 3.19 and the fact that many important properties, such as normability and local boundedness, are preserved under passage to quotient spaces, we obtain the following result. Recall that a topological algebra is said to be locally bounded if it possesses a bounded open neighborhood of zero.

Proposition 3.20. *Let A be a topological algebra and X a complete metrizable irreducible left A-module. Then the following properties hold:*

(a) *If A is a Banach algebra, then X is also a Banach space.*
(b) *If A is complete and locally bounded, then X is also complete and locally bounded.*
(c) *If A is a complete fundamental algebra, then X is also complete and fundamental.*

Proof. Let $x_0 \in X \setminus \{0\}$ as in the previous Proposition. According to Proposition 3.19, $M = \ker(x_0)$ is a closed left ideal and X is topologically isomorphic, as a left A-module, to the quotient space $A - M$ (equipped with the quotient metric defined in Proposition 3.19).

Since topological isomorphism preserves properties such as completeness, normability and local boundedness, it suffices to verify that these properties are inherited by $A - M$ from A. Once these properties are established for $A - M$, the same properties hold for X, because X and $A - M$ are homeomorphic.

More precisely, the inheritance of normability and local boundedness from A to $A - M$ follows from standard results on quotient spaces (see [16, Theorem 1.41(c)]). The completeness of $A - M$ is a consequence of Proposition 3.19 together with [16, Theorem 1.41(d)], as observed therein.

Finally, the fundamental property is inherited by $A - M$ in view of Proposition 3.12. □

A natural next step in the study of modules and representations is to consider the subalgebra

$$\mathcal{D} = \{\, T \in L(X) : T(ax) = a\,T(x),\ (a \in A,\ x \in X)\,\},$$

which is introduced in Schur's Lemma (Proposition 3.21) as a division subalgebra of $L(X)$.

In the case of Banach spaces, it is well known that every element of $\mathcal{D}$ is continuous. Proposition 3.22 extends this result to the setting of complete metrizable modules.

This observation highlights that, for a detailed study of the structure of $\mathcal{D}$ and the associated representations, it is essential to consider an appropriate topology on the space of operators.

Theorem 3.21 ([8, chapter 3, Section 24, Schur's Lemma])**.** *Let X be an irreducible left A-module and let*

$$\mathcal{D} = \{\, T \in L(X) : T(ax) = a\,T(x) \text{ for all } a \in A,\ x \in X\,\}.$$

Then $\mathcal{D}$ contains the identity operator and forms a division subalgebra of $L(X)$.

Proposition 3.22. *Let X be a complete metrizable topological left A-module and let*

$$\mathcal{D} = \{\, T \in L(X) : T(ax) = a\,T(x) \text{ for all } a \in A,\ x \in X\,\}.$$

Then every operator in $\mathcal{D}$ is bounded; that is,

$$\mathcal{D} \subseteq BL(X),$$

where $BL(X)$ denotes the space of bounded linear operators on X.

Proof. Let $x_0 \in X \setminus \{0\}$ and let M and U be as defined in Proposition 3.19. Recall that the quotient space $A - M$ is endowed with the quotient metric $\rho_{_M}$ introduced in Proposition 3.19. Take an operator $T \in \mathcal{D}$ and define

$$z := U^{-1}T(x_0) \in A - M.$$

We aim to show that the map

$$U^{-1}TU : A - M \longrightarrow A - M$$

is a continuous linear operator.

For an arbitrary $a \in A$, we compute

$$\begin{aligned} U^{-1}TU(a + M) &= U^{-1}T(ax_0) \\ &= U^{-1}(aT(x_0)) \\ &= a\,U^{-1}(T(x_0)) \\ &= az. \end{aligned}$$

Next, we prove continuity. Since the operator is linear, it suffices to check continuity at $0_{A-M} = M$. Suppose that $a_n + M \to M$ in $A - M$ with respect to $\rho_{_M}$. By the characterization of convergence in quotient spaces (see Proposition 3.19), there exists a sequence $(m_n)_n \subset M$ such that

$$d(a_n - m_n, 0) \to 0,$$

where d denotes the invariant metric on A.
Hence $a_n - m_n \to 0$ in A and by the continuity of the module multiplication (Lemma 3.18), we have

$$(a_n - m_n)z \to 0 \quad in\, A - M.$$

Since $m_n \in M$ and $z \in A - M$, the definition of the quotient module action gives $m_n z = 0_{A-M}$. Therefore, $a_n z = (a_n - m_n)z$. Consequently

$$\begin{aligned} U^{-1}TU(a_n + M) &= a_n z \\ &= (a_n - m_n)z \\ &\to 0 \quad in\, A - M, \end{aligned}$$

and therefore $U^{-1}TU$ is continuous.
By Proposition 3.19, U is a topological isomorphism; consequently, both U and U^{-1} are bounded linear operators. Since $U^{-1}TU$ is a continuous linear operator on the complete metrizable space $A - M$, it is bounded (see[16, Corollary 2.4]).
Since T can be written as

$$T = U\ \left(U^{-1}TU\right)\ U^{-1},$$

and the composition of bounded linear operators is bounded, we conclude that

$$T \in BL(X).$$

This completes the proof. □

Since Proposition 3.22 shows that the elements of $\mathcal{D}$ are bounded, it is natural to examine the topological structure of the operator space. A fundamental tool in this context is the topology of uniform convergence on bounded sets, introduced in [14]. Below, we recall its definition.

Following [14], a neighborhood in a topological vector space is any set that contains an open set around zero. Let X and Y be topological vector spaces and denote by $\mathrm{CL}(X, Y)$ the space of all continuous linear maps from X to Y.

Let $\mathcal{B}_X$ denote the family of all bounded subsets of X and let $\mathcal{W}_Y$ denote the family of all closed, balanced neighborhoods of Y. For each $B \in \mathcal{B}_X$ and $W \in \mathcal{W}_Y$, define

$$U(B, W) := \{\, T \in \mathrm{CL}(X, Y) : T(B) \subseteq W \,\}.$$

The family

$$\{U(B,W) : B \in \mathcal{B}_X,\, W \in \mathcal{W}_Y\}$$

forms a topology on $\mathrm{CL}(X,Y)$, called the topology of uniform convergence on bounded sets and denoted by t_u [14].

Proposition 3.23 ([14, pp. 262–263])**.** *Let X and Y be complete metrizable vector spaces. Then the following statements hold:*

(a) *The topological space $(\mathrm{CL}(X,Y), t_u)$ is complete.*
(b) *If $(T_n)_n$ is a sequence in $\mathrm{CL}(X,Y)$ converging to T with respect to t_u, then*

$$T_n(x) \longrightarrow T(x) \quad \text{for every } x \in X.$$

In the special case $X = Y$, we simply write

$$\mathrm{CL}(X) := \mathrm{CL}(X,X)$$

instead of $\mathrm{CL}(X,Y)$.

Proposition 3.24 ([14, p. 290, Proposition A.6.24])**.** *Let X be a complete metrizable topological vector space. Then $\mathrm{CL}(X)$, the algebra of all continuous linear operators on X, endowed with the topology of uniform convergence on bounded sets t_u, is a complete topological algebra.*

In semisimple Banach algebras, all norms are equivalent (see [8, Chapter 3, p. 130, Theorem 9]). A fundamental consequence is the uniqueness of the topology. The proof of this result relies on the continuity of every irreducible representation of a semisimple Banach algebra on a normed space. This continuity follows from the classical result that if $\pi(a) \in B_L(X)$ for all $a \in A$, then π is continuous (see [8, Chapter 3, p. 128, Theorem 7]). Motivated by this framework, we seek to establish an analogous result for FSS-algebras. As a first step, we prove the following theorem.

Theorem 3.25. *Let A be a complete metrizable algebra and X a complete metrizable left A-module. Let π denote the representation induced by the module multiplication; that is, for each $a \in A$ and $x \in X$,*

$$\pi(a)x = ax.$$

Then π defines a continuous homomorphism from A into the algebra $(\mathrm{CL}(X), t_u)$. In other words, π is continuous with respect to the topology t_u on $\mathrm{CL}(X)$.

Proof. First, we show that for every $a \in A$, the operator $\pi(a)$ is continuous; that is, $\pi(a) \in \mathrm{CL}(X)$. Let $(x_n)_n$ be a sequence in X converging to $x \in X$. Since the module multiplication is continuous, we have

$$ax_n \to ax.$$

Hence, $\pi(a)$ is indeed a continuous linear operator.

Next, we prove that the mapping

$$\pi : A \longrightarrow (\mathrm{CL}(X), t_u)$$

is continuous. Since π is linear, it suffices to check continuity at 0_A.

Let $B \subseteq X$ be a bounded set and let $W \subseteq X$ a balanced open neighborhood of 0_X. Since X is a topological vector space, balanced neighborhoods of 0_X form a neighborhood base at 0_X. By continuity of the module multiplication at $(0_A, 0_X)$, there exist neighborhoods $V_A \subseteq A$ and $V_X \subseteq X$ such that

$$V_A V_X \subseteq W.$$

Since B is bounded, there exists $\lambda > 0$ such that

$$B \subseteq \lambda V_X.$$

Define

$$V := \frac{1}{\lambda} V_A \subseteq A.$$

Then for every $v \in V$ and $b \in B$, we have

$$\pi(v)(b) = vb \in \left(\frac{1}{\lambda} V_A\right)(\lambda V_X) \subseteq W.$$

Therefore, $\pi(V) \subseteq U(B, W)$, showing that π is continuous with respect to the topology t_u on $\mathrm{CL}(X)$. □

Although Proposition 3.22 and Theorem 3.25 do not constitute exact analogues of the corresponding classical results for Banach algebras, they provide an appropriate framework for investigating the relationships among representations, modules and natural operator topologies in the setting of FSS-algebras.

Finally, we state a complementary result concerning complete metrizable FSS-algebras.

Lemma 3.26. *Let A be a complete metrizable strongly sequential algebra, let $K \subseteq A$ and let $\lambda > 0$. If $x \in \overline{K}$, then there exists an element $k \in K$ such that*

$$\beta(x - k) \leq \lambda.$$

Proof. Let U_A be a neighborhood of zero in A such that $u^n \to 0$ for every $u \in U_A$. Such a neighborhood exists due to the strongly sequential property of A.

Since $x + \lambda U_A$ is a neighborhood of x, it intersects K. Hence, there exist $u \in U_A$ and $k \in K$ such that

$$k = x + \lambda u.$$

It follows that

$$\begin{aligned}\beta(x-k) &= \beta\big(x-(x+\lambda u)\big)\\ &= \beta(-\lambda u)\\ &= \lambda\,\beta(u)\\ &\leq \lambda.\end{aligned}$$

Therefore, the claim is established. □

Proposition 3.27. *Let $T : A \to B$ be an epimorphism from a complete metrizable FSS-algebra A onto a semisimple complete metrizable FSS-algebra B. Then the kernel of T is closed and moreover,*

$$\mathrm{J}(A) \subseteq \ker(T).$$

Proof. Let $T : A \to B$ be an epimorphism onto the semisimple complete metrizable FSS-algebra B and set $K := \ker(T)$. We first show that K is closed.

Take any $a \in \overline{K}$ and $x \in A$. Let $(a_n)_n \subset K$ be a sequence converging to a. By continuity of the multiplication in A, we have

$$xa_n \to xa \quad \text{in } A.$$

Since $xa_n \in K$ for each n, it follows that $xa \in \overline{K}$.

By Lemma 3.26, there exists $k \in K$ such that

$$\beta(xa-k) < 1.$$

Thus, the hypotheses of Proposition 3.5 are satisfied and therefore $xa-k$ is quasi-invertible in A. Hence, there exists $y \in A$ such that

$$(xa-k)\circ y = y \circ (xa-k) = 0,$$

or equivalently,

$$(xa-k)+y-(xa-k)y = y+(xa-k)-y(xa-k) = 0.$$

Therefore,

$$(xa-k)y = y(xa-k) = (xa-k)+y.$$

Applying the homomorphism T and using $T(k)=0$, we get

$$T(xa)T(y) = T(y)T(xa) = T(xa)+T(y),$$

which shows that $T(xa)$ is quasi-invertible in B. Since $T(xa) = T(x)T(a)$ and $T(A) = B$, for every $b \in B$ there exists $x \in A$ such that $b = T(x)$. Consequently, $bT(a) = T(x)T(a) = T(xa) \in q\text{-Inv}(B)$. It follows that

$$BT(a) \subseteq q\text{-Inv}(B).$$

By Proposition 3.13(2), we obtain $T(a) \in J(B)$. Since B is semisimple, $J(B) = \{0\}$, and therefore $T(a) = 0$. Hence $a \in K$, showing that K is closed.

Now let $j \in J(A)$. By Proposition 3.13(1),

$$Aj \subseteq q\text{-Inv}(A).$$

Applying T yields

$$BT(j) = T(A)T(j) = T(Aj) \subseteq q\text{-Inv}(B).$$

Another application of Proposition 3.13(2) gives

$$T(j) \in J(B) = \{0\},$$

so that $T(j) = 0$. Consequently,

$$J(A) \subseteq \ker(T),$$

which completes the proof. □

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[1] Department of Mathematics, Faculty of Science, Payame Noor University (PNU), Tehran, Iran.
Email address: MSG@pnu.ac.ir

[2] Department of Basic Sciences, Faculty of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.
Email address: amorovvatpour@nus.ac.ir

[3] Department of Mathematics, University of Zanjan, University blvd, Iran.
Email address: aliyousefi@znu.ac.ir