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Multiplicative φ-Fixed Point and Common Multiplicative φ-Fixed Point Results with Application to Caputo Fractional Differential Equation in Multiplicative Metric Space

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Abstract. This research paper establishes φ-fixed point Theorems in multiplicative metric spaces. We propose the concept of a multiplicative φ-fixed point, thereby generalizing the classical φ-fixed point framework. A structured family of functions is introduced and employed to derive existence and uniqueness results for $\varrho\varphi$ contraction. In addition, φ-fixed point result under a rational-type contraction is established and the main result is also extended to a common multiplicative φ-fixed point theorem. Representative examples are included to clarify the theoretical constructions. Finally, the theoretical framework is utilized to guarantee the existence of solutions for a Caputo fractional differential equation.

1. Introduction

Metric fixed point theory (FPT) is a fundamental branch of nonlinear functional analysis, valued for both its theoretical importance and practical applications. Since Banach fixed point theorem [5], a vast body of results has been developed over the last century. Progress in this area has primarily followed two directions: weakening the contractive conditions imposed on mappings, and modifying the underlying abstract framework. Consequently, the study of new classes of spaces and their intrinsic properties has attracted significant attention. Notable examples include quasi-metric spaces, b-metric spaces, symmetric

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spaces, fuzzy metric spaces, dislocated metric spaces, partial metric spaces, 2-metric spaces, along with various hybrid generalizations of these structures.

In 2008, Bashirov et al. [6] proposed the notion of a multiplicative metric space (MM-Space), replacing addition and subtraction with multiplication and division to simplify computations. They investigated the properties of multiplicative derivatives and integrals and established the fundamental theorem of multiplicative calculus. After that Özavsar and Cevikel [13] introduced the concept of multiplicative contraction mappings and proved several fixed point (FP) theorems in complete (MM-Space), along with some fundamental topological properties of these spaces. Subsequently, Hxiaoju et al. [7] studied common fixed points (CFP) for weakly commutative mappings in the multiplicative framework. After these developments, a wide variety of existing structures were generalized to the multiplicative setting, leading to the introduction of b-(MM-Space), S-(MM-Space), orthogonal S-(MM-Space), and several other related multiplicative spaces, on which extensive FP results have been established.

In 2014, Jleli et al. [9] introduced the study of fixed points of mappings that coincide with the zeros of a lower semi-continuous (LSC) function φ. They defined the notions of φ-fixed points (φ-FP) and (F, φ)-contractions to establish existence and uniqueness results for φ-FP in complete metric spaces (CMS). Characterizing fixed points via the zeros of the function φ significantly simplifies the process of identifying such points. Subsequently, the study of φ-FP problems has attracted considerable attention, with further contributions reported in works such as [2, 8, 10, 11, 14]. In 2021, Saleh et al. [16] further extended this framework by introducing the concepts of φ-coincidence points (φ-CP) and common φ-FP, and established several related results in metric and partial metric spaces.

Furthermore fractional differential equations have attracted considerable attention because they provide a useful mathematical framework for describing complex systems with memory and hereditary properties. They arise in various applications, including anomalous diffusion, viscoelasticity, control theory, biological systems, and financial models. Unlike classical integer-order differential equations, fractional differential equations can incorporate the influence of past states into the mathematical description of a system. From a mathematical perspective, establishing the existence and uniqueness of solutions to such equations is therefore an important problem. Fixed point theory provides an effective approach to addressing this problem under suitable conditions.

Recent studies have investigated the existence, uniqueness, and numerical approximation of solutions to fractional differential and integro-differential equations involving Hilfer-type derivatives, delay systems, and stochastic fractional models [3, 4, 17, 18]. These developments highlight the growing importance of fractional calculus in analyzing complex dynamical systems and motivate further research into the existence and uniqueness of solutions using fixed point theory.

In this paper, we introduce the concept of a multiplicative φ-FP and define an associated family of functions ϱ to establish existence and uniqueness results. We also obtain fixed point theorems for rational-type contractions and extend the results to common multiplicative φ-FP. Illustrative examples are provided, and an application to the existence of solutions for a Caputo fractional differential equation is presented.

2. Preliminaries

In this section, some important definitions and results are provided for further explanation of the main results.

Definition 2.1. ([6]). Let X be a nonempty set. A mapping

$$d : X \times X \to \mathbb{R}$$

is called a multiplicative metric if it satisfies:

(i) $d(x, y) \geq 1$ for all $x, y \in X$;
(ii) $d(x, y) = 1$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iv) $d(x, z) \leq d(x, y)\, d(y, z)$ for all $x, y, z \in X$.

Then (X, d) is called a multiplicative metric space.

Definition 2.2. (Multiplicative Reverse Triangle Inequality) ([13]). Let (X, d) be a multiplicative metric space. Then, for all $x, y, z \in X$, the following inequality holds:

$$\max\left\{\frac{d(x,z)}{d(y,z)}, \frac{d(y,z)}{d(x,z)}\right\} \leq d(x, y)$$

Equivalently,

$$\left|\frac{d(x,z)}{d(y,z)}\right|^* \leq d(x, y)$$

where $|t|^* = \max\{t, t^{-1}\}$ for $t > 0$.

Definition 2.3. ([13]). Let (X, d) be a multiplicative metric space, $a \in X$ and $r > 1$. The multiplicative open ball centered at a with radius r is

$$B_r(a) = \{x \in X : d(a, x) < r\}.$$

The multiplicative closed ball is defined by

$$\overline{B}_r(a) = \{x \in X : d(a,x) \leq r\}.$$

Definition 2.4. ([13]). Let A be a subset of a multiplicative metric space (X,d). A point $x \in X$ is called a limit point of A if

$$(A \cap B_\varepsilon(x)) \setminus \{x\} \neq \varnothing, \quad \text{for every } \varepsilon > 1.$$

Definition 2.5. ([13]) Let (X,d) and (Y,ρ) be multiplicative metric spaces and $a \in X$. A function $f : (X,d) \to (Y,\rho)$ is said to be multiplicative continuous at a if for every $\varepsilon > 1$ there exists $\delta > 1$ such that

$$d(x,a) < \delta \quad \Longrightarrow \quad \rho(f(x),f(a)) < \varepsilon.$$

Equivalently,

$$f(B(a;\delta)) \subseteq B(f(a);\varepsilon).$$

The function f is multiplicative continuous on X if it is continuous at every point of X.

Remark 2.6. The main difference between ordinary continuity and multiplicative continuity lies in their distance structures. In an ordinary metric space with metric ρ, the distance between convergent points approaches 0, whereas in a multiplicative metric space with metric d, it approaches 1. Moreover, the two notions are equivalent under the logarithmic transformation $\rho(x,y) = \ln d(x,y)$.

Definition 2.7. ([13]). A sequence $\{x_n\}$ in a multiplicative metric space (X,d) is said to be multiplicatively convergent to $x \in X$ if for every $\varepsilon > 1$ there exists $n_0 \in \mathbb{N}^*$ such that

$$d(x_n,x) < \varepsilon, \quad \text{for all } n \geq n_0.$$

We write $x_n \to x$ as $n \to \infty$.

Definition 2.8. ([13]). A sequence $\{x_n\}$ in a multiplicative metric space (X,d) is called a multiplicative Cauchy sequence if for every $\varepsilon > 1$ there exists $n_0 \in \mathbb{N}^*$ such that

$$d(x_n,x_m) < \varepsilon, \quad \text{for all } n,m \geq n_0.$$

Definition 2.9. ([13]). A multiplicative metric space (X,d) is said to be complete if every multiplicative Cauchy sequence in X converges to a point of X.

Definition 2.10. ([13]). Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists $\lambda \in [0,1)$ such that

$$d(f(x),f(y)) \leq (d(x,y))^\lambda, \quad \text{for all } x,y \in X.$$

Definition 2.11. ([9]). Let $\varphi : E \to [0, \infty)$ be a lower semi continous function and $T : E \to E$. A point $\sigma \in E$ is called a φ-fixed point of T if $\sigma = T\sigma$ and $\varphi(\sigma) = 0$.

Definition 2.12. ([9]). A function $F : [0, \infty)^3 \to [0, \infty)$ is said to be a control function if for $\theta_1, \theta_2, \theta_3 \in [0, \infty)$, it satisfies the following conditions:

$(F1)$ $\max\{\theta_1, \theta_2\} \leq F(\theta_1, \theta_2, \theta_3)$;
$(F2)$ $F(0, 0, 0) = 0$;
$(F3)$ F is continuous.

3. Main Results

Definition 3.1. Let (Ξ, ζ) be a Multiplicative metric space, $\varphi : \Xi \to [1, \infty)$ be a given Lower semi-continuous function and $\Gamma : \Xi \to \Xi$ be an operator. The set of all fixed points of the operator Γ is denoted by

$$F_\Gamma = \{\sigma \in \Xi : \Gamma\sigma = \sigma\}.$$

The set of all ones of the function φ is denoted by

$$\theta_\varphi = \{\sigma \in X : \varphi(\sigma) = 1\}.$$

Definition 3.2. An element $\mu \in \Xi$ is said to be a multiplicative φ-fixed point of the operator Γ if and only if

$$\mu \in F_\Gamma \cap \theta_\varphi.$$

Example 3.3. Consider $\varphi = \zeta(\sigma, \Gamma\sigma)$, where ζ is the multiplicative metric on some non-empty space Ξ and Γ is the given self-map on Ξ. Then clearly fixed point of Γ is multiplicative φ-FP of Γ.

Example 3.4. Consider $\varphi = e^{\zeta(\sigma, \Gamma\sigma)}$, where ζ is the usual metric on some non-empty space Ξ and Γ is the given self-map on Ξ. Then clearly FP of Γ is multiplicative φ-FP of Γ

We denote by ϱ the set of all functions$\varrho : [1, \infty)^3 \to [1, \infty)$ satisfying the following conditions:

$(\varrho 1)$ $\max\{\xi_1, \xi_2, \xi_3\} \leq \varrho(\xi_1, \xi_2, \xi_3)$ for all $\xi_1, \xi_2, \xi_3 \in [1, \infty)$,
$(\varrho 2)$ $\varrho(\xi_1, \xi_2, \xi_3) = 1$ if and only if $\xi_1 = \xi_2 = \xi_3 = 1$,
$(\varrho 3)$ ϱ is continuous.

Example 3.5. Essentially, the following functions $\varrho : [1, \infty)^3 \to [1, \infty)$ belong to family ϱ:

(i) $\varrho(\xi_1, \xi_2, \xi_3) = \xi_1.\xi_2.\xi_3$,
(ii) $\varrho(\xi_1, \xi_2, \xi_3) = \max\{\xi_1, \xi_2\} \cdot c$,
(iii) $\varrho(\xi_1, \xi_2, \xi_3) = \xi_1\xi_2$.

Definition 3.6. Let $\varphi : \Xi \to [1, \infty)$, and let $\Gamma, \Upsilon : \Xi \to \Xi$ be two self-mappings on a complete multiplicative metric space Ξ. A point $\sigma \in \Xi$ is called a common multiplicative φ-fixed point of Γ and Υ if

$$\Gamma\sigma = \Upsilon\sigma = \sigma \quad \text{and} \quad \varphi(\sigma) = 1$$

Thus, σ is a common fixed point of both mappings, and it additionally satisfies the condition $\varphi(\sigma) = 1$. Since φ takes values in $[1, \infty)$, this condition means that φ attains its minimum admissible value at σ.

Definition 3.7. Let (Ξ, ζ) be a multiplicative metric space. A mapping $\Gamma : \Xi \to \Xi$ is called a *multiplicative* $\varrho - \varphi$ *contraction* if there exists a real constant $\lambda \in [0, 1)$ such that for all $\sigma_1, \sigma_2 \in \Xi$,

$$\varrho(\zeta(\Gamma(\sigma_1), \Gamma(\sigma_2), \varphi(\Gamma\sigma_1), \varphi(\Gamma\sigma_2)) \leq \varrho(\zeta(\sigma_1, \sigma_2), \varphi(\sigma_1), \varphi(\sigma_2))^{\lambda}. \tag{3.1}$$

Now, based on the definition of multiplicative $\varrho - \varphi$ contraction, we introduce the following Theorem for multiplicative metric spaces.

Theorem 3.8. *Let $\Gamma : \Xi \to \Xi$ be a multiplicative ϱ-φ contraction, defined on a complete MM-Space. Then, $\Gamma : \Xi \to \Xi$ has a unique φ-FP.*

Proof. Consider a point $\sigma_0 \in \Xi$. Now we define a sequence $\{\sigma_n\}$ in Ξ such that

$$\sigma_n = \Gamma(\sigma_{n-1}), \quad n = 1, 2, \ldots$$

As Γ is multiplicative $\varrho - \varphi$ contraction, then for $\sigma = \sigma_n, \mu = \sigma_{n-1}$, we have

(3.2)

$$\begin{aligned} \varrho(\zeta(\sigma_{n+1}, \sigma_n), \varphi(\sigma_{n+1}), \varphi(\sigma_n)) &= \varrho(\zeta(\Gamma\sigma_n, \Gamma\sigma_{n-1}), \varphi(\Gamma\sigma_n), \varphi(\Gamma\sigma_{n-1})) \\ &\leq \varrho(\zeta(\sigma_n, \sigma_{n-1}), \varphi(\sigma_n), \varphi(\sigma_{n-1}))^{\lambda} \\ &\vdots \\ &\leq \varrho(\zeta(\sigma_1, \sigma_0), \varphi(\sigma_1), \varphi(\sigma_0))^{\lambda^n}. \end{aligned}$$

Using $(\varrho 1)$ property, we get

$$\max\{\zeta(\sigma_{n+1}, \sigma_n), \varphi(\sigma_{n+1})\} \leq \varrho(\zeta(\sigma_1, \sigma_0), \varphi(\sigma_1), \varphi(\sigma_0))^{\lambda^n}$$

then

$$\zeta(\sigma_{n+1}, \sigma_n) \leq \varrho(\zeta(\sigma_1, \sigma_0), \varphi(\sigma_1), \varphi(\sigma_0))^{\lambda^n},$$

which gives $\lim_{n\to\infty} \zeta(\sigma_{n+1}, \sigma_n) = 1$ and $\lim_{n\to\infty} \varphi(\sigma_{n+1}) = 1$.

Now to prove that sequence $\{\sigma_n\}$ is multiplicative Cauchy sequence, we take $m, n \in \mathbb{N}^*$ such that $m > n$. Then by property of multiplicative metric and inequality (3.2), we get

$$\begin{aligned} \zeta(\sigma_m, \sigma_n) &\leq \zeta(\sigma_m, \sigma_{m-1}) \cdot \zeta(\sigma_{m-1}, \sigma_{m-2}) \cdots \zeta(\sigma_{n+1}, \sigma_n) \\ &\leq \varrho(\zeta(\sigma_1, \sigma_0), \varphi(\sigma_1))^{\lambda^{m-1+m-2+\cdots+n}} \end{aligned}$$

$$\leq \varrho(\zeta(\sigma_1, \sigma_0), \varphi(\sigma_1))^{\frac{\lambda^n}{1-\lambda}}.$$

This implies that

$$\zeta(\sigma_m, \sigma_n) \to 1 \quad \text{as } m, n \to \infty.$$

Hence, the sequence $\{\sigma_n\}$ is multiplicative Cauchy. Now by the completeness of Ξ, there exists $\varpi \in \Xi$ such that

$$\sigma_n \to \varpi \quad \text{as } n \to \infty. \tag{3.3}$$

Since $\sigma_{n+1} \to \varpi$ as $n \to \infty$ and φ is LSC, we have

$$\varphi(\varpi) \leq \liminf_{n\to\infty} \varphi(\sigma_{n+1}).$$

Moreover, since $\varphi(\sigma_{n+1}) \geq 1$ for every $n \in \mathbb{N}$, it follows that

$$1 \leq \varphi(\varpi) \leq \liminf_{n\to\infty} \varphi(\sigma_{n+1}).$$

Combining this inequality with the previously established upper bound

$$\limsup_{n\to\infty} \varphi(\sigma_{n+1}) \leq 1,$$

we obtain

$$1 \leq \varphi(\varpi) \leq \liminf_{n\to\infty} \varphi(\sigma_{n+1}) \leq \limsup_{n\to\infty} \varphi(\sigma_{n+1}) \leq 1.$$

Consequently,

$$\varphi(\varpi) = 1 \quad \text{and} \quad \lim_{n\to\infty} \varphi(\sigma_{n+1}) = 1. \tag{3.4}$$

Moreover from inequalities (3.3), (3.4) and ($\varrho 2$) property of function ϱ, we obtain

$$\begin{aligned}\zeta(\Gamma\varpi, \varpi) &\leq \zeta(\Gamma\varpi, \Gamma\sigma_n) \cdot \zeta(\Gamma\sigma_n, \varpi) \\ &\leq \varrho(\zeta(\varpi, \sigma_n), \varphi(\varpi), \varphi(\sigma_n))^{\lambda} \cdot \zeta(\sigma_{n+1}, \varpi) \to 1 \quad \text{as } n \to \infty.\end{aligned}$$

Thus,

$$\zeta(\Gamma\varpi, \varpi) = 1,$$

which implies that $\Gamma\varpi = \varpi$. Hence, ϖ is a multiplicative φ-FP of Γ.

Now, if there exists another multiplicative φ-FP, $\varpi' \in \Xi$ of mapping Γ, then

$$\begin{aligned}\zeta(\varpi, \varpi') = \zeta(\Gamma\varpi, \Gamma\varpi') &\leq \varrho(\zeta(\Gamma\varpi, \Gamma\varpi'), \varphi(\Gamma\varpi), \varphi(\Gamma\varpi')) \\ &\leq \varrho(\zeta(\varpi, \varpi'), \varphi(\varpi), \varphi(\varpi'))^{\lambda}.\end{aligned}$$

Therefore

$$\varrho(\zeta(\varpi, \varpi'), 1, 1) \leq \varrho(\zeta(\varpi, \varpi'), 1, 1)^{\lambda}.$$

Hence $\varrho(\zeta(\varpi, \varpi'), 1, 1) = 1$ implies $\zeta(\varpi, \varpi') = 1$. Thus $\varpi = \varpi'$.

This implies that ϖ is the unique multiplicative φ-FP of Γ. □

Theorem 3.9. *Let (Ξ, ζ) be a complete MM-Space and Γ is a continuous self mapping defined on Ξ such that for all $\sigma, \mu \in \Xi$ and $\lambda \in [0,1)$, it satisfies*

$$\varrho(\zeta(\Gamma\sigma, \Gamma\mu), \varphi(\Gamma\sigma), \varphi(\Gamma\mu)) \leq \varrho(\mathfrak{S}(\sigma, \mu), \varphi(\sigma), \varphi(\mu))^{\lambda}, \tag{3.5}$$

where

$$\mathfrak{S}(\sigma, \mu) = \max\left\{\zeta(\sigma, \mu), \frac{\zeta(\sigma, \Gamma\sigma)(1 + \zeta(\mu, \Gamma\mu))}{1 + \zeta(\Gamma\sigma, \Gamma\mu)}\right\},$$

$\varphi : \Xi \to [0, \infty)$ is a LSC function and ϱ is a control function. Then, $\Gamma : \Xi \to \Xi$ has a unique multiplicative φ-FP.

Proof. let the sequence σ_n be defined by $\sigma_n = \Gamma(\sigma_{n-1}) = \Gamma^n(\sigma_0)$ for all $n \in \mathbb{N}^*$. If for any $m \in \mathbb{N}^*$,$\sigma_m = \sigma_{m-1}$, then $\Gamma\sigma_m = \Gamma\sigma_{m-1}$, that is $\sigma_{m+1} = \sigma_m$. Therefore, σ_m is a FP of Γ. Therefore, we can assume that $\sigma_{m+1} \neq \sigma_m$ for all $m \in \mathbb{N}^*$. Hence, we consider that $\zeta(\sigma_n, \sigma_{n+1}) > 0$ for all n$\in \mathbb{N}^*$ and take $\sigma = \sigma_{n-1}, \mu = \sigma_n$ in (3.5), which gives

$$\varrho(\zeta(\Gamma\sigma_{n-1}, \Gamma\sigma_n), \varphi(\Gamma\sigma_{n-1}), \varphi(\Gamma\sigma_n)) \leq \varrho(\mathfrak{S}(\sigma_{n-1}, \sigma_n), \varphi(\sigma_{n-1}), \varphi(\sigma_n))^{\lambda}$$

then

$$\varrho(\zeta(\sigma_n, \sigma_{n+1}), \varphi(\sigma_n), \varphi(\sigma_{n+1})) \leq \varrho(\mathfrak{S}(\sigma_{n-1}, \sigma_n), \varphi(\sigma_{n-1}), \varphi(\sigma_n))^{\lambda},$$

where

$$\begin{aligned}\mathfrak{S}(\sigma_{n-1}, \sigma_n) &= \max\left\{\zeta(\sigma_{n-1}, \sigma_n), \frac{\zeta(\sigma_{n-1}, \sigma_n)(1 + \zeta(\sigma_n, \sigma_{n+1}))}{1 + \zeta(\sigma_n, \sigma_{n+1})}\right\} \\ &= \zeta(\sigma_{n-1}, \sigma_n).\end{aligned}$$

Then, proceeding as in Theorem (3.8), we can easily prove existence of multiplicative φ-FP.

Uniqueness: Let ϖ' and ϖ'' be two multiplicative φ-FP of Γ. Setting $\sigma = \varpi'$ and $\mu = \varpi''$ in equation (3.5), we have

$$\varrho(\zeta(\Gamma\varpi', \Gamma\varpi''), \varphi(\Gamma\varpi'), \varphi(\Gamma\varpi'')) \leq \varrho(\mathfrak{S}(\varpi', \varpi''), \varphi(\varpi'), \varphi(\varpi''))^{\lambda},$$

where

$$\mathfrak{S}(\varpi', \varpi'') = \max\left\{\zeta(\varpi', \varpi''), \frac{\zeta(\varpi', \Gamma\varpi')(1 + \zeta(\varpi'', \Gamma\varpi''))}{1 + \zeta(\Gamma\varpi', \Gamma\varpi'')}\right\}.$$

As $\Gamma\varpi' = \varpi'$ and $\Gamma\varpi'' = \varpi''$, so we get

$$\begin{aligned}\mathfrak{S}(\varpi', \varpi'') &= \max\left\{\zeta(\varpi', \varpi''), \frac{2}{1 + \zeta(\varpi', \varpi'')}\right\} \\ &= \zeta(\varpi', \varpi'').\end{aligned}$$

Hence, we obtain

$$\varrho(\zeta(\Gamma\varpi', \Gamma\varpi''), \varphi(\Gamma\varpi'), \varphi(\Gamma\varpi'')) \leq \varrho(\zeta(\varpi', \varpi''), \varphi(\varpi'), \varphi\varpi''))^{\lambda}.$$

As $\varphi(\varpi') = \varphi(\varpi'') = 1$, thus we have

$$\varrho(\zeta(\varpi', \varpi''), 1, 1) \leq \varrho(\zeta(\varpi', \varpi''), 1, 1)^{\lambda},$$

which implies $\zeta(\varpi', \varpi'') = 1$. Hence $\varpi' = \varpi''$, hence ϖ'' is unique multiplicative φ-FP of Γ. □

Common Multiplicative φ-Fixed Point Result. Here, we extend multiplicative φ-FP result to multiplicative common φ-FP result for two weakly compatible mappings.

Theorem 3.10. *Let $\Upsilon, \Gamma : \Xi \to \Xi$ be two continuous self mappings defined on (Ξ, ζ), a complete MM-Space such that Υ is bijective mapping and for all $\sigma, \mu \in \Xi$ and $\lambda \in [0, 1)$, they satisfy*

$$\varrho(\zeta(\Gamma\sigma, \Gamma\mu), \varphi(\Gamma\sigma), \varphi(\Gamma\mu)) \leq \varrho(\zeta(\Upsilon\sigma, \Upsilon\mu), \varphi(\Upsilon\sigma), \varphi(\Upsilon\mu))^{\lambda} \tag{3.6}$$

where, ϱ is a control function and $\varphi : \Xi \to [0, \infty)$ is a LSC function. If $\Gamma(\Xi) \subset \Upsilon(\Xi)$ and $\Upsilon(\Xi)$ is a complete subspace of Ξ. Then Γ and Υ have a unique common multiplicative φ-FP.

Proof. Let $\sigma_0 \in \Xi$ be an arbitrary point in Ξ. Since $\Gamma(\Xi) \subset \Upsilon(\Xi)$, there exists $\sigma_1 \in \Xi$ such that $\Upsilon\sigma_1 = \Gamma\sigma_0$. In this way, we construct two distinct sequences $\{\Gamma\sigma_n\}$ and $\{\Upsilon\sigma_n\}$ such that $\Upsilon\sigma_{n+1} = \Gamma\sigma_n$ for all $n \in \mathbb{N}^*$. If for some $n \in \mathbb{N}^*$, we have $\Upsilon\sigma_n = \Upsilon\sigma_{n+1}$ that is $\Upsilon\sigma_n = \Gamma\sigma_n$, then σ_n is a CFP of Γ and Υ. So on the contrary, let $\Upsilon\sigma_n \neq \Upsilon\sigma_{n+1}$ $\forall n \in \mathbb{N}^*$. If we take $\sigma = \sigma_{n-1}$ and $\mu = \sigma_n$ in(3.6) for each $n \in \mathbb{N}^*$, we have

$$\begin{aligned}&\varrho(\zeta(\Gamma\sigma_{n-1}, \Gamma\sigma_n), \varphi(\Gamma\sigma_{n-1}), \varphi(\Gamma\sigma_n)\\ &\quad \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\sigma_n), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_n))^{\lambda}\end{aligned}$$

then

$$\begin{aligned}&\varrho(\zeta(\Gamma\sigma_{n-2}, \Gamma\sigma_{n-1}), \varphi(\Gamma\sigma_{n-2}), \varphi(\Gamma\sigma_{n-1}))^{\lambda}\\ &\leq \varrho(\zeta(\Upsilon\sigma_{n-2}, \Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_{n-2}), \varphi(\Upsilon\sigma_{n-1}))^{\lambda^2}\\ &\quad \vdots\\ &\leq \varrho(\zeta(\Upsilon\sigma_0, \Upsilon\sigma_1), \varphi(\Upsilon\sigma_0), \varphi(\Upsilon\sigma_1))^{\lambda^n}.\end{aligned} \tag{3.7}$$

Since $\lambda < 1$, so $\varrho(\zeta(\Upsilon\sigma_0, \Upsilon\sigma_1), \varphi(\Upsilon\sigma_0), \varphi(\Upsilon\sigma_1))^{\lambda^n} = 1$ as $n \to \infty$. Therefore by property $(\varrho 1)$, we get $\lim_{n\to\infty} \zeta(\Upsilon\sigma_n, \Upsilon\sigma_{n+1}) = 1$ and $\lim_{n\to\infty} \varphi(\Upsilon\sigma_n) = 1$. Now our aim is to prove that $\{\Upsilon\sigma_n\}$ is a multiplicative Cauchy sequence. For this, let $\epsilon > 1$ and since

$$\lim_{n\to\infty} \varrho(\zeta(\Upsilon\sigma_0, \Upsilon\sigma_1), \varphi(\Upsilon\sigma_0), \varphi(\Upsilon\sigma_1))^{\lambda^n} = 1,$$

hence there exists $N \in \mathbb{N}^*$ such that

$$\varrho(\zeta(\Upsilon\sigma_0, \Upsilon\sigma_1), \varphi(\Upsilon\sigma_0), \varphi(\Upsilon\sigma_1))^{\lambda^n} < \epsilon, \quad \forall n \geq N.$$

Therefore by (3.7), we have

$$\varrho(\zeta(\Upsilon\sigma_n, \Upsilon\sigma_{n+1}), \varphi(\Upsilon\sigma_n), \varphi(\Upsilon\sigma_{n+1})) < \epsilon, \quad \forall n \geq N. \tag{3.8}$$

Now let $m, n \in \mathbb{N}^*$ and $m > n$. We will prove now that there exists $N_1 \in \mathbb{N}^*$ such that

$$\zeta(\Upsilon\sigma_n, \Upsilon\sigma_m) \leq \epsilon, \quad \forall m \geq n \geq N_1 \in \mathbb{N}^*. \tag{3.9}$$

If $m = n + 1$, then by property $(\varrho 1)$, (3.9) is true by (3.8). Moreover if $\sigma_n = \sigma_m$, then (3.9) is trivially true. therefore without loss of generality, we can take $\sigma_n \neq \sigma_m$. Suppose (3.9) is true for $m = k$ i.e.

$$\lim_{n\to\infty} \zeta(\Upsilon\sigma_n, \Upsilon\sigma_k) = 1$$

Hence, by using (3.6) for $\sigma = \sigma_{n-1}$ and $\mu = \sigma_k$, we get

$$\begin{aligned} &\varrho(\zeta(\Gamma\sigma_{n-1}, \Gamma\sigma_k), \varphi(\Gamma\sigma_{n-1}), \varphi(\Gamma\sigma_k)) \\ &\quad \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\sigma_k), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_k))^{\lambda} \end{aligned} \tag{3.10}$$

then

$$\begin{aligned} &\varrho(\zeta(\Upsilon\sigma_n, \Upsilon\sigma_{k+1}), \varphi(\Upsilon\sigma_n), \varphi(\Gamma\sigma_{k+1})) \\ &\quad \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\sigma_k), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_k))^{\lambda}. \end{aligned} \tag{3.11}$$

Therefore

$$\zeta(\Upsilon\sigma_n, \Upsilon\sigma_{k+1}) \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\sigma_k), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_k))^{\lambda}.$$

Moreover, we have already proved that $\lim_{n\to\infty} \varphi(\Upsilon\sigma_n) = 1$, therefore $\lim_{n\to\infty} \varphi(\Upsilon\sigma_{n-1}) = \lim_{n\to\infty} \varphi(\Upsilon\sigma_k) = 1$, as $m \geq n$ and $k = m - 1$. Hence, (3.10) reduces to

$$\lim_{n\to\infty} \zeta(\Upsilon\sigma_n, \Upsilon\sigma_{k+1}) \leq \lim_{n\to\infty} \varrho(1, 1, 1)^{\lambda} = 1,$$

Since $\{\Upsilon\sigma_n\}$ is a Cauchy sequence and $\Upsilon(\Xi)$ is complete, there exists $\varpi \in \Upsilon(\Xi)$ such that

$$\lim_{n\to\infty} \Upsilon\sigma_n = \varpi.$$

Since Υ is bijective, there exists a unique $z \in \Xi$ such that $\Upsilon z = \varpi$. By the continuity of Υ, we have

$$\lim_{n\to\infty} \Upsilon(\Upsilon\sigma_n) = \Upsilon\varpi.$$

Furthermore by inequality (3.11), we have

$$\varphi(\Upsilon\sigma_n) \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\sigma_k), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\sigma_k))^{\lambda} \tag{3.12}$$

then

$$\lim_{n\to\infty} \varphi(\Upsilon\sigma_n) \leq \lim_{n\to\infty} \varrho(1, 1, 1)^{\lambda} = 1.$$

Since φ is LSC, hence $\varphi(\varpi) \leq \liminf_{n\to\infty} \varphi(\Upsilon\sigma_n)$. then

$$\varphi(\varpi) = 1 \quad (\text{because}\varphi(\varpi) \in [1, \infty)).$$

Now we will show that $\Upsilon\varpi = \Gamma\varpi$. For this, we have

$$\zeta(\Upsilon\varpi, \Gamma\varpi) = \lim_{n\to\infty} \zeta(\Upsilon\sigma_n, \Gamma\varpi) = \lim_{n\to\infty} \zeta(\Gamma\sigma_{n-1}, \Gamma\varpi).$$

Next by setting $\sigma = \sigma_{n-1}$ and $\mu = \varpi$ in (3.6), we get

$$\begin{aligned}&\varrho(\zeta(\Gamma\sigma_{n-1}, \Gamma\varpi), \varphi(\Gamma\sigma_{n-1}), \varphi(\Gamma\varpi))\\ &\quad \leq \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\varpi), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\varpi))^{\lambda},\end{aligned}$$

which imlies

$$\begin{aligned}&\lim_{n\to\infty} \varrho(\zeta(\Gamma\sigma_{n-1}, \Gamma\varpi), \varphi(\Gamma\sigma_{n-1}), \varphi(\Gamma\varpi))\\ &\quad \leq \lim_{n\to\infty} \varrho(\zeta(\Upsilon\sigma_{n-1}, \Upsilon\varpi), \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\varpi))^{\lambda}.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \zeta(\Gamma\sigma_{n-1}, \Gamma\varpi) \leq \varrho\left(\lim_{n\to\infty} \zeta(\Upsilon\sigma_{n-1}, \Upsilon\varpi), \lim_{n\to\infty} \varphi(\Upsilon\sigma_{n-1}), \varphi(\Upsilon\varpi)\right)^{\lambda}.$$

Since $\Upsilon\sigma_n$ converges to ϖ, therefore $\lim_{n\to\infty} \zeta(\Upsilon\sigma_{n-1}, \Upsilon\varpi) = 1$ and using (3.12), we have

$$\lim_{n\to\infty} \zeta(\Gamma\sigma_{n-1}, \Gamma\varpi) \leq \varrho(1, 1, 1)^{\lambda},$$

gives $\zeta(\Upsilon\varpi, \Gamma\varpi) = 1$. Hence $\Upsilon\varpi = \Gamma\varpi = \varpi$.

Therefore, we obtain that ϖ is the multiplicative common φ-FP of mappings Γ and Υ.

Uniqueness: Let ϖ and π be two multiplicative common φ-FP of mappings Γ and Υ. Then by taking $\sigma = \varpi$ and $\mu = \pi$ in (3.6), we have

$$\varrho(\zeta(\Gamma\varpi, \Gamma\pi), \varphi(\Gamma\varpi), \varphi(\Gamma\pi)) \leq \varrho(\zeta(\Upsilon\varpi, \Upsilon\pi), \varphi(\Upsilon\varpi), \varphi(\Upsilon\pi))^{\lambda}$$

then

$$\varrho(\zeta(\Gamma\varpi, \Gamma\pi), 1, 1) \leq \varrho(\zeta(\Upsilon\varpi, \Upsilon\pi), 1, 1)^{\lambda}.$$

Consequently $\varrho(\zeta(\Upsilon\varpi, \Upsilon\pi), 1, 1) \leq \varrho(\zeta(\Upsilon\varpi, \Upsilon\pi), 1, 1)^{\lambda}$. Therefore

$$\zeta(\Upsilon\varpi, \Upsilon\pi) = 1 \quad \Longrightarrow \quad \Upsilon\varpi = \Upsilon\pi. \tag{3.13}$$

Since Υ is a bijective self-map, it is injective. Therefore,

$$\Upsilon\varpi = \Upsilon\pi \quad \Longrightarrow \quad \varpi = \pi.$$

Hence, the multiplicative common φ-fixed point of Γ and Υ is unique. Therefore ϖ is a unique multiplicative common φ-FP of mappings Γ and Υ. □

Corollary 3.11. *Let $\Upsilon, \Gamma : \Xi \to \Xi$ be two continuous self-mappings on a complete multiplicative metric space (Ξ, ζ), where Υ is bijective. Suppose that, for all $\sigma, \mu \in \Xi$ and some $\lambda \in [0, 1)$,*

$$\zeta(\Gamma\sigma, \Gamma\mu) \leq \zeta(\Upsilon\sigma, \Upsilon\mu)^{\lambda}.$$

If $\Gamma(\Xi) \subseteq \Upsilon(\Xi)$ and $\Upsilon(\Xi)$ is a complete subspace of Ξ, then Γ and Υ have a unique common multiplicative fixed point, provided the remaining assumptions of Theorem 3.7 hold.

Example 3.12. Let $\Xi = \{\theta_1, \theta_2, \theta_3\}$ and $\Gamma, \Upsilon : \Xi \to \Xi$ be defined as

$$\Gamma\theta_1 = \theta_1, \qquad \Gamma\theta_2 = \theta_3, \qquad \Gamma\theta_3 = \theta_1$$

and

$$\Upsilon\theta_1 = \theta_1, \qquad \Upsilon\theta_2 = \theta_2, \qquad \Upsilon\theta_3 = \theta_2.$$

Assume that metric ζ is considered as

$$\zeta(\theta_1, \theta_2) = 2, \qquad \zeta(\theta_1, \theta_3) = 3, \qquad \zeta(\theta_2, \theta_3) = 4$$

and $\zeta(\theta_i, \theta_i) = 1$, $\zeta(\theta_i, \theta_j) = \zeta(\theta_j, \theta_i)$ for all $i, j \in \{1, 2, 3\}$. It can be easily verified here that the defined metric is multiplicative.

Now consider $\varrho(\xi_1, \xi_2, \xi_3) = \xi_1.\xi_2.\xi_3$ and $\varphi(\theta_1) = \varphi(\theta_3) = 1, \varphi(\theta_2) = 2$. Since the left–hand side of the inequalities (3.1), (3.5), (3.6) in Theorems (3.8), (3.9) and (3.10) is the same, we verify the corresponding right–hand sides separately under different cases, thereby satisfying all three inequalities.

Case 1: $\sigma = \theta_1, \mu = \theta_2$

$$\begin{aligned}\varrho(\zeta(\Gamma\theta_1, \Gamma\theta_2), \varphi(\Gamma\theta_1), \varphi(\Gamma\theta_2)) &= \zeta(\theta_1, \theta_3) \times \varphi(\theta_1) \times \varphi(\theta_3) \\ &= 3.\end{aligned}$$

Now, for inequality (3.1),

$$\varrho(\zeta(\theta_1, \theta_2), \varphi(\theta_1), \varphi(\theta_2))^{\lambda} = (2 \times 1 \times 2)^{\lambda} = 4^{\lambda}.$$

For inequality (3.5),

$$\begin{aligned}\mathfrak{S}(\theta_1, \theta_2) &= \max\left\{\zeta(\theta_1, \theta_2), \frac{\zeta(\theta_1, \Gamma\theta_1)(1 + \zeta(\theta_2, \Gamma\theta_2))}{1 + \zeta(\Gamma\theta_1, \Gamma\theta_2)}\right\} \\ &= \max\left\{2, \frac{1(1+4)}{1+3}\right\} \\ &= \max\left\{2, \frac{5}{4}\right\} = 2.\end{aligned}$$

Therefore,

$$\varrho(\mathfrak{S}(\theta_1, \theta_2), \varphi(\theta_1), \varphi(\theta_2))^{\lambda} = (2 \times 1 \times 2)^{\lambda} = 4^{\lambda}.$$

Finally, for inequality (3.6),

$$\begin{aligned}\varrho(\zeta(\Upsilon\theta_1,\Upsilon\theta_2),\varphi(\Upsilon\theta_1),\varphi(\Upsilon\theta_2))^\lambda &= \varrho(\zeta(\theta_1,\theta_2),\varphi(\theta_1),\varphi(\theta_2))^\lambda\\ &= (2\times 1\times 2)^\lambda = 4^\lambda.\end{aligned}$$

Hence, all three inequalities are satisfied whenever $3 \le 4^\lambda$, that is,

$$\lambda \ge \frac{\ln 3}{\ln 4}.$$

Since $\lambda \in [0,1)$, the required range is

$$\lambda \in \left[\frac{\ln 3}{\ln 4}, 1\right).$$

Case 2: $\sigma = \theta_1, \mu = \theta_3$

$$\begin{aligned}\varrho(\zeta(\Gamma(\theta_1),\Gamma(\theta_3)),\varphi(\Gamma\theta_1),\varphi(\Gamma\theta_3)) &= \zeta(\theta_1,\theta_1)\times\varphi(\theta_1)\times\varphi(\theta_1)\\ &= 1.\end{aligned}$$

Clearly all the inequalities are satisfied directly in this case.

Case 3: $\sigma = \theta_2, \mu = \theta_3$

$$\begin{aligned}\varrho(\zeta(\theta_2,\Gamma\theta_3),\varphi(\Gamma\theta_2),\varphi(\Gamma\theta_3)) &= \zeta(\theta_2,\theta_1)\times\varphi(\theta_3)\times\varphi(\theta_1)\\ &= 3.\end{aligned}$$

Now, for inequality (3.1),

$$\varrho(\zeta(\theta_2,\theta_3),\varphi(\theta_2),\varphi(\theta_3))^\lambda = (4\times 2\times 1)^\lambda = 8^\lambda.$$

For inequality (3.5),

$$\begin{aligned}\mathfrak{S}(\theta_2,\theta_3) &= \max\left\{\zeta(\theta_2,\theta_3), \frac{\zeta(\theta_2,\Gamma\theta_2)(1+\zeta(\theta_3,\Gamma\theta_3))}{1+\zeta(\Gamma\theta_2,\Gamma\theta_3)}\right\}\\ &= \max\left\{4, \frac{4(1+3)}{1+3}\right\} = 4.\end{aligned}$$

Hence,

$$\varrho(\mathfrak{S}(\theta_2,\theta_3),\varphi(\theta_2),\varphi(\theta_3))^\lambda = (4\times 2\times 1)^\lambda = 8^\lambda.$$

Finally, for inequality (3.6),

$$\begin{aligned}\varrho(\zeta(\Upsilon\theta_2,\Upsilon\theta_3),\varphi(\Upsilon\theta_2),\varphi(\Upsilon\theta_3))^\lambda &= \varrho(\zeta(\theta_2,\theta_2),\varphi(\theta_2),\varphi(\theta_2))^\lambda\\ &= (1\times 2\times 2)^\lambda = 4^\lambda.\end{aligned}$$

Therefore, the required inequalities hold in this case whenever

$$3 \le 8^\lambda, \qquad 3 \le 8^\lambda, \qquad 3 \le 4^\lambda.$$

Thus, the common restriction is

$$\lambda \ge \frac{\ln 3}{\ln 4}.$$

Hence for $\lambda \geq \max\left\{\ln\left(\frac{3}{4}\right), \ln\left(\frac{3}{8}\right)\right\}$, all the three inequalities hold true. Clearly $\sigma = \theta_1$ is the unique multiplicative φ-FP of Γ and unique common φ-FP of Γ and Υ.

Example 3.13. As in the previous example, here also we show that all three theorems hold true by using a single example:
Here we consider, $\Xi = [1, \infty)$ and $\zeta(\sigma, \mu) = \max\left\{\frac{\sigma}{\mu}, \frac{\mu}{\sigma}\right\}$. Assume that $\varphi(\sigma) = \zeta(\sigma, \Gamma\sigma)$ which implies that $\varphi(\Gamma\sigma) = \zeta(\Gamma\sigma, \Gamma(\Gamma\sigma))$ and $\varrho(\xi_1, \xi_2, \xi_3) = \xi_1.\xi_2.\xi_3$.

To verify Theorems (3.8) and (3.9), for all $\sigma \in \Xi$, we define the mapping Γ in the following manner.

$$\Gamma\sigma = \sigma^{\frac{1}{2}}.$$

Then we have

$$\begin{aligned}
&\varrho(\zeta(\Gamma\sigma, \Gamma\mu), \varphi(\Gamma\sigma), \varphi(\Gamma\mu)) \\
&\quad = \varrho\left(\zeta\left(\sigma^{\frac{1}{2}}, \mu^{\frac{1}{2}}\right), \varphi\left(\sigma^{\frac{1}{2}}\right), \varphi\left(\mu^{\frac{1}{2}}\right)\right) \\
&\quad = \max\left\{\sqrt{\frac{\sigma}{\mu}}, \sqrt{\frac{\mu}{\sigma}}\right\} \times \max\left\{\frac{\sigma^{\frac{1}{2}}}{\sigma^{\frac{1}{4}}}, \frac{\sigma^{\frac{1}{4}}}{\sigma^{\frac{1}{2}}}\right\} \times \max\left\{\frac{\mu^{\frac{1}{2}}}{\mu^{\frac{1}{4}}}, \frac{\mu^{\frac{1}{4}}}{\mu^{\frac{1}{2}}}\right\}
\end{aligned}$$

As $\sigma, \mu \in [1, \infty)$, therefore we get

$$\varrho(\zeta(\Gamma\sigma, \Gamma\mu), \varphi(\Gamma\sigma), \varphi(\Gamma\mu)) = (\zeta(\sigma, \mu))^{\frac{1}{2}}\sigma^{\frac{1}{4}}\mu^{\frac{1}{4}}.$$

Now

$$\begin{aligned}
&\varrho(\zeta(\sigma, \mu), \varphi(\sigma), \varphi(\mu))^{\lambda} \\
&\quad = \left(\zeta(\sigma, \mu) \times \max\left\{\frac{\sigma}{\sigma^{\frac{1}{2}}}, \frac{\sigma^{\frac{1}{2}}}{\sigma}\right\} \times \max\left\{\frac{\mu}{\mu^{\frac{1}{2}}}, \frac{\mu^{\frac{1}{2}}}{\mu}\right\}\right)^{\lambda} \\
&\quad = \left(\zeta(\sigma, \mu)\sigma^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{\lambda}.
\end{aligned}$$

Hence for $\lambda \geq \frac{1}{2}$, inequality (3.1) is satisfied.

For theorem (3.9), we have

$$\mathfrak{S}(\sigma, \mu) = \max\left\{\zeta(\sigma, \mu), \frac{\zeta(\sigma, \Gamma\sigma)(1 + \zeta(\mu, \Gamma\mu))}{1 + \zeta(\Gamma\sigma, \Gamma\mu)}\right\} \geq \zeta(\sigma, \mu)$$

then

$$\varrho(\mathfrak{S}(\sigma, \mu), \varphi(\sigma), \varphi(\mu))^{\lambda} \geq \left(\zeta(\sigma, \mu)\sigma^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{\lambda}.$$

Again for $\lambda \geq \frac{1}{2}$, inequality (3.5) is satisfied.

Now, for Theorem (3.10), we consider the mappings Γ and Υ for all $\sigma \in \Xi$ as follows:

$$\Gamma\sigma = \sigma^{\frac{1}{4}} \quad \text{and} \quad \Upsilon(\sigma) = \sigma^{\frac{1}{2}}.$$

Then we have

$$\begin{aligned}
&\varrho(\zeta(\Gamma\sigma,\Gamma\mu),\varphi(\Gamma\sigma),\varphi(\Gamma\mu))\\
&\quad=\varrho\left(\zeta\left(\sigma^{\frac{1}{4}},\mu^{\frac{1}{4}}\right),\varphi\left(\sigma^{\frac{1}{4}}\right),\varphi\left(\mu^{\frac{1}{4}}\right)\right)\\
&\quad=\zeta(\sigma,\mu)^{\frac{1}{4}}\times\max\left\{\frac{\sigma^{\frac{1}{4}}}{\sigma^{\frac{1}{16}}},\frac{\sigma^{\frac{1}{16}}}{\sigma^{\frac{1}{4}}}\right\}\times\max\left\{\frac{\mu^{\frac{1}{4}}}{\mu^{\frac{1}{16}}},\frac{\mu^{\frac{1}{16}}}{\mu^{\frac{1}{4}}}\right\}\\
&\quad=\varrho(\zeta(\Gamma\sigma,\Gamma\mu),\varphi(\Gamma\sigma),\varphi(\Gamma\mu))\\
&\quad=(\zeta(\sigma,\mu))^{\frac{1}{4}}\sigma^{\frac{3}{16}}\mu^{\frac{3}{16}}.
\end{aligned}$$

and

$$\begin{aligned}
&\varrho(\zeta(\Upsilon\sigma,\Upsilon\mu),\varphi(\Upsilon\sigma),\varphi(\Upsilon\mu))^{\lambda}\\
&\quad=\varrho\left(\zeta\left(\sigma^{\frac{1}{2}},\mu^{\frac{1}{2}}\right),\varphi\left(\sigma^{\frac{1}{2}}\right),\varphi\left(\mu^{\frac{1}{2}}\right)\right)^{\lambda}\\
&\quad=\left(\zeta(\sigma,\mu)^{\frac{1}{2}}\times\max\left\{\frac{\sigma^{\frac{1}{2}}}{\sigma^{\frac{1}{4}}},\frac{\sigma^{\frac{1}{4}}}{\sigma^{\frac{1}{2}}}\right\}\times\max\left\{\frac{\mu^{\frac{1}{2}}}{\mu^{\frac{1}{4}}},\frac{\mu^{\frac{1}{4}}}{\mu^{\frac{1}{2}}}\right\}\right)^{\lambda}\\
&\quad=\left((\zeta(\sigma,\mu))^{\frac{1}{2}}\sigma^{\frac{1}{4}}\mu^{\frac{1}{4}}\right)^{\lambda}.
\end{aligned}$$

Hence here as well for $\lambda\geq\frac{1}{2}$, inequality (3.6) is satisfied.

Clearly $\sigma=1$ is the unique multiplicative φ-FP of Γ and unique common φ-FP of Γ and Υ.

4. Application

Preliminaries and Theoretical Background. In this section, we present some fundamental concepts and definitions from fractional calculus which are essential for the analysis of the proposed problem.

Definition 4.1. Let $f\in C([0,\infty),\mathbb{R})$ and $\mathfrak{n}>0$. The Riemann-Liouville fractional integral of order $\mathfrak{n}$ is defined as

$$(I^{\mathfrak{n}}f)(\tau)=\frac{1}{\Gamma(\mathfrak{n})}\int_0^t(\tau-\omega)^{\mathfrak{n}-1}f(\omega)\,d\omega,\quad\tau>0,$$

provided the integral exists.

Definition 4.2. Let $j-1<\mathfrak{n}<j$, where $j\in\mathbb{N}^*$. The Caputo fractional derivative of order $\mathfrak{n}$ of a function $\gamma:[0,1]\times\mathbb{R}\to[0,\infty)$ is given by

$$({}^{C}D^{\mathfrak{n}}\gamma)(\tau)=\frac{1}{\Gamma(j-\mathfrak{n})}\int_0^t(\tau-\omega)^{j-\mathfrak{n}-1}\gamma^{(j)}(\omega)\,d\omega,\quad\tau>0.$$

Fractional derivatives play an important role in modeling systems that exhibit memory and hereditary properties, since they allow the present state of a system to depend on its entire past history rather than only on

local behavior. This nonlocal characteristic makes fractional operators particularly suitable for describing complex dynamics that cannot be accurately captured by classical integer-order derivatives.

Recent studies have further advanced the theoretical and computational aspects of fractional differential equations. Odibat [12] investigated the analytical properties and numerical approximation of an adaptive tempered Caputo fractional derivative, while Ali et al. [1] developed a new iterative process with applications to signal processing and Caputo-type nonlinear fractional differential equations. Furthermore, Ren et al. [15] established positive-solution results for a Caputo fractional differential equation with infinite-point boundary conditions using fixed point techniques. These developments motivate further applications of fixed point theory to nonlinear fractional differential equations.

As a result, fractional derivatives have been extensively applied in various fields. In physics, they are used to model anomalous diffusion and viscoelastic materials, while in control theory, fractional-order systems often provide improved stability and robustness. Additionally, applications in biology, economics, and finance highlight their effectiveness in capturing long-term dependence and persistent memory effects.

Definition 4.3. Let $\Xi = \{\sigma : \sigma \in C([0,1], \mathbb{R})\}$ and define multiplicative metric ζ given by

$$\zeta(\sigma, \mu) = \exp(\sup_{\tau \in [0,1]} |\sigma(\tau) - \mu(\tau)|), \quad \forall \sigma, \mu \in \Xi.$$

Then (Ξ, ζ) is a complete MM-space.

Theorem 4.4. *Consider the nonlinear FDE*

$$^{C}D^{\hbar}\sigma(\tau) = \gamma(\tau, \sigma(\tau)), \quad 0 < \tau < 1,\ 1 < \hbar \leq 2, \tag{4.1}$$

subject to the integral boundary conditions

$$\sigma(0) = 0, \qquad I\sigma(1) = \sigma'(0).$$

Let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function. Let the following conditions be fulfilled:

(i) *The function $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is CTS.*

(ii) *Suppose that mapping γ satisfies*

$$|\gamma(\tau, \sigma) - \gamma(\tau, \mu)| \leq \frac{\Gamma(\hbar + 1)}{8} |\sigma - \mu|,$$

for all $\sigma, \mu \in \Xi$ and for all $\tau \in [0,1]$.

(iii) *Define the operator $\Upsilon : \Xi \to \Xi$ by*

$$(\Upsilon\sigma)(\tau) = \frac{1}{\Gamma(\hbar)} \int_0^t (\tau - \omega)^{\hbar - 1} \gamma(\omega, \sigma(\omega))\, d\omega$$

$$+\frac{2\tau}{\Gamma(\hbar)}\int_0^1\left(\int_0^\omega(\omega-\upsilon)^{\hbar-1}\gamma(\upsilon,\sigma(\upsilon))d\upsilon\right)d\omega,$$

for all $\sigma,\mu\in\Xi$.

(iv) *Assume that* Υ *is bijective. Then* (4.1) *has at least one solution. Then problem admits at least one solution.*

Proof. Now by condition (ii), for $\sigma,\mu\in\Xi$, we obtain

$$\begin{aligned}
&|\Upsilon\sigma(\tau)-\Upsilon\mu(\tau)|\\
&\leq\left|\frac{1}{\Gamma(\hbar)}\int_0^\tau(\tau-\omega)^{\hbar-1}(\gamma(\omega,\sigma(\omega))-\gamma(\omega,\mu(\omega)))d\omega\right|\\
&\quad+\left|\frac{2\tau}{\Gamma(\hbar)}\int_0^1\left(\int_0^\omega(\omega-\upsilon)^{\hbar-1}(\gamma(\upsilon,\sigma(\upsilon))-\gamma(\upsilon,\mu(\upsilon)))d\upsilon\right)d\omega\right|\\
&\leq\frac{1}{\Gamma(\hbar)}\int_0^\tau|(\tau-\omega)^{\hbar-1}||(\gamma(\omega,\sigma(\omega))-\gamma(\omega,\mu(\omega)))|d\omega\\
&\quad+\frac{2\tau}{\Gamma(\hbar)}\int_0^1\left(\int_0^\omega(\omega-\upsilon)^{\hbar-1}|\gamma(\upsilon,\sigma(\upsilon))-\gamma(\upsilon,\mu(\upsilon))|d\upsilon\right)d\omega
\end{aligned}$$

then

$$\begin{aligned}
|\Upsilon\sigma(\tau)-\Upsilon\mu(\tau)|&\leq\frac{\Gamma(\hbar+1)}{8\Gamma(\hbar)}\int_0^\tau(\tau-\omega)^{\hbar-1}|\sigma(\omega)-\mu(\omega)|\,d\omega\\
&\quad+\frac{\tau\Gamma(\hbar+1)}{4\Gamma(\hbar)}\int_0^1\int_0^\omega(\omega-\upsilon)^{\hbar-1}|\sigma(\upsilon)-\mu(\upsilon)|\,d\upsilon\,d\omega\\
&\leq\frac{\Gamma(\hbar+1)}{8\Gamma(\hbar)}\|\sigma-\mu\|_\infty\frac{\tau^\hbar}{\hbar}+\frac{\tau\Gamma(\hbar+1)}{4\Gamma(\hbar)}\|\sigma-\mu\|_\infty\frac{1}{\hbar(\hbar+1)}\\
&=\left(\frac{\tau^\hbar}{8}+\frac{\tau}{4(\hbar+1)}\right)\|\sigma-\mu\|_\infty\\
&\leq\frac{1}{4}\|\sigma-\mu\|_\infty,
\end{aligned}$$

Hence Consequently, we obtain

$$\begin{aligned}
\zeta(\Upsilon\sigma,\Upsilon\mu)&=\exp\left(\sup_{\tau\in[0,1]}|(\Upsilon\sigma)(\tau)-(\Upsilon\mu)(\tau)|\right)\\
&\leq\exp\left(\frac{1}{4}\sup_{\tau\in[0,1]}|\sigma(\tau)-\mu(\tau)|\right)\\
&\leq\exp\left(\frac{3}{8}\sup_{\tau\in[0,1]}|\sigma(\tau)-\mu(\tau)|\right).
\end{aligned}$$

Thus, the required contractive estimate is satisfied. By choosing

$$\varrho(\xi_1, \xi_2, \xi_3) = \xi_1\xi_2\xi_3, \qquad \varphi(\sigma) = 1, \quad \text{for all } \sigma \in \Xi,$$

and taking$\lambda = \frac{3}{8}$, inequality (3.1) is satisfied. Hence, provided that the remaining hypotheses of Theorem (3.8) hold, the operator Υ admits a unique φ-fixed point. Therefore, the given fractional differential problem has a unique solution. □

5. Conclusion

This paper extends the notion of φ-fixed points from classical metric spaces to the setting of multiplicative metric spaces. By introducing a suitable family of functions, we establish existence and uniqueness results for multiplicative φ-fixed points, including results for rational-type contractions and common multiplicative φ-fixed points. Several examples are provided to illustrate the applicability of the obtained results. Furthermore, an application to a fractional differential equation involving the Caputo fractional derivative demonstrates the relevance of the developed fixed point framework to fractional differential equations.

6. Future Scope

The present work can be extended in several directions. Similar results may be developed in other generalized multiplicative settings, such as b-MM-spaces, S-MM-spaces, and orthogonal MM-spaces. Moreover, the introduced family of functions may be applied to multi-valued mappings and to the study of further classes of fractional differential and integral equations.

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