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Some Fixed Point Results in Extended Rectangular Quasi b-Metric Spaces for Contractive Mappings

Nisha Balhara[1], Savita Rathee[2*], Divya Kaushik[3] and Diksha Goyal[4]

Abstract. In this paper, we introduce the concept of extended rectangular quasi b-metric spaces, which generalizes both quasi b-metric and rectangular quasi b-metric structures. Within this framework, we establish several fixed point results for self-mappings satisfying almost contraction conditions. Furthermore, we extend these results to Reich-type contractions and prove the existence of common fixed points for such mappings. Several illustrative examples are provided to demonstrate the applicability and effectiveness of the proposed theorems.

1. Introduction

The generalization of classical metric spaces has played a vital role in advancing fixed point theory and nonlinear analysis. Many real-world problems arising in optimization, differential equations, and dynamical systems cannot be effectively treated within the framework of standard metric spaces. To overcome these limitations, several generalized structures have been introduced over the past decades. Among the earliest contributions, Czerwik [6] proposed the concept of a b-metric space, where the triangular inequality is replaced by a relaxed inequality involving a constant factor. This generalization preserves many useful features of metric spaces while allowing greater flexibility in analytical applications.

In 2000, Branciari [5] proposed a significant generalization of metric spaces by replacing the classical triangular inequality with a more

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* Corresponding author.

flexible quadrilateral-type inequality involving four points, thereby introducing the concept of rectangular metric spaces along with an extension of the Banach contraction principle. In quasi-b-metric and extended quasi-b-metric spaces, Aydi [3] further investigated contractive conditions and common fixed point theorems, emphasizing their relevance to nonlinear mappings. Subsequently, George et al. [7] further developed this framework by defining rectangular b-metric spaces as a natural generalization of rectangular metric spaces, and established corresponding versions of the Banach contraction mapping theorem within this setting. Asim et al.[2] broaden the framework of rectangular b-metric spaces by introducing an extended rectangular b-metric structure and employ this setting to establish new fixed point results. They obtained theorems not only generalize but also refine several related results available in the literature.

Generalized metric structures, such as extended b-metric and quasi b-metric spaces, have proved to be powerful tools in establishing existence and uniqueness results for solutions of integral and differential equations. Motivated by this, Shatanawi et al.[15] introduced the notions of extended (α, ξ)-contractions and generalized extended (α, ξ)-contractions for self-mappings in the setting of extended quasi b-metric spaces, and also investigated the corresponding fixed point results. The study of fixed point results under generalized contraction principles has been actively extended to broader spaces including intuitionistic fuzzy, bi-b-metric, partial b-metric, and v-generalized b-metric spaces[4, 9, 12, 13].

Overall, the development of rectangular quasi-b-metric spaces reflects a natural progression in the generalization of distance structures. Their rich geometry and weaker constraints provide a powerful analytical setting for establishing new fixed point results and solving a variety of nonlinear problems. This motivates further investigation into their properties and the exploration of new classes of contractions within this generalized framework.

2. Preliminaries

We need the following symbols and class of functions:

- $R^+ = [0, \infty)$;
- R is the set of real numbers;
- N is the set of natural numbers;
- $\Psi = \{\psi : [0, \infty) \to [0, \infty)$, such that,$\psi$ is non decreasing, continuous, $\sum_{k=1}^{\infty} s^k \psi^k(t) < \infty$, $s\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$ if and only if $t = 0$, where ψ^k is the kth iterate if ψ and $s \geq 1\}$.

Definition 2.1 ([6]). Let Ξ be a nonempty set and $s \geq 1$ be a given real number. A function $\delta : \Xi \times \Xi \to [0, \infty)$ is a b-metric if and only if for all $\theta, y, z \in \Xi$, the following conditions are satisfied:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq s[\delta(\theta, z) + \delta(z, y)]$.

The pair (Ξ, δ) is called a b-metric space.

Definition 2.2 ([14]). Let Ξ be a non empty and $s \geq 1$.
Let $\delta : \Xi \times \Xi \to [0, \infty)$ be a function which satisfies:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq s[\delta(\theta, z) + \delta(z, y)]$.

Then δ is called a quasi b-metric and the pair (Ξ, δ) is called a quasi b-metric space. The number s is called the coefficient of (Ξ, δ).

Definition 2.3 ([8]). Let (Ξ, δ, s) be a quasi b-metric space and let $(\theta_n) \subseteq \Xi$ be a sequence.

(i) The sequence (θ_n) is called a *left Cauchy sequence* if

$$\delta(\theta_n, \theta_m) \to 0 \quad \text{as } n, m \to +\infty.$$

(ii) The quasi b-metric space (Ξ, δ, s) is said to be *left complete* if every left Cauchy sequence converges to some point $\theta \in \Xi$.

Remark. Any quasi metric space or any b-metric is a quasi b-metric space, but the converse is not true in general.

Example. Let $\Xi = \{1, 2, 3\}$ be a set and $\delta : \Xi \times \Xi \to [0, \infty)$ by

$$\delta(1, 1) = \delta(2, 2) = \delta(3, 3) = 0;$$
$$\delta(1, 2) = \delta(1, 3) = \delta(2, 3) = 6;$$
$$\delta(2, 1) = \delta(3, 1) = \delta(3, 2) = 5.$$

Then (Ξ, δ) is a quasi b-metric space with s ≥ 1 but it is not a b-metric space.

Definition 2.4 ([5]). Let Ξ be a nonempty set and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq \delta(\theta, u) + \delta(u, v) + \delta(v, y)$ for all $\theta, y \in \Xi$ and all distinct points $u, v \in \Xi$.

Then δ is called a rectangular metric on Ξ, and the pair (Ξ, δ) is called a rectangular metric space.

Definition 2.5 ([7]). Let Ξ be a non empty set, $s \geq 1$ be a given real number, and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq s[\delta(\theta, u) + \delta(u, v) + \delta(v, y)]$ for all $\theta, y \in \Xi$ and all distinct points $u, v \in \Xi$.

Then δ is called a rectangular b-metric on Ξ, and the pair (Ξ, δ) is called a rectangular b-metric space.

Example ([7]). Let $\Xi = N$, $\alpha > 0$ and $\delta : \Xi \times \Xi \to [0, \infty)$ such that:

(i) $\delta(\theta, y) = \delta(y, \theta)$ for all $\theta, y \in \Xi$;
(ii) $\delta(\theta, y) = 0$, if and only if $\theta \neq y$;
(iii) $\delta(\theta, y) = 4\alpha$, if $\theta, y \in \{1, 2\}$ and $\theta \neq y$;
(iv) $\delta(\theta, y) = \alpha$, if $\theta, y \notin \{1, 2\}$ and $\theta \neq y$.

Then (Ξ, δ) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (Ξ, δ) is not a rectangular metric space, as
$\delta(1, 2) = 4\alpha \not\leq 3\alpha = \delta(1, 3) + \delta(3, 4) + \delta(4, 2)$.

Definition 2.6 ([11]). Let X be a nonempty set. A function $\delta : \Xi \times \Xi \to [0, \infty)$ is a p-metric if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $t \leq \Omega(t)$ such that for all $\theta, z, y \in \Xi$, the following conditions hold:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq \Omega[\delta(\theta, z) + \delta(z, y)]$.

Then, the pair $(\Xi, \delta)$is called a p-metric space or an extended b-metric space. A b-metric is a p-metric, when $\Omega(t) = st$, $s \geq 1$ while a metric is a p-metric, when $\Omega(t) = t$.

Definition 2.7 ([10]). Let Ξ be a nonempty set, $s \geq 1$ and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq \delta(\theta, u) + \delta(u, v) + \delta(v, y)$ for all $\theta, y \in \Xi$ and all distinct points $u, v \in \Xi$.

Then δ is called a rectangular quasi metric on Ξ, and the pair (Ξ, δ) is called a rectangular quasi metric space.

Definition 2.8 ([1]). Let Ξ be a nonempty set, $s \geq 1$ and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq s\{\delta(\theta, u) + \delta(u, v) + \delta(v, y)\}$ for all $\theta, y \in \Xi$ and all distinct points $u, v \in \Xi$.

Then δ is called a rectangular quasi b-metric on Ξ, and the pair (Ξ, δ) is called a rectangular quasi b-metric space.

Definition 2.9 ([2])**.** Let (Ξ, δ) be an extended rectangular b-metric space and a self-mapping $\Upsilon : \Xi \to \Xi$ is called orbitally continuous if $\lim_{k\to\infty} \Upsilon^{nk}\theta = \theta$ for some $\theta \in \Xi$ implies $\lim_{k\to\infty} \Upsilon(\Upsilon^{nk}\theta) = \Upsilon\theta$. Besides, (Ξ, δ) is called Υ-orbitally complete if every cauchy sequence which is obtained in $\theta, \Upsilon\theta, \ldots, \Upsilon^n\theta, \ldots$ for some $\theta \in \Xi$ converges in Ξ.

3. Main Results

In this section, we introduce another type of generalized metric space, which we refer as extended rectangular quasi b-metric space (ERQbMS).

Definition 3.1. Let Ξ be a nonempty set,and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $t \leq \Omega(t)$ for all $t > 0$ and $\Omega(0) = 0$. A mapping $\delta : \Xi \times \Xi \to [0, \infty)$ is said to be an extended rectangular quasi b-metric on Ξ, if δ satisfies the following conditions:

(i) $\delta(\theta, \phi) = 0$ iff $\theta = \phi$;
(ii) $\delta(\theta, \phi) \leq \Omega\{\delta(\theta, u) + \delta(u, v) + \delta(v, \phi)\}$ $\forall$ $\theta, \phi \in \Xi$ and all distinct points $u, v \in \Xi$.

Then, the pair (Ξ, δ) is said to be an extended rectangular quasi b-metric space (ERQbMS).

Remark.
(i) If $\Omega(t) = st$ with $s \geq 1$, then (Ξ, δ) is a rectangular quasi b-metric space that is an extension of quasi b-metric space as well as rectangular b-metric space. Therefore, it is an extension of the quasi metric space and the rectangular metric space.
(ii) Every extended rectangular b-metric space is an extended rectangular quasi b-metric space.

Example. Consider $\Xi = \{1, 2, 3, 4\}$ and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function defined by

$$\delta(1,1) = \delta(2,2) = \delta(3,3) = \delta(4,4) = 0;$$
$$\delta(1,2) = \delta(4,2) = \delta(3,4) = 1;$$
$$\delta(2,3) = \delta(3,1) = \delta(4,1) = 2;$$
$$\delta(2,1) = \delta(4,3) = 3;$$
$$\delta(1,3) = \delta(3,2) = 4;$$
$$\delta(2,4) = 5, \delta(1,4) = 7.$$

Now, we show that (Ξ, δ) is an extented rectangular quasi b-metric space for $\Omega(t) = 3t$. Here,first condition is trivial. Now, for second condition, we have

Case 1:

$$\delta(1,4) \leq \Omega\{\delta(1,2) + \delta(2,3) + \delta(3,4)\}$$
$$7 \leq 3\{1+4+1\} \quad \Rightarrow \quad 7 \leq 18.$$

Also,

$$\delta(1,4) \leq \Omega\{\delta(1,3) + \delta(3,2) + \delta(2,4)\}$$
$$7 \leq 3\{4+4+5\} \quad \Rightarrow \quad 7 \leq 39.$$

Case 2:

$$\delta(2,4) \leq \Omega\{\delta(2,1) + \delta(1,3) + \delta(3,4)\}$$
$$5 \leq 3\{3+4+1\} \quad \Rightarrow \quad 5 \leq 24.$$

Also,

$$\delta(2,4) \leq \Omega\{\delta(2,3) + \delta(3,1) + \delta(1,4)\}$$
$$5 \leq 3\{2+2+7\} \quad \Rightarrow \quad 5 \leq 33.$$

Similarly, we can discuss other cases. Thus, $\forall\, \theta, y \in \Xi$ and for all distinct point $u, v \in \Xi$ we have

$$\delta(\theta, y) \leq \Omega\{\delta(\theta, u) + \delta(u, v) + \delta(v, y)\}.$$

Hence, (Ξ, δ) is an extented rectangular quasi b-metric space.

Definition 3.2. A sequence $\{\theta_n\}$ in ERQbMS (Ξ, δ) is said to be left convergent to $\theta \in \Xi$ if $\lim\limits_{n\to\infty} \delta\{\theta_n, \theta\} = 0$.

Definition 3.3. Let (Ξ, δ) be a ERQbMS and sequence $\{\theta_n\}$ in (Ξ, δ) is said to be left Cauchy if $\lim\limits_{n,m\to\infty} \delta\{\theta_n, \theta_m\} = 0$.

Definition 3.4. An ERQbMS (Ξ, δ) is said to be a left complete if every left cauchy in Ξ is convergent to some point in Ξ.

Theorem 3.5. *Let (Ξ, δ) be a ERQbMS with nontrivial function Ω and $\Upsilon : \Xi \to \Xi$ satisfies, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \Upsilon\phi) \leq \Gamma\delta(\theta, \phi) + L\delta(\phi, \Upsilon\theta), \tag{3.1}$$

where $\Gamma \geq 0$,$\Gamma + L < 1$ and $L \geq 0$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *Υ is orbitally sequential continuous.*

Then, Υ admits a unique fixed point.

Proof. Let $\theta_0 \in \Xi$ be arbitrary. Consider the sequence $\{\theta_n\}$ by $\theta_{n+1} = \Upsilon\theta_n$ for all $n \geq 0$. We shall show that $\{\theta_n\}$ is a cauchy sequence. If $\{\theta_n\} = \{\theta_{n+1}\}$ then $\{\theta_n\}$ is a fixed point of Υ. So, assume

that $\{\theta_n\} \neq \{\theta_{n+1}\}$ for all $n \geq 0$. Define $\delta(\theta_n, \theta_{n+1}) = S_n$, it follows from (3.1) that

$$\begin{aligned}
&\delta(\theta_n, \theta_{n+1}) = \delta(\Upsilon\theta_{n-1}, \Upsilon\theta_n) \leq \Gamma\delta(\theta_{n-1}, \theta_n) + L\delta(\theta_n, \Upsilon\theta_{n-1}) \\
&\delta(\theta_n, \theta_{n+1}) \leq \Gamma\delta(\theta_{n-1}, \theta_n) + L\delta(\theta_n, \theta_n), \\
&\delta(\theta_n, \theta_{n+1}) \leq \Gamma\delta(\theta_{n-1}, \theta_n) \\
\Rightarrow \quad &S_n \leq \Lambda S_{n-1}, \text{ where } \Lambda = \Gamma.
\end{aligned}$$

Repeating this process, we get $\Rightarrow S_n \leq \Lambda^n S_0$. Now for $m > n$,

$$\begin{aligned}
\delta(\theta_n, \theta_m) &\leq \Omega\{\delta(\theta_n, \theta_{n+1}) + \delta(\theta_{n+1}, \theta_{n+2}) + \ldots + \delta(\theta_{m-1}, \theta_m)\} \\
&\leq \Omega\{\Lambda^n(\delta(\theta_0, \theta_1)) + \Lambda^{n+1}(\delta(\theta_0, \theta_1)) + \ldots + \Lambda^{m-1}(\delta(\theta_0, \theta_1))\} \\
&\leq \Omega\left\{(\Lambda^n + \Lambda^{n+1} + \Lambda^{n+2} + \ldots + \Lambda^m)(\delta(\theta_0, \theta_1))\right\} \text{ here } \Lambda < 1 \\
&\leq \Omega\sum_{i=n}^{m} S_i \leq \Omega\sum_{i=n}^{\infty} \Lambda^i S_0 \leq \Omega\left\{\frac{S_0\Lambda^n}{1-\Lambda}\right\}.
\end{aligned}$$

As $n \to \infty$ then $\delta(\theta_n, \theta_m) \to 0$. Hence $\{\theta_n\}$ is cauchy and by completeness it converges to some point $u \in \Xi$ implies that $\lim_{n\to\infty}(\theta_n) = u$.

Now, we shall show that u is fixed point of Υ. Since $\{\theta_n\} \to u$ and using picard iteration $\Upsilon(\theta_n) = \theta_{n+1} \to u$ after using orbital sequential continuity $\Upsilon(\theta_n) \to \Upsilon(u)$ but due to uniqueness property, $\Upsilon(u) = u$. Thus, u is a fixed point of Υ. Assume v be another fixed point of Υ. Then it follows

$$\begin{aligned}
\delta(u, v) = \delta(\Upsilon u, \Upsilon v) &\leq [\Gamma\delta(u, v) + L\delta(v, \Upsilon u) \\
&\leq [\Gamma\delta(u, v) + L\delta(v, u)
\end{aligned} \tag{3.2}$$

$$\delta(u, v) \leq [\Gamma\delta(u, v) + L\delta(v, u)$$

$$\begin{aligned}
\delta(v, u) = \delta(\Upsilon v, \Upsilon u) &\leq [\Gamma\delta(v, u) + L\delta(u, \Upsilon v) \\
&\leq [\Gamma\delta(v, u) + L\delta(u, v)
\end{aligned} \tag{3.3}$$

$$\delta(v, u) \leq [\Gamma\delta(v, u) + L\delta(u, v)$$

from equation (3.2) and (3.3)

$$(1 - \Gamma - L)[\delta(u, v) - \delta(v, u)] \leq 0 \quad \Rightarrow \quad [\delta(u, v) - \delta(v, u)] \leq 0$$

because $\Gamma + L < 1$ and distance never be

$$\text{negative} \quad \Rightarrow \quad \delta(u, v) = 0.$$

Therefore, we get $u = v$. Thus fixed point is unique. □

Example. Consider $\Xi = A \cup B$, where $A = \left\{\frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\right\}$ and $B = \left\{\frac{2}{3}\right\}$. Define $\delta : \Xi \times \Xi \to [0, \infty)$ such that

$$\delta\left(\frac{1}{7}, \frac{1}{9}\right) = \delta\left(\frac{1}{10}, \frac{1}{8}\right) = 0.3;\ \delta\left(\frac{1}{7}, \frac{1}{10}\right) = \delta\left(\frac{1}{8}, \frac{1}{9}\right) = 0.01;$$
$$\delta\left(\frac{1}{7}, \frac{1}{8}\right) = \delta\left(\frac{1}{9}, \frac{1}{10}\right) = 0.14;\ \delta\left(\frac{1}{9}, \frac{1}{7}\right) = \delta\left(\frac{1}{8}, \frac{1}{10}\right) = 0.23;$$
$$\delta\left(\frac{1}{10}, \frac{1}{7}\right) = \delta\left(\frac{1}{9}, \frac{1}{8}\right) = 0.012;\ \delta\left(\frac{1}{8}, \frac{1}{7}\right) = \delta\left(\frac{1}{10}, \frac{1}{9}\right) = 0.013;$$
$$\delta\left(\frac{1}{8}, \frac{2}{3}\right) = \delta\left(\frac{2}{3}, \frac{1}{8}\right) = 0.66;\ \delta\left(\frac{1}{10}, \frac{2}{3}\right) = 0.2,\ \delta\left(\frac{2}{3}, \frac{1}{10}\right) = 0.5;$$
$$\delta(u, v) = |u - v|^2 + |u - v| \text{ otherwise.}$$

Then (Ξ, δ) is ERQbMS with $\Omega(t) = 17t$ and let $\Upsilon : \Xi \to \Xi$ is orbitally sequential continuous mapping defined by

$$\Upsilon(\theta) = \begin{cases} \frac{1}{8}; & \text{if } \theta \in A \\ \frac{1}{10}; & \text{if } \theta \in B. \end{cases}$$

Hence, Υ satisfies all the conditions of above theorem and therefore, Υ has a unique fixed point which is $\theta = \frac{1}{8}$.

Theorem 3.6. *Let (Ξ, δ) be a ERQbMS with $\Omega(t)$=st, where $s \geq 1$ and $s\alpha < 1$, and $\Upsilon : \Xi \to \Xi$ satisfies, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \Upsilon\phi) \leq \alpha\delta(\theta, \Upsilon\theta) + \beta\delta(\phi, \Upsilon\phi) + \Gamma\delta(\theta, \phi), \tag{3.4}$$

where $\alpha, \beta, \Gamma \geq 0$ and $\alpha + \beta + \Gamma < 1$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *Υ is orbitally sequential continuous.*

Then, Υ has a unique fixed point.

Proof. Let $\theta_0 \in \Xi$ be arbitrary. Define the sequence $\{\theta_n\}$ by $\theta_{n+1} = \Upsilon\theta_n$ for all $n \geq 0$. We shall show that $\{\theta_n\}$ is a cauchy sequence. If $\{\theta_n\} = \{\theta_{n+1}\}$ then $\{\theta_n\}$ is a fixed point of Υ. So, suppose that $\{\theta_n\} \neq \{\theta_{n+1}\}$ for all $n \geq 0$. Define $\delta(\theta_n, \theta_{n+1}) = S_n$, it follows from equation (3.4) that

$$\begin{aligned}
\delta(\theta_n, \theta_{n+1}) = \delta(\Upsilon\theta_{n-1}, \Upsilon\theta_n) &\leq \alpha\delta(\theta_{n-1}, \Upsilon\theta_{n-1}) \\
&\quad + \beta\delta(\theta_n, \Upsilon\theta_n) + \Gamma\delta(\theta_{n-1}, \theta_n) \\
\delta(\theta_n, \theta_{n+1}) &\leq \alpha\delta(\theta_{n-1}, \theta_n) + \beta\delta(\theta_n, \theta_{n+1}) + \Gamma\delta(\theta_{n-1}, \theta_n) \\
(1 + \beta)\delta(\theta_n, \theta_{n+1}) &\leq (\alpha + \Gamma)\delta(\theta_{n-1}, \theta_n) \\
\delta(\theta_n, \theta_{n+1}) &\leq \frac{(\alpha + \Gamma)}{(1 + \beta)}\delta(\theta_{n-1}, \theta_n)
\end{aligned}$$

$$\Rightarrow \quad \delta(\theta_n, \theta_{n+1}) \leq \Lambda\delta(\theta_{n-1}, \theta_n), \text{ where } \Lambda = \frac{(\alpha+\Gamma)}{(1+\beta)} \text{ and } \Lambda \in [0,1)$$

$$\Rightarrow \quad S_n \leq \Lambda S_{n-1}.$$

Repeating this process, we get $\Rightarrow S_n \leq \Lambda^n S_0$. Now for $m > n$,

$$\begin{aligned}\delta(\theta_n, \theta_m) &\leq \Omega\left(\delta(\theta_n, \theta_{n+1}) + \delta(\theta_{n+1}, \theta_{n+2}) + \ldots + \delta(\theta_{m-1}, \theta_m)\right)\\ &\leq \Omega\left(\Lambda^n\left(\delta(\theta_0, \theta_1)\right) + \Lambda^{n+1}\left(\delta(\theta_0, \theta_1)\right) + \ldots + \Lambda^{m-1}\left(\delta(\theta_0, \theta_1)\right)\right)\\ &\leq \Omega\left(\Lambda^n + \Lambda^{n+1} + \Lambda^{n+2} + \ldots + \Lambda^m\}\left(\delta(\theta_0, \theta_1)\right)\right) \text{ here } \Lambda < 1\end{aligned}$$

$$\delta(\theta_n, \theta_m) \leq \Omega\left(\sum_{i=n}^{m} S_i\right) \leq \Omega\left(\sum_{i=n}^{\infty} \Lambda^i S_0\right) \leq \Omega\left(\frac{S_0\Lambda^n}{1-\Lambda}\right).$$

As $n \to \infty$ then $\delta(\theta_n, \theta_m) \to 0$. Hence $\{\theta_n\}$ is cauchy and by completeness it converges to some point $u \in \Xi$ implies that $\lim_{n\to\infty}(\theta_n) = u$.

Now, we shall show that u is fixed point of Υ. Since $\{\theta_n\} \to u$ and using picard iteration $\Upsilon(\theta_n) = \theta_{n+1} \to u$ after using orbital sequential continuity $\Upsilon(\theta_n) \to \Upsilon(u)$ but due to uniqueness property, $\Upsilon(u) = u$. Thus, u is a fixed point of Υ. Assume v be another fixed point of Υ. Then it follows

$$\begin{aligned}\delta(u, v) = \delta(\Upsilon u, \Upsilon v) &\leq \Omega[\alpha\delta(u, v) + \beta\delta(u, \Upsilon u) + \gamma\delta(v, \Upsilon v)]\\ &\leq \Omega[\alpha\delta(u, v) + \beta\delta(u, u) + \gamma\delta(v, v)]\\ &= \Omega[\alpha\delta(u, v)]\end{aligned}$$

implies that

$$\delta(u, v) \leq \Omega[\alpha\delta(u, v)] = s\alpha\delta(u, v)$$

$$\begin{aligned}(1 - s\alpha)\delta(u, v) \leq 0 \text{ where } s\alpha < 1 \quad &\Rightarrow \quad (1 - \Omega\alpha) \leq 0 \text{ not possible}\\ &\Rightarrow \quad \delta(u, v) = 0.\end{aligned}$$

Therefore, we get $u = v$. Thus fixed point is unique. □

Example. Let $\Xi = \{1, 2, 3\}$ and define $\delta : \Xi \times \Xi \to [0, \infty)$ be a function defined by

$$\begin{aligned}&\delta(1,1) = \delta(2,2) = \delta(3,3) = 0;\\ &\delta(1,2) = 2, \delta(2,1) = 1;\\ &\delta(1,3) = 3, \delta(3,1) = 1;\\ &\delta(2,3) = 2, \delta(3,2) = 1.\end{aligned}$$

Then (Ξ, δ) is an extended rectangular quasi b-metric space with $\Omega(t) = 2t$ and also let $\Upsilon : \Xi \to \Xi$ is orbitally sequential continuous mapping define as $\Upsilon(\theta) = 2$ for all $\theta \in \Xi$ Hence, Υ satisfies all the

conditions of the above theorem and therefore, Υ has a unique fixed point which is $\theta = 2$.

Corollary 3.7. *Let (Ξ, δ) be a ERQbMS with nontrivial function Ω and $\Upsilon : \Xi \to \Xi$ satisfies, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \Upsilon\phi) \leq \Gamma\delta(\theta, \phi)$$

where $\Gamma \in [0, 1)$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *Υ is orbitally sequential continuous.*

Then, Υ admits a unique fixed point.

Corollary 3.8. *Let (Ξ, δ) be a ERQbMS with nontrivial function Ω and $\Upsilon : \Xi \to \Xi$ satisfies, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \Upsilon\phi) \leq \alpha\{\delta(\theta, \Upsilon\theta) + \delta(\phi, \Upsilon\phi\}$$

where $0 \leq \alpha < \frac{1}{2}$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *Υ is orbitally sequential continuous.*

Then, Υ admits a unique fixed point.

Theorem 3.9. *Let (Ξ, δ) be a ERQbMS with nontrivial function Ω and Υ, $\mathcal{S} : \Xi \to \Xi$ are two self mappings such that $\Upsilon(\Xi) \subset \mathcal{S}(\Xi)$ and satisfying, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \mathcal{S}\phi) \leq \alpha\delta(\theta, \Upsilon\theta) + \beta\delta(\phi, \mathcal{S}\phi) + \Gamma\delta(\theta, \phi) \tag{3.5}$$

where $\alpha, \beta, \Gamma \geq 0$ and $\alpha + \beta + \Gamma < 1$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *$\Upsilon, \mathcal{S}$ are orbitally sequential continuous,*
(iii) *Ω satisfies a local linear bound $\Omega(t) \leq Ct$ for small t, and $C\left\{k(\frac{\alpha^2+\alpha\Gamma}{1-\beta} + \beta) + \Gamma\right\} < 1$,*
(iv) *there exist a $k > 0$ such that $\delta(\theta, \phi) \leq k\delta(\phi, \theta)$, for all $\theta, \phi \in \Xi$.*

Then, Υ and $\mathcal{S}$ have a common fixed point.

Proof. Let an arbitrary $\theta_0 \in \Xi$ and define the sequence $\{\theta_n\}$ by $\theta_{2n+1} = \Upsilon\theta_{2n}$, $\mathcal{S}(\theta_{2n+1}) = \theta_{2n+2}$ for all $n \geq 0$.

Define $T_n = \delta(\theta_{n+1}, \theta_n)$, $R_n = \delta(\theta_n, \theta_{n+1})$ for all $n \geq 0$.

$$\begin{aligned} R_{2n+1} &= \delta(\Upsilon\theta_{2n}, \mathcal{S}\theta_{2n+1}) \\ &\leq \alpha\delta(\theta_{2n}, \Upsilon\theta_{2n}) + \beta\delta(\theta_{2n+1}, \mathcal{S}\theta_{2n+1}) + \Gamma\delta(\theta_{2n}, \theta_{2n+1}) \\ &\leq \alpha\delta(\theta_{2n}, \theta_{2n+1}) + \beta\delta(\theta_{2n+1}, \theta_{2n+2}) + \Gamma\delta(\theta_{2n}, \theta_{2n+1}) \\ &\leq \alpha R_{2n} + \beta R_{2n+1} + \Gamma R_{2n} \end{aligned} \tag{3.6}$$

$$(1-\beta)R_{2n+1} \leq (\alpha+\Gamma)R_{2n} \quad \Rightarrow \quad R_{2n+1} \leq \frac{\alpha+\Gamma}{1-\beta}R_{2n}$$

$$R_{2n+1} \leq \Lambda R_{2n}, \quad \text{where } \Lambda = \frac{\alpha+\Gamma}{1-\beta}.$$

Let $\theta = \theta_{2n+1}$, $\phi = \theta_{2n}$

$$\delta(\Upsilon\theta_{2n+1}, \mathcal{S}\theta_{2n}) \leq \alpha\delta(\theta_{2n+1}, \Upsilon\theta_{2n+1}) + \beta\delta(\theta_{2n}, \mathcal{S}\theta_{2n}) + \Gamma\delta(\theta_{2n+1}, \theta_{2n})$$
$$\delta(\theta_{2n+2}, \theta_{2n+1}) \leq \alpha\delta(\theta_{2n+1}, \theta_{2n+2}) + \beta\delta(\theta_{2n}, \theta_{2n+1}) + \Gamma\delta(\theta_{2n+1}, \theta_{2n})$$
$$T_{2n+1} \leq \alpha R_{2n+1} + \beta R_{2n} + \Gamma T_{2n}.$$

Now, using equation (3.6), we get

$$T_{2n+1} \leq \alpha\Lambda R_{2n} + \beta R_{2n} + \Gamma T_{2n}$$
$$T_{2n+1} \leq (\alpha\Lambda + \beta)R_{2n} + \Gamma T_{2n} \tag{3.7}$$

by assumption (iv),

$$R_{2n} = \delta(\theta_{2n}, \theta_{2n+1}) \leq k\delta(\theta_{2n+1}, \theta_{2n}) = kT_{2n}$$
$$\Rightarrow \quad R_{2n} \leq kT_{2n} \ \text{ for all } n \geq 0. \tag{3.8}$$

Now, we use equation (3.8) in equation (3.7), we have

$$T_{2n+1} \leq (\alpha\Lambda + \beta)kT_{2n} + \Gamma T_{2n}$$
$$\leq \{(\alpha\Lambda + \beta)k + \Gamma\}T_{2n}.$$

Using value of Λ in above inequality, we have

$$T_{2n+1} \leq \left\{\left(\alpha\frac{\alpha+\Gamma}{1-\beta} + \beta\right)k + \Gamma\right\}T_{2n}$$

$$\Rightarrow \quad T_{2n+1} \leq \tilde{r}T_{2n}, \text{ where } \tilde{r} = \left\{\left(\alpha\frac{\alpha+\Gamma}{1-\beta} + \beta\right)k + \Gamma\right\} \text{ and } 0 \leq \tilde{r} \leq 1.$$

$$T_{2n+1} \leq \tilde{r}T_{2n} \leq \tilde{r}^2T_{2n} \leq \cdots \leq \tilde{r}^{n+1}T_0.$$

Letting as $n \to \infty$ then $\tilde{r}^{n+1} \to 0$.

$$R_n = \delta(\theta_n, \theta_{n+1}) \leq k\delta(\theta_{n+1}, \theta_n) = kT_n \to 0 \text{ as } n \to \infty \Rightarrow R_n \to 0.$$

We know that $R_{n+1} \leq \Lambda R_n$, $\Lambda = \frac{\alpha+\Gamma}{1-\beta}$. So, repeating this process we get $R_n \leq \Lambda^n R_0$. Hence,

$$\sum_{n=0}^{\infty} R_n \leq R_0 \sum_{n=0}^{\infty} \Lambda^n = \frac{R_0}{1-\Lambda} < \infty.$$

Now, using extended rectangular inequality,

$$\delta(\theta_n, \theta_m) \leq \Omega\left(\sum_{k=n}^{m-1} R_k\right) \leq \Omega\left(\sum_{k=n}^{\infty} R_k\right), \quad m > n.$$

Therefore, we have the tail sum tends to zero as $n \to \infty$. By monotonicity of Ω, it follow that $\delta(\theta_n, \theta_m) \to 0$, uniformly in $m > n$. Hence, $\{\theta_n\}$ is left cauchy and by left completeness of Ξ, there exist $\theta \in \Xi$ such that $\delta(\theta_n, \theta) \to 0$.

Now, we show that θ is common fixed point of both the mapping Υ and $\mathcal{S}$. Therefore,

$$\begin{aligned}\delta(\Upsilon\theta_{2n}, \mathcal{S}\theta) &= \delta(\theta_{2n+1}, \mathcal{S}\theta)\\ &\leq \alpha\delta(\theta_{2n}, \Upsilon\theta_{2n}) + \beta(\theta, \mathcal{S}\theta) + \Gamma\delta(\theta_{2n}, \theta)\\ &\leq \alpha\delta(\theta_{2n}, \theta_{2n+1}) + \beta(\theta, \mathcal{S}\theta) + \Gamma\delta(\theta_{2n}, \theta)\\ &\leq \alpha R_{2n} + \beta(\theta, \mathcal{S}\theta) + \Gamma\delta(\theta_{2n}, \theta)\\ &\leq \beta(\theta, \mathcal{S}\theta).\end{aligned}$$

Taking limit supremum $n \to \infty$, we get

$$\limsup_{n\to\infty} \delta(\theta_{2n+1}, \mathcal{S}\theta) \leq \beta\delta(\theta, \mathcal{S}\theta). \tag{3.9}$$

Now, by extended rectangular inequality,

$$\delta(\theta, \mathcal{S}\theta) \leq \Omega\left\{\delta(\theta, \theta_{2n+1}) + \delta(\theta_{2n+1}, \mathcal{S}\theta) + \delta(\mathcal{S}\theta, \mathcal{S}\theta)\right\}$$

$$\delta(\theta, \mathcal{S}\theta) \leq \Omega\left\{\delta(\theta, \theta_{2n+1}) + \delta(\theta_{2n+1}, \mathcal{S}\theta)\right\}. \tag{3.10}$$

Again taking limit supremum $n \to \infty$, we have

$$\delta(\theta, \mathcal{S}\theta) \leq \Omega \limsup_{n\to\infty}\left\{\delta(\theta, \theta_{2n+1}) + \delta(\theta_{2n+1}, \mathcal{S}\theta)\right\}$$

$$\delta(\theta, \mathcal{S}\theta) \leq \Omega\left\{\delta(\theta, \theta) + \limsup_{n\to\infty} \delta(\theta_{2n+1}, \mathcal{S}\theta)\right\}$$

$$\delta(\theta, \mathcal{S}\theta) \leq \Omega \limsup_{n\to\infty} \delta(\theta_{2n+1}, \mathcal{S}\theta). \tag{3.11}$$

Now, using (3.9) in equation (3.11)

$$\begin{aligned}\delta(\theta, \mathcal{S}\theta) &\leq \Omega \limsup_{n\to\infty} \delta(\theta_{2n+1}, \mathcal{S}\theta)\\ &\leq \Omega\beta\delta(\theta, \mathcal{S}\theta)\\ &\leq C\beta\delta(\theta, \mathcal{S}\theta)\\ \Rightarrow \quad \delta(\theta, \mathcal{S}\theta) &= 0 \text{ by using linear bound property of } \Omega\\ \mathcal{S}\theta &= \theta.\end{aligned}$$

Similarly, we can prove that $\Upsilon\theta = \theta$. Suppose θ and ϕ be two common fixed points of the mapping Υ and $\mathcal{S}$. Therefore, $\mathcal{S}\theta = \Upsilon\theta = \theta$ and $\mathcal{S}\phi = \Upsilon\phi = \phi$. Then, we obtain

$$\begin{aligned}\delta(\theta, \phi) &= \delta(\Upsilon\theta, \mathcal{S}\phi)\\ &\leq \alpha\delta(\theta, \Upsilon\theta) + \beta\delta(\phi, \mathcal{S}\phi) + \Gamma\delta(\theta, \phi)\\ &\leq \alpha\delta(\theta, \theta) + \beta\delta(\phi, \phi) + \Gamma\delta(\theta, \phi)\end{aligned}$$

$$\leq \Gamma\delta(\theta, \phi).$$

Since $\Gamma < 1$ implies that $\delta(\theta, \phi) = 0 \Rightarrow \theta = \phi$. □

Corollary 3.10. *Let (Ξ, δ) be a ERQbMS with nontrivial function Ω and $\Upsilon : \Xi \to \Xi$ satisfies, for all $\theta, \phi \in \Xi$,*

$$\delta(\Upsilon\theta, \mathcal{S}\phi) \leq \alpha\delta(\theta, \Upsilon\theta) + \beta\delta(\phi, \mathcal{S}\phi) + \Gamma\delta(\theta, \phi) \tag{3.12}$$

where $\alpha, \beta, \Gamma \geq 0$ and $\alpha + \beta + \Gamma < 1$. Assume that

(i) *(Ξ, δ) is left complete,*
(ii) *Υ is orbitally sequential continuous.*

Then, Υ has a unique fixed point.

Proof. Here, we take $\mathcal{S} = \Upsilon$ then it become Theorem 3.6. □

4. Conclusion

In this study, we proposed a new distance space of the Ω-rectangular type without assuming symmetry. We proved the existence and uniqueness of a fixed point for single-valued mappings using a Reich-type contraction along the orbit and an orbital sequential symmetry requirement. The convergence of the iterative sequence is guaranteed by completeness, and its left Cauchy nature is guaranteed by the Ω-condition. For two single-valued mappings with an appropriate Reich-type contractive condition, we also get a common fixed point result. These results extend several classical fixed point theorems by removing symmetry assumptions and working in a more general Ω-controlled setting.

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[1] Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India.

Email address: nishadevi.rs.maths@mdurohtak.ac.in

[2] Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India.
Email address: savitarathee.math@mdurohtak.ac.in

[3] Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India.
Email address: divya.rs.maths@mdurohtak.ac.in

[4] Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India.
Email address: diksha.rs.maths@mdurohtak.ac.in